Problem One: Order Relations

i. What three properties does a binary relation have to have to be a partial order?

ii. Consider the power set \( P(A) \) of any set \( A \). Prove that \( \subseteq \) is a partial order over \( P(A) \).

iii. Let \( \mid \) be the divisibility relation. We write \( x \mid y \) if \( x \) divides \( y \); that is, there exists some integer \( k \in \mathbb{Z} \) such that \( y = xk \). Prove that \( \mid \) is a partial order over \( \mathbb{N} \), but that it is not a partial order over \( \mathbb{Z} \).

Problem Two: Combining Relations

Suppose that \( (A, \leq_A) \) and \( (B, \leq_B) \) are ordered sets such that \( \leq_A \) is a total order and \( \leq_B \) is a total order. Consider the set \( A \times B \). Define a relationship \( \leq_{A \times B} \) on \( A \times B \) such that \( (a_1, b_1) \leq_{A \times B} (a_2, b_2) \) iff at least one of \( a_1 \leq_A a_2 \) and \( b_1 \leq_B b_2 \) is true.

i. Is \( \leq_{A \times B} \) reflexive? If so, prove it. If not, give a counterexample.

ii. Is \( \leq_{A \times B} \) antisymmetric? If so, prove it. If not, give a counterexample.

iii. Is \( \leq_{A \times B} \) transitive? If so, prove it. If not, give a counterexample.

iv. Is \( \leq_{A \times B} \) total? If so, prove it. If not, give a counterexample.

v. Based on your results from (i), (ii), (iii), and (iv), is \( \leq_{A \times B} \) a total order?

Problem Three: Functions and Cardinality

Prove that for any sets \( A \) and \( B \), \( |A \times B| = |B \times A| \).

Problem Four: The Pigeonhole Principle

Prove that if 501 distinct natural numbers in the range 0 to 999 are chosen, there must some pair whose sum is exactly 999.