Linearized Kalman Filter for Relative Spacecraft Motion

Martin Kamme*
Stanford University, Palo Alto, California, 94305

Satellite rendezvous can be modeled as a POMDP where the position and velocity of the Chaser satellite relative to the Target satellite are the uncertain states. This report will attempt to develop a Linearized Kalman Filter for relative spacecraft motion that can generate beliefs to be used when approaching satellite rendezvous as a POMDP. The results show that the Linearized Kalman Filter performs well when the model dynamics are accurate and the in plane relative separation is larger than 100m. Once the relative separation is within 100m, the error of the LKF with bad measurements places the Chaser satellite dangerously close to the Target satellite.

I. Nomenclature

C	=	Chaser satellite reference frame
$\mathbf{H}$	=	partial derivative of observations with respect to the state
$\mathbf{I}$	=	identity matrix
$\mathbf{K}_{k+1}$	=	Kalman gain
LKF	=	Linearized Kalman Filter
O	=	observation
$\hat{\mathbf{P}}_\mathbf{0}$	=	best initial estimate of the covariance matrix
$\bar{\mathbf{P}}_\mathbf{k}$	=	current estimate of error covariance matrix
$\bar{\mathbf{P}}_\mathbf{k+1}$	=	predicted error covariance matrix
$\hat{\mathbf{P}}_\mathbf{k+1}$	=	estimated error covariance matrix
$\mathbf{Q}$	=	process noise matrix
$\mathbf{R}$	=	measurement noise matrix
RSW	=	satellite local coordinate system
$\mathcal{T}$	=	Target satellite reference frame
$\mathcal{U}$	=	noise statistics
$\hat{\mathbf{X}}_\mathbf{0}$	=	best initial estimate of the state
$\hat{\mathbf{X}}_\mathbf{k}$	=	current estimate of the state
$\bar{\mathbf{X}}_\mathbf{k+1}$	=	predicted state
$\hat{\mathbf{X}}_\mathbf{k+1}$	=	estimate of the state
x	=	position in x of chaser satellite relative to target frame
$\dot{x}$	=	velocity in x of chaser satellite relative to target frame
y	=	position in y of chaser satellite relative to target frame
$\dot{y}$	=	velocity in y of chaser satellite relative to target frame
$\mathbf{z}$	=	observations
$\boldsymbol{\phi}$	=	state transition matrix
ω	=	angular rate of target satellite

II. Introduction

INTEREST in autonomous satellite rendezvous began with the Russian Cosmos spacecraft pair in the 60s [1] and is undergoing exponential growth with the emergence of the satellite servicing, assembly, and disposal industries. Autonomous satellite rendezvous is a notoriously difficult task that has been executed successfully by a select set of actors [2]. A significant amount of the difficulty can be attributed to the vast amount of sources of uncertainty. Relevant

*Masters Student, Department of Aeronautics and Astronautics to the discussion of POMDPs is the state uncertainty; explicitly the uncertainty of the relative position, velocity, attitude, and angular rate of the Chaser satellite relative to the Target [3].

A Partially Observable Markov Decision Process is a Markov Decision Process with state uncertainty and there are multiple ways to approach the state uncertainty of the problem [4]. This report will develop a Linearized Kalman Filter that will generate beliefs over range observations. Before setting up the filter, it is necessary to review relative satellite motion and the Linearized Clohessy Wilthsire equations. The figure below presents an illustration of the RSW coordinate system as well as the Target and Chaser reference frames. The *RSW* coordinate system defines *R* as parallel to the position vector, *S* in the direction of velocity, and *W* normal to the orbit plane [5]. *All variables in this report will be expressed relative to the Target frame in RSW coordinates*.

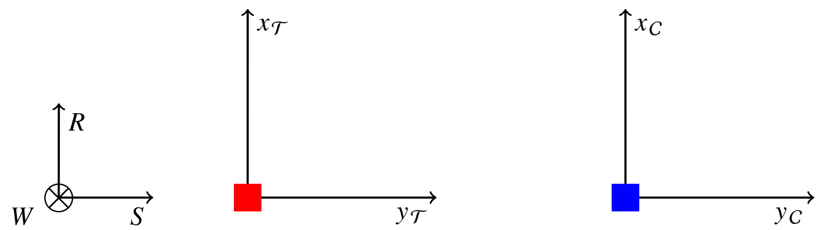


Fig. 1 System of Target Satellite T and Chaser Satellite C

The Target satellite is a non cooperative agent and thus the chaser will be performing the actions. This report will only develop the LKF for generating observations and will not explore the action space. However, it is necessary to perform a single action to be able to analyze the performance of the LKF as the Chaser coasts through multiple relative states. The action that will be considered in this report is a burn parallel to the velocity vector. If the satellites are orbiting close to one another in the same circular orbit with Chaser satellite slightly ahead of the Target satellite, a Chaser burn in the velocity vector will result in a spacecraft "hop". When the Chaser satellite intersects the orbital path of the target satellite, it will perform the exact same burn but anti-parallel to its velocity vector, placing the Chaser again in the same orbit as the Target satellite but at a smaller relative distance than before the initial burn. In the frame of the target satellite, this will look something like what is presented in 2, hence the term "hop". The chaser satellite will "hop" from its initial relative position (shown in blue) to the final relative position (shown in green).

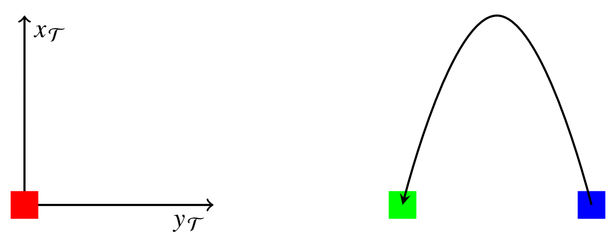


Fig. 2 Chaser Hop Maneuver

This report will only consider the rendezvous phase of proximity operations. This covers relative separation of 10km all the way up to 100m. After 100m, the docking phase begins which is arguably an entirely different problem [3]. To further narrow the scope of the problem, this report will only evaluate the LKF over a single hop maneuver of the chaser from 10 km to 5km. The coast time and thus the deltaV of the hop maneuver will also be fixed.

The equations that describe the relative motion of close orbiting satellites in circular orbits are known as the Clohessy-Wilshire equations [5] and are presented below.

$$\ddot{x} - 2\omega\dot{y} - 3\omega^2 x = f_x \tag{1}$$

$$\ddot{y} + 2\omega\dot{x} = f_y \tag{2}$$

$$\ddot{z} + \omega^2 z = f_z \tag{3}$$

This report will only explore maneuvers in the orbital plane; and since motion in the z direction is decoupled from x and y, z will be dropped.

III. Problem Formulation

Developing the Linearized Kalman filter requires that the system has linear-Gaussian dynamics and observations [4]. Fortunately, the close orbiting and circular orbit assumptions of the Clohessy Wilthsire approach results in linearized equations of motion. The additional following assumptions are necessary to develop and analyze the LKF:

1) No orbital perturbations such as drag, earth aspherocity, solar radiation pressure etc
2) DeltaV from thruster is instantaneous
3) Target satellite only makes range measurements of the Chaser satellite
4) The range measurements can be broken down into x and y measurements

As shown by Vallado in [5], the process for Linearized Kalman Filtering is made up of a prediction and an update.

Prediction

$$\mathbf{\bar{X}_{k+1}} = \boldsymbol{\phi}\mathbf{\hat{X}_k}$$
$$\mathbf{\bar{P}_{k+1}} = \boldsymbol{\phi}\mathbf{\hat{P}_k}\boldsymbol{\phi}^T + \mathbf{Q}$$

Update

$$\mathbf{K}_{k+1} = \mathbf{\bar{P}_{k+1}}\mathbf{H}_{k+1}^T[\mathbf{H}_{k+1}\mathbf{\bar{P}_{k+1}}\mathbf{H}_{k+1}^T + \mathbf{R}]^{-1}$$
$$\mathbf{\hat{X}_{k+1}} = \mathbf{\bar{X}_{k+1}} + \mathbf{K}_{k+1}[\mathbf{z} - \mathbf{H}_{k+1}\mathbf{\bar{X}_{k+1}}]$$
$$\mathbf{\hat{P}_{k+1}} = [\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1}]\mathbf{\bar{P}_{k+1}}$$

The initial state matrix, initial covariance, process noise matrix, and measurement noise matrix are set as

$$\mathbf{\hat{X}}_0 = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} \mathbf{\hat{P}}_0 = \begin{bmatrix} \sigma_x^2 & 0 & 0 & 0 \\ 0 & \sigma_y^2 & 0 & 0 \\ 0 & 0 & \sigma_{\dot{x}}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\dot{y}}^2 \end{bmatrix} \mathbf{Q} = \begin{bmatrix} \mathcal{U}_x^2 & 0 & 0 & 0 \\ 0 & \mathcal{U}_y^2 & 0 & 0 \\ 0 & 0 & \mathcal{U}_{\dot{x}}^2 & 0 \\ 0 & 0 & 0 & \mathcal{U}_{\dot{y}}^2 \end{bmatrix} \mathbf{R} = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \tag{4}$$

σ can be interpreted as the quality of the range measurements and $\mathcal{U}$ can be interpreted as the uncertainty related to the dynamics model. These two quantities, along with the number of measurements and the initial conditions are what will drive the performance of the LKF. The observation matrix **z** and the partial derivative matrix of the observations **H** are straight forward because the measurements are only range measurements.

$$\mathbf{z} = \begin{bmatrix} O_x \\ O_y \end{bmatrix} \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{5}$$

The remaining matrix ϕ is the state transition matrix and requires a solution to the Clohessy Wiltshire equations. Vallado [5] presents a nice solution assuming that the Chaser satellite imparts no external forces.

$$x(t) = \frac{\dot{x}_0}{\omega}SIN(\omega t) - (3x_0 + \frac{2\dot{y}_0}{\omega})COS(\omega t) + (4x_0 + \frac{y_0}{\omega}) \tag{6}$$
$$y(t) = (6x_0 + \frac{4\dot{y}_0}{\omega})SIN(\omega t) + \frac{2\dot{x}_0}{\omega}COS(\omega t) - (6\omega x_0 + 3\dot{y}_0)t + (y_0 - \frac{2\dot{x}_0}{\omega}) \tag{7}$$
$$\dot{x}(t) = \dot{x}_0 COS(\omega t) + (3\omega x_0 + 2\dot{y}_0)SIN(\omega t) \tag{8}$$
$$\dot{y}(t) = (6\omega x_0 + 4\dot{y}_0)COS(\omega t) - 2\dot{x}_0 SIN(\omega t) - (6\omega x_0 + 3\dot{y}_0) \tag{9}$$

These can be formulated nicely into the state transition matrix.

$$\phi = \begin{bmatrix} 4 - 3COS(\omega t) & 0 & \frac{1}{\omega}SIN(\omega t) & \frac{-2}{\omega}COS(\omega t) + \frac{2}{\omega} \\ 6SIN(\omega t) - 6\omega t & 1 & \frac{2}{\omega}COS(\omega t) - \frac{2}{\omega} & \frac{4}{\omega}SIN(\omega t) - 3t \\ 3\omega SIN(\omega t) & 0 & COS(\omega t) & 2SIN(\omega t) \\ 6\omega COS(\omega t) - 6\omega & 0 & -2SIN(\omega t) & 4COS(\omega t) - 3 \end{bmatrix} \tag{10}$$

With all the matrices set up, the analysis steps can be formulated:

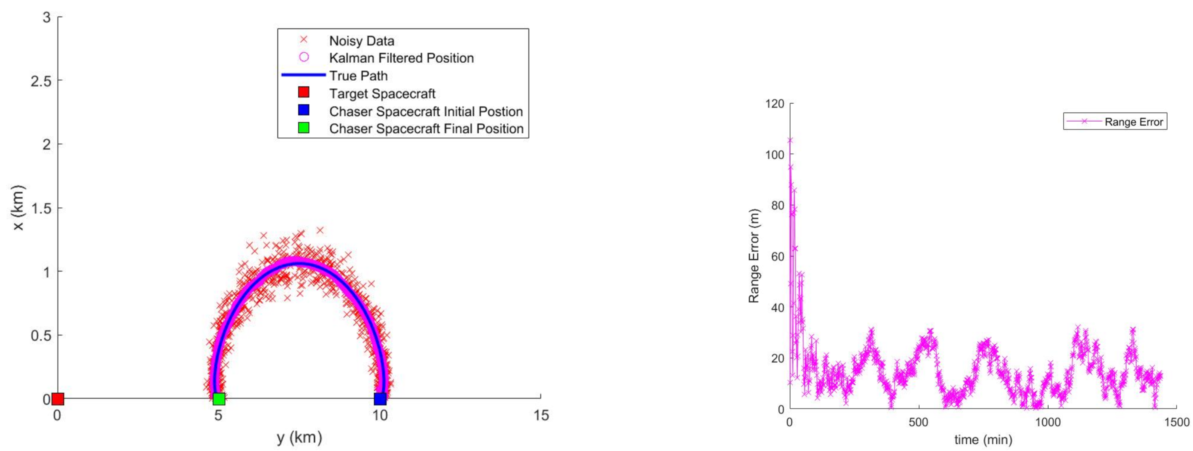
1) initialize relative position to [0 , 10] and velocity to [0 , 0]
2) with desired next separation and coast time, perform thruster burn
3) simulate the true trajectory with a Clohessy Wiltshire propagator

4) apply random Gaussian noise to the true trajectory to generate simulated noisy observations
5) propagate the trajectory again but now with the LKF and the noisy observations
6) compare the estimated states from the LKF to those from the ideal trajectory
7) repeat with different combinations of measurement quality σ and dynamics model quality $\mathcal{U}$

IV. Results

A. Bad Measurements, Good Dynamics

$\sigma_x = \sigma_y = .1km$ and $\mathcal{U}_x = \mathcal{U}_y = .001km$ and $\mathcal{U}_{\dot{x}} = \mathcal{U}_{\dot{y}} = 1e^{-7}km/s$



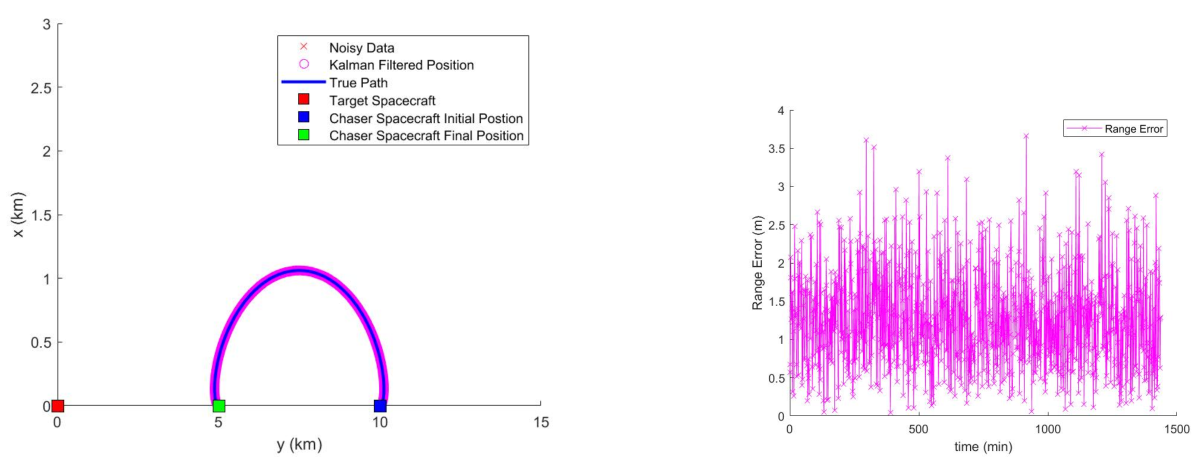
(a) Relative Motion (b) Range Error

Fig. 3 Bad Measurements and Good Dynamics

3a and 3b demonstrate where the Kalman Filter preforms the best. The noisy measurements are off by hundreds of meters but with a good dynamics model the LKF is able to bring down the range error down to around 20m.

B. Good Measurements, Bad Dynamics

$\sigma_x = \sigma_y = .001km$ and $\mathcal{U}_x = \mathcal{U}_y = .1km$ and $\mathcal{U}_{\dot{x}} = \mathcal{U}_{\dot{y}} = 1e^{-4}km/s$



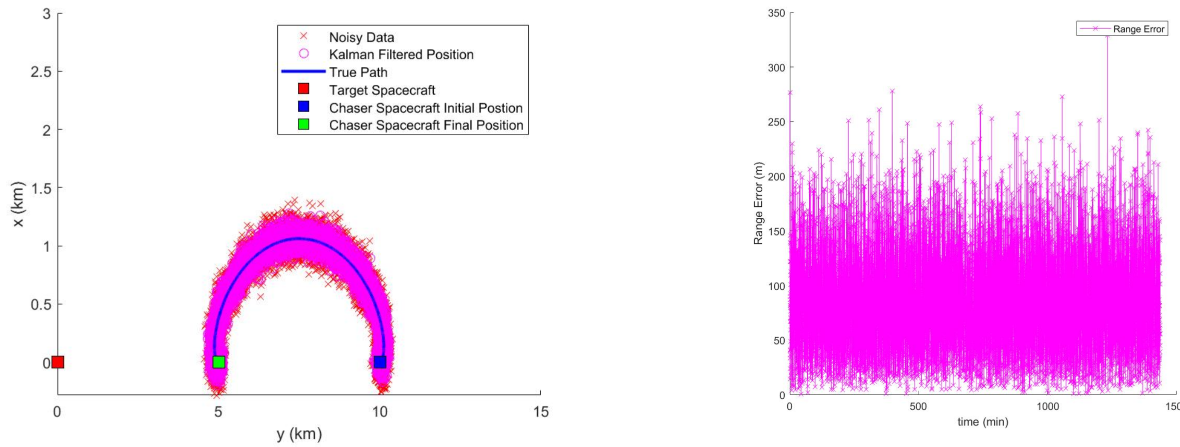
(a) Relative Motion (b) Range Error

Fig. 4 Good Measurements and Bad Dynamics

The results of the LKF with good measurements and bad dynamics are rather uninteresting. Upon inspection, it can be seen the the LKF places the state estimate exactly at the observed location because the measurements are weighted so much more over the dynamics model. This result also explains why no simulation was run with good measurements and good dynamics: the results would be identical except for a slightly lower range error.

C. Bad Measurements, Bad Dynamics

$\sigma_x = \sigma_y = .1km$ and $\mathcal{U}_x = \mathcal{U}_y = .1km$ and $\mathcal{U}_{\dot{x}} = \mathcal{U}_{\dot{y}} = 1e^{-4}km/s$



(a) Relative Motion

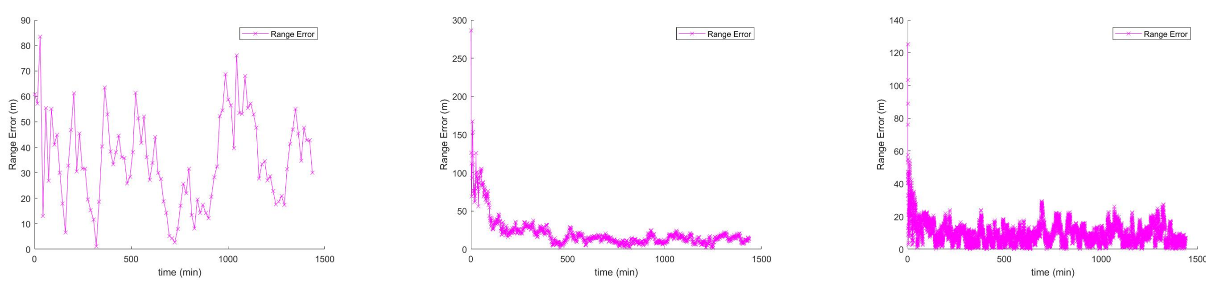
(b) Range Error

Fig. 5 Bad Measurements and Bad Dynamics

5a and 5b show where the LKF breaks down. With bad measurements and bad dynamics, the LKF is unable to reliably estimate the state and cannot reduce the range error.

D. Observations

Run the Bad Measurements and Good Dynamics again with 100, 1000, and 10000 observations.



(a) 100

(b) 1000

(c) 10000

Fig. 6 Range Error for Different Number of Observations

6a, 6b, and 6c show that the number of observations has an effect on the convergence on the LKF but only up to a certain point. 1000 observations is definitely better than 100 observations but 10000 does not necessarily give additional information of convergence over 1000 observations.

V. Conclusion

The goal of this report was to develop a Linearized Kalman Filter that could generate beliefs for use in a satellite rendezvous POMDP. The LKF developed here is a recommendable tool for working with relative in plane satellite separations of larger than 100m, thus it is useful for the rendezvous phase of proximity operations [3]. When the filter is given bad measurements, it is able to significantly reduce the range error if the model dynamics are accurate. The filter also preforms better with more observations but only up to a certain point. The filter does not work well when very noisy measurements are combined with a very uncertain dynamics model. Moving forward, the LKF can be expanded to include out of plane motion and velocity measurements as well. For larger separations or even elliptical orbits, an entirely new approach would need to be developed.

References

[1] NASA, "An Assessment of the Technology of Automated Rendezvous and Capture in Space," 1998.

[2] LoCrasto, H., and Axelrad, P., "Framework for autonomous rendezvous and docking in low earth orbit," *FAA Center of Excellence for Commerical Space Transportation Research*, 2013.

[3] Jewison, C., "Guidance and Control for Multi-Stage Rendezvous and Docking Operations, in the Presence of Uncertainty," , 2017.

[4] Kochenderfer, M., Wheeler, T., and Wray, K., *Algorithms for Decision Making*, The MIT Press, 2020.

[5] Vallado, D., and McClain, W., *Fundamentals of Astrodynamics and Applications*, 4^{th} ed., Microcosm Press, Hawthorne, 2013, Chap. 10.