AA278A Homework 2: Stability of Hybrid Systems
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Assigned May 3; Due May 19

If you make use of results from lecture notes or elsewhere, please state the result and the reference.

Problem 1: When Globally Quadratic Lyapunov Theory Fails.

Consider the linear hybrid system example from Lecture 6:

- \( Q = \{ q_1, q_2 \}, X = \mathbb{R}^2 \)
- \( \text{Init} = Q \times \{ x \in X : ||x|| > 0 \} \)
- \( f(q_1, x) = A_1 x \) and \( f(q_2, x) = A_2 x \), with:
  \[
  A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}
  \]
- \( \text{Dom} = \{ q_1, \{ x \in \mathbb{R}^2 : x_1x_2 \geq 0 \} \} \cup \{ q_2, \{ x \in \mathbb{R}^2 : x_1x_2 \leq 0 \} \} \)
- \( R(q_1, \{ x \in \mathbb{R}^2 : x_1x_2 \leq 0 \}) = (q_2, x) \) and \( R(q_2, \{ x \in \mathbb{R}^2 : x_1x_2 \geq 0 \}) = (q_1, x) \)

Simulation indicates the equilibrium \( x_e = 0 \) to be stable; yet there is no solution \( P \) to the LMI conditions:

\[
\frac{P = P^T > 0}{A_i^T P + P A_i < 0}
\]

for \( i = 1, 2 \). As we saw in class, this makes sense, since when the system matrices \( A_1 \) and \( A_2 \) are interchanged, \( x_e = 0 \) is unstable.

Using the piecewise quadratic Lyapunov function theorem (Theorem 9) of Lecture 6, prove that \( x_e = 0 \) of the hybrid system described above is asymptotically stable. HINT 1: The same Lyapunov function actually works across both discrete states. HINT 2: This problem may be done very quickly using MATLAB’s LMI toolbox (instructions at the end of Lecture 6).

Problem 2: Stabilizing unstable systems through switching.

Give an example of a hybrid automaton that has unstable dynamics in each discrete mode, but for which the equilibrium \( x_e = 0 \) is stable. Illustrate your example through Matlab simulation. Prove, using one of the stability theorems from class, that your example is stable.
Problem 3.
Consider the following switching system

\[ \dot{x} = A_q x \]

where \( q \in \{1, 2\} \) and

\[
A_1 = \begin{bmatrix}
-a_1 & b_1 \\
0 & -c_1
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
-a_2 & b_2 \\
0 & -c_2
\end{bmatrix}
\]

Assume that \( a_i, b_i, \) and \( c_i, \) for \( i = 1, 2 \) are real numbers and that \( a_i, c_i > 0. \) Show that the switched system is asymptotically stable.

Problem 4: Common Lyapunov Function for Commuting A-matrices. Assume that \( K > 1 \) matrices \( A_q, q \in \{1, \ldots, K\} \) are given. Consider the switched linear system \( \dot{x} = A_q x \) where \( \sigma : \mathbb{R}^n \to Q \) such that for all \( (q, x) \in Q \times \mathbb{R}^n, f(q, x) = A_q x. \) Assume that the matrices \( A_q \) are stable (ie. with eigenvalues in the open left half of the complex plane).

Also, assume that, for all \( i, j \in \{1, \ldots, K\}, A_i A_j = A_j A_i. \) Now, let \( P_1, \ldots, P_m \) be the unique symmetric positive definite matrices that satisfy the Lyapunov equations:

\[
A_i^T P_1 + P_1 A_i = -I \tag{3}
\]

\[
A_i^T P_i + P_i A_i = -P_{i-1}, \quad i = 2, \ldots, m \tag{4}
\]

Derive an explicit integral formula for \( P_m \) which only depends on the \( A_i, \) \( i = 1, \ldots, m. \) Then show that the function \( V(q, x) = x^T P_m x \) is a common Lyapunov function for the systems \( \dot{x} = A_i x, \) \( i = 1, \ldots, m. \)