Graphs and Relations
Friday Four Square!
Today at 4:15PM outside Gates.
Announcements

• Problem Set 1 due right now.
• Problem Set 2 out.
  • Checkpoint due Monday, April 16.
  • Assignment due Friday, April 20.
  • Play around with induction and its applications!
Mathematical Structures

• Just as there are common data structures in programming, there are common mathematical structures in discrete math.

• So far, we've seen simple structures like sets and natural numbers, but there are many other important structures out there.

• For the next week, we'll explore several of them.
Some Formalisms
Ordered and Unordered Pairs

- An **unordered pair** is a set \{a, b\} of two elements (remember that sets are unordered).
  - \{0, 1\} = \{1, 0\}

- An **ordered pair** \((a, b)\) is a pair of elements in a specific order.
  - \((0, 1) \neq (1, 0)\).
  - Two ordered pairs are equal iff each of their components are equal.

- An **unordered tuple** is a set \{a_0, a_1, \ldots, a_{n-1}\} of n elements.

- An **ordered tuple** \((a_0, a_1, \ldots, a_{n-1})\) is a collection of n elements in a specific order.
  - This is sometimes called a **sequence**.
  - As with ordered pairs, two ordered tuples are equal iff each of their elements are equal.
The Cartesian Product

- Recall: The **power set** \( \mathcal{P}(S) \) of a set is the set of all its subsets.
- The **Cartesian Product** of \( A \times B \) of two sets is defined as

\[
A \times B \equiv \{ (a, b) \mid a \in A \text{ and } b \in B \}
\]
The Cartesian Product

• Recall: The **power set** $\mathcal{P}(S)$ of a set is the set of all its subsets.

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$$A \times B \equiv \{ (a, b) | a \in A \text{ and } b \in B \}$$

$\{ 0, 1, 2 \}$  $\{ a, b, c \}$

A  B
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$$A \times B \equiv \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

$$\{0, 1, 2\} \times \{a, b, c\} = \{ (0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}$$
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  \]

\[
\begin{array}{c}
\{0, 1, 2\} \\
A
\end{array} \times \begin{array}{c}
\{a, b, c\} \\
B
\end{array} = \begin{array}{ccc}
a & b & c \\
0 & 1 & 2
\end{array}
\]
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\[
\begin{align*}
\{0, 1, 2\} & \times \{a, b, c\} = \\
\begin{array}{c|ccc}
\hline
& a & b & c \\
\hline
0 & (0, a) & (0, b) & (0, c) \\
1 & (1, a) & (1, b) & (1, c) \\
2 & (2, a) & (2, b) & (2, c) \\
\hline
\end{array}
\end{align*}
\]
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● We denote $A^k \equiv A \times A \times \ldots \times A$

$$\{0, 1, 2\} \times \{a, b, c\} = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
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- We denote $A^k \equiv A \times A \times \ldots \times A$ $k$ times

\[
\begin{align*}
\{0, 1, 2\} \times \{0, 1, 2\} &= \{(0, 0), (0, 1), (0, 2), \\
&(1, 0), (1, 1), (1, 2), \\
&(2, 0), (2, 1), (2, 2)\}
\end{align*}
\]
The Cartesian Product

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• The **Cartesian Product** of $A \times B$ of two sets is defined as

  $$A \times B \equiv \{ (a, b) | a \in A \text{ and } b \in B \}$$

• We denote $A^k \equiv A \times A \times \ldots \times A$

  $\begin{align*}
  \{0, 1, 2\}^2 &= \{(0, 0), (0, 1), (0, 2), \\
  &\quad (1, 0), (1, 1), (1, 2), \\
  &\quad (2, 0), (2, 1), (2, 2)\}
  \end{align*}$
Graphs
A graph is a mathematical structure for representing relationships.
Each graph is a set of *vertices* (or *nodes*) connected by *edges* (or *arcs*).
Start with x and y

y = 0?

Answer is x

x = y
y = x \% y
Formalisms

• A **graph** is an ordered pair $G = (V, E)$ where
  
  • $V$ is a set of the **vertices** (nodes) of the graph.
  • $E$ is a set of the **edges** (arcs) of the graph.

• Each edge is an pair of the **start** and **end** (or **source** and **sink**) of the edge.
Directed and Undirected Graphs

- A graph is **directed** if its edges are ordered pairs.
- A graph is **undirected** if the edges are unordered pairs.
- An undirected graph is a special case of a directed graph (just add edges both ways).
Navigating a Graph
Navigating a Graph

A → B → C → E → F → D
Navigating a Graph

A B D F
A path from $v_0$ to $v_n$ is a sequence of edges $((v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n))$.

The length of a path is the number of edges it contains.
Navigating a Graph
A node v is **reachable** from node u if there is a path from u to v.
Navigating a Graph
Navigating a Graph

A

B

C

D

E

F
Navigating a Graph

A

B

C

D

E

F

B D B
Navigating a Graph

B → D 
D → B
B → E
E → C
C → A
A → F
F → D
D → B
B → D

B D B D B
Navigating a Graph

A → B
B → D
D → F
F → E
E → C
C → A
Navigating a Graph
Navigating a Graph

A → B → D → B → D → F

A → F → E → C → B → D → B → D → F
A cycle in a graph is a path

$\left((v_0, v_1), (v_1, v_2), \ldots, (v_n, v_0)\right)$

that starts and ends at the same node.
A **simple path** is a path that does not contain a cycle.

A **simple cycle** is a cycle that does not contain a smaller cycle.
Properties of Nodes

A → B → D → F → C → E
The **indegree** of a node is the number of edges entering that node.

The **outdegree** of a node is the number of edges leaving that node.

In an undirected graph, these are the same and are called the **degree** of the node.
Summary of Terminology

- A **path** is a series of edges connecting two nodes.
  - The **length** of a path is the number of edges in the path.
  - A node v is **reachable** from u if there is a path from u to v.
- A **cycle** is a path from a node to itself.
- A **simple path** is a path without a cycle.
- A **simple cycle** is a cycle that does not contain a nested cycle.
- The **indegree** and **outdegree** of a node are the number of edges entering/leaving it.
A directed acyclic graph (DAG) is a directed graph with no cycles.
Examples of DAGs
Examples of DAGs

- Indian
- Mediterranean
- Mexican
- Chinese
- Italian
- American
- Dorm
Examples of DAGs
Examples of DAGs
Traversing a DAG

- Graph
  - Path
    - Path Length
    - Simple Path
  - Cycle
    - Simple Cycle
  - Reachability
- Degree
Traversing a DAG

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Traversing a DAG
Traversing a DAG

Path

Simple Path

Reachability

Simple Cycle

Degree

Graph
Traversing a DAG
Traversing a DAG

Diagram showing the relationships between different concepts related to traversing a DAG, including:

- Path
- Simple Path
- Reachability
- Simple Cycle
- Degree

Other concepts mentioned:

- Graph
- Cycle
Traversing a DAG

Path
- Path Length
- Simple Path
- Reachability

Degree

Graph

Cycle

Simple Cycle
Traversing a DAG

Path

Path Length
Simple Path
Reachability

Graph
Cycle
Simple Cycle
Degree
Traversing a DAG

- Graph
- Cycle
- Simple Cycle
- Degree
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- Path Length
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Traversing a DAG

- Graph
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Traversing a DAG
Traversing a DAG

Reachability

Graph
Cycle
Simple Cycle
Degree
Path
Path Length
Simple Path
Traversing a DAG
Traversing a DAG

Graph

Path
  - Path Length
  - Simple Path
  - Reachability

Cycle
  - Simple Cycle

Degree

- Graph
- Cycle
- Simple Cycle
- Degree
- Path
- Path Length
- Simple Path
- Reachability
Topological Sort

- Order the nodes of a DAG so no node is picked before its predecessors.

- Algorithm:
  - Find a node with no outgoing edges (outdegree 0)
  - Remove it from the graph.
  - Add it to the resulting ordering.

- There may be many valid orderings:
Topological Sort

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  1 2 3
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```
1 2 3
1 2 3
1 3 2
```
Topological Sort

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Algorithm:
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  1 2 3
  1
  2 3
  1
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- There may be many valid orderings:
Relations
Relations

• A **binary relation** is a property that describes whether two objects are related in some way.

• Examples:
  • Less-than: \( x < y \)
  • Divisibility: \( x \) divides \( y \) evenly
  • Friendship: \( x \) is a friend of \( y \)
  • Tastiness: \( x \) is tastier than \( y \)

• If we have a binary relation \( R \), we write \( aRb \) if \( a \) is related to \( b \).
  • \( a = b \)
  • \( a < b \)
  • \( a \) “is tastier than” \( b \)
Relations as Sets

- Formally, a relation is a set of ordered pairs representing the pairs for which the relation is true.
  - Equality: \{ (0, 0), (1, 1), (2, 2), \ldots \}  
  - Less-than: \{ (0, 1), (0, 2), \ldots, (1, 2), (1, 3), \ldots \}  
- Formally, we have that \( aRb \equiv (a, b) \in R \)
- The binary relations we'll discuss today will be binary relations over a set \( A \).
  - Each relation is a subset of \( A^2 \).
Binary Relations and Graphs

• Each (directed) graph defines a binary relation:
  • $aRb \text{ iff } (a, b) \text{ is an edge.}$

• Each binary relation defines a graph:
  • $(a, b) \text{ is an edge iff } aRb.$

• Example: Less-than
An Important Question

- Why study binary relations and graphs separately?

  - **Simplicity:**
    - Certain operations feel more “natural” on binary relations than on graphs and vice-versa.
    - Converting a relation to a graph might result in an overly complex graph.

  - **Terminology:**
    - Vocabulary for graphs often different from that for relations.
Equivalence Relations
“x and y have the same color”

“x = y”

“x and y have the same shape”

“x and y have the same area”

“x and y are programs that produce the same output”
Informally

An **equivalence relation** is a relation that indicates when objects have some trait in common.

Do **not** use this definition in proofs! It’s just an intuition!
Properties of Equivalence Relations

\[ xRy \equiv x \text{ and } y \text{ have the same shape. } \]
Properties of Equivalence Relations

\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]

Recall: This symbol means “is defined as”
Properties of Equivalence Relations

\[ x \mathcal{R} y \iff x \text{ and } y \text{ have the same shape. } \]
Properties of Equivalence Relations

\[ xRy \equiv x \text{ and } y \text{ have the same shape. } \]
Properties of Equivalence Relations

\( xRy \equiv x \text{ and } y \text{ have the same shape. } \)
Properties of Equivalence Relations

$xRy \equiv x$ and $y$ have the same shape.
Properties of Equivalence Relations

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Properties of Equivalence Relations

\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]

\[ xRy \]

\[ yRx \]
A binary relation $R$ over a set $A$ is called \textbf{symmetric} iff for any $x \in A$ and $y \in A$, if $xRy$, then $yRx$.

This definition (and others like it) can be used in formal proofs.
Properties of Equivalence Relations

\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]
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Properties of Equivalence Relations

\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]

\[ xRx \]
Reflexivity

A binary relation $R$ over a set $A$ is called reflexive iff

For any $x \in A$, $xRx$. 
Some Reflexive Relations

• Equality:
  • For any $x$, $x = x$.

• Not greater than:
  • For any integer $x$, $x \leq x$.

• Subset:
  • For any set $S$, $S \subseteq S$. 
Properties of Equivalence Relations

\[ x \mathbin{R} y \equiv x \text{ and } y \text{ have the same shape.} \]
Properties of Equivalence Relations

\[ x R y \equiv x \text{ and } y \text{ have the same shape.} \]
Properties of Equivalence Relations

\( xRy \equiv x \text{ and } y \text{ have the same shape.} \)
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\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]

\[ xRy \quad \text{and} \quad yRz \]
Properties of Equivalence Relations

\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]

\[ xRy \quad \text{and} \quad yRz \]

\[ xRz \]
Transitivity

A binary relation R over a set A is called **transitive** iff

For any $x, y, \text{ and } z$, if $xRy$ and $yRz$, then $xRz$. 
Some Transitive Relations

- **Equality:**
  - \( x = y \) and \( y = z \) implies \( x = z \).

- **Less-than:**
  - \( x < y \) and \( y < z \) implies \( x < z \).

- **Subset:**
  - \( S \subseteq T \) and \( T \subseteq U \) implies \( S \subseteq U \).
Equivalence Relations

A binary relation $R$ over a set $A$ is called an equivalence relation if it is

- reflexive,
- symmetric, and
- transitive.
xRy ≡ x and y have the same shape.
\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]
This arrow denotes $R$.

$xRy \equiv x$ and $y$ have the same shape.
xRy ≡ x and y have the same shape.
\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]
What property says this edge must be here?

$xRy \equiv x \text{ and } y \text{ have the same shape.}$
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xRy ≡ x and y have the same shape.
\( xRy \equiv x \) and \( y \) have the same shape.
What property says this edge must be here?

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What property says these edges must be here?

\[ xRy \equiv x \text{ and } y \text{ have the same shape.} \]
xRy $\equiv$ x and y have the same shape.
xRy ≡ x and y have the same shape.
xRy \equiv x \text{ and } y \text{ are the same shape.}
xRy ≡ x and y are the same shape.
xRy \equiv x \text{ and } y \text{ are the same shape.}
\[ xRy \equiv x \text{ and } y \text{ are the same shape. } \]
xRy \equiv x \text{ and } y \text{ are the same shape.}
xRy ≡ x and y are the same color.
xRy ≡ x and y are the same color.
\[ xRy \equiv \text{x and y are the same color.} \]
\( x R y \equiv x \text{ and } y \text{ are the same color.} \)
\( xRy \equiv x = y \)
$x R y \equiv x = y$
\[ x \mathbin{R} y \equiv x = y \]
Equivalence Classes

- Given an equivalence relation \( R \) over a set \( A \), for any \( a \in A \), the **equivalence class of \( a \)** is the set

  \[
  [x]_R \equiv \{ a \mid a \in A \text{ and } xRa \}
  \]

- Informally, the set of all elements equal to \( a \).
- \( R \) **partitions** the set \( A \) into a set of equivalence classes.
Theorem: Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

Proof: We need to show that every $a \in A$ belongs to at least one equivalence class and to at most one equivalence class.

To see that every element of $A$ belongs to at least one equivalence class, consider any $a \in A$ and the equivalence class $[a]_R = \{ x \in A \mid aRx \}$. Since $R$ is an equivalence relation, $R$ is reflexive, so $aRa$. Thus $a \in [a]_R$.

Since our choice of $a$ was arbitrary, every $a \in A$ belongs to at least one equivalence class – namely, $[a]_R$.

To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R = [y]_R$. To do this, we prove that if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R \subseteq [y]_R$. By interchanging $[x]_R$ and $[y]_R$, we can conclude that $[y]_R \subseteq [x]_R$, from which we have $[x]_R = [y]_R$. ■
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Theorem: Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

Proof: We need to show that every $a \in A$ belongs to at least one equivalence class and to at most one equivalence class. To see that every element of $A$ belongs to at least one equivalence class, consider any $a \in A$ and the equivalence class $[a]_R = \{x \mid x \in A \text{ and } aRx\}$. Since $R$ is an equivalence relation, $R$ is reflexive, so $aRa$. Thus $a \in [a]_R$. Since our choice of $a$ was arbitrary, every $a \in A$ belongs to at least one equivalence class – namely, $[a]_R$.

To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $[a]_R = [x]_R$ and $[a]_R = [y]_R$, then $[x]_R = [y]_R$. To do this, we prove that if $[a]_R = [x]_R$ and $[a]_R = [y]_R$, then $[x]_R \subseteq [y]_R$. By interchanging $[x]_R$ and $[y]_R$, we can conclude that $[y]_R \subseteq [x]_R$, from which we have $[x]_R = [y]_R$. Assume that $[a]_R = [x]_R$ and $[a]_R = [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. By transitivity, from $aRx$ and $xRt$ we know $aRt$. Since $a \in [y]_R$, we also know $yRa$. By transitivity, from $yRa$ and $aRt$ we know $yRt$. Thus, $t \in [y]_R$. Since our choice of $t$ was arbitrary, $[x]_R \subseteq [y]_R$. By our above reasoning, $[x]_R = [y]_R$. ■
Existence and Uniqueness

- The proof we are attempting is a type of proof called an **existence and uniqueness** proof.
- We need to show that for any \(a \in A\), there *exists* an equivalence class containing \(a\) and that this equivalence class is **unique**.
- These are two completely separate steps.
Proving Existence

• To prove **existence**, we need to show that for any $a \in A$, that $a$ belongs to at least one equivalence class.

• This is just a proof of an existential statement.

• Can we find an equivalence class containing $a$?
Theorem: Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

Proof: We need to show that every $a \in A$ belongs to at least one equivalence class and to at most one equivalence class.

To see that every element of $A$ belongs to at least one equivalence class, consider any $a \in A$ and the equivalence class $\left[ a \right]_R = \{ x \mid x \in A \text{ and } aRx \}$. Since $R$ is an equivalence relation, $R$ is reflexive, so $aRa$. Thus $a \in a \left[ a \right]_R$. Since our choice of $a$ was arbitrary, every $a \in A$ belongs to at least one equivalence class – namely, $\left[ a \right]_R$.

To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $a \left[ a \right]_R \subseteq x \left[ x \right]_R$ and $a \left[ a \right]_R \subseteq y \left[ y \right]_R$, then $x \left[ x \right]_R = y \left[ y \right]_R$. To do this, we prove that if $a \left[ a \right]_R \subseteq x \left[ x \right]_R$ and $a \left[ a \right]_R \subseteq y \left[ y \right]_R$, then $x \left[ x \right]_R \subseteq y \left[ y \right]_R$. By interchanging $x \left[ x \right]_R$ and $y \left[ y \right]_R$, we can conclude that $y \left[ y \right]_R \subseteq x \left[ x \right]_R$, from which we have $x \left[ x \right]_R = y \left[ y \right]_R$. ■
Theorem: Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

Proof: We need to show that every $a \in A$ belongs to at least one equivalence class and to at most one equivalence class.

Assume that $a \in x R$ and $a \in y R$. Consider any $t \in x R$. Since $t \in x R$, we know $x R t$. Since $a \in x R$, we have that $x R a$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $x R a$ we know $a R x$. By transitivity, from $a R x$ and $x R t$ we know $a R t$. Since $a \in y R$, we also know $y R a$. By transitivity, from $y R a$ and $a R t$ we know $y R t$. Thus, $t \in y R$. Since our choice of $t$ was arbitrary, $x R \subseteq y R$. By our above reasoning, $x R = y R$. ■
**Theorem:** Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

**Proof:** We need to show that every $a \in A$ belongs to at least one equivalence class and to at most one equivalence class.

To see that every element of $A$ belongs to at least one equivalence class, consider any $a \in A$ and the equivalence class $[a]_R = \{ x \mid x \in A \text{ and } aRx \}$. 

Since $R$ is an equivalence relation, $R$ is reflexive, so $aRa$. Thus $a \in [a]_R$.

Since our choice of $a$ was arbitrary, every $a \in A$ belongs to at least one equivalence class – namely, $[a]_R$.

To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $a \in x_R$ and $a \in y_R$, then $x_R = y_R$. To do this, we prove that if $a \in x_R$ and $a \in y_R$, then $x_R \subseteq y_R$. By interchanging $x_R$ and $y_R$, we can conclude that $y_R \subseteq x_R$, from which we have $x_R = y_R$. 

Assume that $a \in x_R$ and $a \in y_R$. Consider any $t \in x_R$. Since $t \in x_R$, we know $xRt$. Since $a \in x_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. By transitivity, from $aRx$ and $xRt$ we know $aRt$. Since $a \in y_R$, we also know $yRa$. By transitivity, from $yRa$ and $aRt$ we know $yRt$. Thus, $t \in y_R$. Since our choice of $t$ was arbitrary, $x_R \subseteq y_R$. By our above reasoning, $x_R = y_R$. ■
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Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. By transitivity, from $aRx$ and $xRt$ we know $aRt$. Since $a \in [y]_R$, we also know $yRa$. By transitivity, from $yRa$ and $aRt$ we know $yRt$. Thus, $t \in [y]_R$. Since our choice of $t$ was arbitrary, $[x]_R \subseteq [y]_R$. By our above reasoning, $[x]_R = [y]_R$. ■
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Assume that $a\in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. By transitivity, from $aRx$ and $xRt$ we know $aRt$. Since $a \in [y]_R$, we also know $yRa$. By transitivity, from $yRa$ and $aRt$ we know $yRt$. Thus, $t \in [y]_R$. Since our choice of $t$ was arbitrary, $[x]_R \subseteq [y]_R$. By our above reasoning, $[x]_R = [y]_R$. ■
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To see that every $a \in A$ belongs to at most one equivalence class, ?????
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To see that every $a \in A$ belongs to at most one equivalence class, how do we prove this?
Proving Uniqueness

To prove that there is a unique object with some property, we can do the following:

- Consider any two arbitrary objects $x$ and $y$ with that property.
- Show that $x = y$.
- Conclude, therefore, that there is only one object with that property, and we just gave it two different names.
**Theorem:** Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

**Proof:** We need to show that every $a \in A$ belongs to at least one equivalence class and to at most one equivalence class.

To see that every element of $A$ belongs to at least one equivalence class, consider any $a \in A$ and the equivalence class $[a]_R = \{ x \mid x \in A \text{ and } aRx \}$. Since $R$ is an equivalence relation, $R$ is reflexive, so $aRa$. Thus $a \in [a]_R$. Since our choice of $a$ was arbitrary, every $a \in A$ belongs to at least one equivalence class – namely, $[a]_R$.

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To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R = [y]_R$. 

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To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R = [y]_R$. To do this, we prove that if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R \subseteq [y]_R$. By interchanging $[x]_R$ and $[y]_R$, we can conclude that $[y]_R \subseteq [x]_R$, from which we have $[x]_R = [y]_R$.

Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$.
**Theorem:** Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

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Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. 
**Theorem**: Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

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To see that every element of $A$ belongs to at least one equivalence class, consider any $a \in A$ and the equivalence class $[a]_R = \{ x \in A \mid xR a \}$. Since $R$ is an equivalence relation, $R$ is reflexive, so $aRa$. Thus $a \in [a]_R$. Since our choice of $a$ was arbitrary, every $a \in A$ belongs to at least one equivalence class – namely, $[a]_R$.

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Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xR t$. Since $a \in [x]_R$, we have that $xRa$. 

**Theorem:** Let $R$ be an equivalence relation over a set $A$. Then every element of $A$ belongs to exactly one equivalence class.

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To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R = [y]_R$. To do this, we prove that if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R \subseteq [y]_R$. By interchanging $[x]_R$ and $[y]_R$, we can conclude that $[y]_R \subseteq [x]_R$, from which we have $[x]_R = [y]_R$.

Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive.
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To see that every $a \in A$ belongs to at most one equivalence class, we show that for any $a \in A$, if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R = [y]_R$. To do this, we prove that if $a \in [x]_R$ and $a \in [y]_R$, then $[x]_R \subseteq [y]_R$. By interchanging $[x]_R$ and $[y]_R$, we can conclude that $[y]_R \subseteq [x]_R$, from which we have $[x]_R = [y]_R$.

Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. Since $R$ is transitive, from $aRx$ and $xRt$ we know $aRt$. Thus, $t \in [a]_R$. Since $t$ was arbitrary, $[x]_R \subseteq [a]_R$. By our above reasoning, $[a]_R = [y]_R$. ■
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Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. By transitivity, from $aRx$ and $xRt$ we know $aRt$. 

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Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. By transitivity, from $aRx$ and $xRt$ we know $aRt$. Since $a \in [y]_R$, we also know $yRa$. 

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Assume that $a \in [x]_R$ and $a \in [y]_R$. Consider any $t \in [x]_R$. Since $t \in [x]_R$, we know $xRt$. Since $a \in [x]_R$, we have that $xRa$. Since $R$ is an equivalence relation, $R$ is symmetric and transitive. By symmetry, from $xRa$ we know $aRx$. By transitivity, from $aRx$ and $xRt$ we know $aRt$. Since $a \in [y]_R$, we also know $yRa$. By transitivity, from $yRa$ and $aRt$ we know $yRt$. ■
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This proof helps to justify our definition of equivalence relations. We need all three of the properties we've listed in order for this proof to work, and we don't need any others.

Since $R$ is an equivalence relation, $R$ is reflexive

$\begin{align*}
\text{Assume that } a &\in [x]_R \text{ and } a \in [y]_R. \text{ Consider any } t \in [x]_R. \text{ Since } t \in [x]_R, \text{ we know } xRt. \text{ Since } a \in [x]_R, \text{ we have that } xRa. \text{ By symmetry, from } xRa \text{ we know } aRx. \\
\text{By transitivity, from } aRx \text{ and } xRt \text{ we know } aRt. \text{ Since } a \in [y]_R, \text{ we also know } yRa. \text{ By transitivity, from } yRa \text{ and } aRt \text{ we know } yRt. \text{ Thus, } t \in [y]_R. \text{ Since our choice of } t \text{ was arbitrary, } [x]_R \subseteq [y]_R. \text{ By our above reasoning, } [x]_R = [y]_R.\end{align*}$

Since $R$ is an equivalence relation, $R$ is symmetric and transitive
Order Relations
“x is larger than y”

“x runs faster than y”

“x is a subset of y”

“x is a part of y”

“x is tastier than y”

“x divides y”
Informally

An **order relation** is a relation that ranks elements against one another.

Again, **do not** use this definition in proofs! It's just an intuition!
Properties of Order Relations

\[ x \leq y \]
Properties of Order Relations

\[ x \leq y \]

\[ 1 \leq 5 \quad \text{and} \quad 5 \leq 8 \]
Properties of Order Relations

\[ x \leq y \]

\[ 1 \leq 5 \quad \text{and} \quad 5 \leq 8 \]

\[ 1 \leq 8 \]
Properties of Order Relations

\[ x \leq y \]

\[ 42 \leq 99 \quad \text{and} \quad 99 \leq 137 \]
Properties of Order Relations

\[ x \leq y \]

\[ 42 \leq 99 \quad \text{and} \quad 99 \leq 137 \]

\[ 42 \leq 137 \]
Properties of Order Relations

\( x \leq y \)

\( x \leq y \quad \text{and} \quad y \leq z \)
Properties of Order Relations

\[ x \leq y \]

\[ x \leq y \quad \text{and} \quad y \leq z \]

\[ x \leq z \]
Properties of Order Relations

\[ x \leq y \]

\[ x \leq y \quad \text{and} \quad y \leq z \]

\[ x \leq z \]

Transitivity
Properties of Order Relations

\[ x \leq y \]
Properties of Order Relations

\[ x \leq y \]

\[ 1 \leq 1 \]
Properties of Order Relations

\[ x \leq y \]

\[ 42 \leq 42 \]
Properties of Order Relations

\[ x \leq y \]

\[ 137 \leq 137 \]
Properties of Order Relations

\[ x \leq y \]

\[ x \leq x \]
Properties of Order Relations

\[ x \leq y \]

\[ x \leq x \]

Reflexivity
Properties of Order Relations

\[ x \leq y \]
Properties of Order Relations

\[ x \leq y \]

\[ 19 \leq 21 \]
Properties of Order Relations

\[ x \leq y \]

\[ 19 \leq 21 \]

\[ 21 \leq 19 \text{?} \]
Properties of Order Relations

\[ x \leq y \]

\[ 19 \leq 21 \]

\[ 21 \leq 19 \text{?} \]
Properties of Order Relations

\[ x \leq y \]

\[ 42 \leq 137 \]
Properties of Order Relations

\[ x \leq y \]

\[ 42 \leq 137 \]

\[ 137 \leq 42? \]
Properties of Order Relations

\[ x \leq y \]

\[ 42 \leq 137 \]

\[ 137 \not\leq 42 \]?
Properties of Order Relations

\[ x \leq y \]

\[ 137 \leq 137 \]
Properties of Order Relations

\[ x \leq y \]

\[ 137 \leq 137 \]

\[ 137 \leq 137? \]
Properties of Order Relations

\[ x \leq y \]

\[ 137 \leq 137 \]

\[ 137 \leq 137 \]
Antisymmetry

A binary relation $R$ over a set $A$ is called \textbf{antisymmetric} iff

For any $x \in A$ and $y \in A$, if $xRy$ and $yRx$, then $x = y$.

Equivalently:

For any $x \in A$ and $y \in A$, if $xRy$ and $y \neq x$, then $yR\overline{x}$.
An Intuition for Antisymmetry
An Intuition for Antisymmetry
An Intuition for Antisymmetry

Self-loops allowed
An Intuition for Antisymmetry

Self-loops allowed

Only one edge between nodes
An Important Detail

• A binary relation $R$ over a set $A$ is antisymmetric iff for any $x \in A$ and $y \in A$, if $xRy$ and $yRx$, then $x = y$.

• Is the relation $<$ over real numbers antisymmetric?
An Important Detail

- A binary relation $R$ over a set $A$ is antisymmetric iff for any $x \in A$ and $y \in A$, if $xRy$ and $yRx$, then $x = y$.

- Is the relation $<$ over real numbers antisymmetric?

- **Yes**: This is vacuously true.
  - It's never possible for $x < y$ and $y < x$ to be true simultaneously.
  - The claim “if $xRy$ and $yRx$, then $x = y$” is thus vacuously true.
Partial Orders

• A binary relation $R$ is a **partial order** if it is
  • **reflexive**,  
  • **antisymmetric**, and  
  • **transitive**.

• A pair $(S, R)$, where $R$ is a partial order over $S$, is called a **partially ordered set** or **poset**.
Partial Orders

- A binary relation $R$ is a \textbf{partial order} if it is
  - reflexive,
  - \textbf{antisymmetric}, and
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Why “partial”?
## 2008 Summer Olympics

<table>
<thead>
<tr>
<th></th>
<th>Gold</th>
<th>Silver</th>
<th>Bronze</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>51</td>
<td>21</td>
<td>28</td>
<td>100</td>
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<td>38</td>
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<td>47</td>
</tr>
<tr>
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Define the relationship

$$(\text{gold}_0, \text{total}_0) \mathcal{R} (\text{gold}_1, \text{total}_1)$$

to be true when

$$\text{gold}_0 \leq \text{gold}_1 \text{ and } \text{total}_0 \leq \text{total}_1$$
\[(g, t)R(g', t') \equiv g \leq g' \text{ and } t \leq t'\]
(g, t)R(g', t') ≡ g ≤ g' and t ≤ t'
\((g, t) R (g', t') \equiv g \leq g' \text{ and } t \leq t'\)
$(g, t) R (g', t') \equiv g \leq g' \text{ and } t \leq t'$
$(g, t) R (g', t') \equiv g \leq g' \text{ and } t \leq t'$
\[(g, t)R(g', t') \equiv g \leq g' \text{ and } t \leq t'\]
Neither did better than the other by our metric.

\[(g, t)R(g', t') \equiv g \leq g' \text{ and } t \leq t'\]
Partial and Total Orders

● A relation $R$ over a set $A$ is called **total** iff for any $x \in A$ and $y \in A$, either $xRy$ or $yRx$.
  • Could both be true?

● A **partial order** is called a **total order** if it is total.

● Examples:
  • Integers ordered by $\leq$.
  • Strings ordered alphabetically.
More Medals

China

Australia

United States

Fewer Medals

Russia
More Medals

Fewer Medals
Hasse Diagrams

- A **Hasse diagram** is a graphical representation of a partial order.
- No self-loops: by **reflexivity**, we can always add them back in.
- Higher elements are bigger than lower elements: by **antisymmetry**, the edges can only go in one direction.
- No redundant edges: by **transitivity**, we can infer the missing edges.
A Hasse diagram is a graphical representation of a partial order.

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This is a good justification for our definition! These drawings encode the structure we’d like, and the three properties we’ve picked guarantee us that they mean what we want them to mean.
Hasse Artichokes
Hasse Artichokes
Hass Avocado
One Final Type of Order
Is this relation reflexive?
Irreflexivity

- Let $R$ be a binary relation over $A$.
- $R$ is **irreflexive** iff for any $a \in A$, $aRa$ is false.
  - $x$ is heavier than $y$
  - $x < y$
  - $x \neq y$
- Note that irreflexive does **not** mean “not reflexive.”
- Reflexive: Every element is **always** related to itself.
- Irreflexive: Every element is **never** related to itself.
Strict Orders

- A binary relation $R$ over a set $A$ is called a strict order iff it is
  - irreflexive,
  - antisymmetric, and
  - transitive.
Turning Things Around
Turning Things Around
Turning Things Around
Inverses

• Given a relation $R$, the **inverse relation of $R$** (denoted $R^{-1}$) is the relation

$$R^{-1} = \{(b, a) \mid aRb\}$$

• Example: The inverse of $\leq$ is $\geq$, since $a \geq b$ iff $b \leq a$.

• Note: inverse relations are **not** the same the opposite of the original relation.
  - The inverse of $\leq$ is **not** $>$.  

• We will see this used more next lecture when we talk about functions.
Important Terms for Today

- Cartesian Product
- Ordered Pair
- Graph
- Path
- Connectivity
- Cycle
- Degree
- DAG
- Topological Sort
- Relation
- Reflexivity
- Symmetry
- Transitivity
- Antisymmetry
- Irreflexivity
- Totality
- Equivalence relation
- Equivalence class
- Partial order
- Hasse Diagram
- Total order
- Strict order
- Inverse relation