The Universal Turing Machine

Where We Are

- The **Church-Turing thesis** tells us that all effective models of computation are no more powerful than a Turing machine.
- We have a family of programming languages (**WB*n***) that are equivalent to Turing machines.
- Let's start exploring what we can do with this new model of computation.

Important Ideas for Today

- The material from today will lay the groundwork for the next few weeks.
- Key concepts:
  - **Encodings**.
  - **Universal machines.**
  - **High-level specifications**
  - **Nondeterministic Turing machines.**
  - **Decidable and recognizable languages**.

Encodings

An Interesting Observation

- **WB** programs give us a way to check whether a string is contained within some language.
- **WB** programs themselves are strings.
- This means that we can run **WB** programs, taking other **WB** programs as input!
- Major questions:
  - Is this specific to **WB** programs?
  - What can we do with this knowledge?

There is a subtle flaw in this reasoning:

"Because **WB** programs are strings, **WB** programs can be run on the source code of other **WB** programs."

What is it?

The Alphabet Problem

- **WB** programs are described using multiple characters: letters, digits, colons, newlines, etc., plus potentially all tape symbols being used.
- Not all **WB** programs are written for languages over this alphabet; in fact, most do not.
- We cannot directly write the source of a **WB** program onto the input tape of another **WB** program.
- Can we fix this?

A Better Encoding

- We will restrict ourselves to talking about languages over alphabets containing at least two symbols in them.
- Let's assume, without loss of generality, that we're dealing with the alphabet $\mathbb{B}$ = { `0`, `1` }.
- It's always possible to encode a string with a large alphabet using a string with just the binary alphabet.
  - Think about how real computers work.

Encoding **WB** Programs

- Determine how many different characters are needed to represent the program.
  - This includes letters for keywords, etc. along with tape symbols.
- Build a fixed-length binary encoding for each symbol in some canonical order.
  - Perhaps write out keyword symbols first, then the blank symbol, then the rest of the input symbols, etc.
- Replace each character in the program with the binary encoding.

Notation for Encodings

- If *P* is a **WB** program, we will denote its binary encoding as ***⟨P⟩***.
- Don't worry too much about the details about how exactly you would compute ⟨*P*⟩; the important part is that there's at least one way to do it.

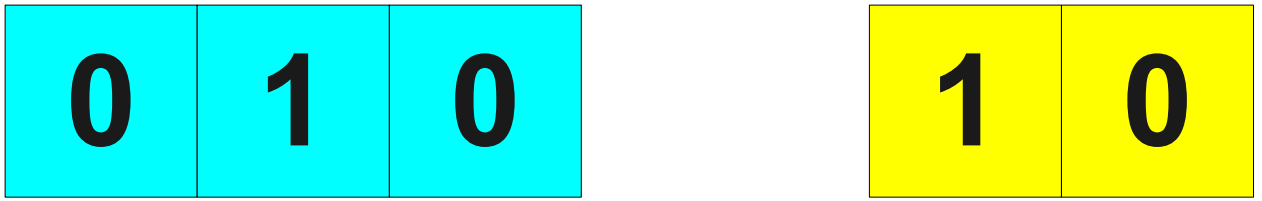
Encoding Other Automata

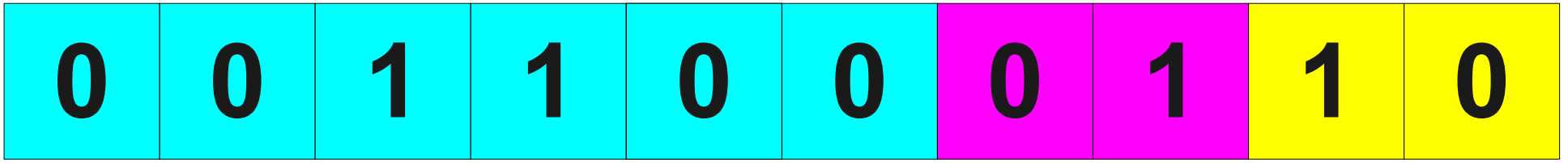
- Using similar techniques, we can encode all the other automata we've seen so far as binary strings.
- For example, if M is a Turing machine, $\langle M \rangle$ refers to a binary encoding of it.
- More generally, if we have any object O we want to encode as a binary string, we can write it out as $\langle O \rangle$.

Encoding Multiple Objects

- Suppose that we want to provide an encoding of multiple objects.
  - Several **WB** programs.
  - A **WB** program and a string.
  - A graph and a path in the graph.
  - A rock and a hard place.
- There are many ways that we can do this.

One Encoding Scheme





Encoding Multiple Objects

- Given several different objects O_1, O_2, …, O_n, we can represent the encoding of those *n* objects as $\langle O_1, O_2, \ldots, O_n \rangle$.
- Example:
  - If *P* is a **WB** program and *w* is a string, then ⟨*P*, *w*⟩ is an encoding of that program and that string.
  - If *G* is a context-free grammar and *P* is a PDA, then ⟨*G*, *P*⟩ is an encoding of that grammar and that PDA.

Universal Machines

Universal Machines and Programs

- **Theorem**: There is a Turing machine U_{TM} called the **universal Turing machine** that, when run on $\langle M, w \rangle$, where M is a Turing machine and w is a string, simulates M running on w.
- **Theorem**: There is a **WB** program U_{WB} called the **universal WB program** that, when run on $\langle P, w \rangle$, where P is a **WB** program and w is a string, simulates P running on w.

Why This Matters

- **As a historically significant achievement.**
  - The universal Turing machine might be the very first “complicated” algorithm ever designed for a computer.
  - Motivation for the “stored-program” model of computers.
- **As a justification for the Church-Turing thesis.**
  - All sufficiently powerful models of computation can simulate one another.
- **As a stepping stone to building elaborate models of computation.**
  - More on that later today.
- **As a stepping stone to finding unsolvable problems.**
  - More on that on Friday.

Sketch of the Universal **WB** Program

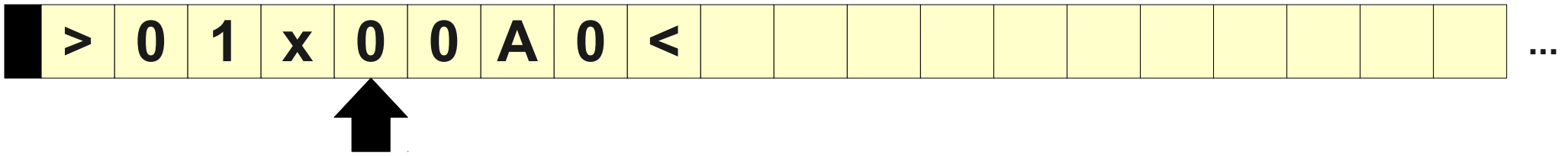
- It is a bit easier to understand the universal **WB** program than the universal TM.
- We will write the universal **WB** program in **WB6** (finite variables, multiple tracks, multiple tapes, multiple stacks) for simplicity, and can then "compile" it down into normal **WB**.

Sketch of the Universal **WB** Program

- To run a **WB** program, at each point in time we need to track three pieces of information:
  - The contents of the tape.
  - The location of the read head.
  - The index of the current instruction.
- Our program will enter a loop and repeatedly do the following:
  - Find the next instruction to execute.
  - Simulate that execution on the simulated tape.

Sketch of the Universal **WB** Program

Simulated tape of the program being executed.



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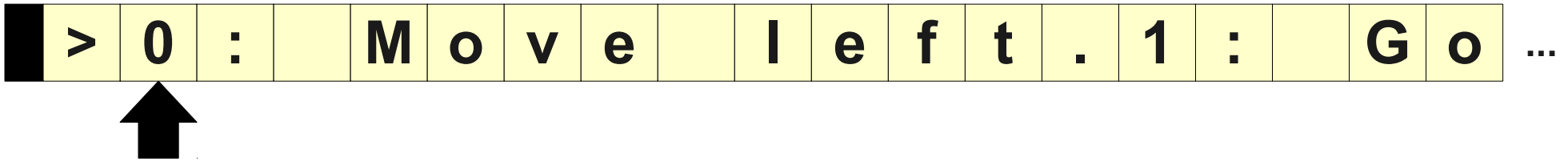
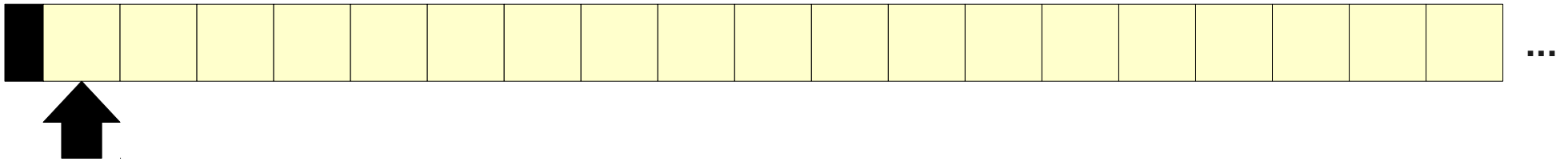


Scratch tape for intermediate computation.



Variables for intermediate storage.

Instr **Letter**

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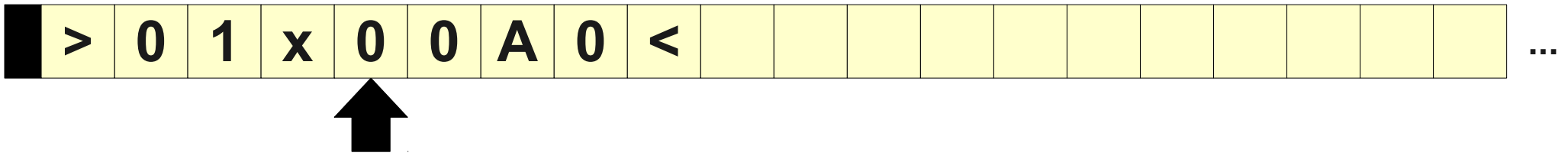

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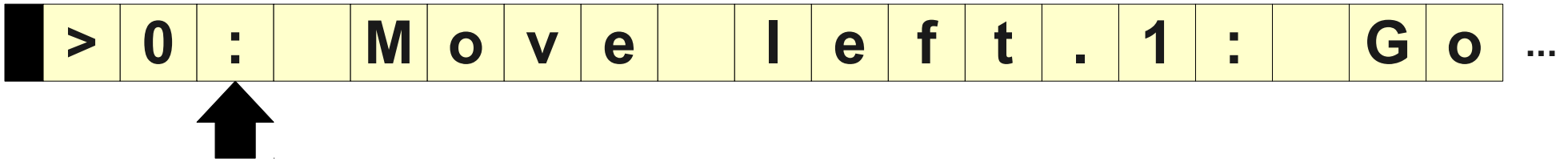

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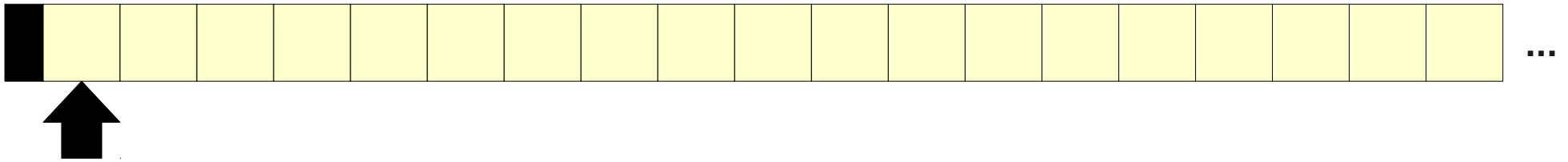

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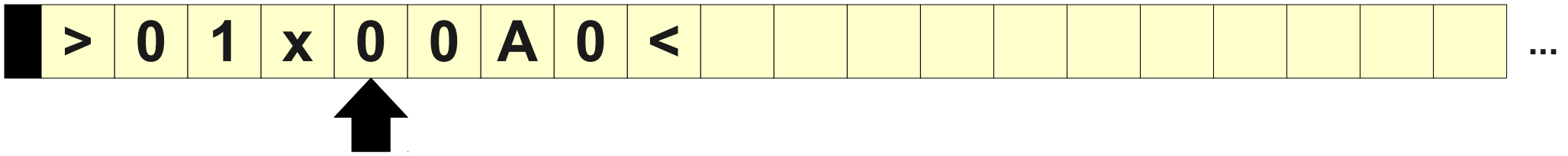

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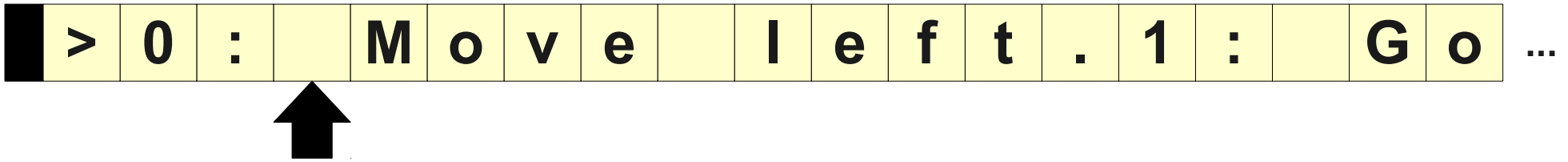

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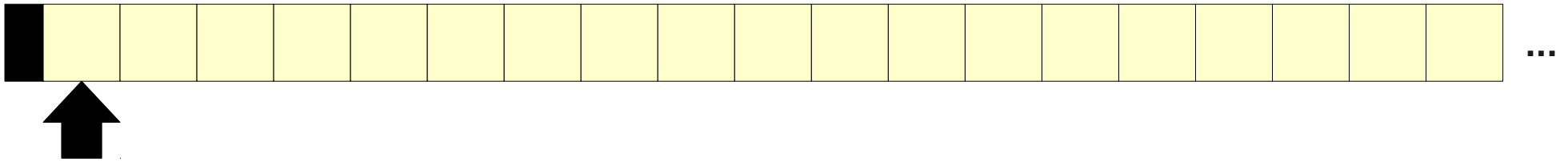

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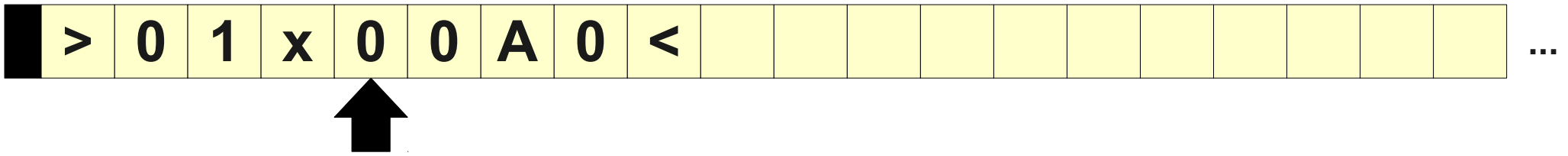

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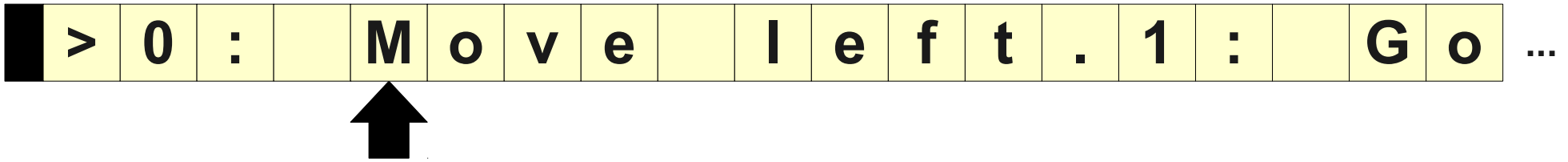

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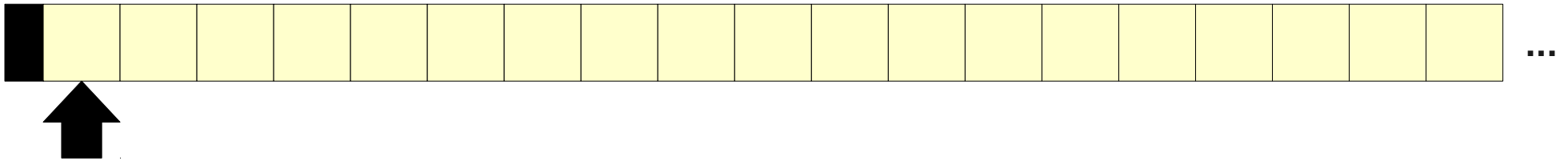

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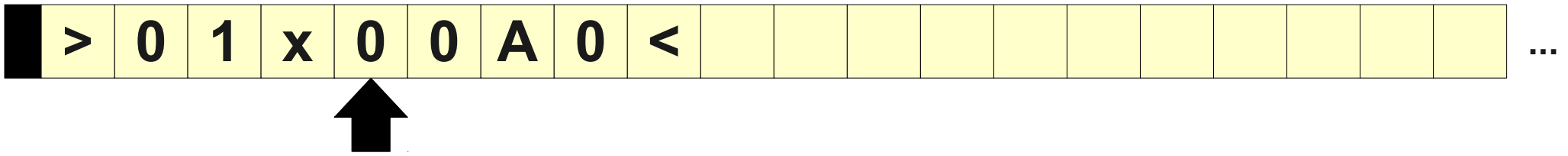

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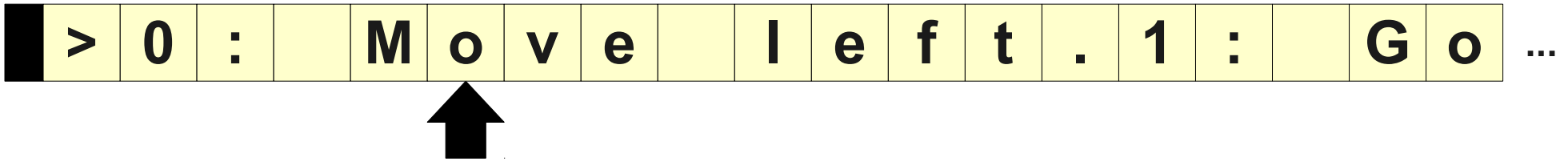

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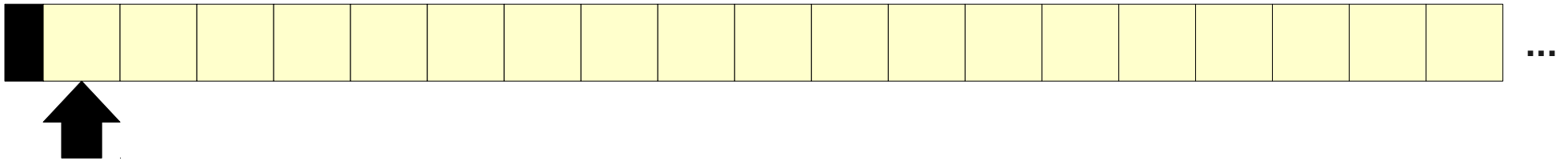

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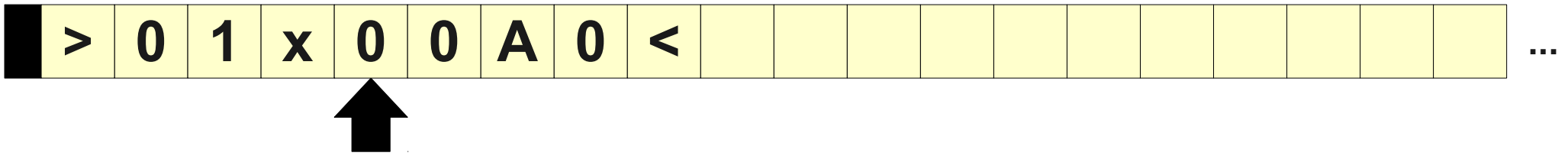

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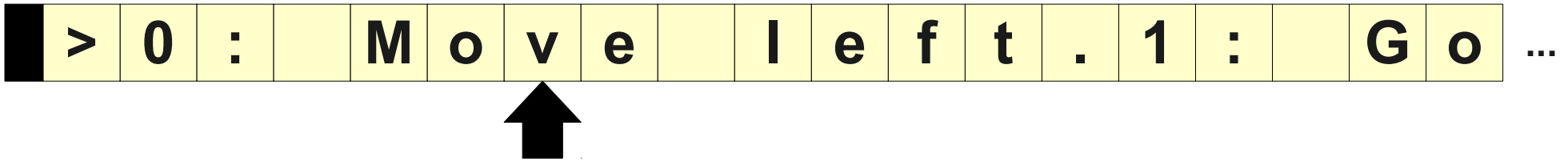

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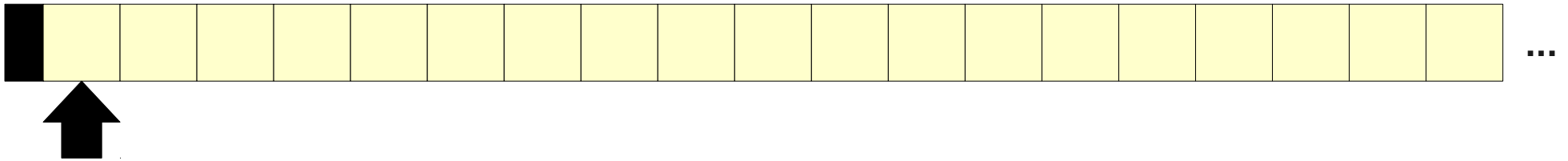

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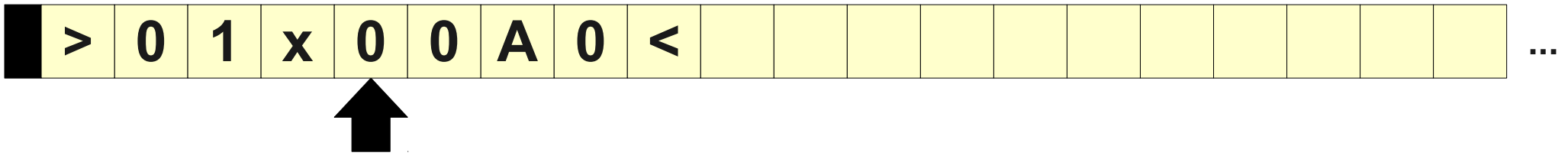

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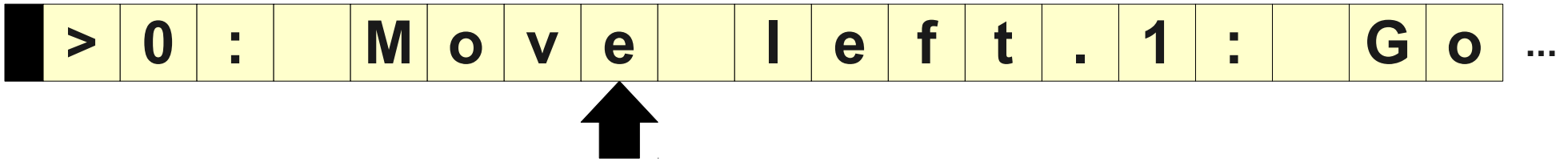

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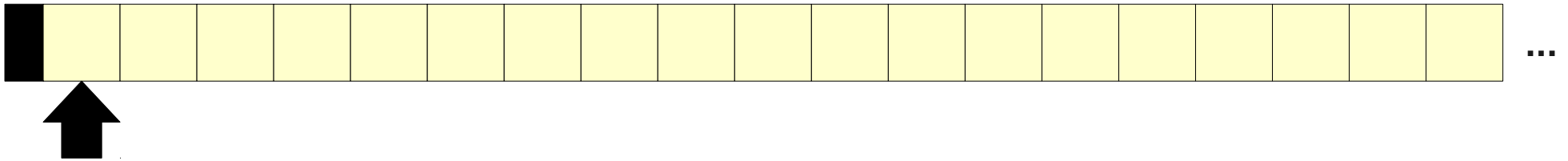

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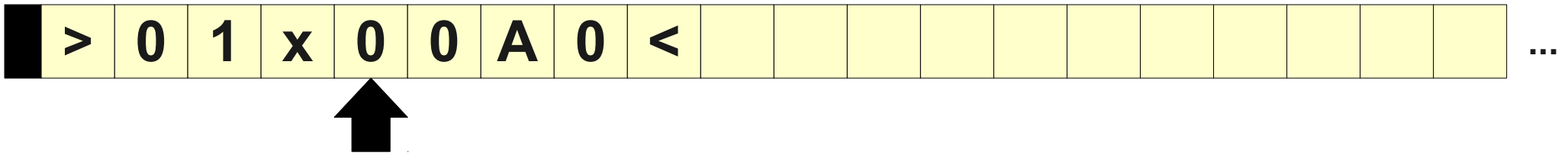

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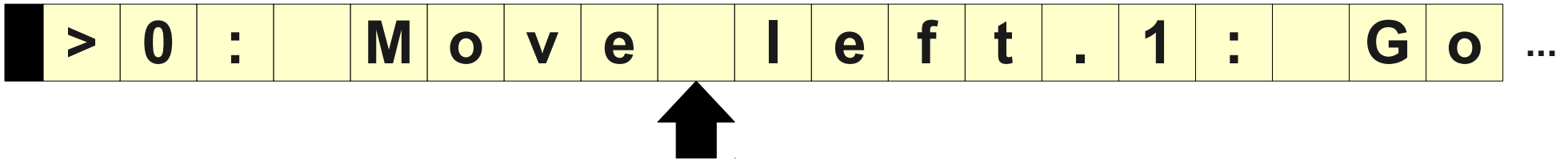

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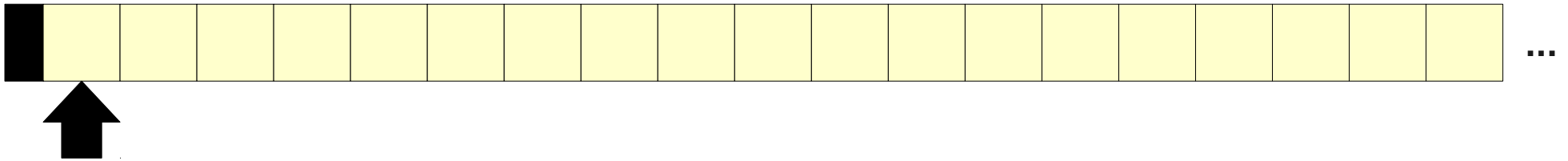

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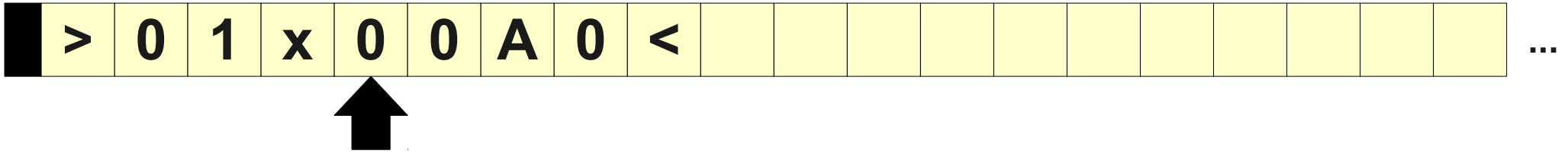

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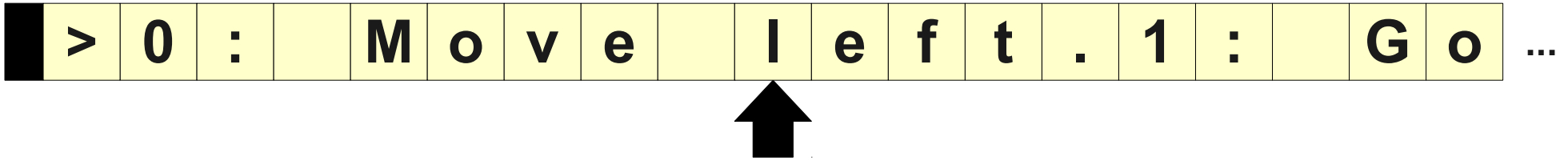

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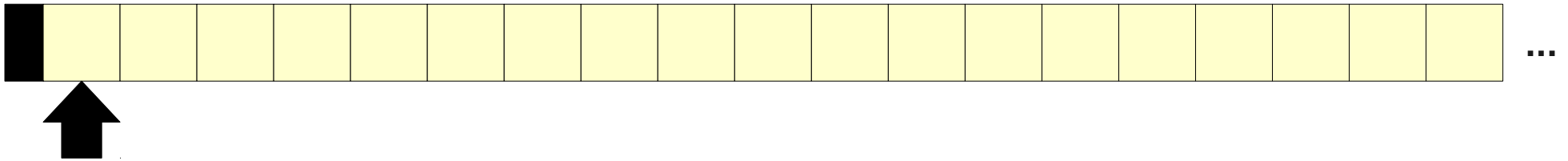

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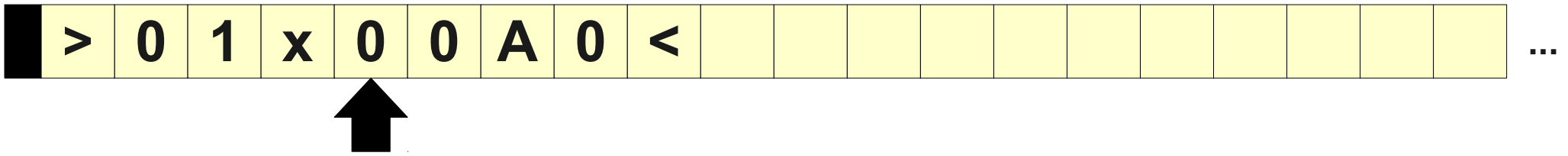

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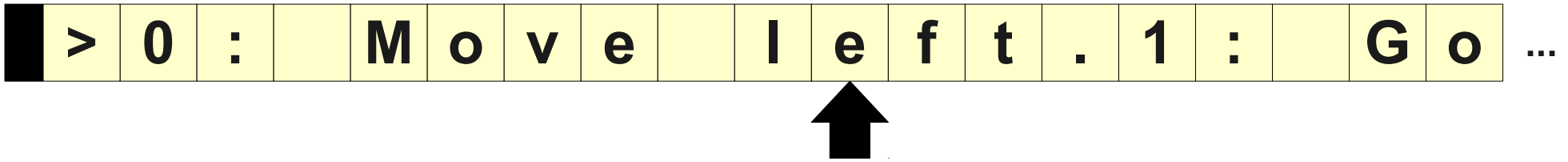

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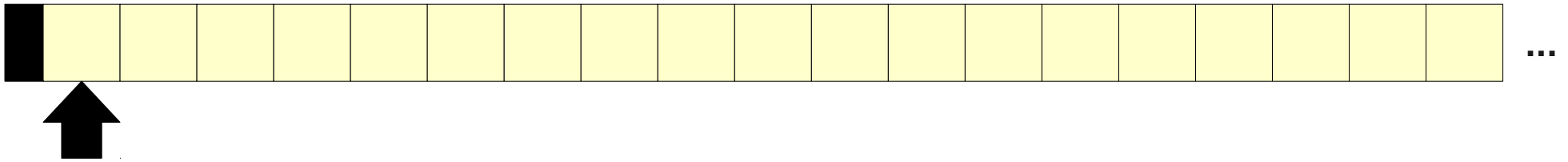

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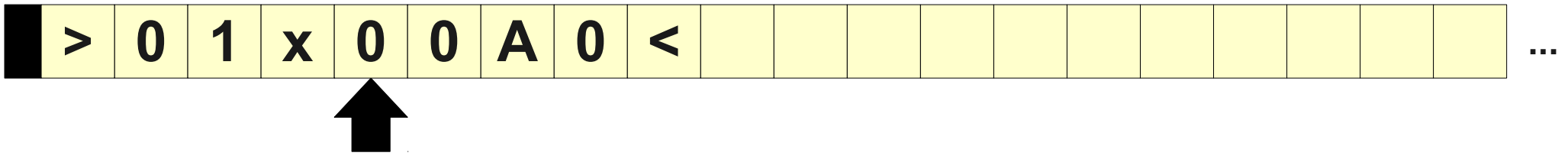

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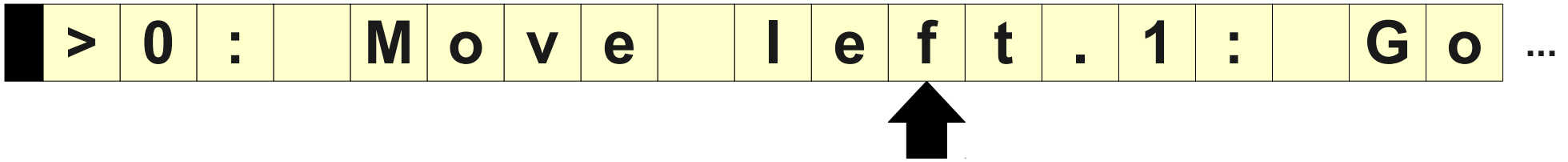

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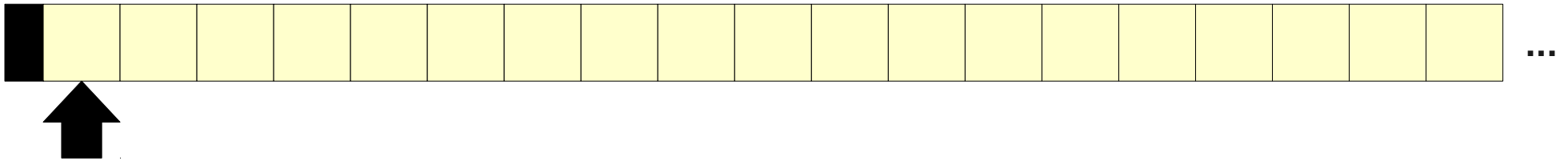

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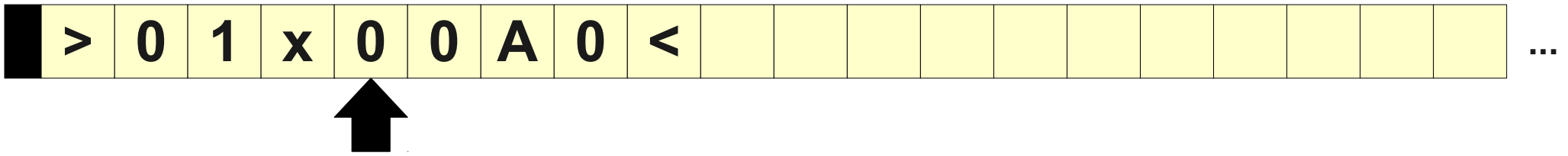

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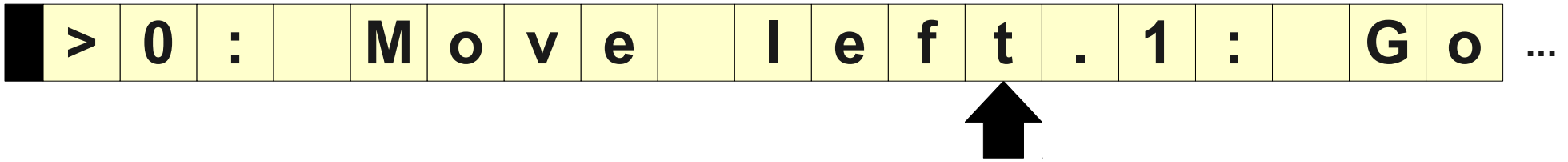

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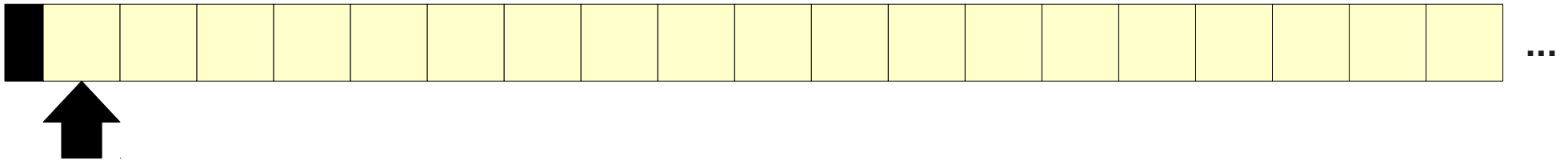

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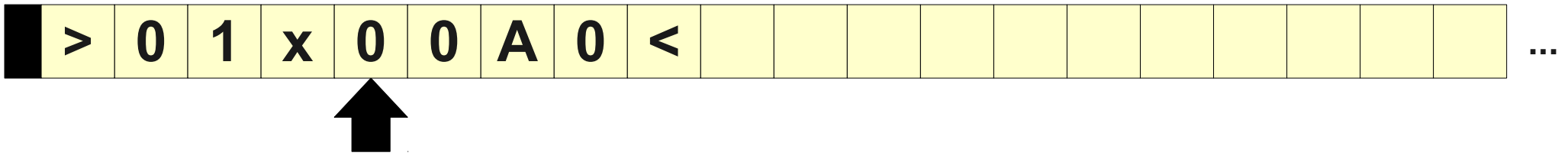

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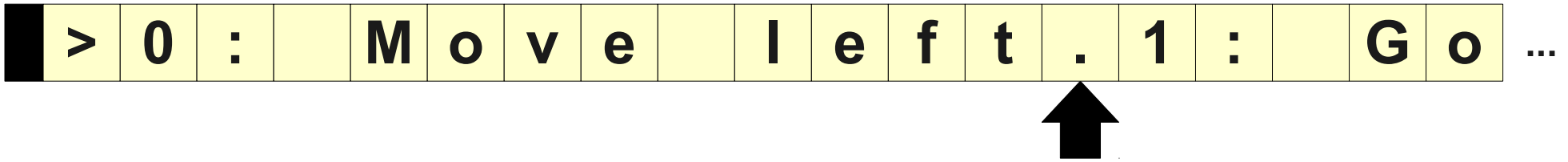

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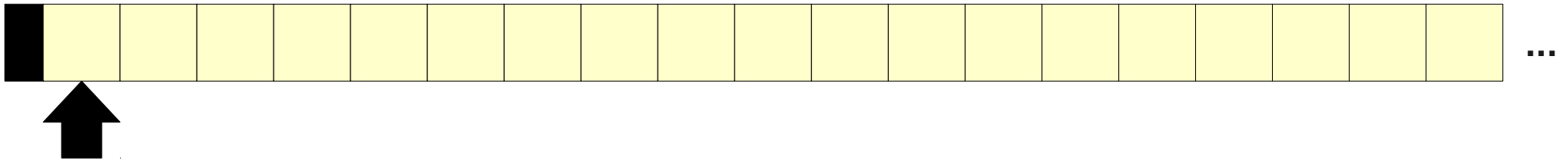

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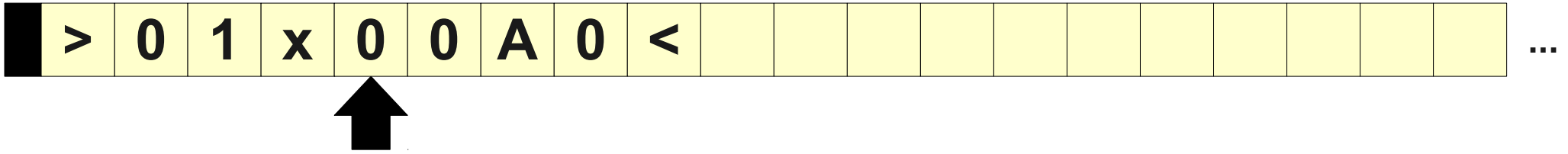

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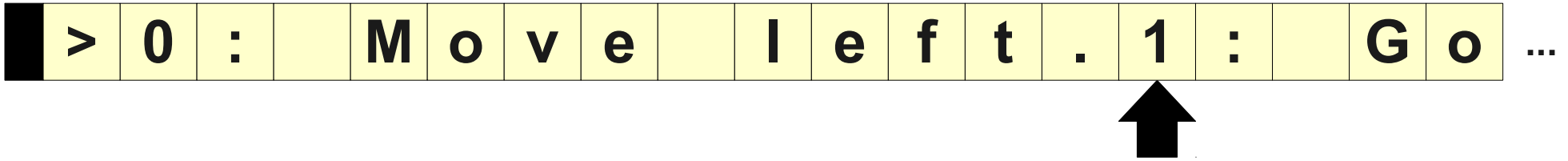

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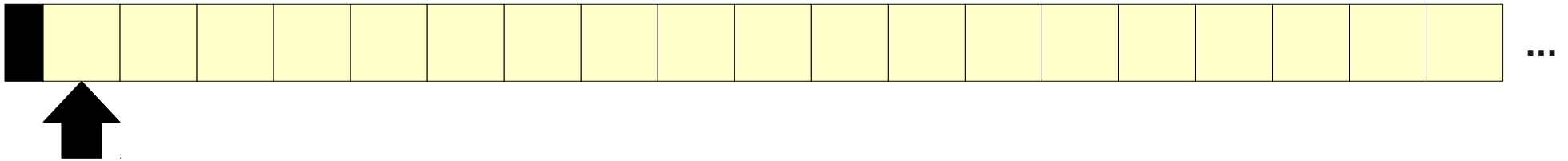

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Instr **ML** **Letter**

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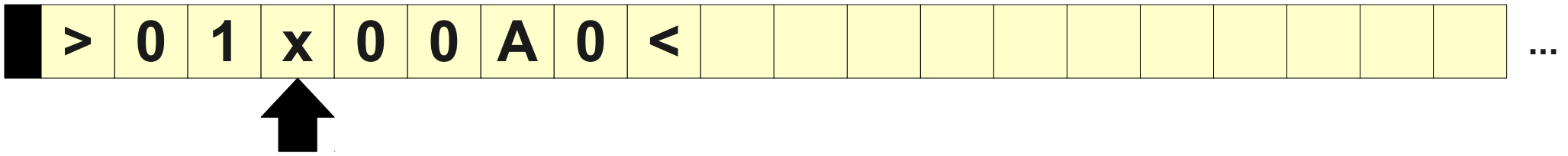
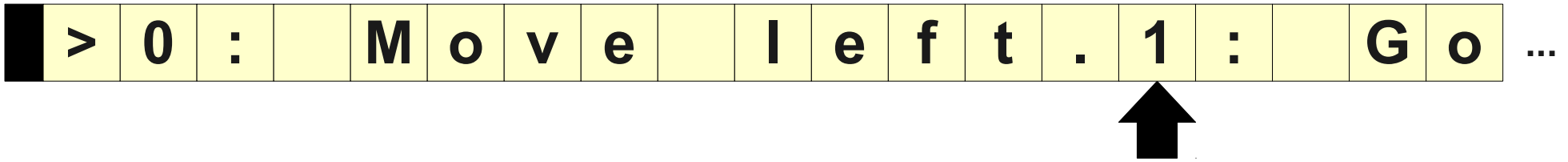
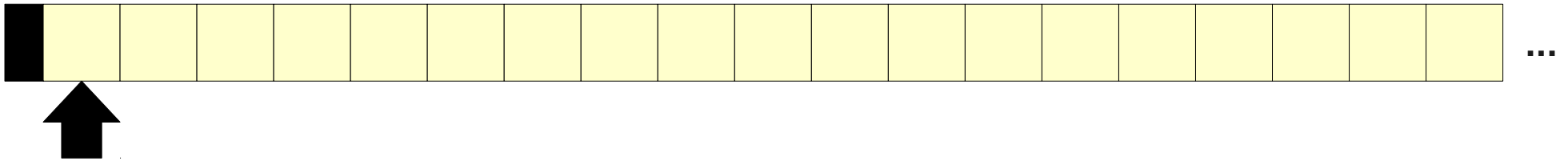


Program tape holding the program being executed.



Scratch tape for intermediate computation.

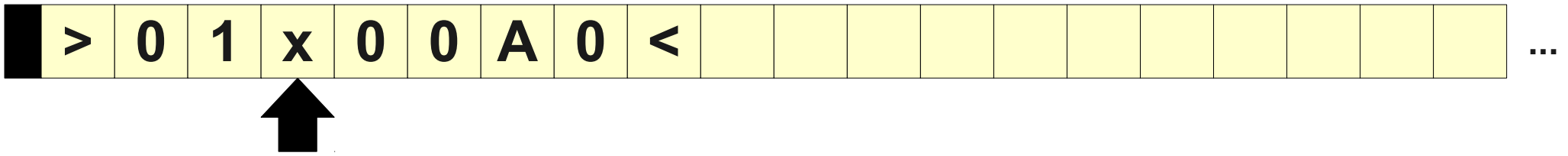


Variables for intermediate storage.

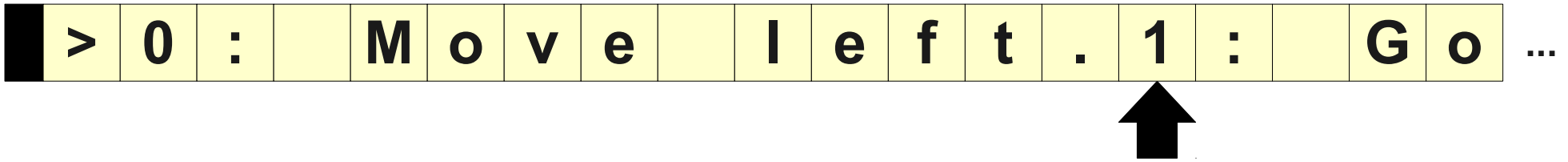
Instr **ML** **Letter**

Sketch of the Universal **WB** Program

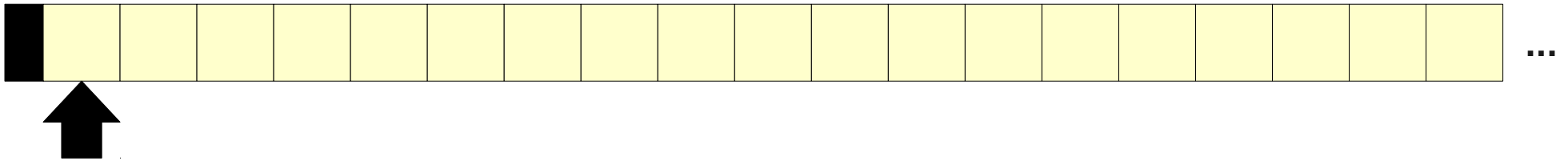
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

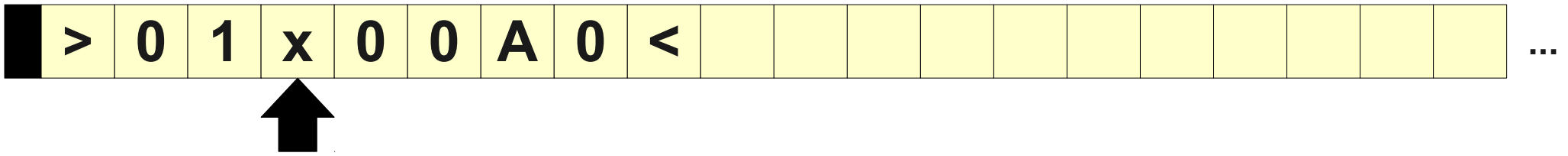


Variables for intermediate storage.

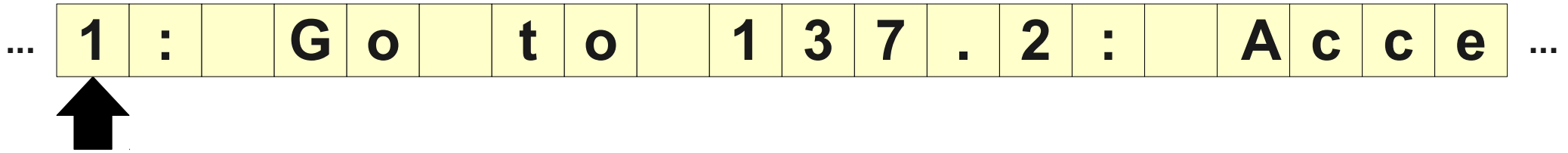
Instr **Letter**

Sketch of the Universal **WB** Program

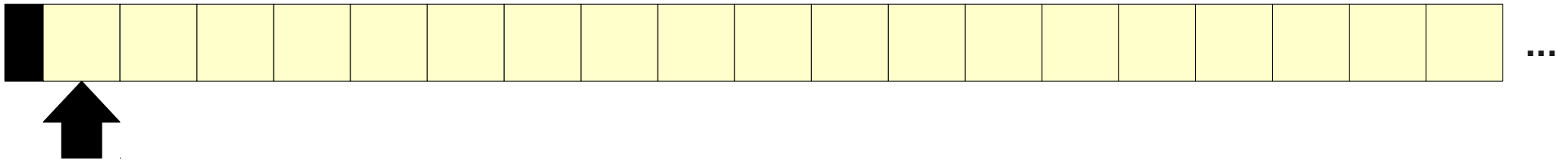
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

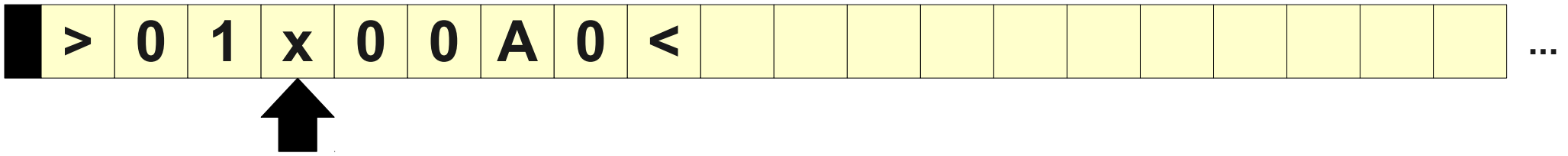


Variables for intermediate storage.

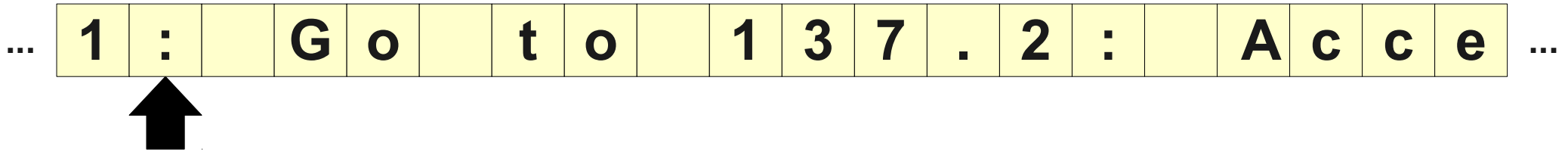
Instr **Letter**

Sketch of the Universal **WB** Program

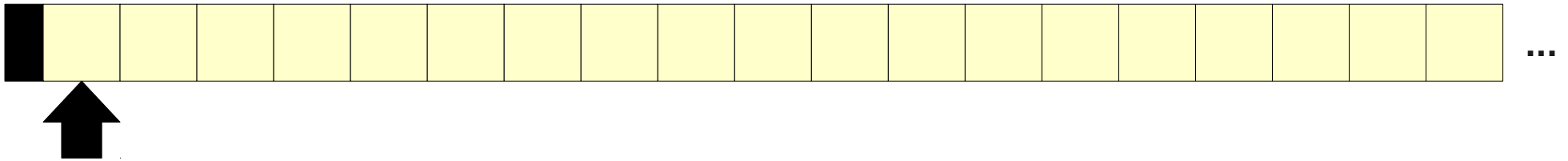
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

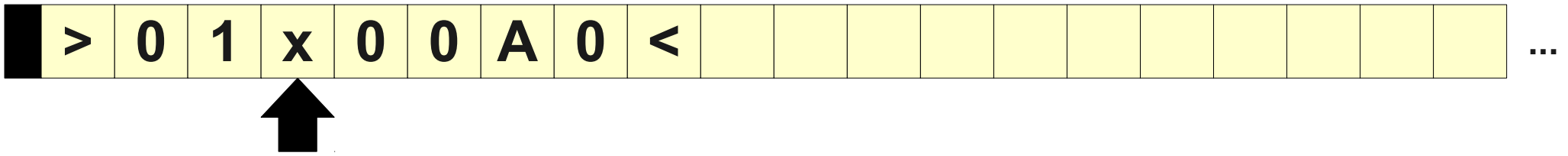


Variables for intermediate storage.

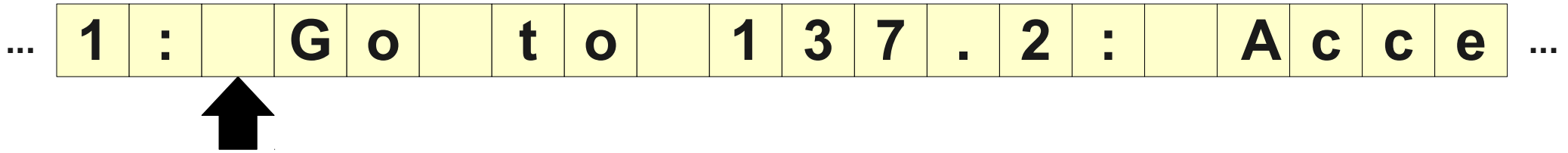
Instr **Letter**

Sketch of the Universal **WB** Program

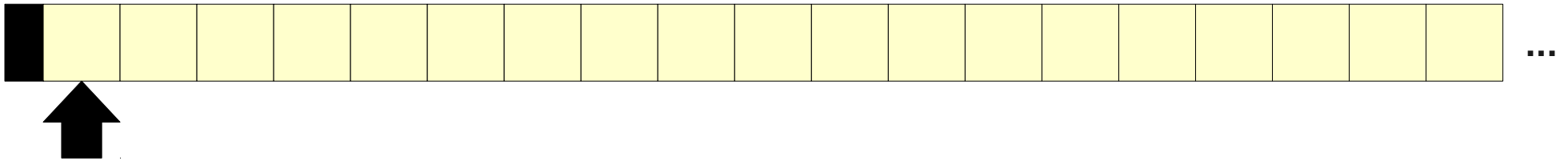
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.



Variables for intermediate storage.

Instr **Letter**

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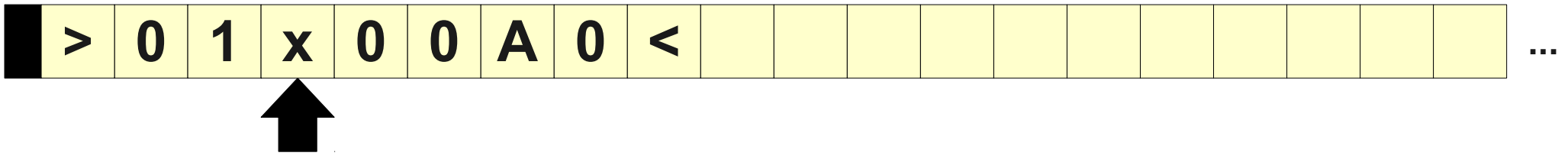
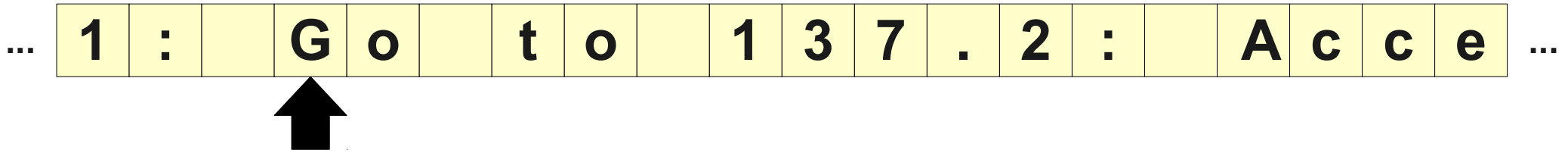
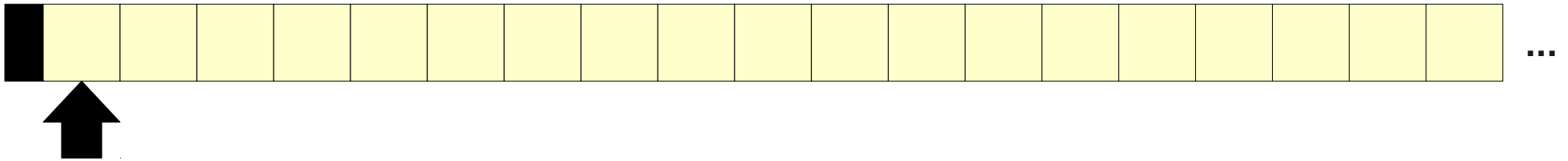
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

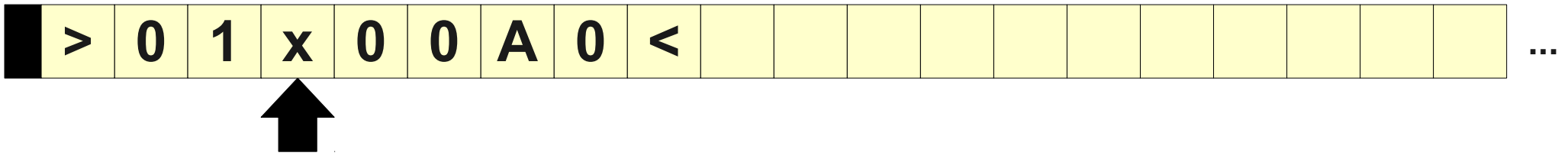


Variables for intermediate storage.

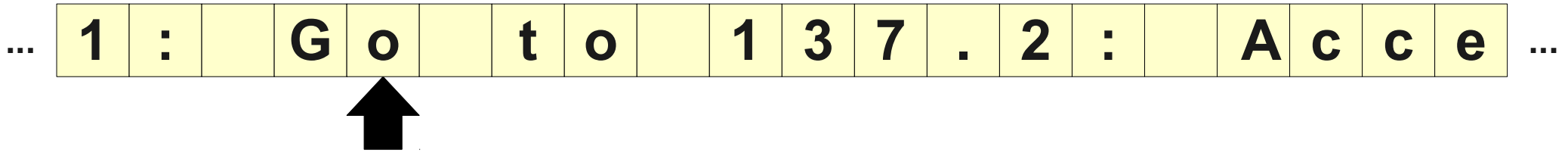
Instr **Letter**

Sketch of the Universal **WB** Program

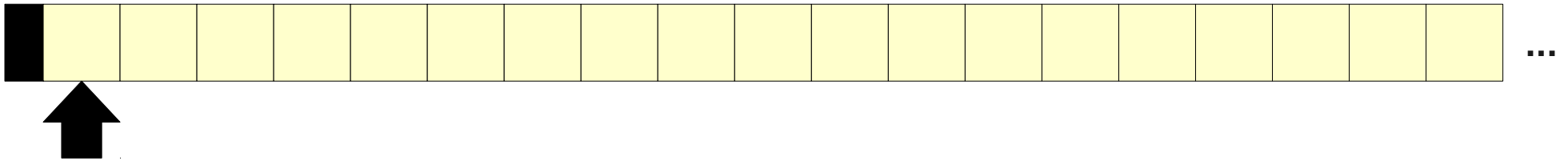
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

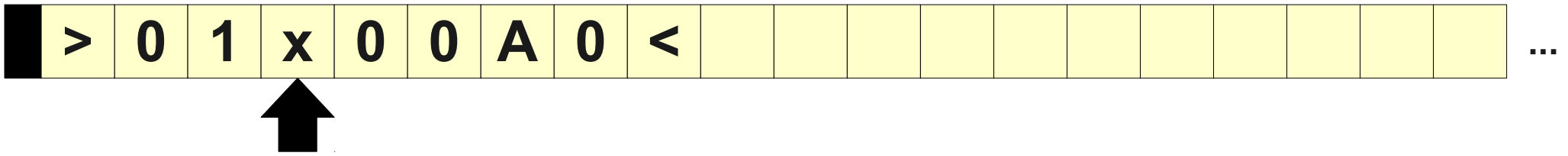


Variables for intermediate storage.

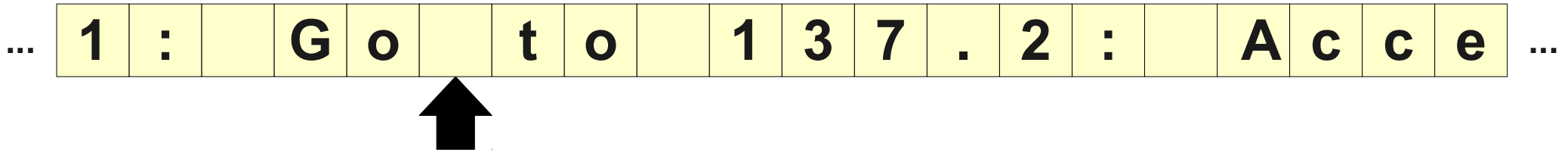
Instr **Letter**

Sketch of the Universal **WB** Program

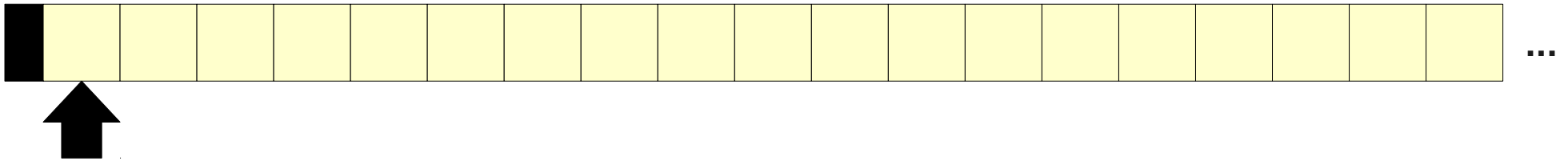
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

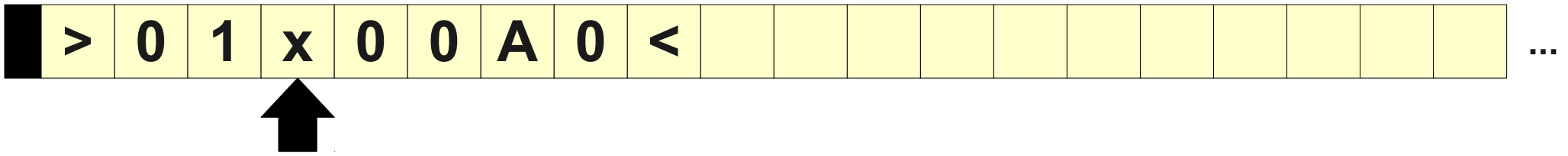


Variables for intermediate storage.

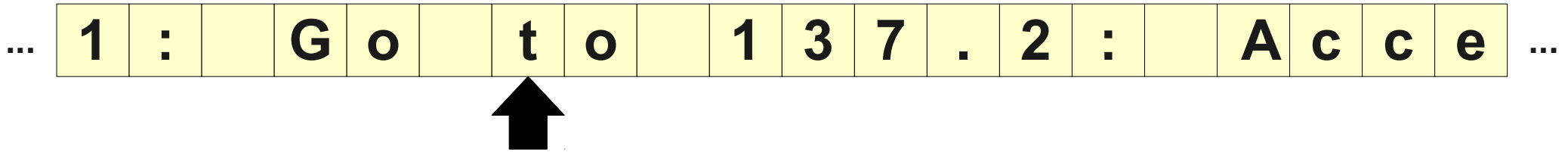
Instr **Letter**

Sketch of the Universal **WB** Program

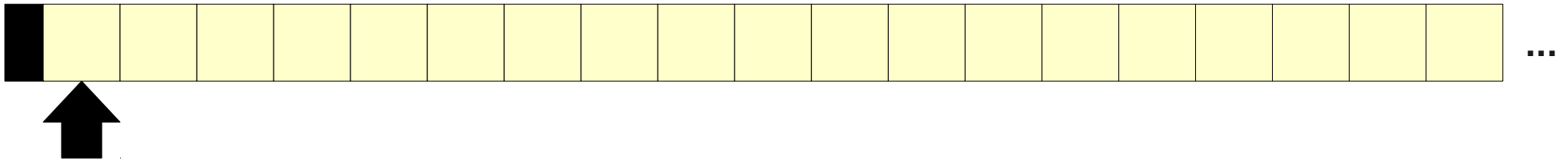
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

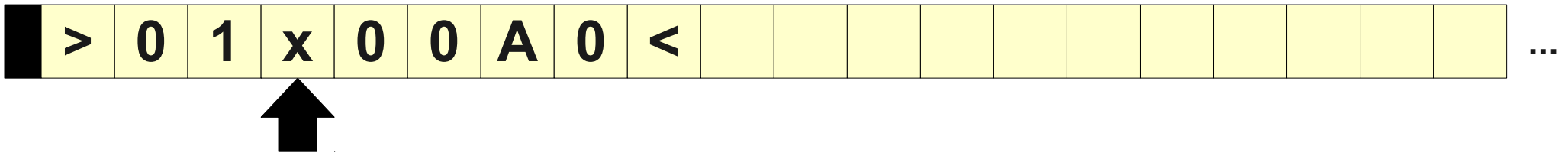


Variables for intermediate storage.

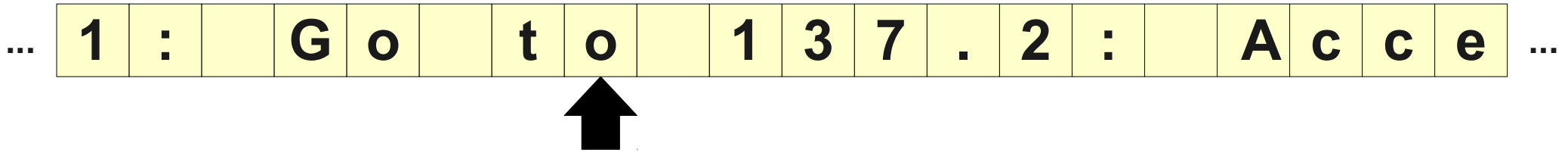
Instr **Letter**

Sketch of the Universal **WB** Program

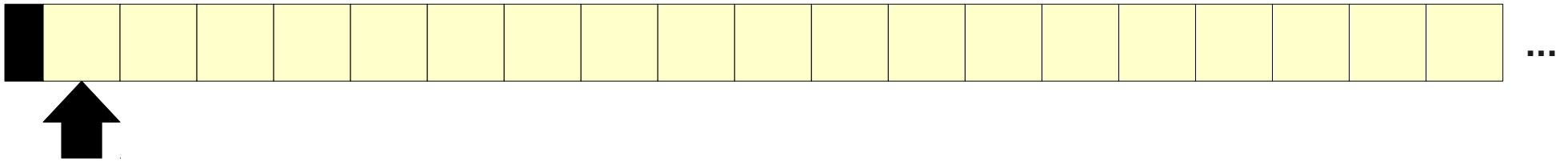
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

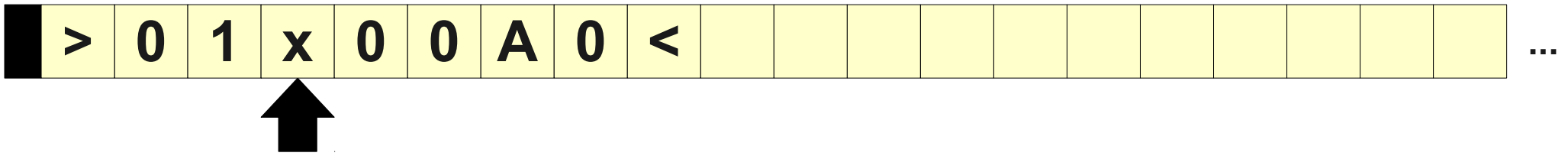


Variables for intermediate storage.

Instr **Letter**

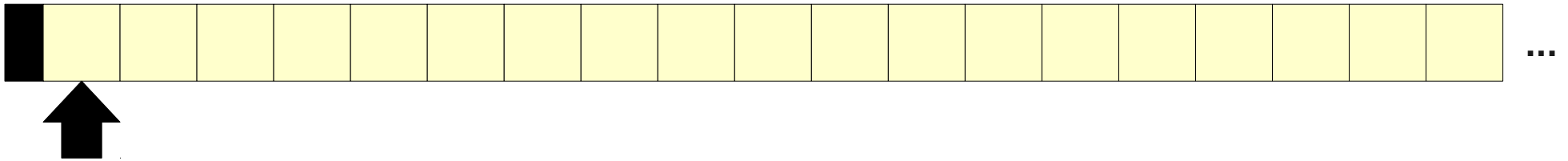
Sketch of the Universal **WB** Program

Simulated tape of the program being executed.



Program tape holding the program being executed.

Scratch tape for intermediate computation.

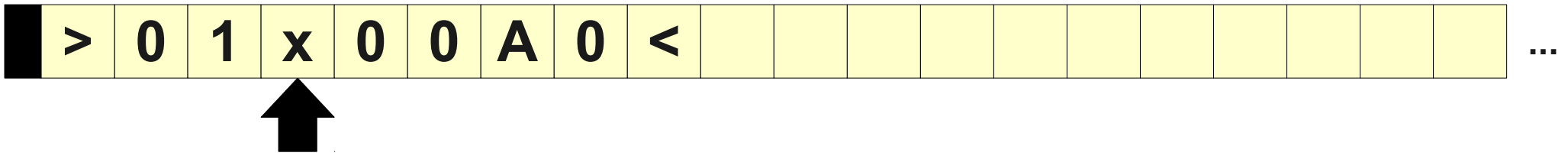


Variables for intermediate storage.

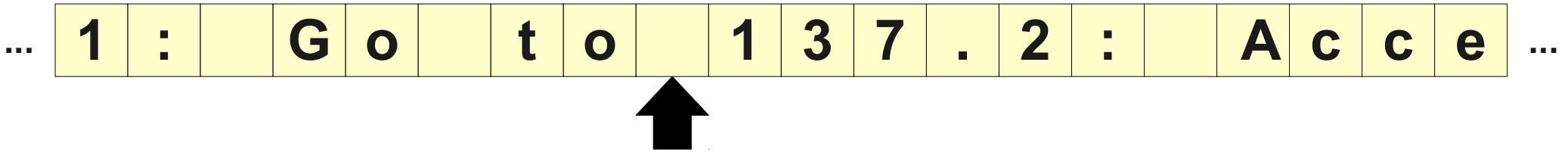
Instr **Letter**

Sketch of the Universal **WB** Program

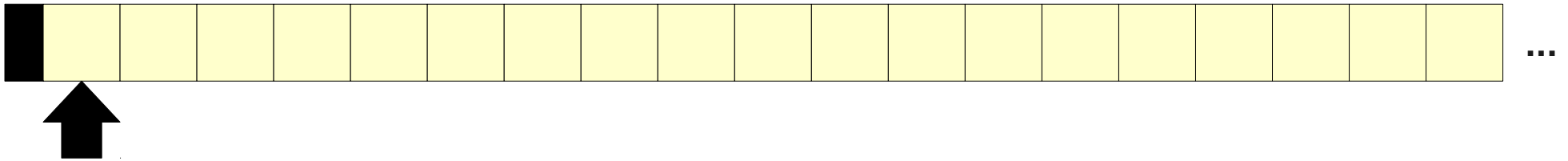
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

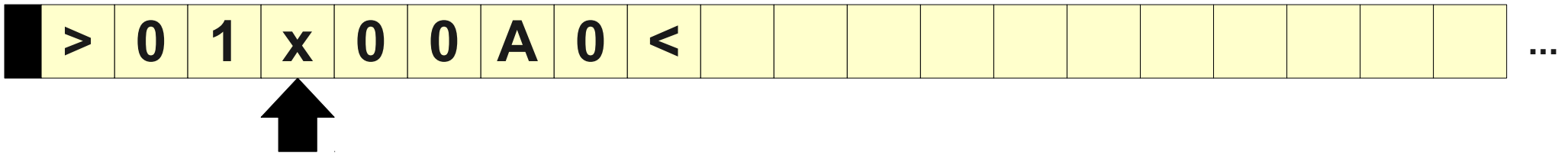


Variables for intermediate storage.

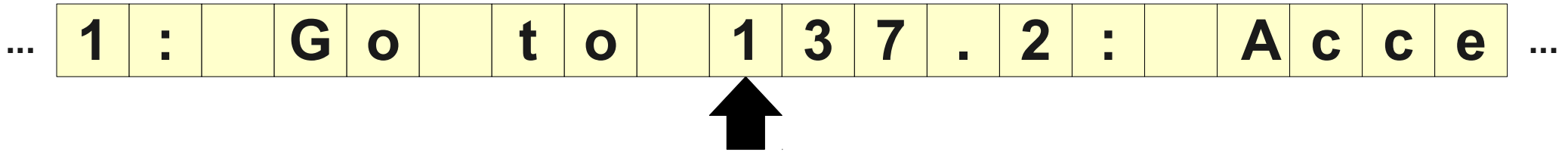
Instr **GoTo** **Letter**

Sketch of the Universal **WB** Program

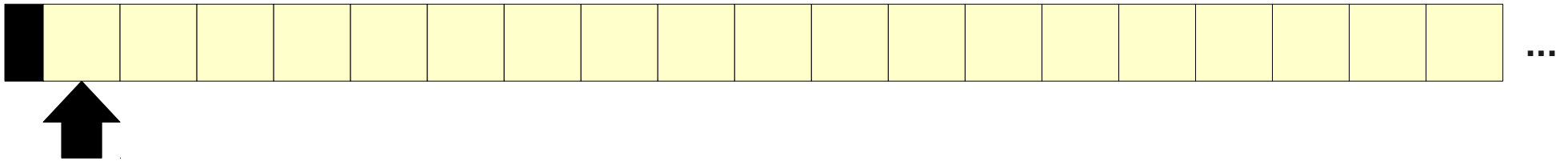
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

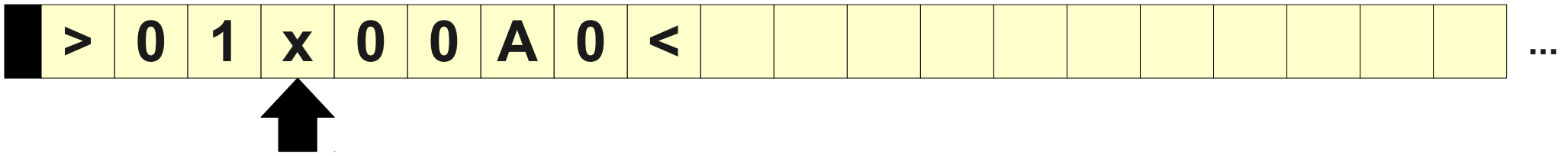


Variables for intermediate storage.

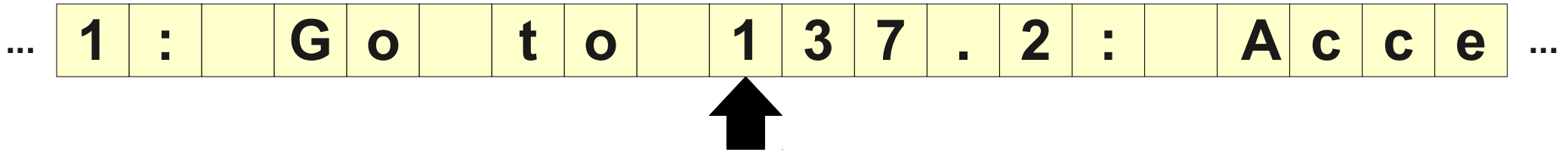
Instr **GoTo** **Letter**

Sketch of the Universal **WB** Program

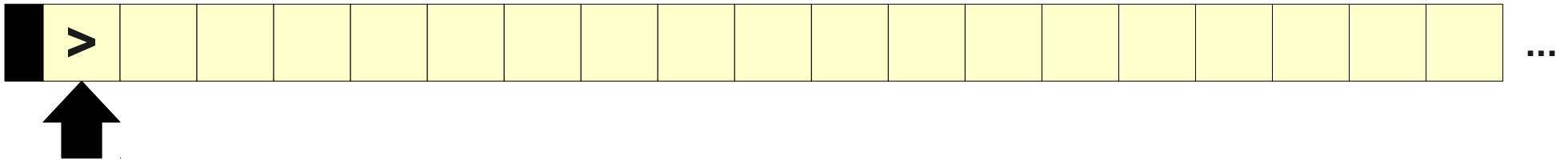
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

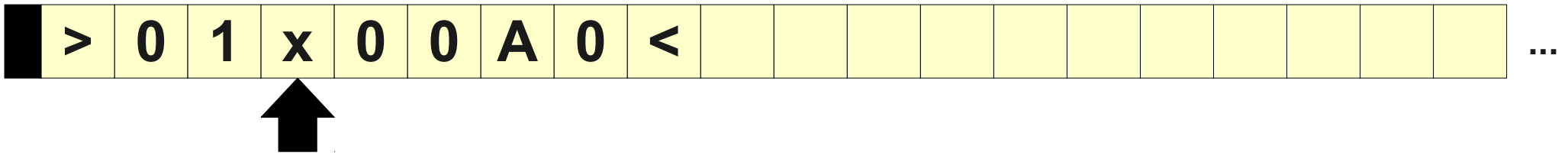


Variables for intermediate storage.

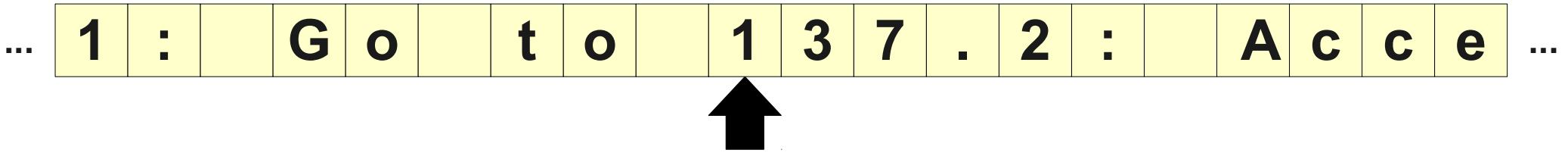
Instr **GoTo** **Letter**

Sketch of the Universal **WB** Program

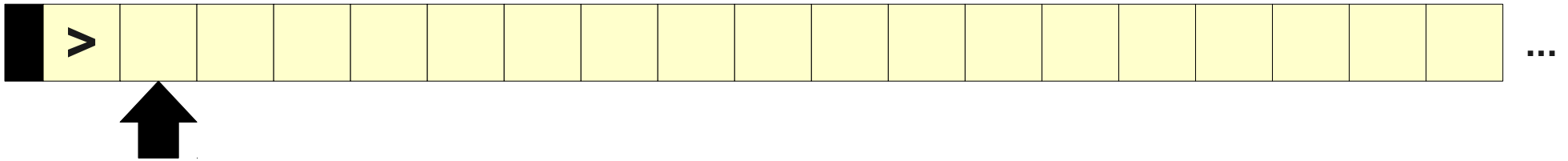
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

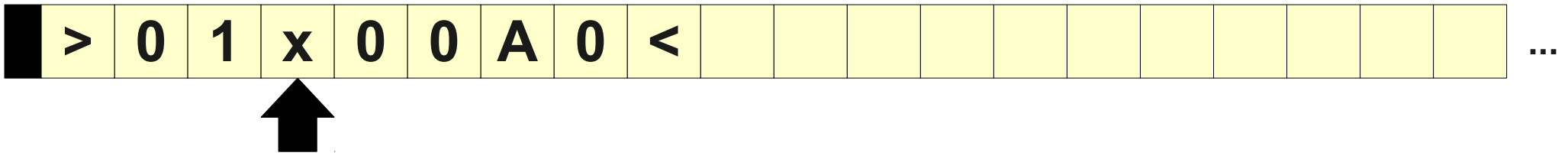


Variables for intermediate storage.

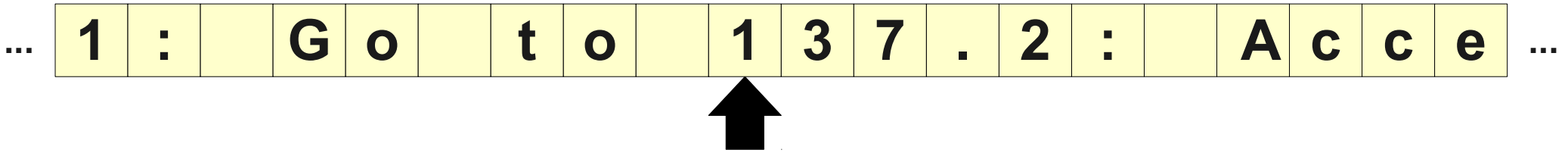
Instr **GoTo** **Letter**

Sketch of the Universal **WB** Program

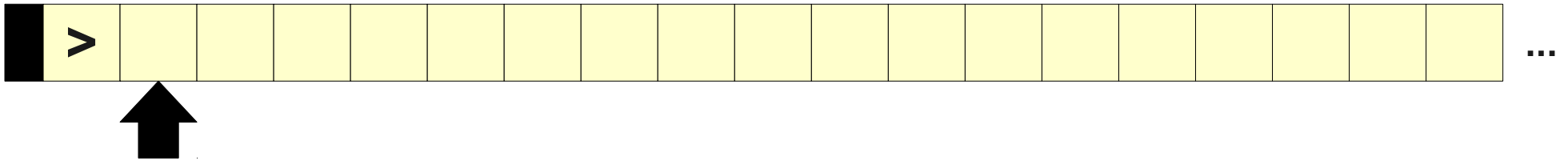
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

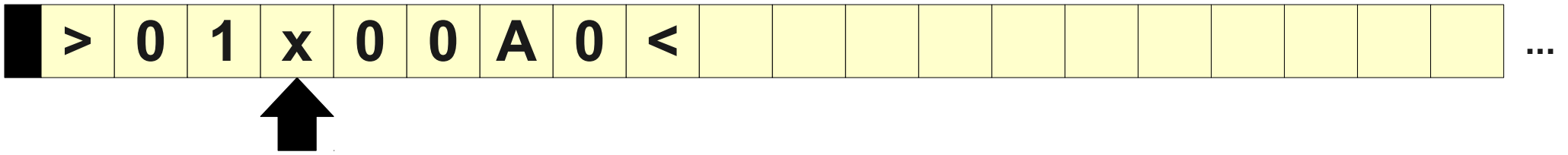


Variables for intermediate storage.

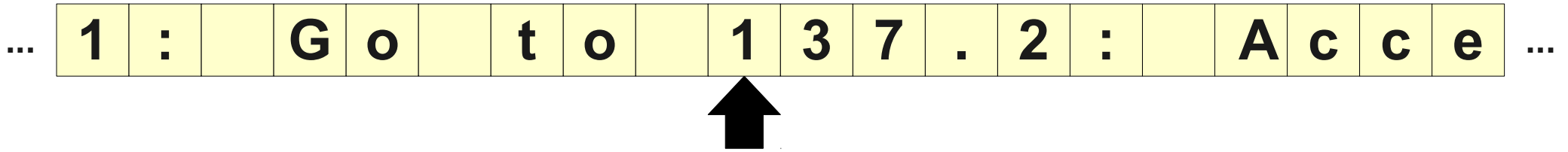
Instr GoTo **Letter** 1

Sketch of the Universal **WB** Program

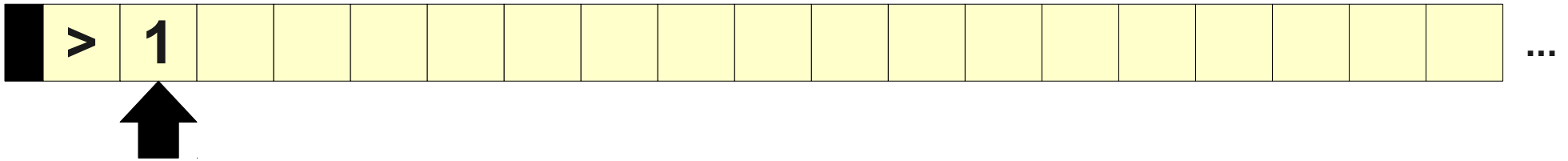
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

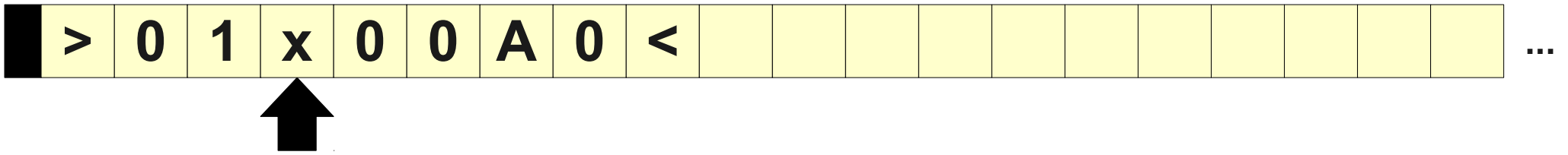


Variables for intermediate storage.

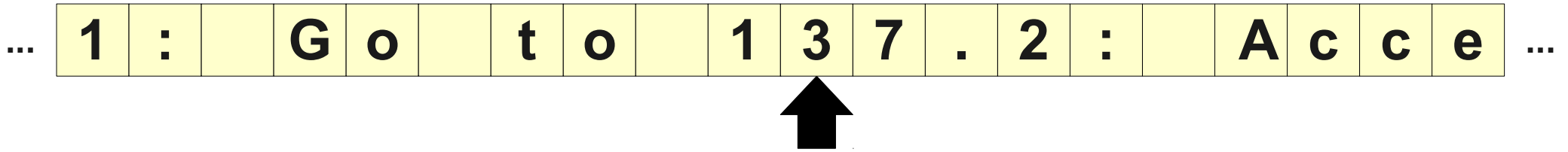
Instr GoTo **Letter** 1

Sketch of the Universal **WB** Program

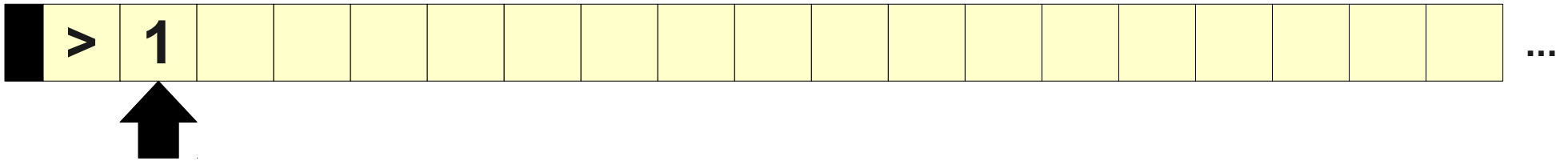
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

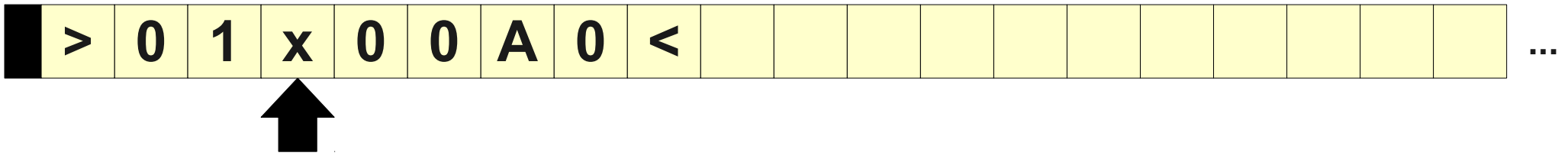


Variables for intermediate storage.

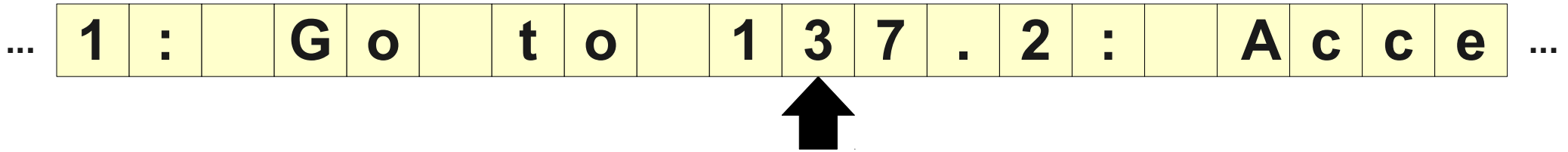
Instr GoTo **Letter** 1

Sketch of the Universal **WB** Program

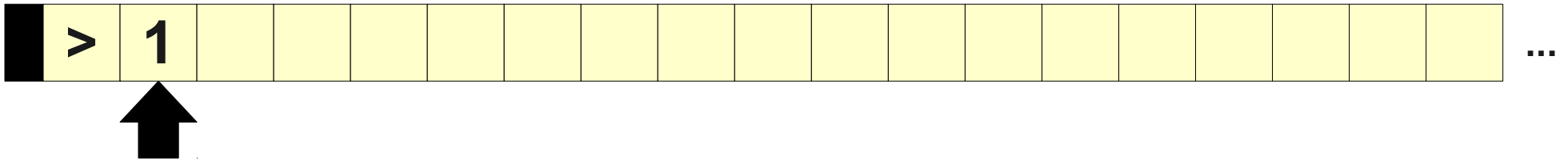
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

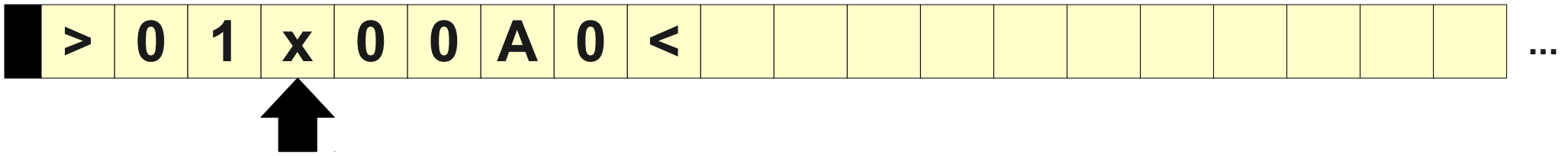


Variables for intermediate storage.

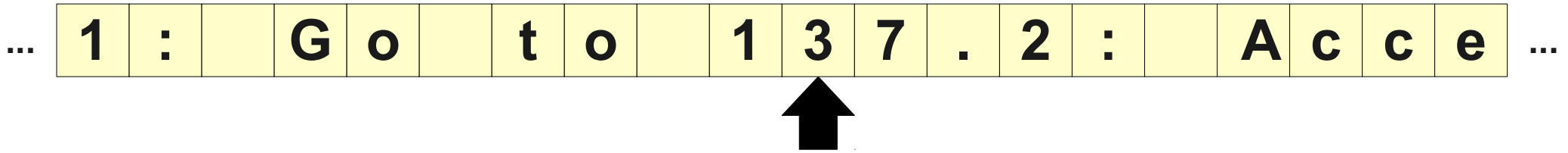
Instr GoTo **Letter** 3

Sketch of the Universal **WB** Program

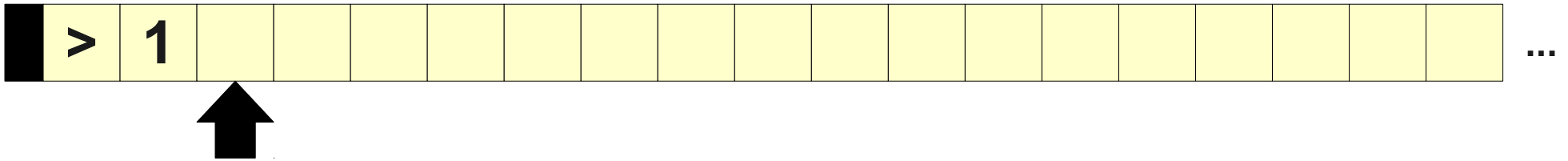
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

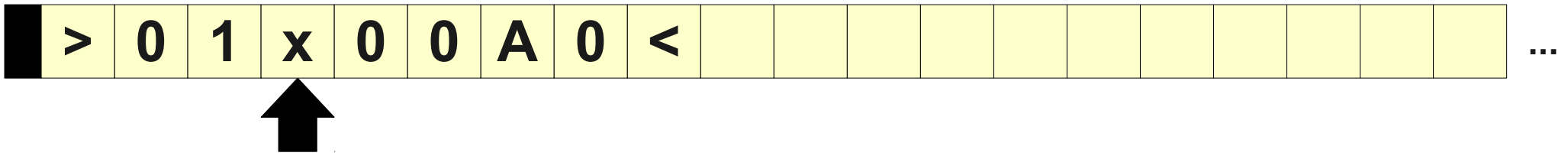


Variables for intermediate storage.

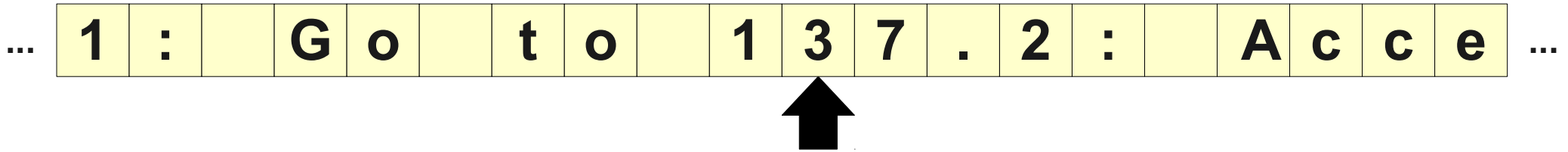
Instr GoTo **Letter** 3

Sketch of the Universal **WB** Program

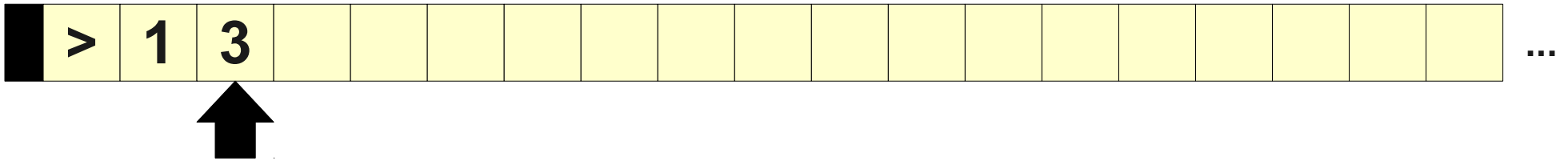
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

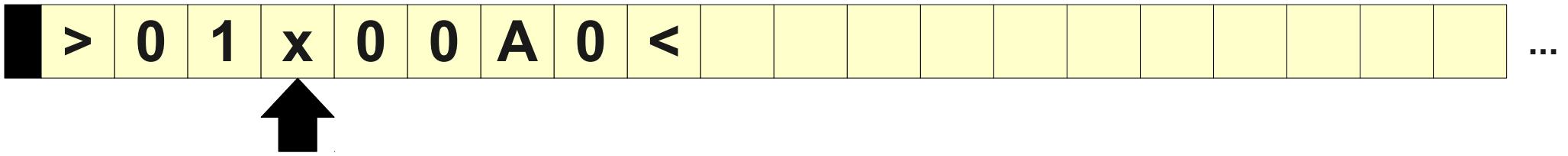


Variables for intermediate storage.

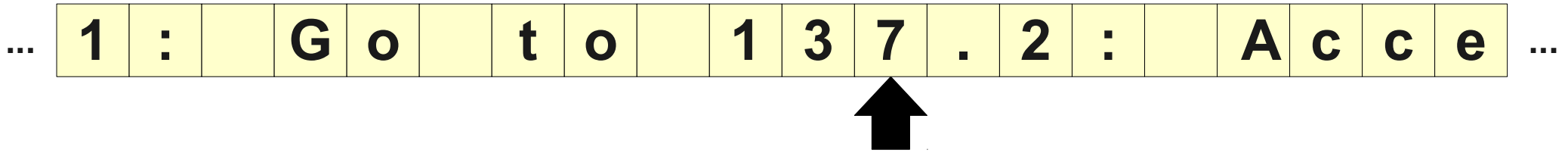
Instr **GoTo** **Letter** **3**

Sketch of the Universal **WB** Program

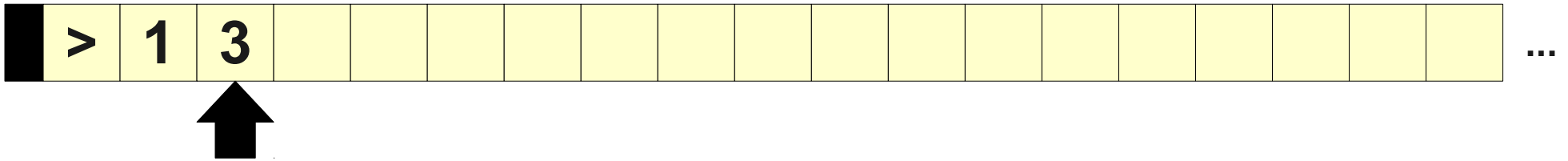
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

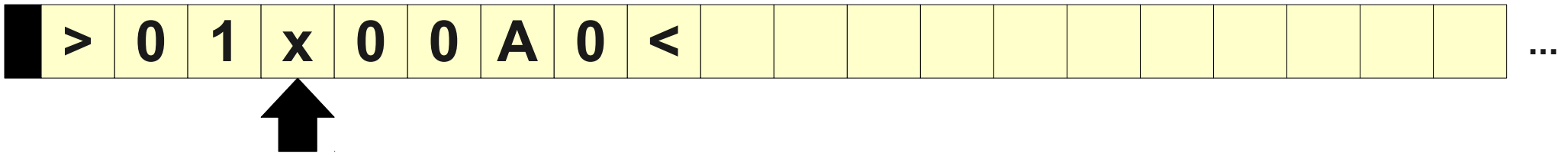


Variables for intermediate storage.

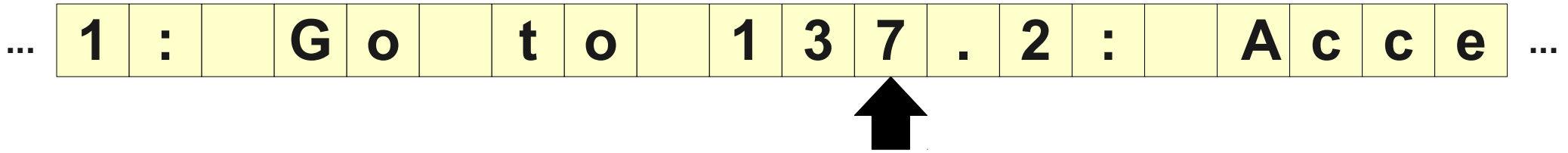
Instr GoTo **Letter** 3

Sketch of the Universal **WB** Program

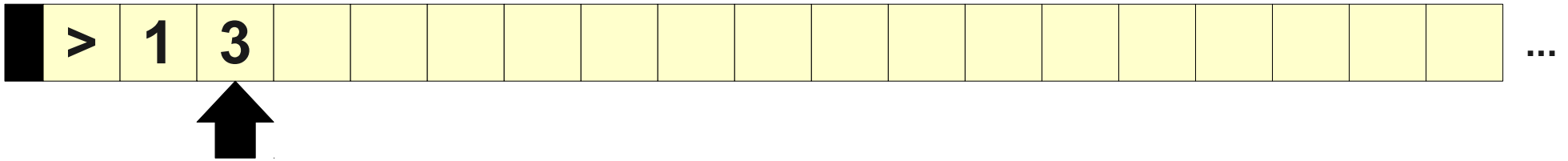
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

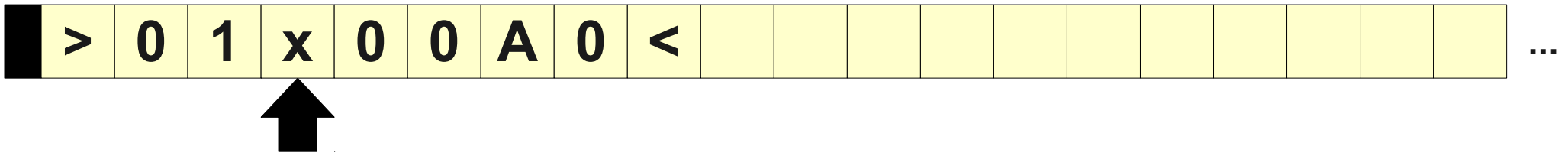


Variables for intermediate storage.

Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

Simulated tape of the program being executed.



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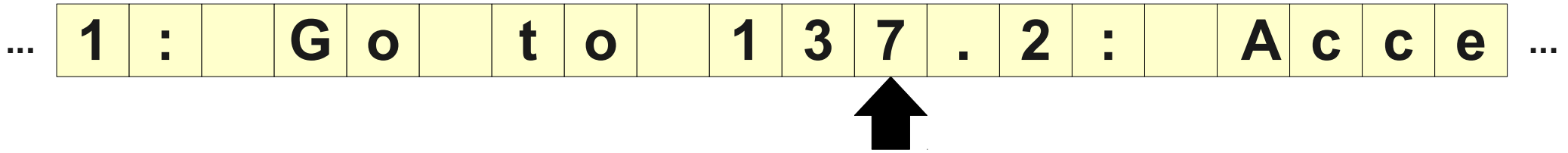
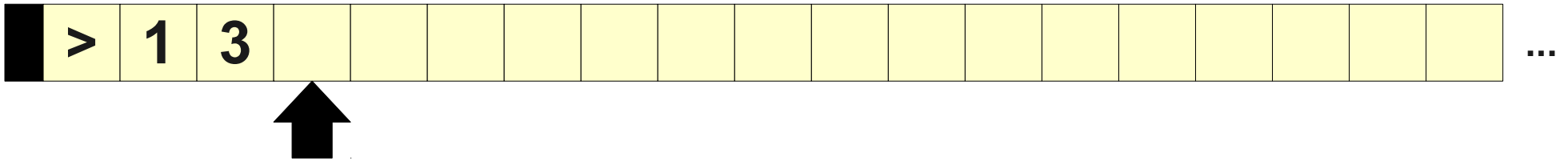


Scratch tape for intermediate computation.

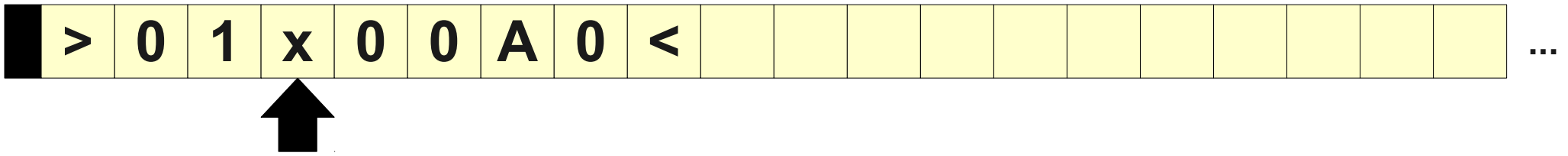


Variables for intermediate storage.

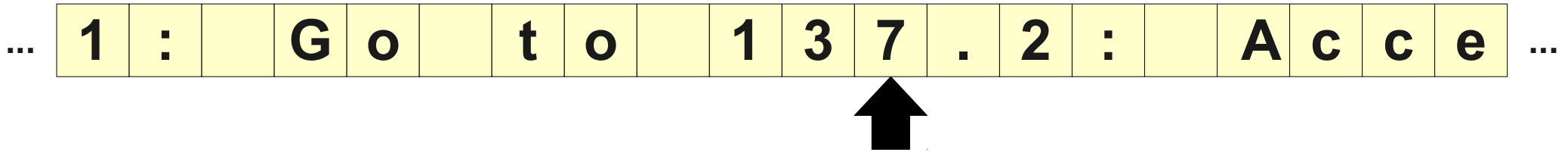
Instr GoTo **Letter** 7

Sketch of the Universal **WB** Program

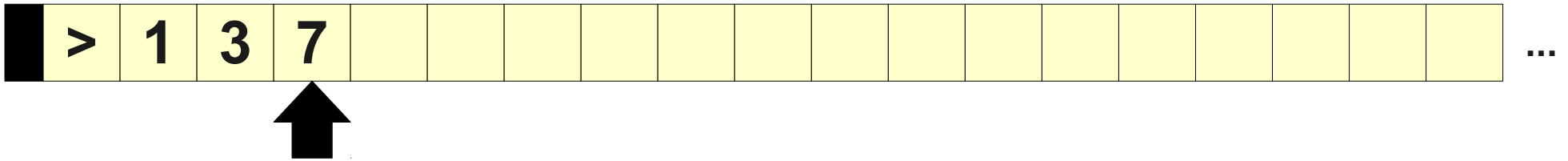
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

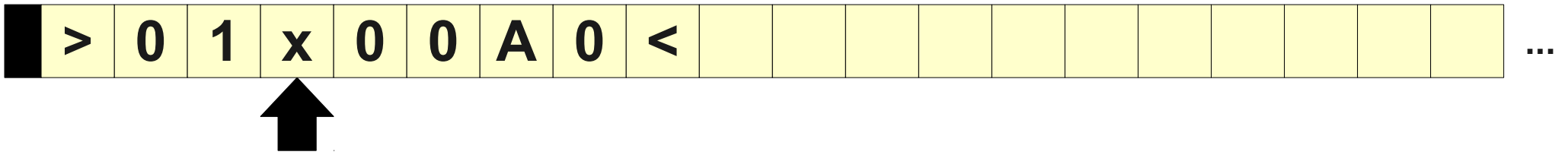


Variables for intermediate storage.

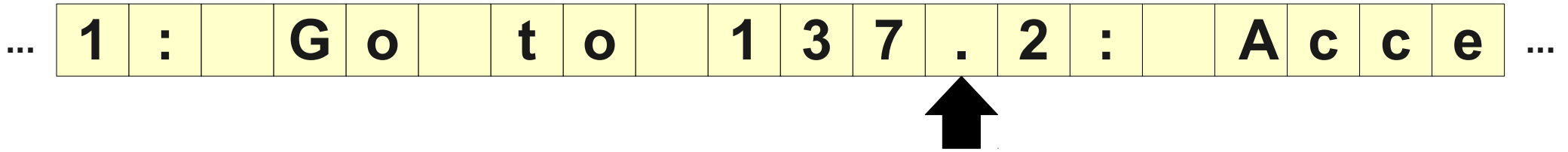
Instr GoTo **Letter** 7

Sketch of the Universal **WB** Program

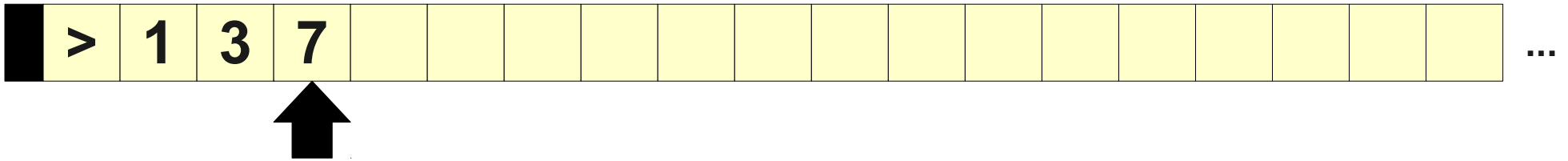
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

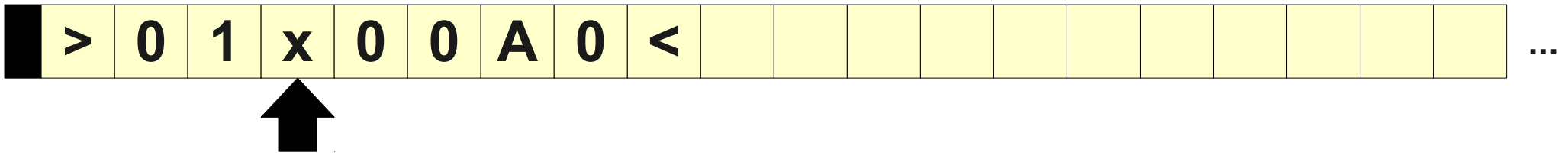


Variables for intermediate storage.

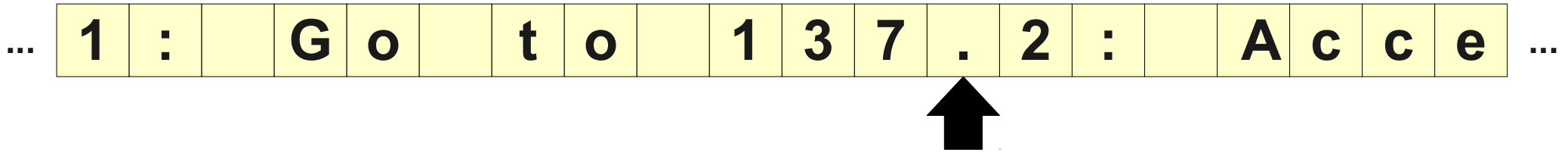
Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

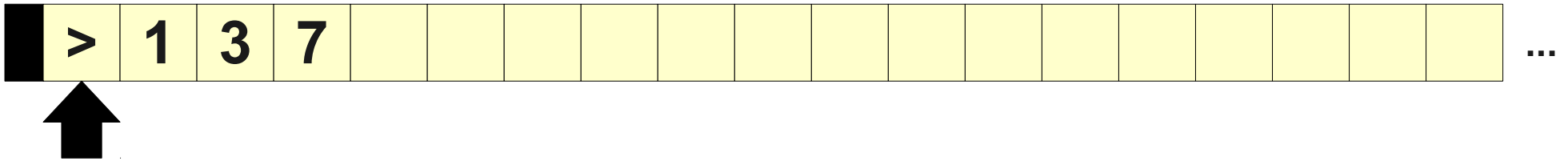
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

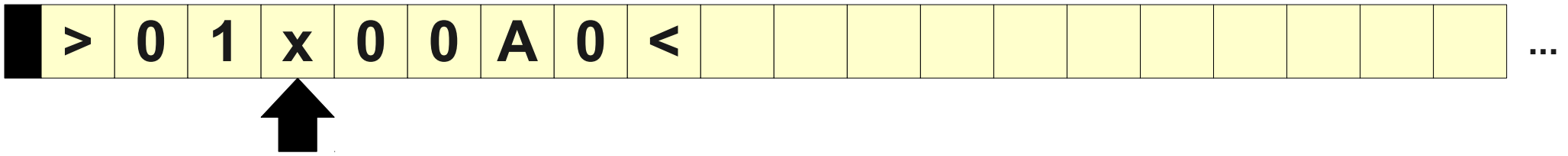


Variables for intermediate storage.

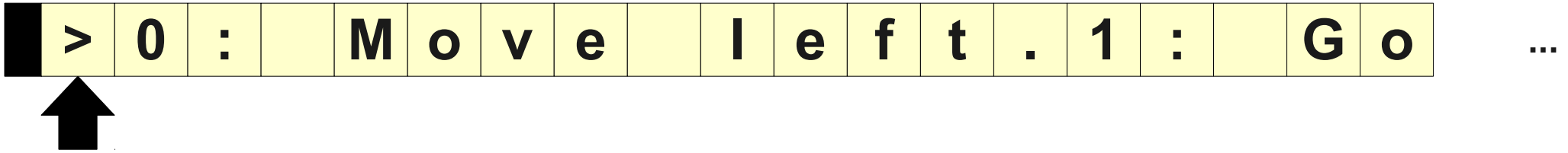
Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

Simulated tape of the program being executed.



Program tape holding the program being executed.



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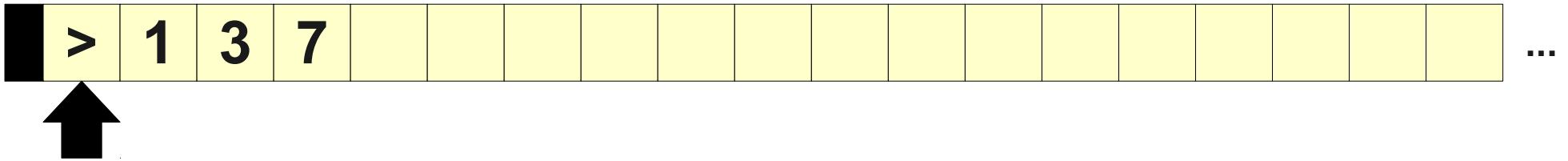
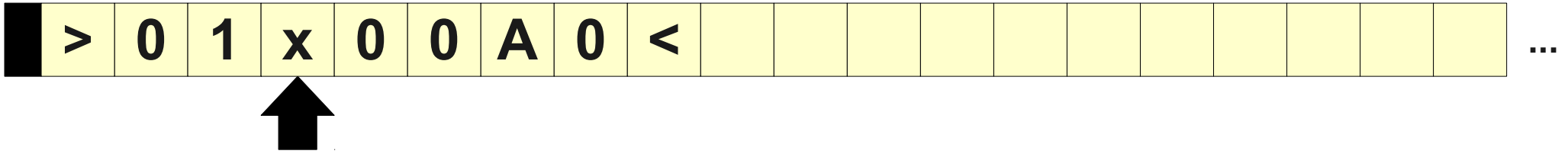


Variables for intermediate storage.

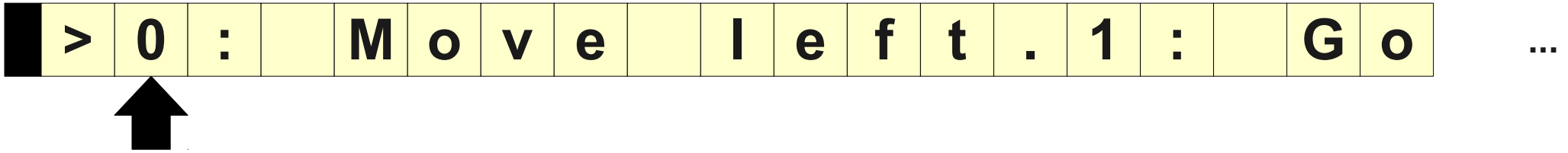
Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

Simulated tape of the program being executed.



Program tape holding the program being executed.



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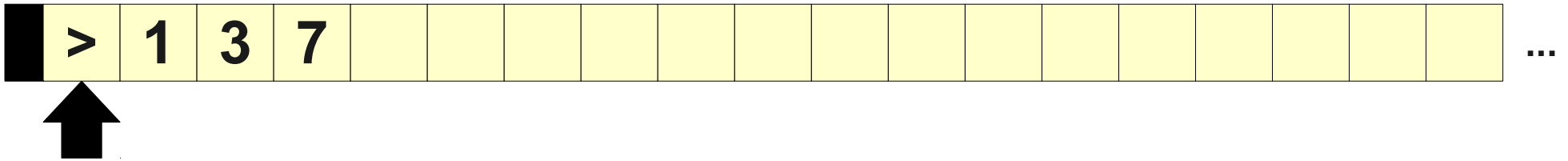
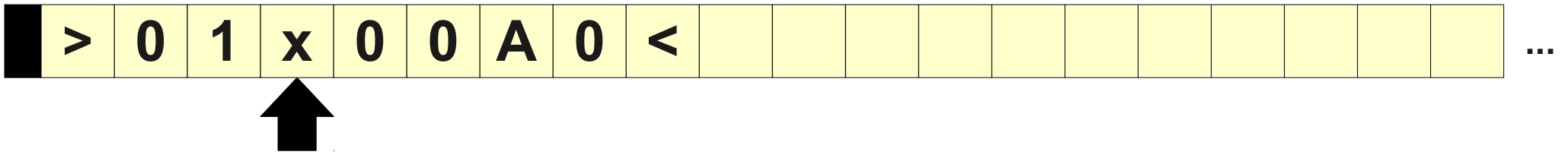


Variables for intermediate storage.

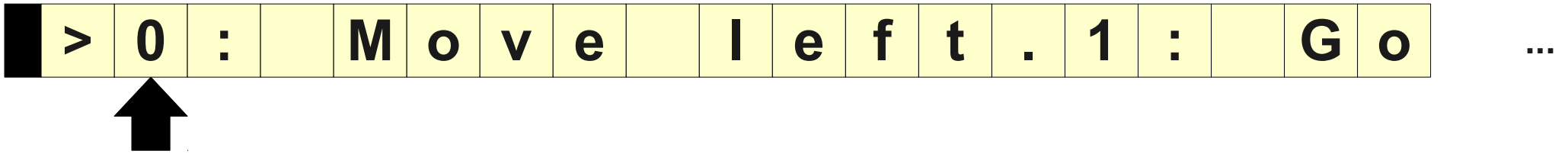
Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

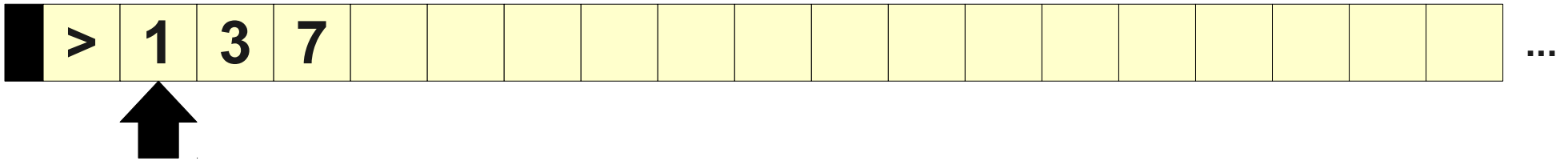
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

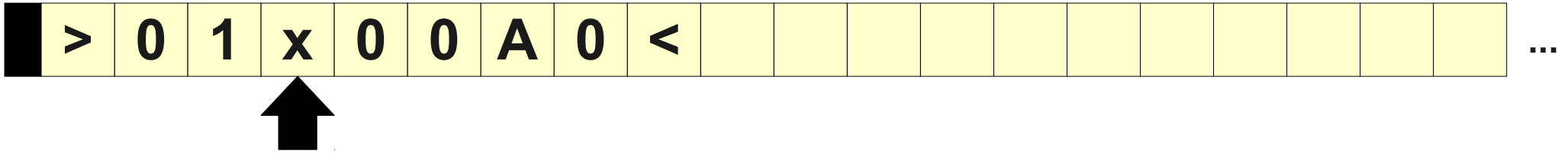


Variables for intermediate storage.

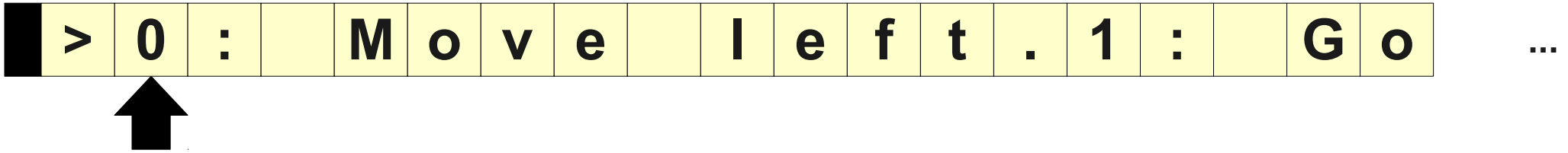
Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

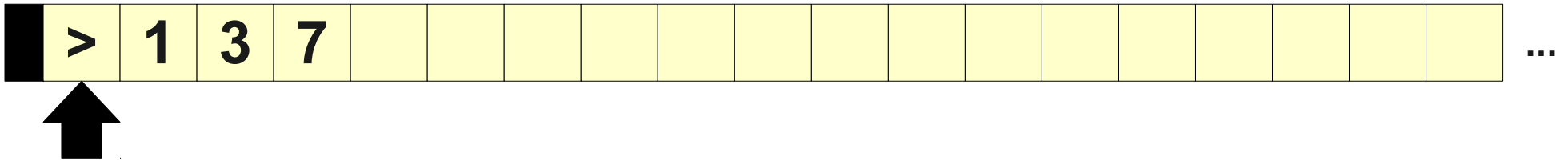
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.



Variables for intermediate storage.

Instr **GoTo** **Letter** **7**

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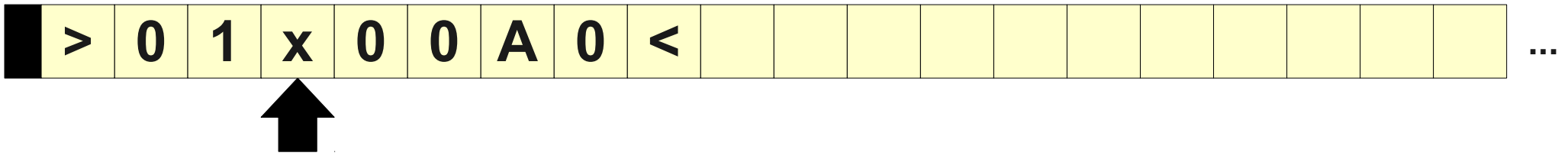
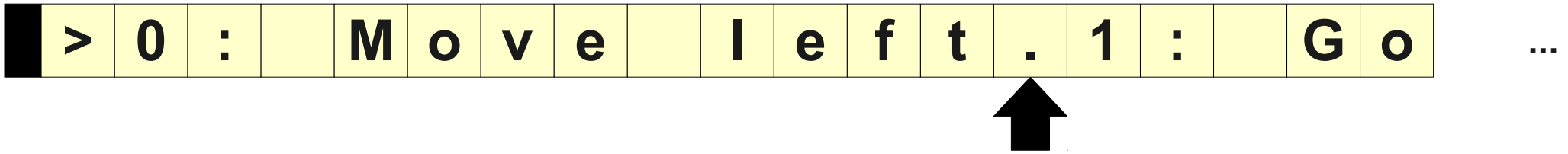
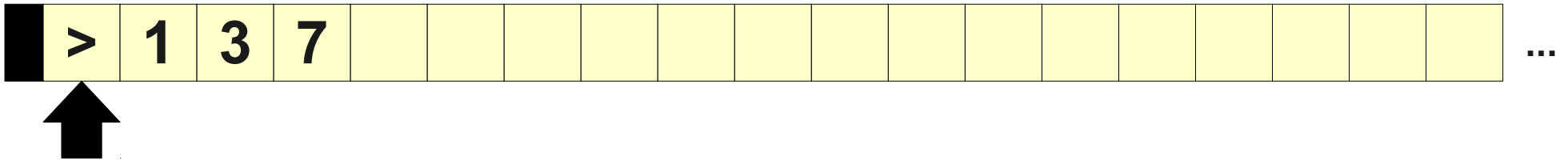
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

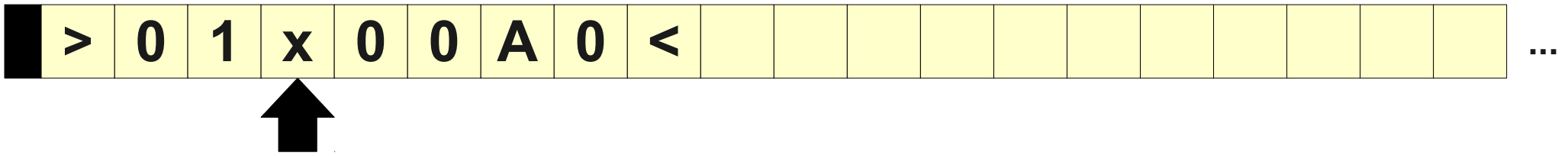


Variables for intermediate storage.

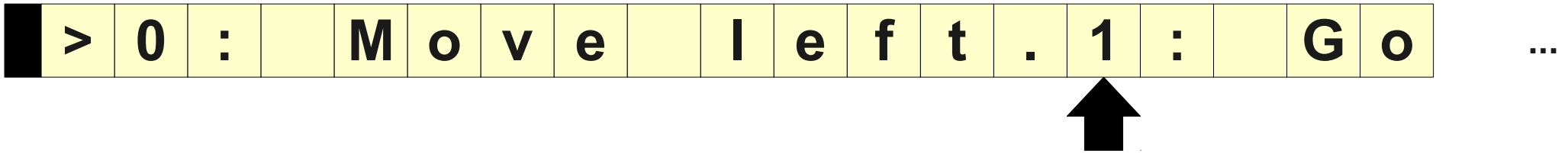
Instr GoTo **Letter** 7

Sketch of the Universal **WB** Program

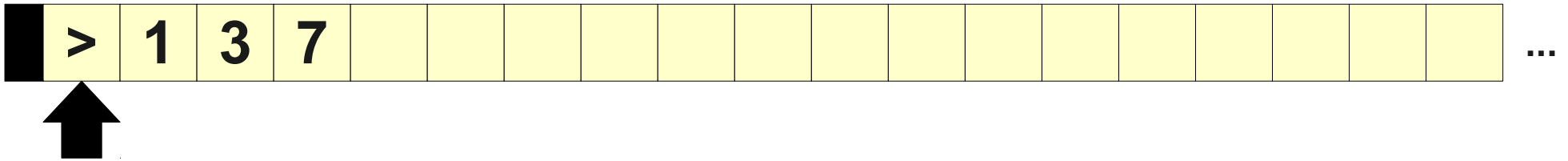
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

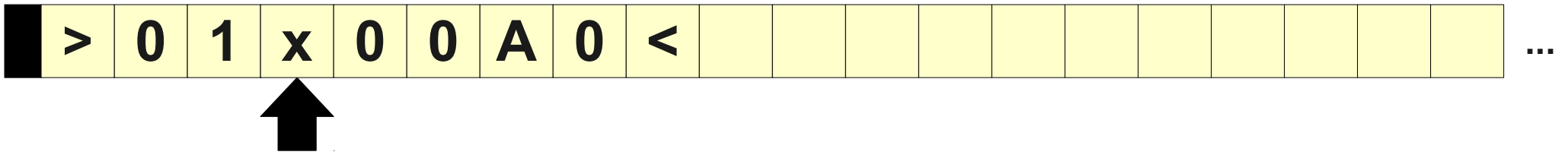


Variables for intermediate storage.

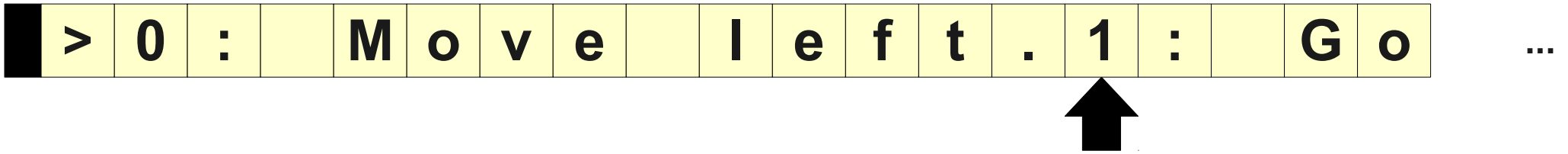
Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

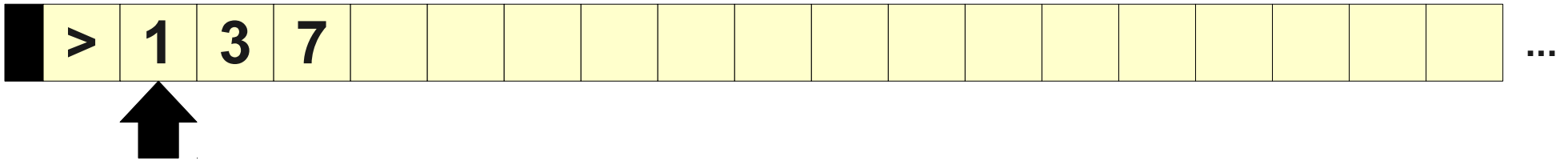
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

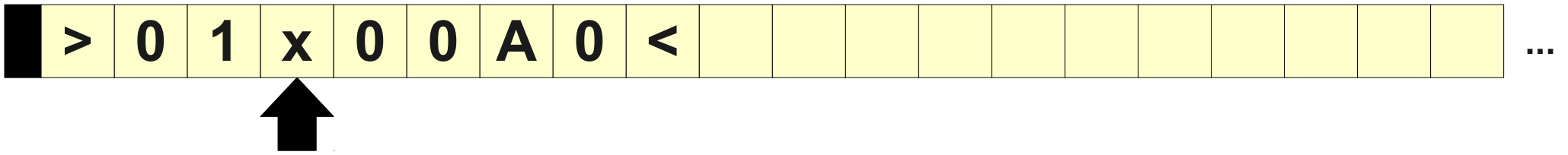


Variables for intermediate storage.

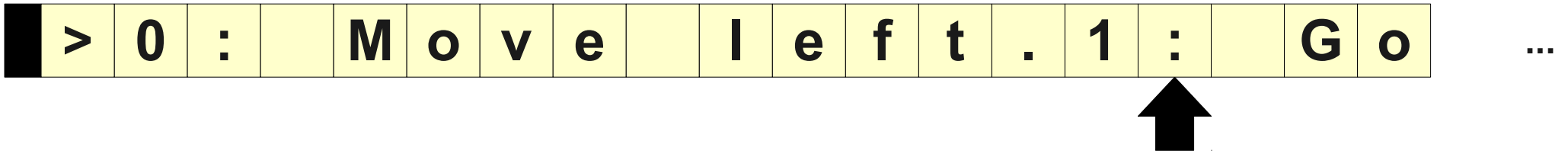
Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

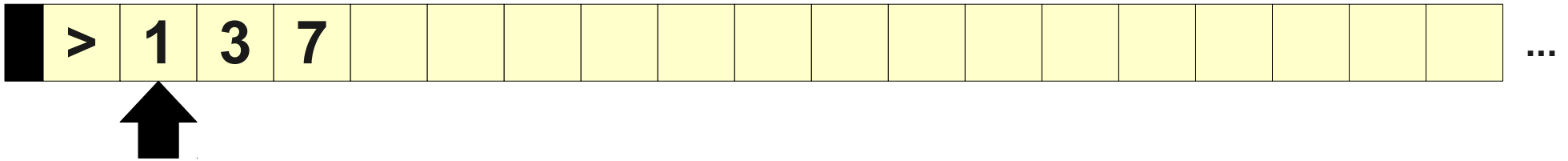
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.

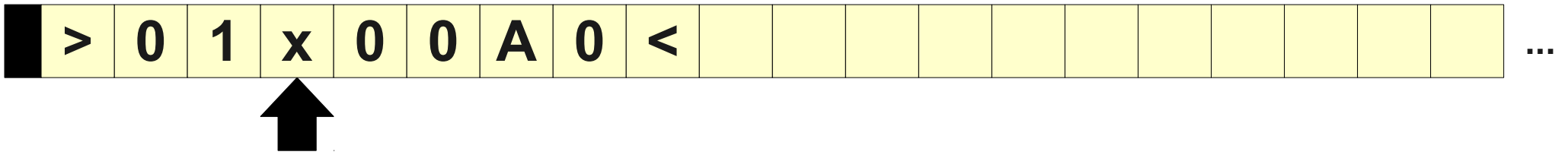


Variables for intermediate storage.

Instr **GoTo** **Letter** **7**

Sketch of the Universal **WB** Program

Simulated tape of the program being executed.



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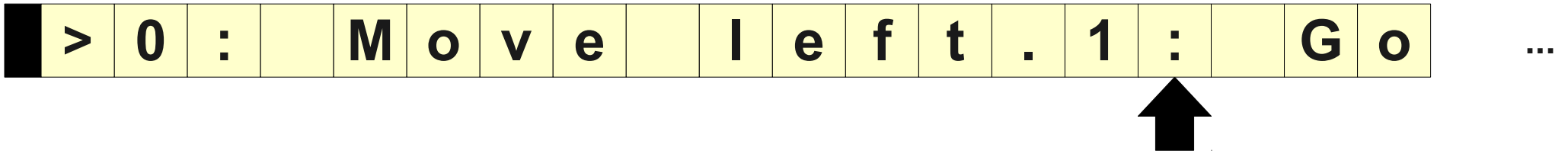
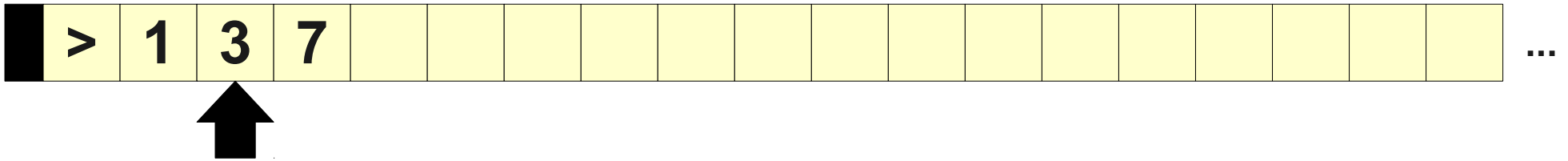


Scratch tape for intermediate computation.



Variables for intermediate storage.

Instr **GoTo** **Letter** **7**

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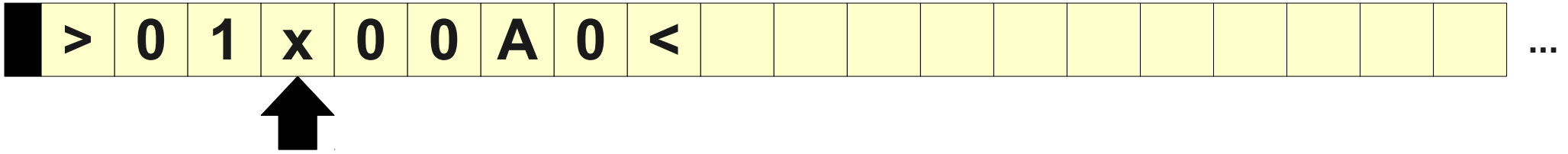
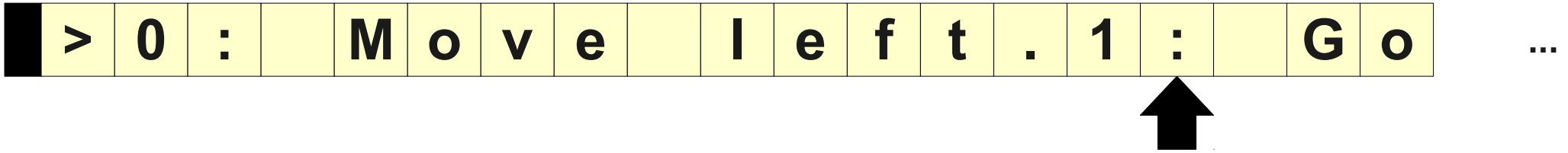
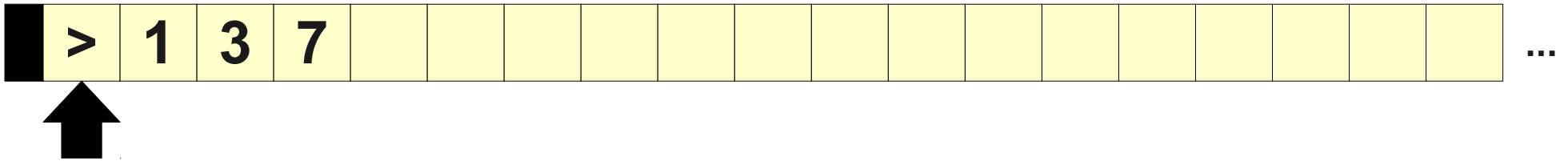


Program tape holding the program being executed.



Scratch tape for intermediate computation.



Variables for intermediate storage.

Instr GoTo **Letter** 7

The Language of U_{TM}

- From a formal language perspective, what is the language of a universal machine?
- The universal Turing machine accepts all strings of the form $\langle M, w \rangle$, where M is a Turing machine that accepts string w.
- This language is called $\mathbf{A_{TM}}$:

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts } w. \}$$

- The universal **WB** program has language $\mathbf{A_{WB}}$:

$$A_{WB} = \{ \langle P, w \rangle \mid P \text{ is a } \mathbf{WB} \text{ program that accepts } w. \}$$

- Both A_{TM} and A_{WB} are recursively enumerable.

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Instrumenting U_{TM}

- Each universal machine has the following structure:
  - Find the next instruction to execute.
  - Determine how to execute that instruction.
  - Simulate the execution of that instruction.
  - Repeat.
- In-between each of those steps, we can add extra logic to modify the behavior of the simulated machine.

Tricks with Universal Machines

- Suppose that we have a TM (or **WB** program) and a string and want to see if that TM accepts the string within *n* steps.
- Can this be done with a Turing machine or **WB** program?
- As a language problem:

$$A_{FAST} = \{ \langle M, w, n \rangle \mid M \text{ is a TM that accepts } w \text{ in at most } n \text{ steps.} \}$$

- Is A_{FAST} recursively enumerable?

Sketch of a Program for A_{FAST}

- We could write a program for A_{FAST} using the following logic:
- "On input *x*:
  - Check that *x* has the form ⟨*M*, *w*, *n*⟩ for a TM *M*, string *w*, and natural number *n*. If not, reject the input string.
  - Copy *n* to a separate tape.
  - Using the universal Turing machine, run *M* on *w*. Before executing any step of *M*, decrement *n*.
  - If *n* reaches 0, reject.
  - Otherwise, if *M* accepts *w*, accept the input string *x*.
  - Otherwise, if *M* rejects *w*, reject the input string *x*."

Tricks with Universal Machines

- Suppose we have a **WB** program *P* and want to see if it accepts string *w* without ever executing the same instruction twice.
- Can we design a TM or **WB** program to determine this?
- As a language problem:

$$A_{ONCE} = \{ \langle P, w \rangle \mid P \text{ is a } \textbf{WB} \text{ program that accepts } w \text{ without executing the same instruction twice. } \}$$

- Is A_{ONCE} recursively enumerable?

Sketch of a Program for A_{ONCE}

- We could write a program for A_{ONCE} using the following logic:
- "On input *x*:
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  - Use the universal **WB** program to run *P* on *w*.
  - After executing each instruction, replace that instruction with **`Reject`**.
  - If *P* accepts *w*, accept the input string *x*.
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High-Level Machine Specifications

- The machines we have described in the past few examples all the form
  - "On input *x*:
    - Description of what to do on *x*."
- These descriptions are called **high-level descriptions**.
- Because of the Church-Turing thesis, if we can know that every step could be done by a computer, then we know a TM/**WB** program exists for the description.
- Unless explicitly asked to write a **WB** program or Turing machine, feel free to use high-level descriptions in your proofs.

Sketch of a Program for A_{FAST}

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A Nice Shorthand

- Most interesting programs require their input to be structured in some fashion.
- As a new notation, we will allow ourselves to write high-level descriptions like this:

  "On input $\langle O_1, O_2, \ldots, O_n \rangle$:

  Do something with $O_1, \ldots, O_n$."

  to mean

  "On input *x*:

  Check that *x* has the form $\langle O_1, O_2, \ldots, O_n \rangle$. If not, reject.

  Do something with $O_1, \ldots, O_n$."

Nondeterministic Turing Machines

Nondeterministic Turing Machines

- A **nondeterministic Turing machine** (abbreviated **NTM**) is a Turing machine in which there may be multiple transitions defined for a particular state/input combination.
- If **any** possible set of choices causes the machine to accept, it accepts.
- Otherwise, if there is at least one choice that loops forever, the NTM loops forever.
- Otherwise, all choices cause the machine to reject, and the NTM rejects.

Nondeterministic Algorithms

- A natural number greater than 0 is **composite** if it is not prime.
- Let Σ = { `1` } and consider the language

$$\textit{COMPOSITE} = \{ \texttt{1}^n \mid n \text{ is composite} \}$$

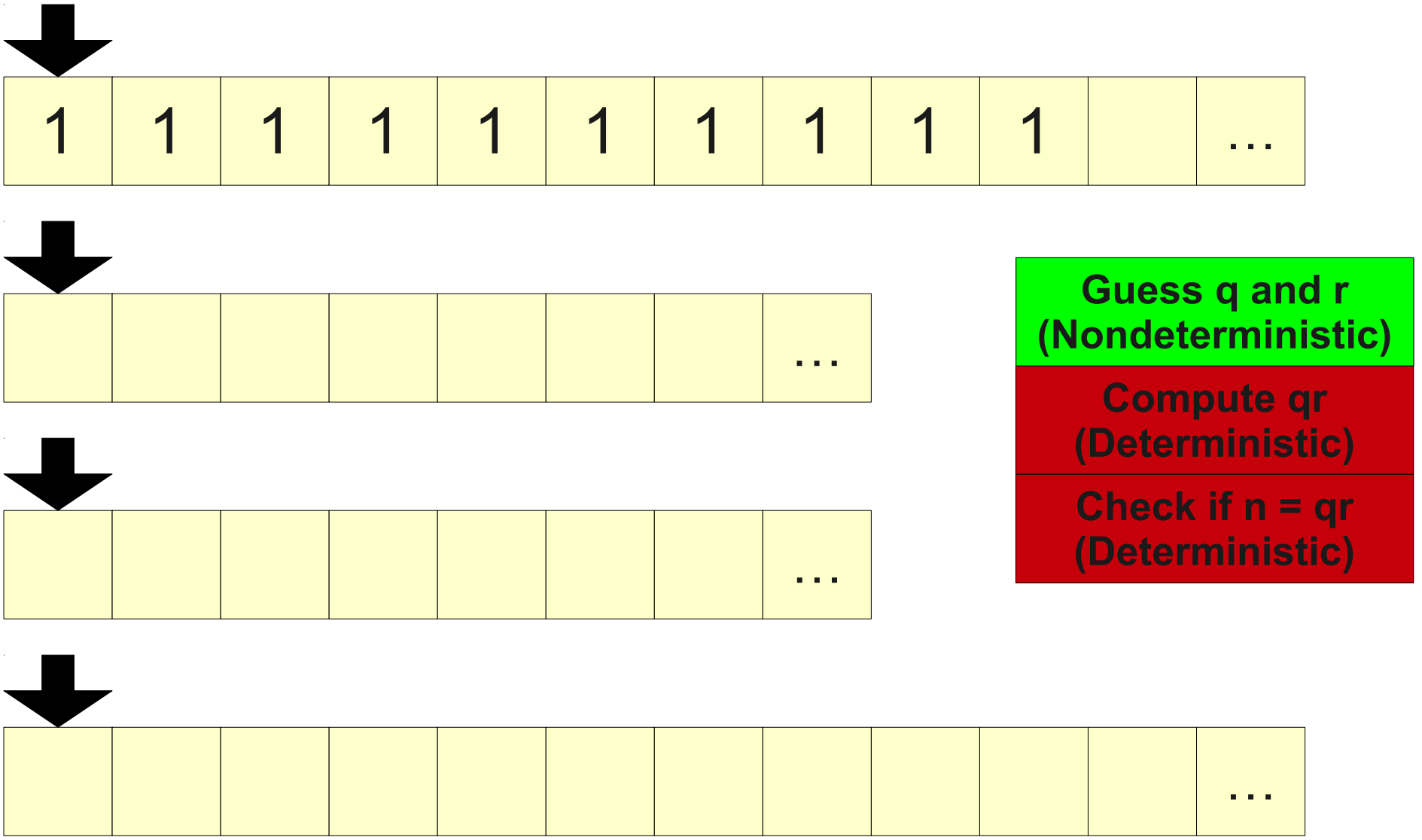
Nondeterministic Algorithms

- A natural number greater than 0 is **composite** if it is not prime.
- Let Σ = { 1 } and consider the language

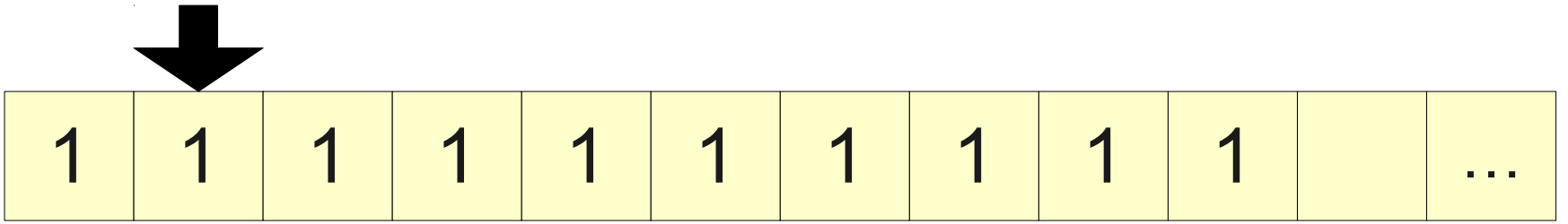
$$COMPOSITE = \{ 1^n \mid n \text{ is composite} \}$$

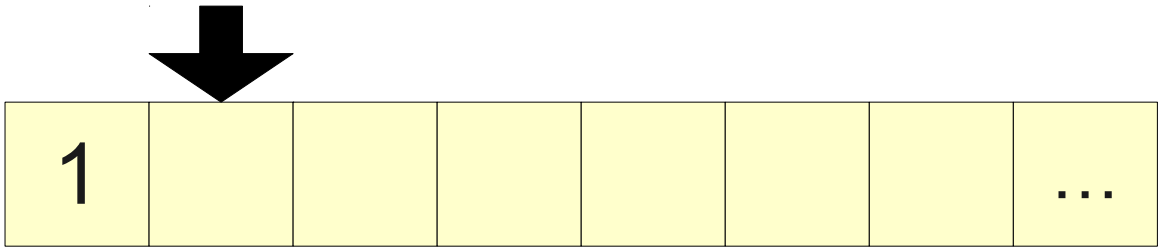
- We can build a **multitape, nondeterministic TM** for *COMPOSITE* as follows:
- M = “On input 1^n:
  - **Nondeterministically** write out q 1s on a second tape ($2 \leq q < n$)
  - **Nondeterministically** write out r 1s on a third tape ($2 \leq r < n$)
  - **Deterministically** check if qr = n.
  - If so, accept.
  - Otherwise, reject”

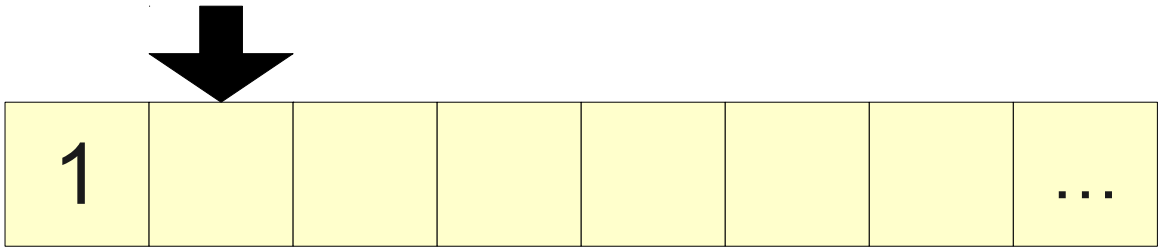
Nondeterministic Algorithms

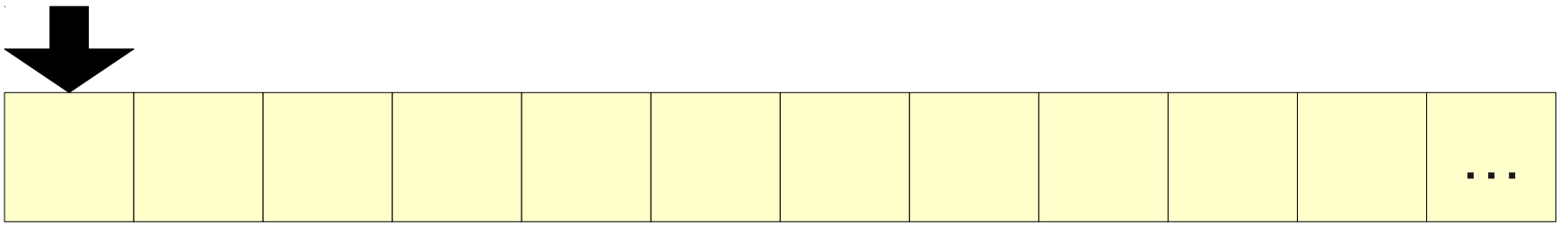


Nondeterministic Algorithms









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Compute qr (Deterministic)

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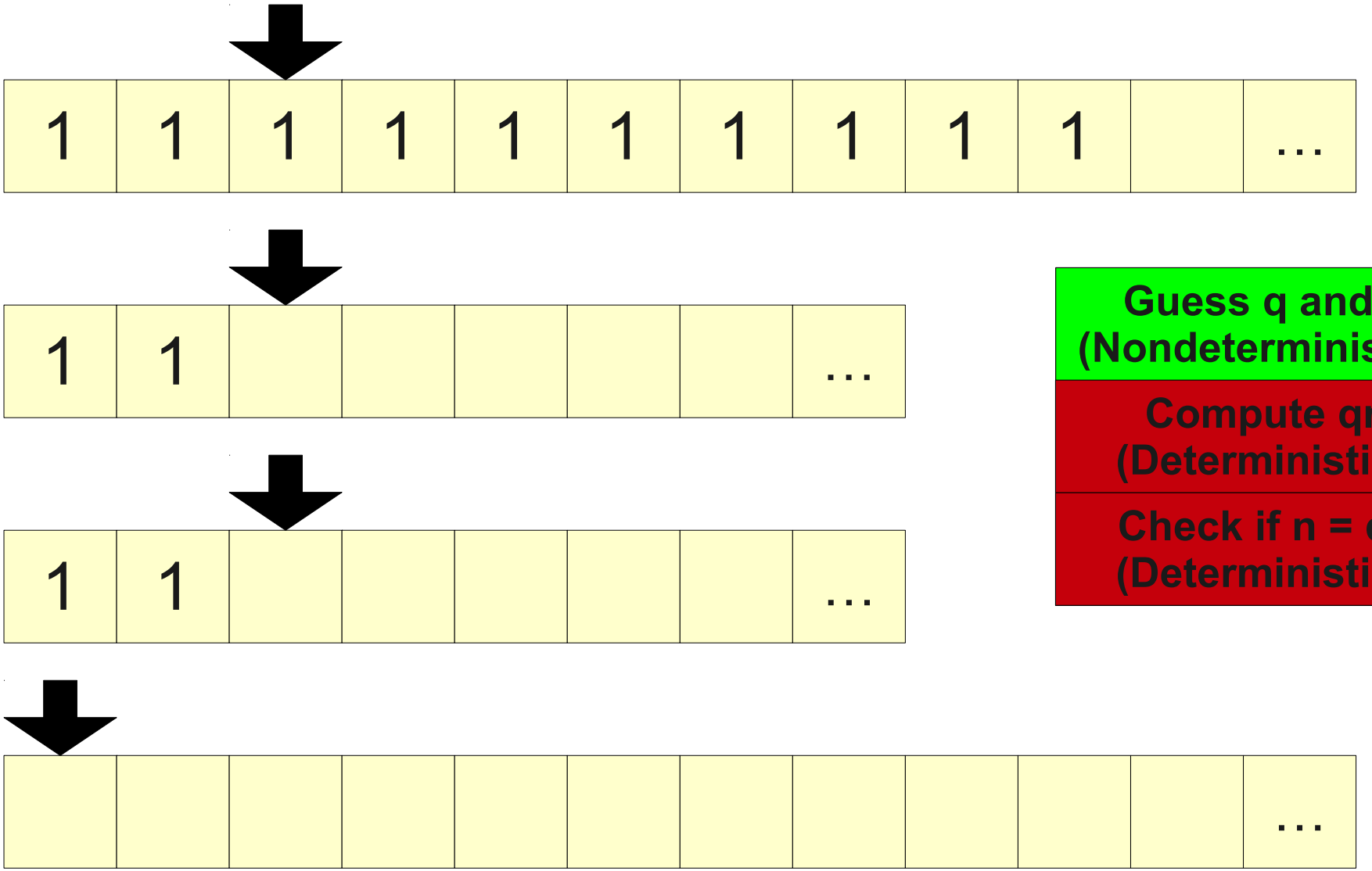
Nondeterministic Algorithms



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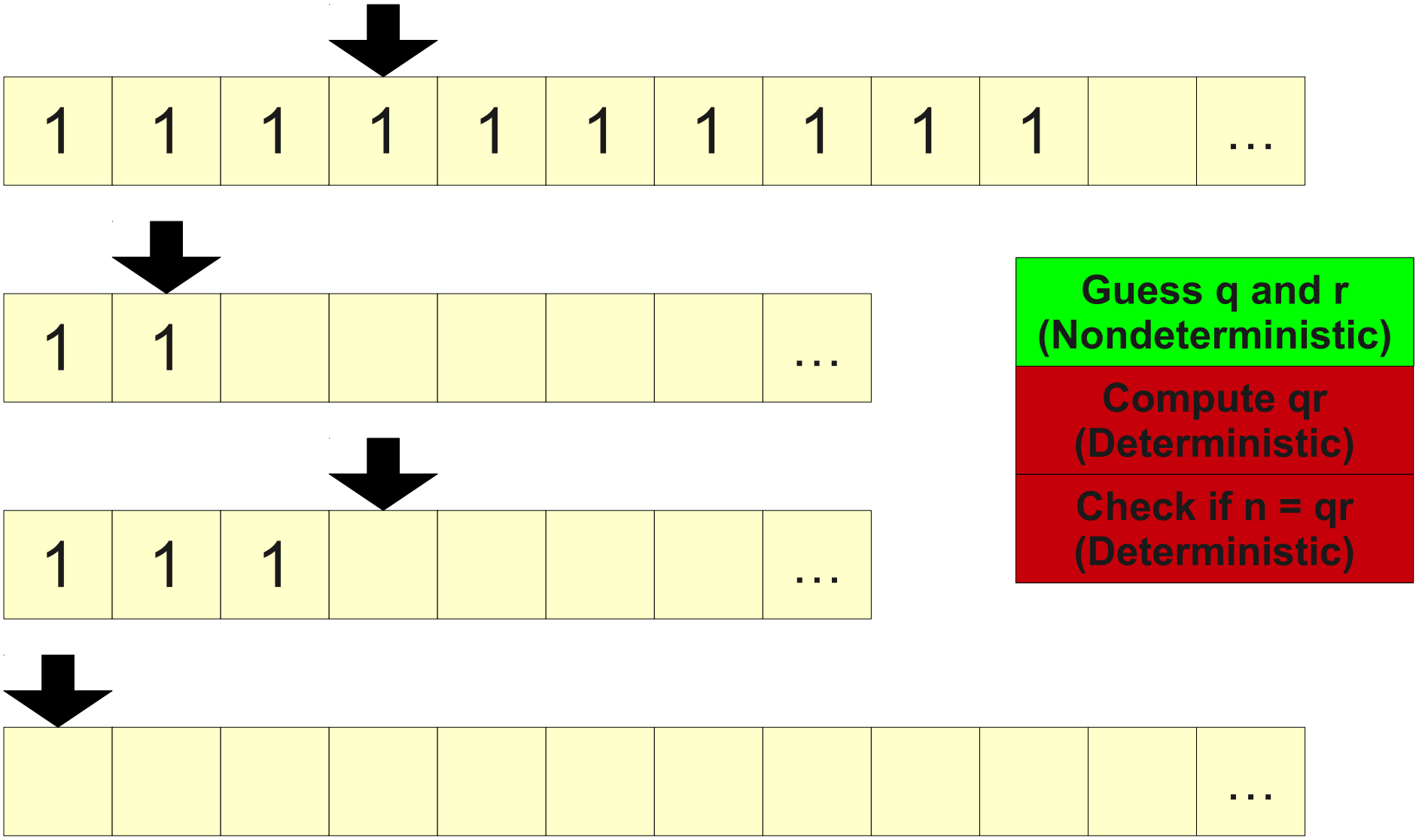

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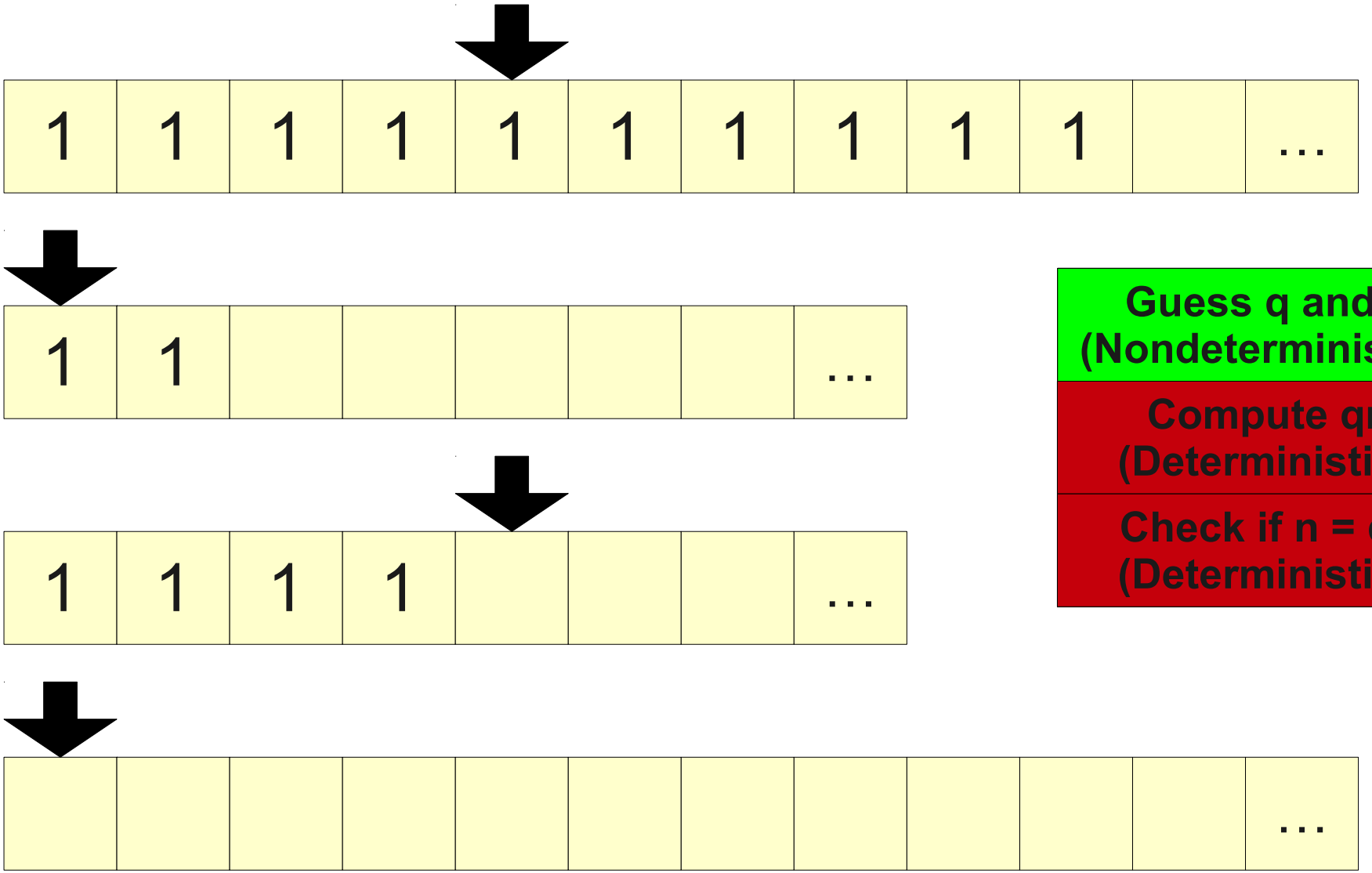

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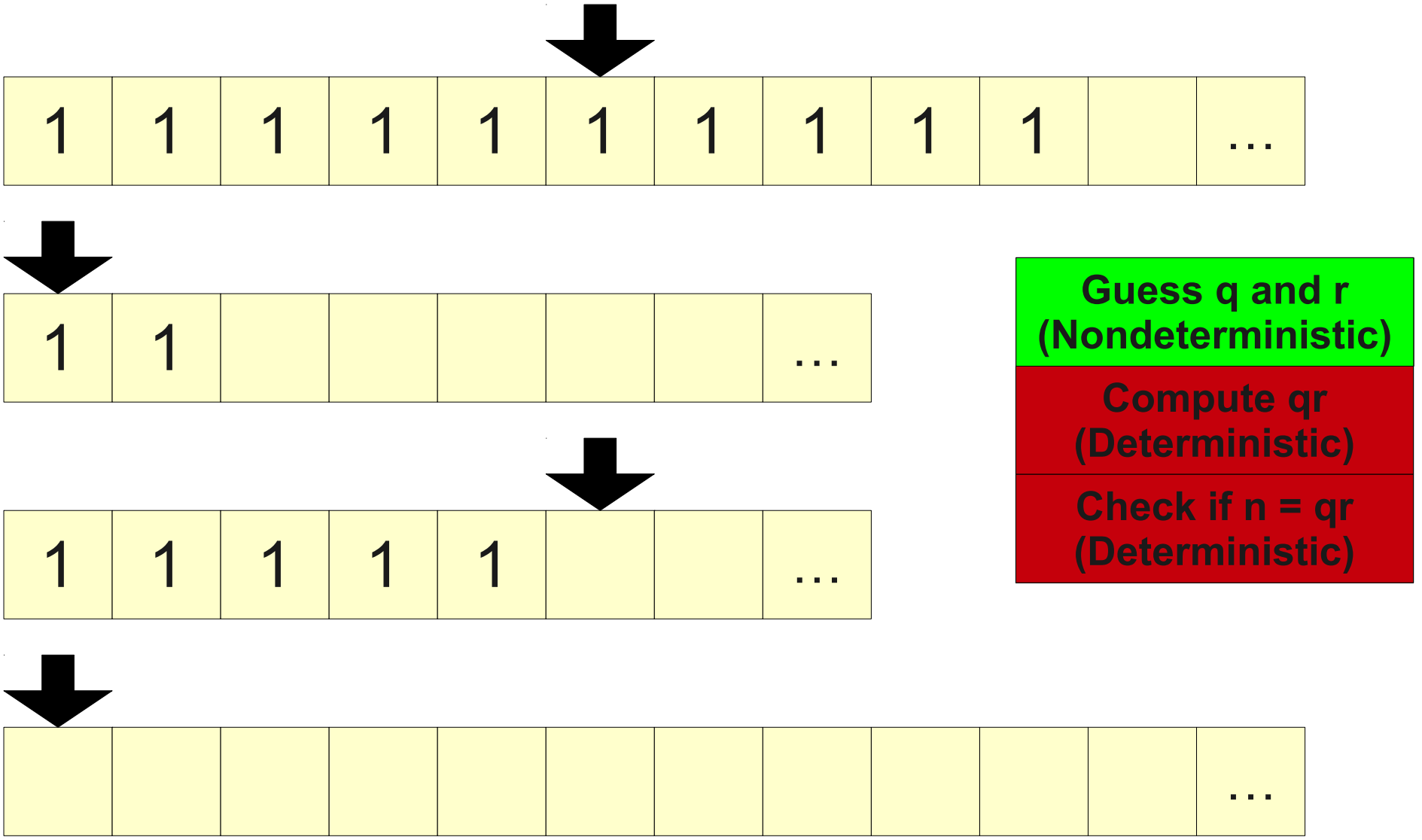

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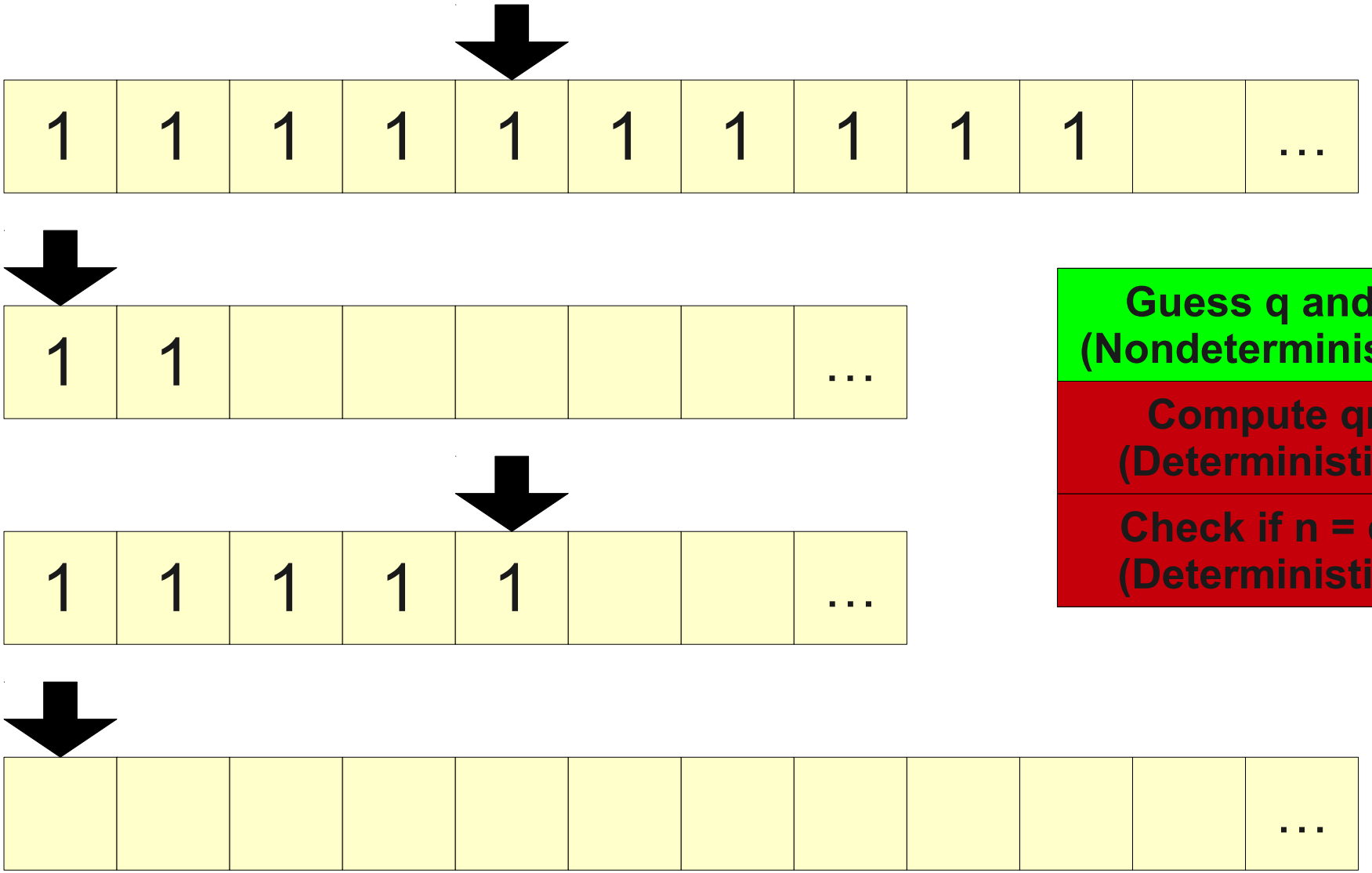

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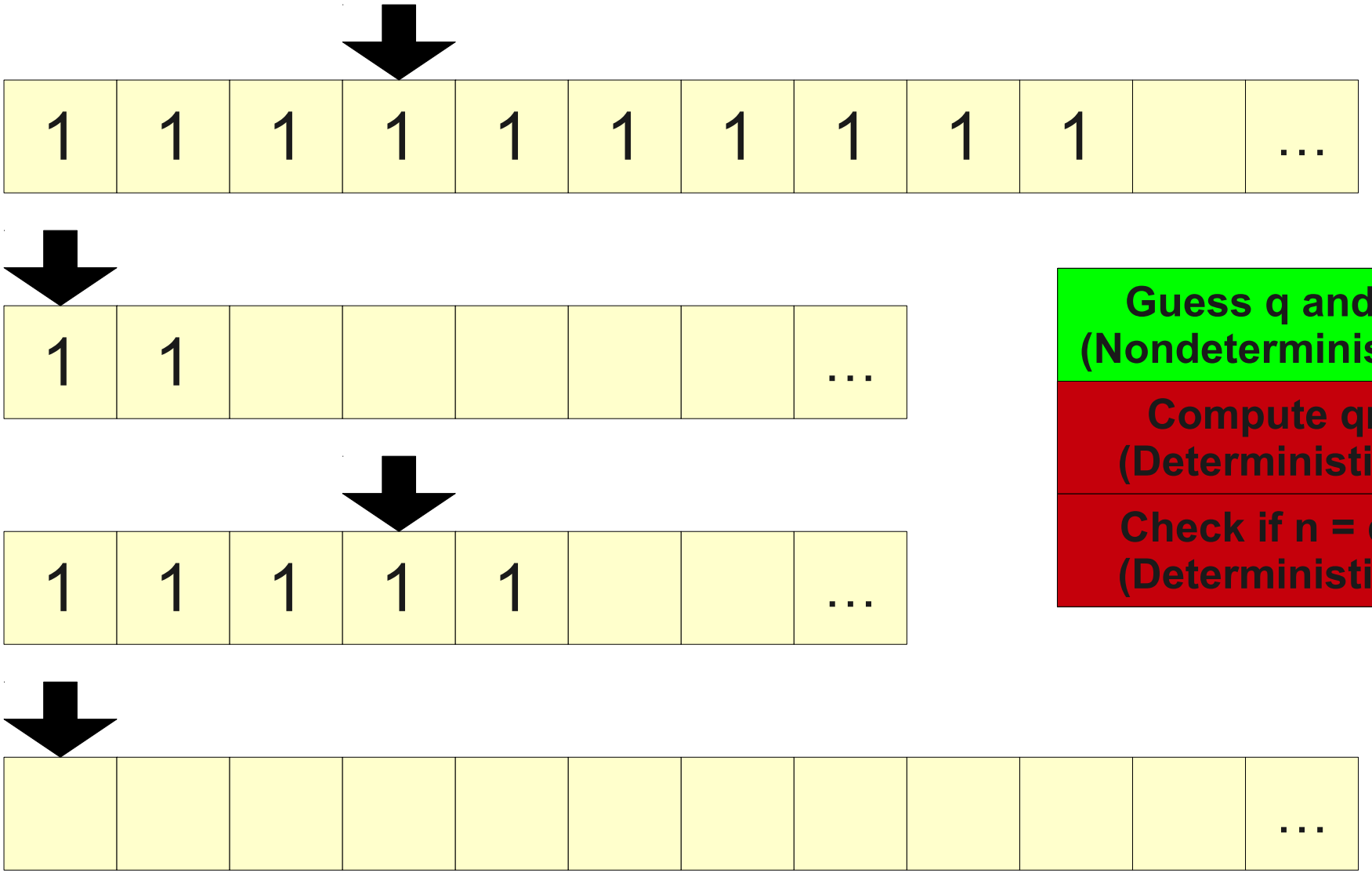

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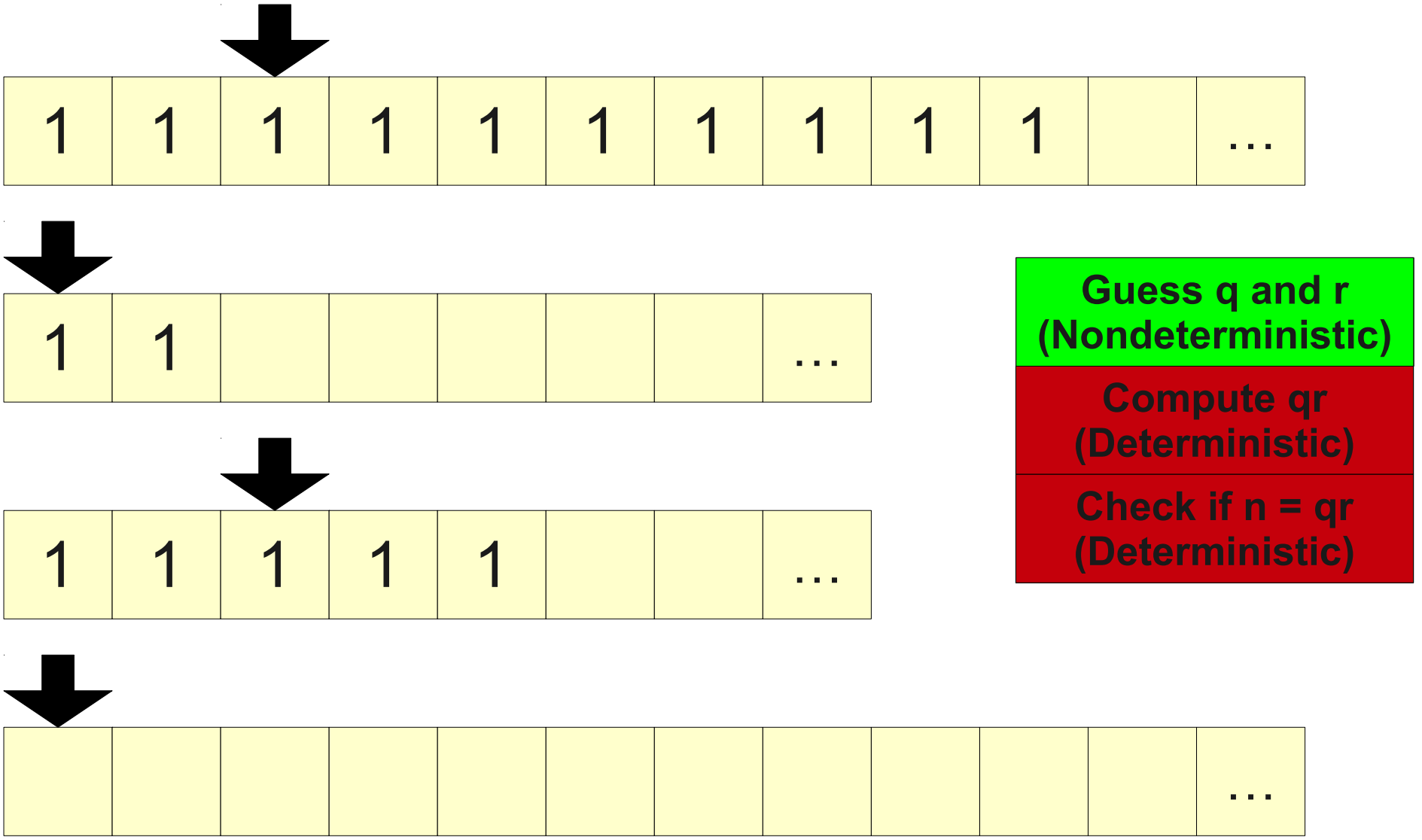

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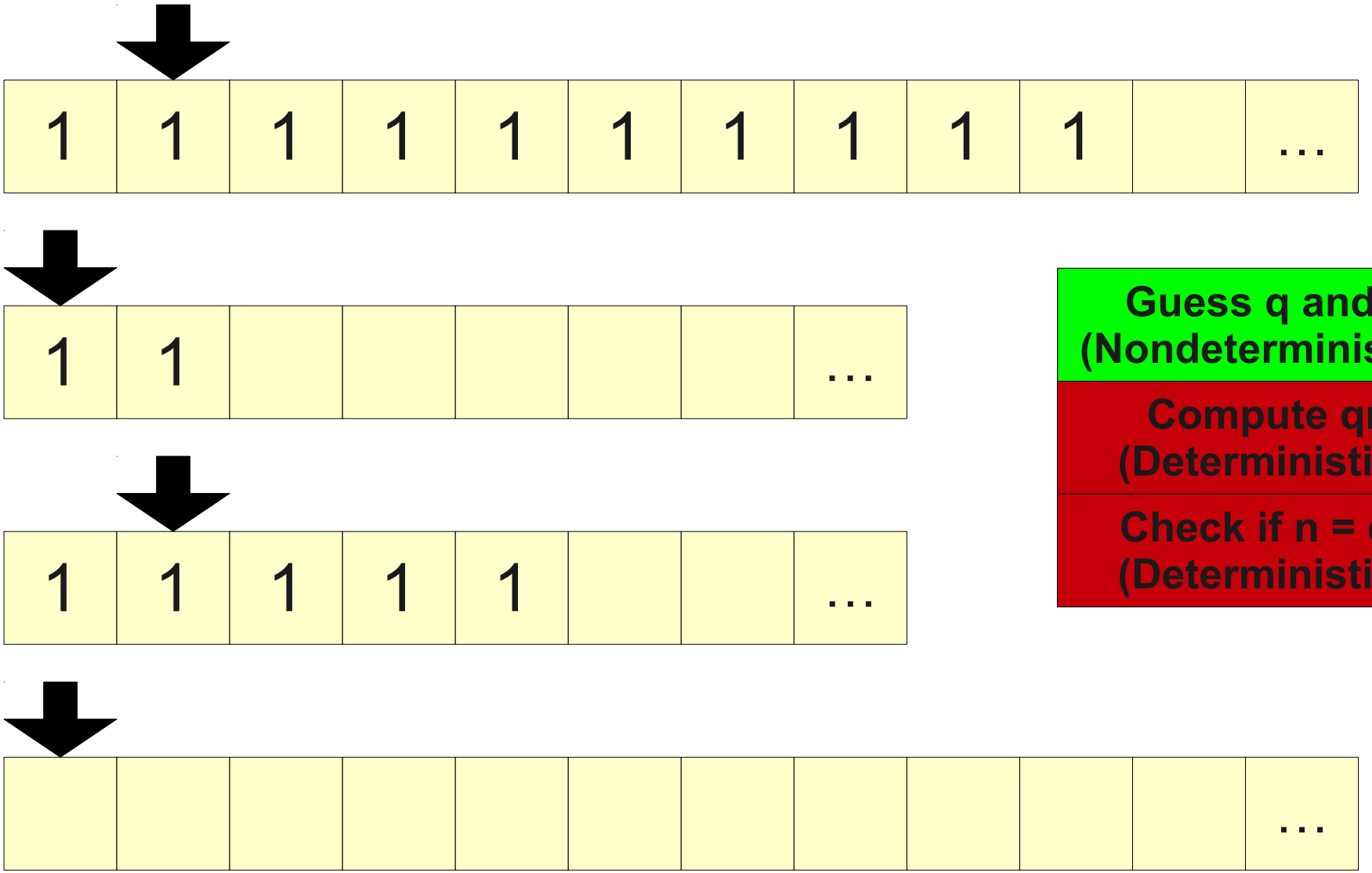

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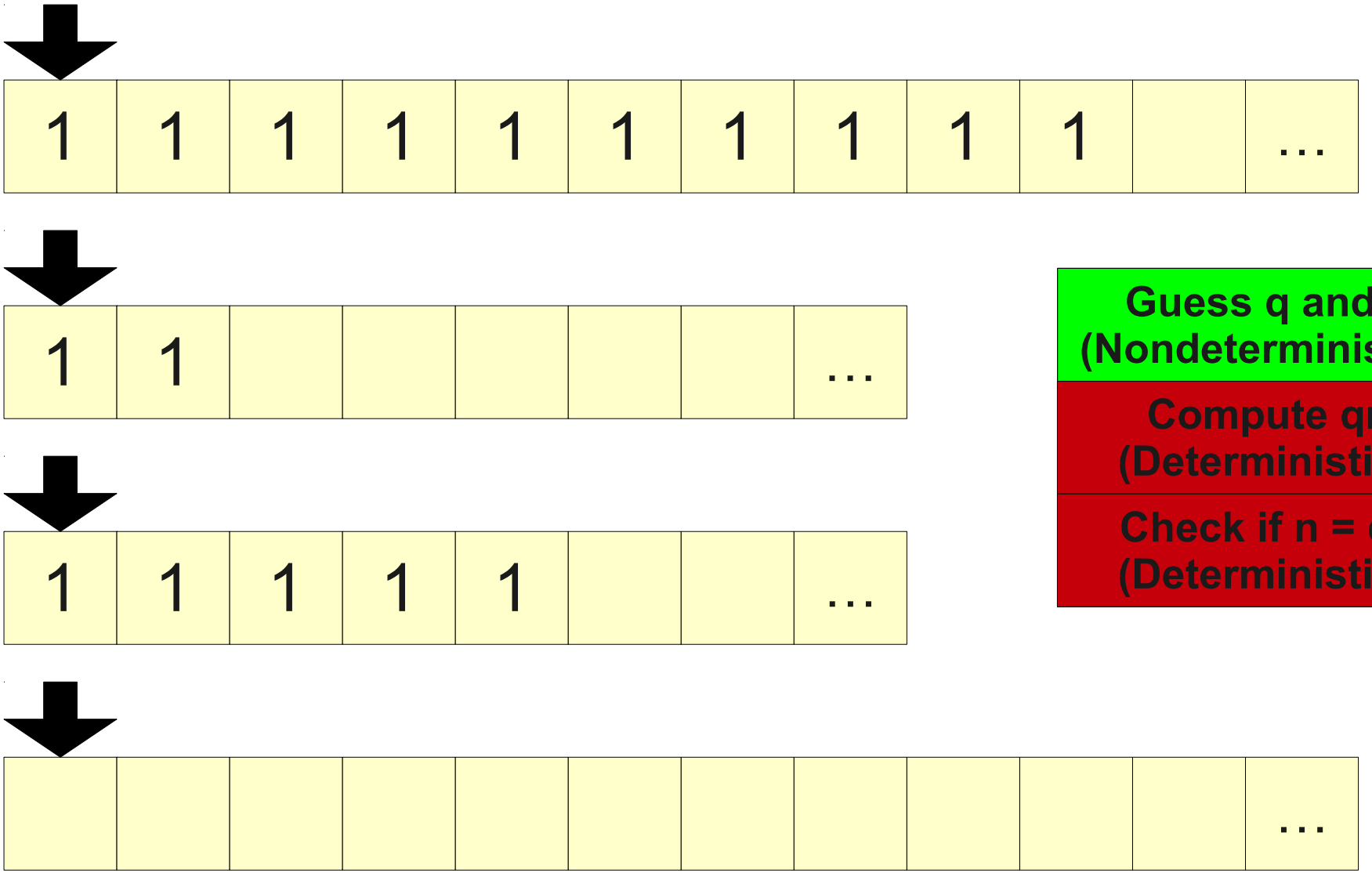

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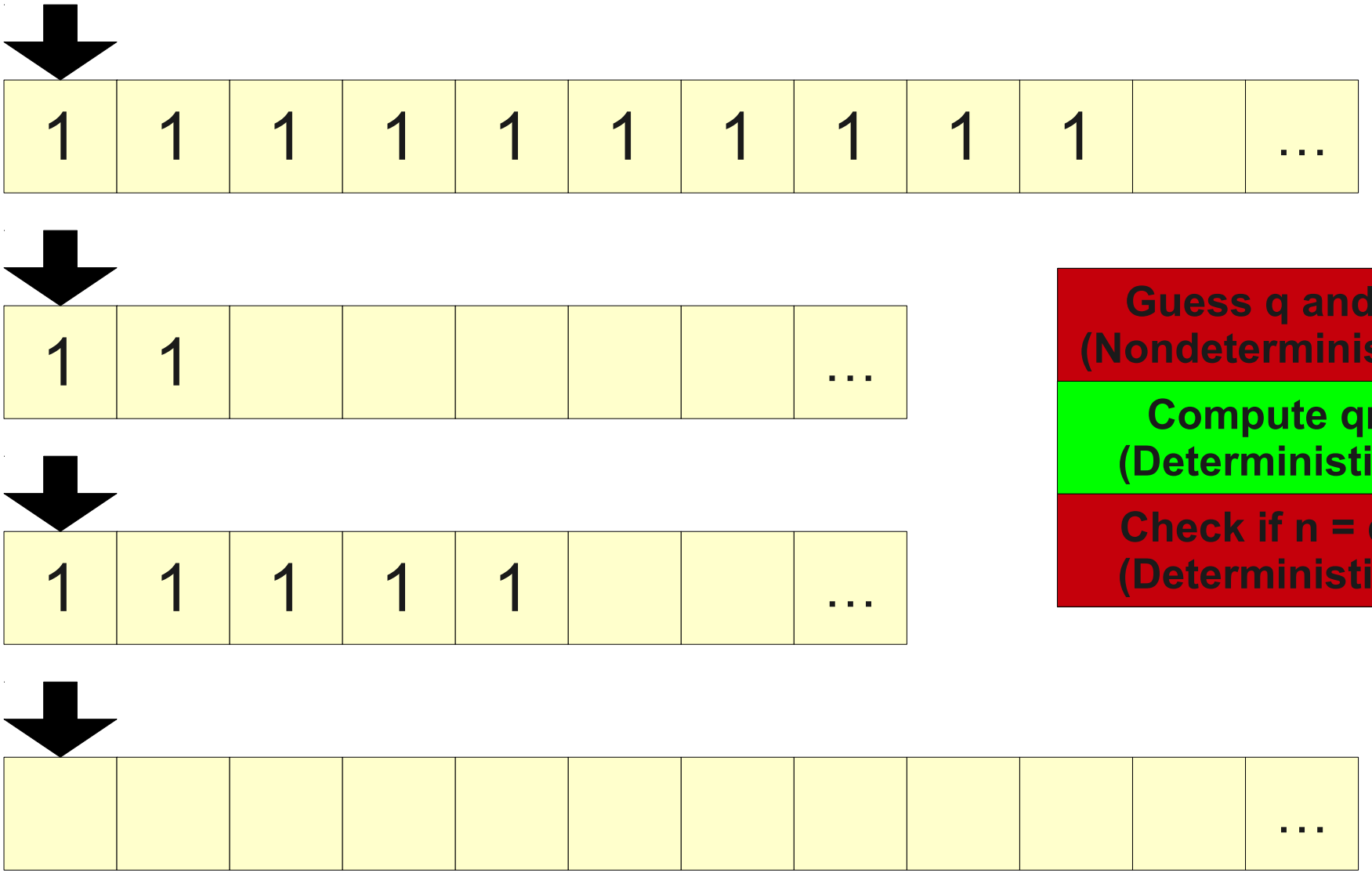

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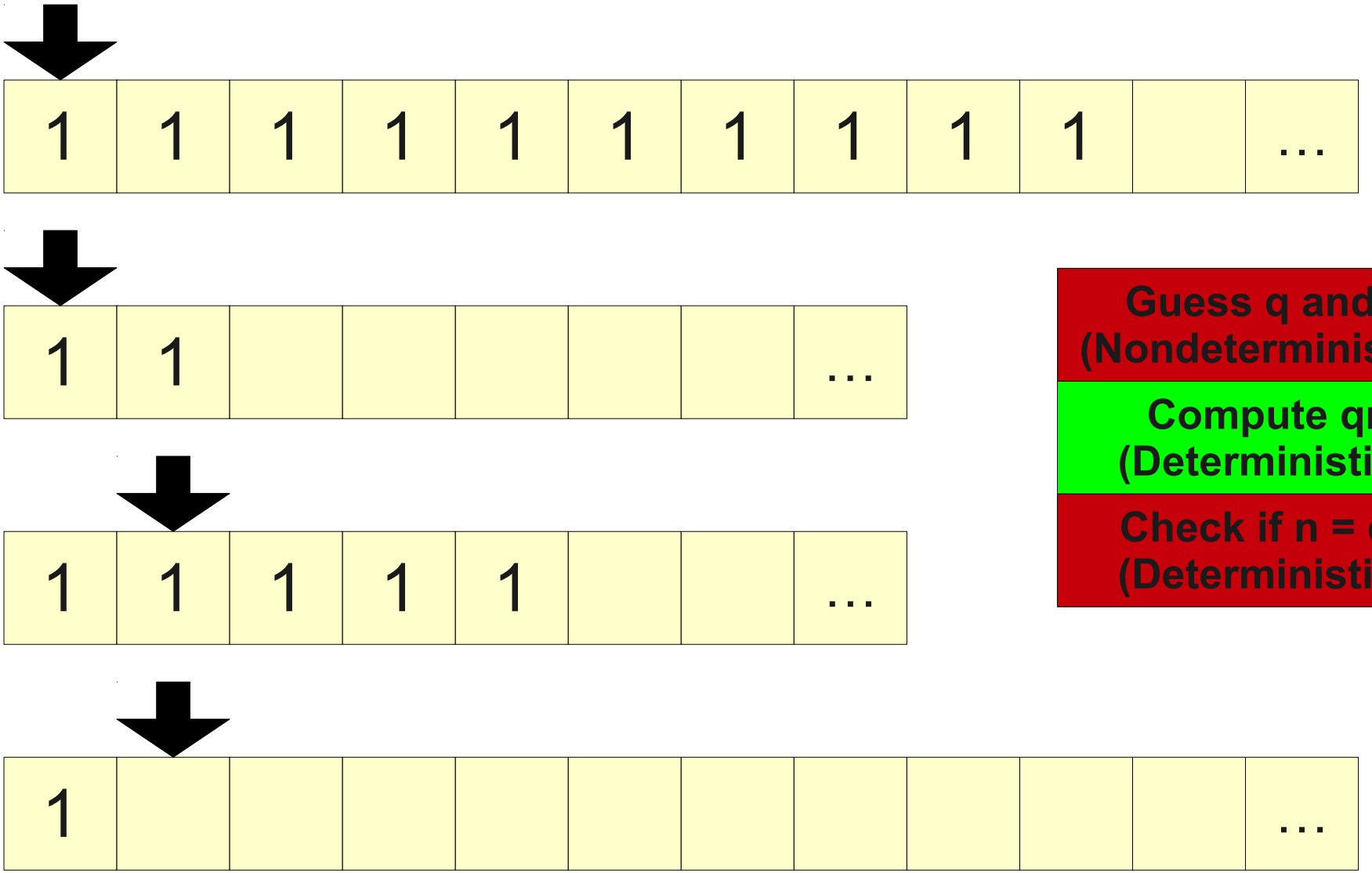

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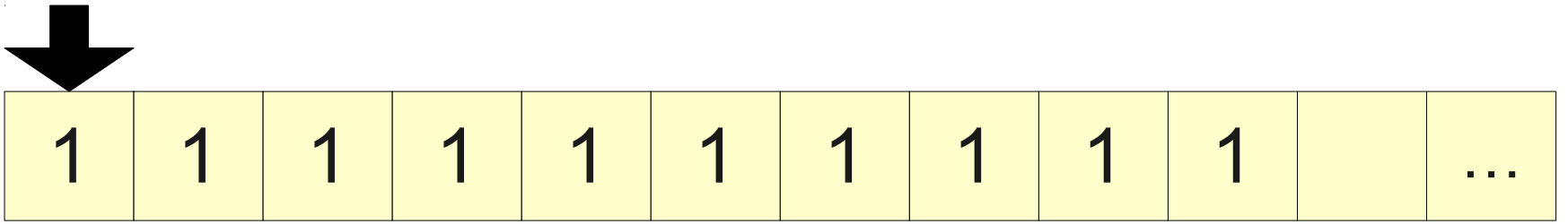

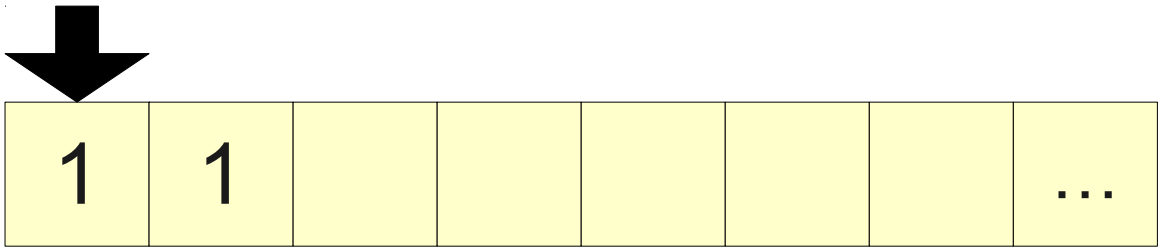

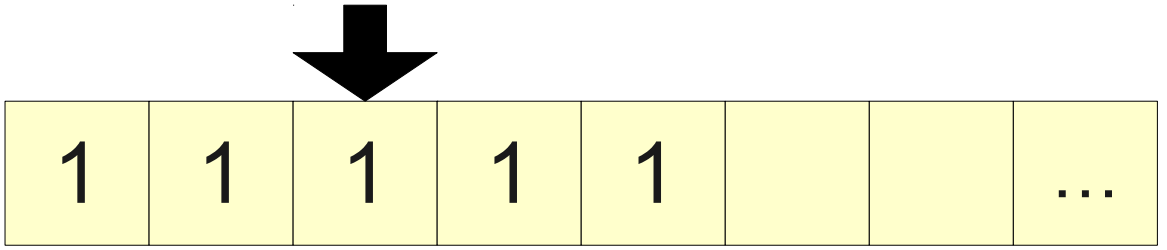

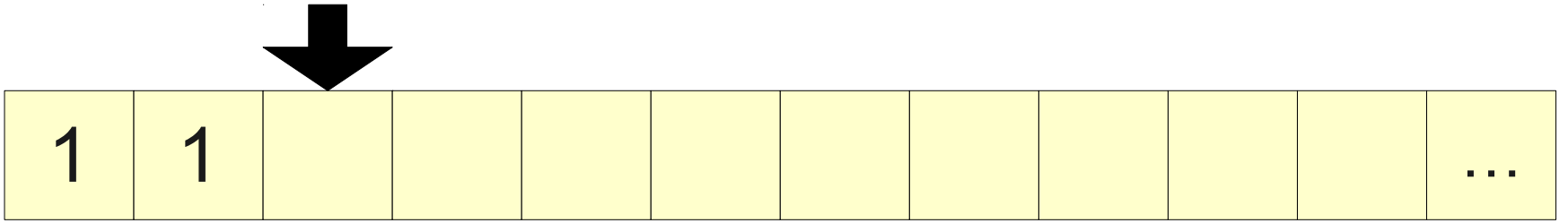

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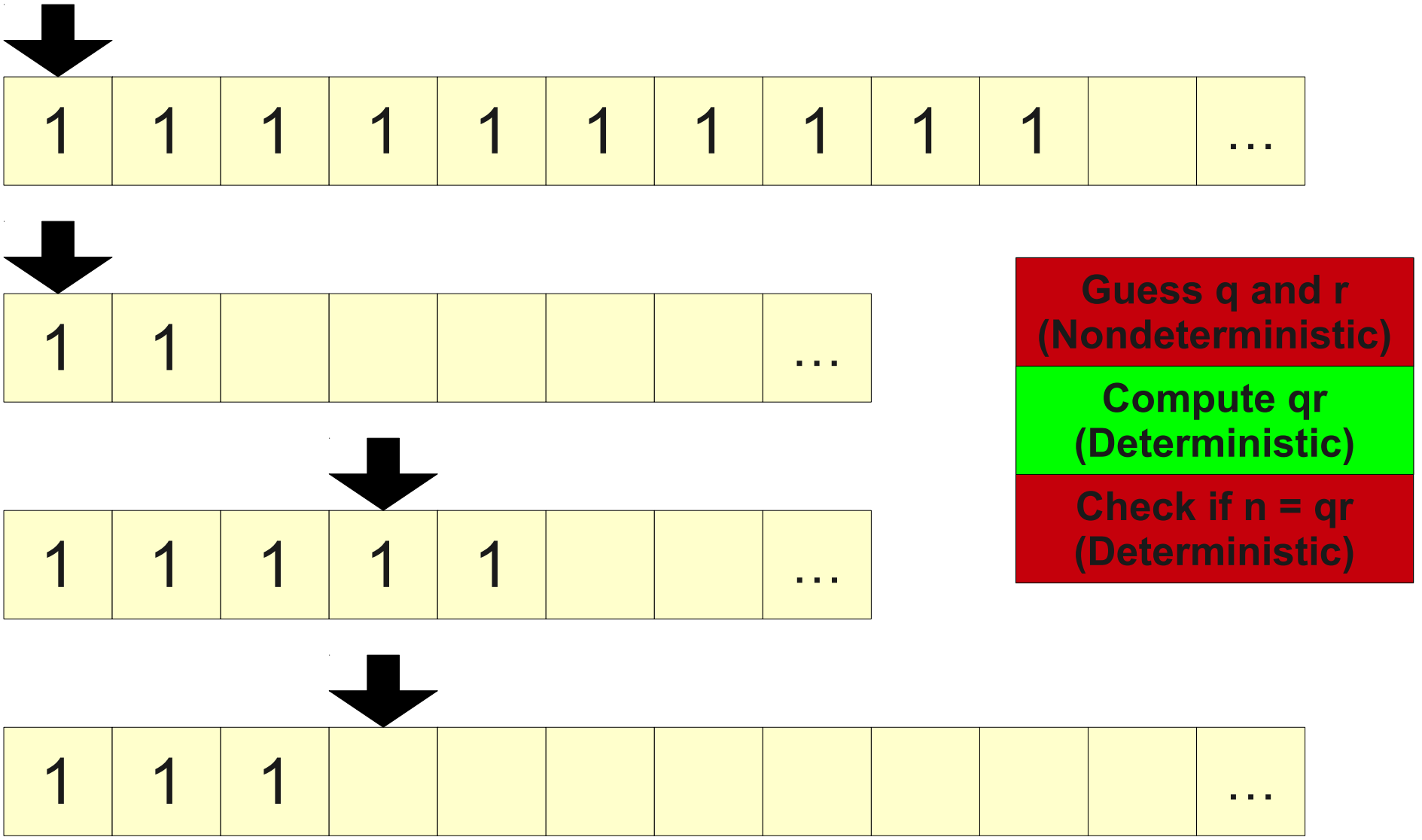

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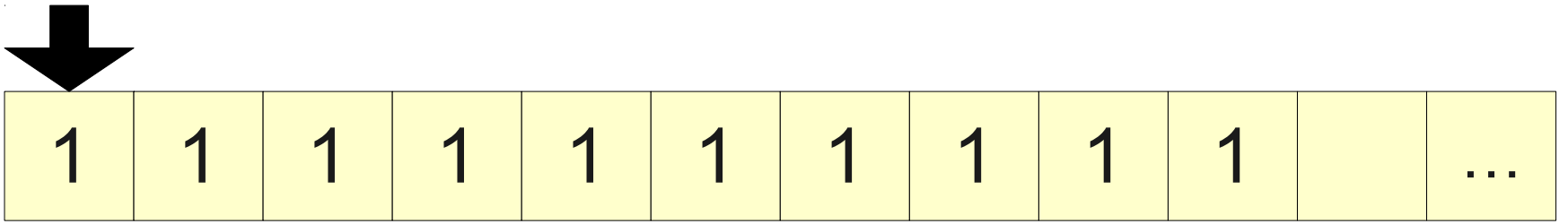

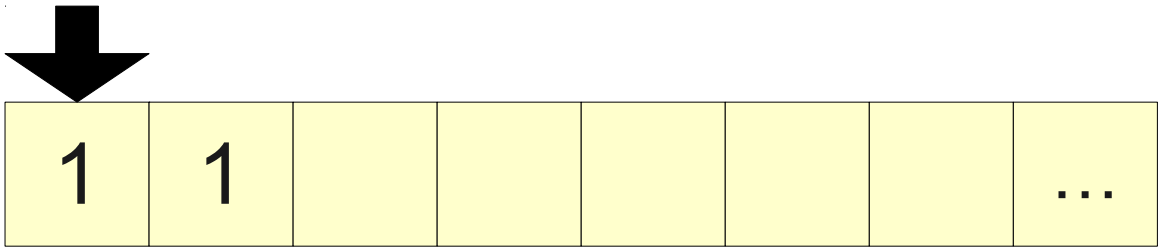

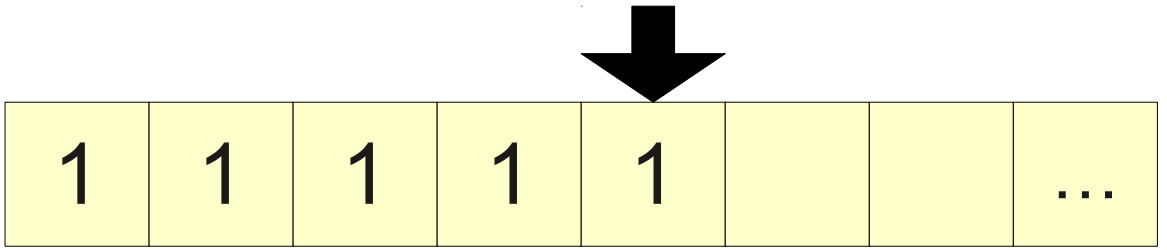

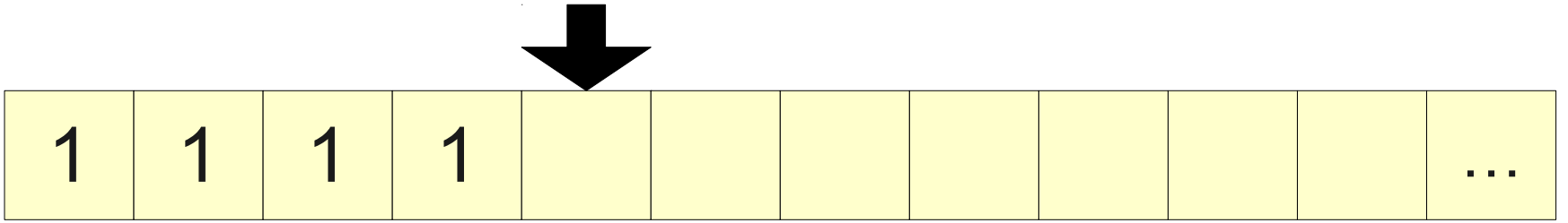

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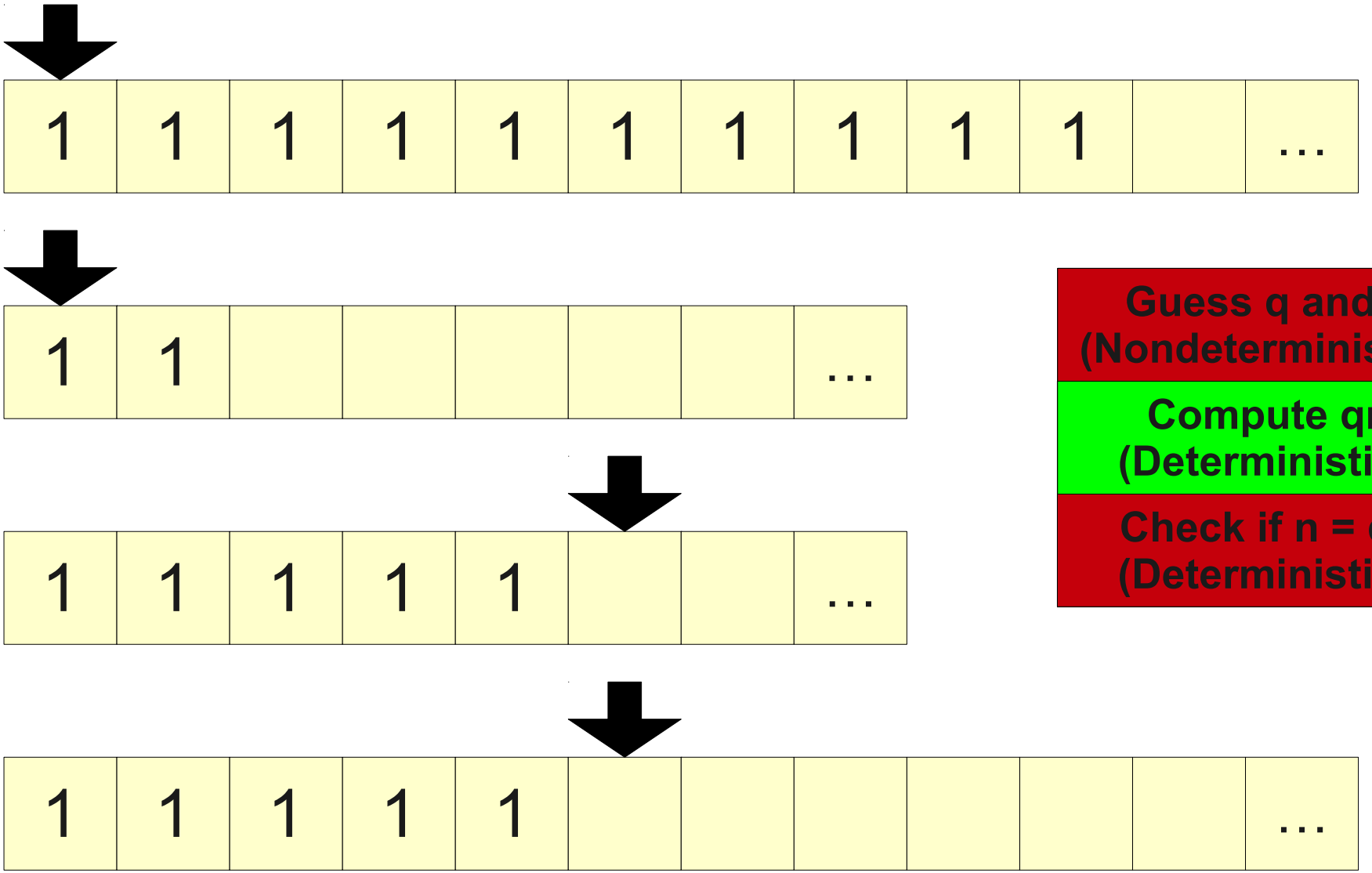

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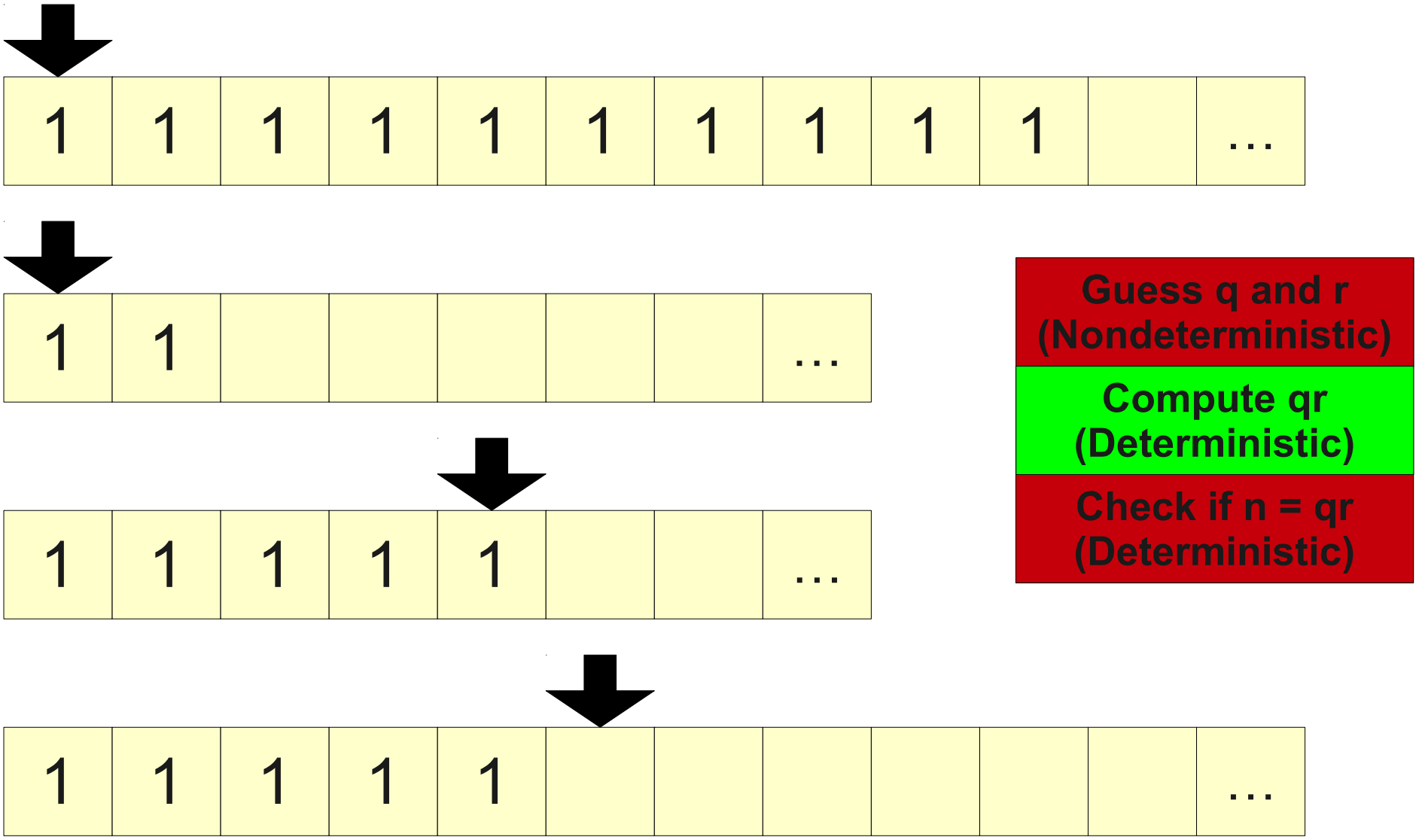

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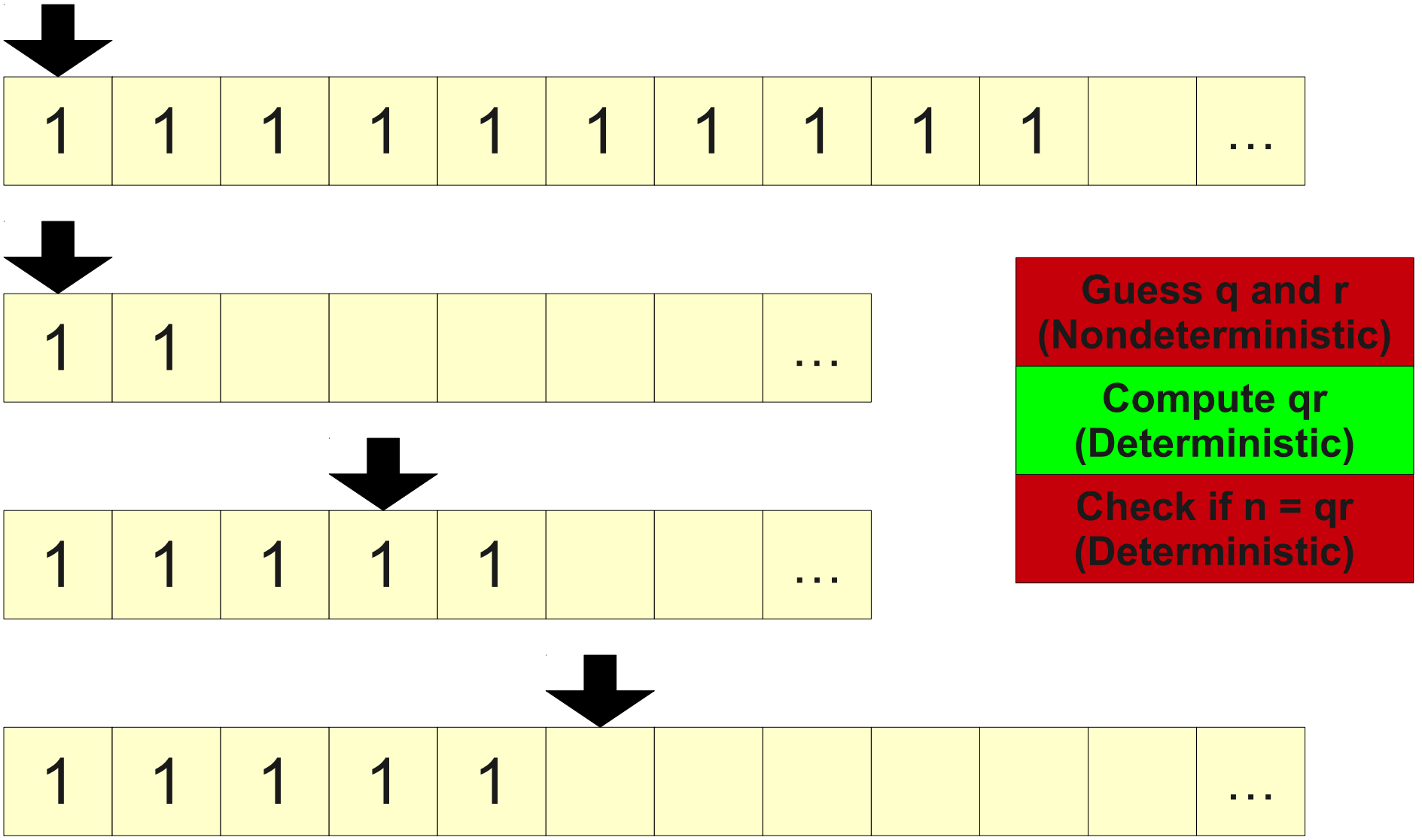

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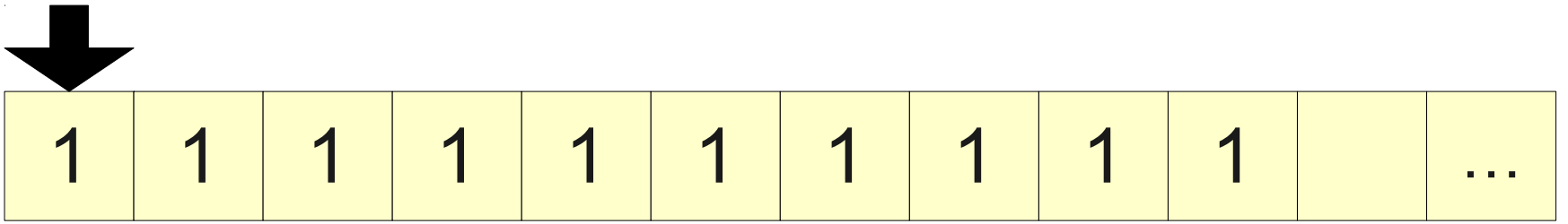

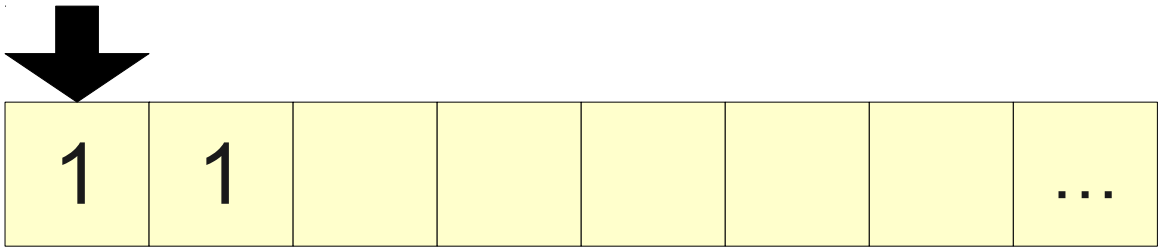

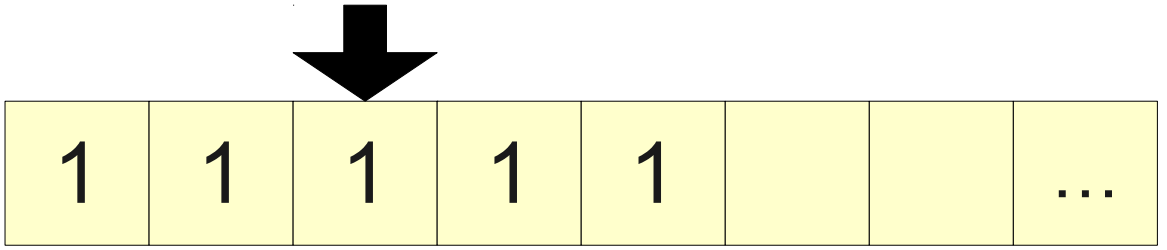

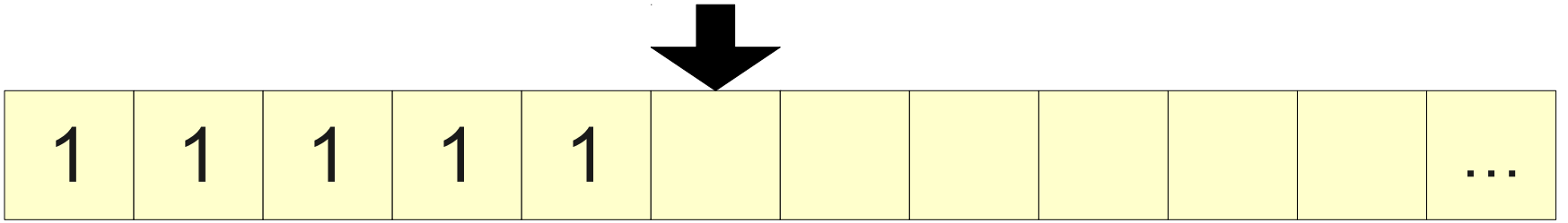

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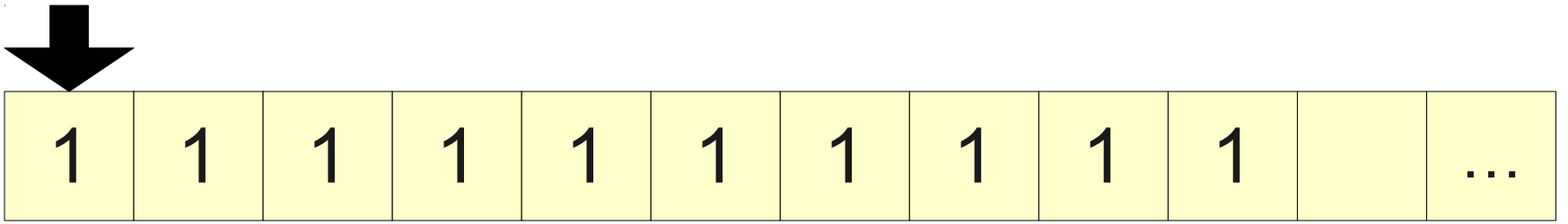

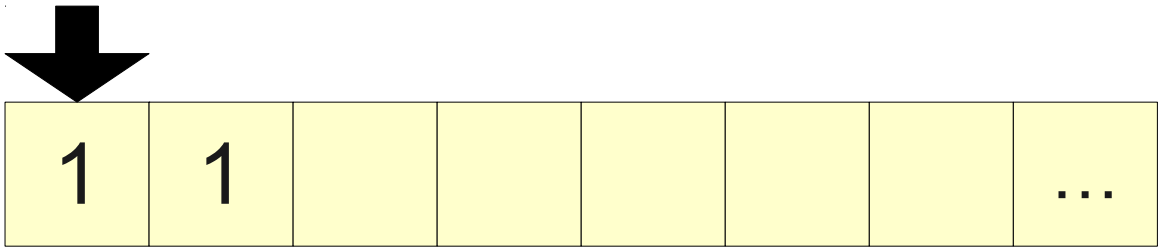

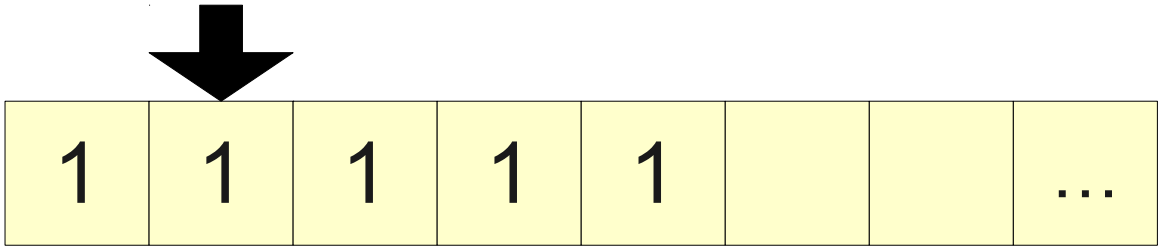

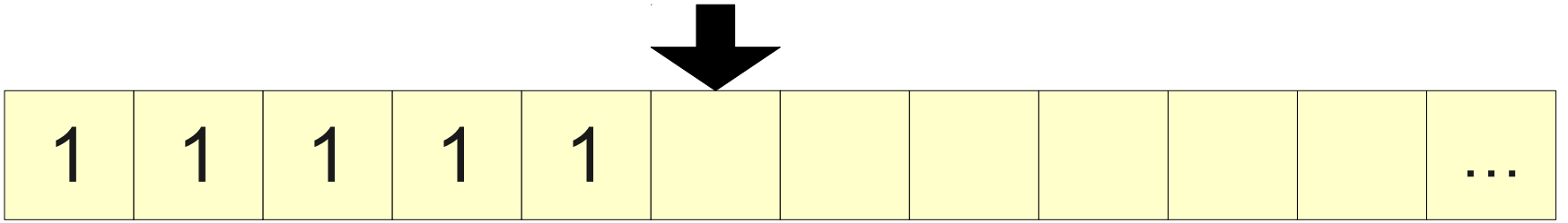

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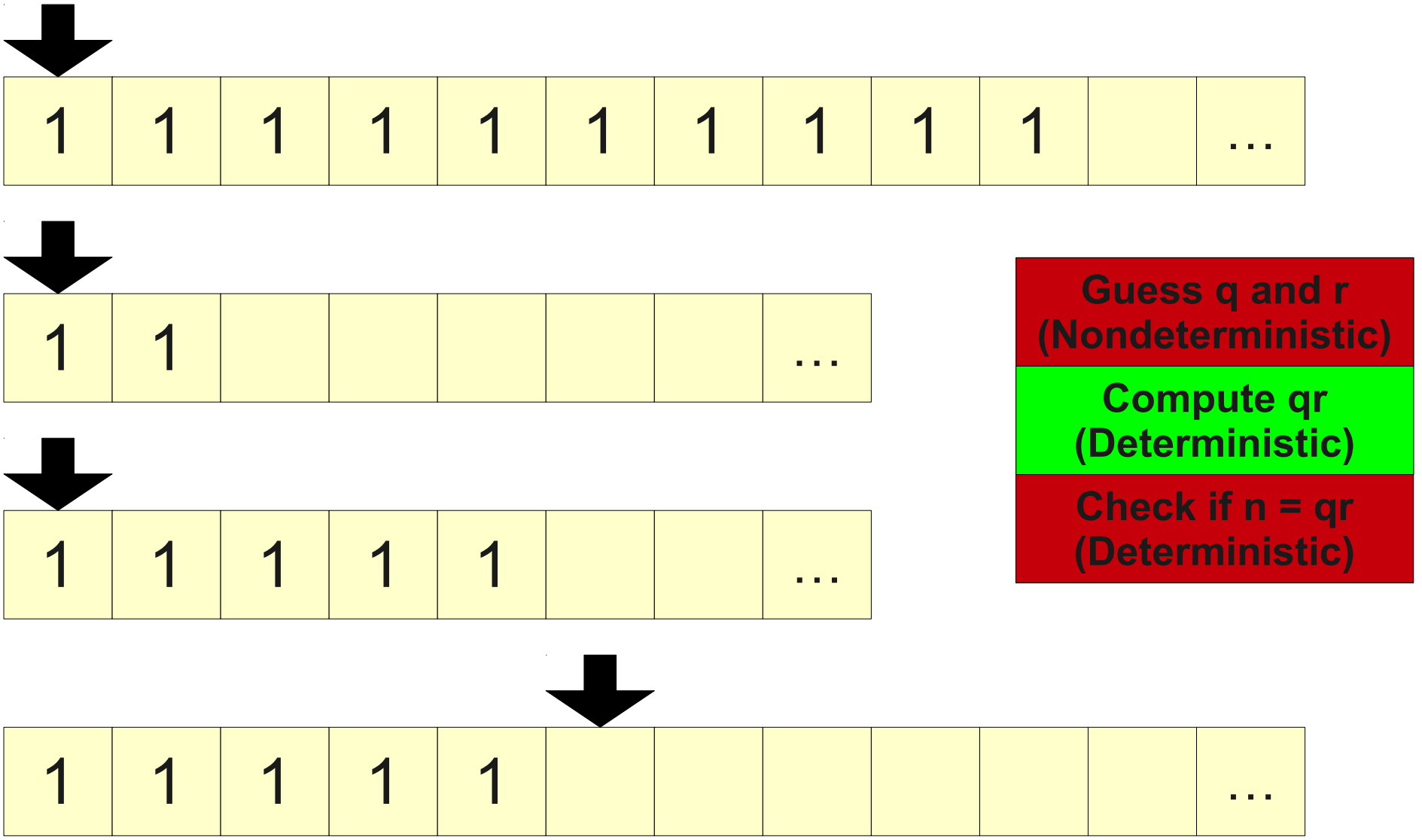
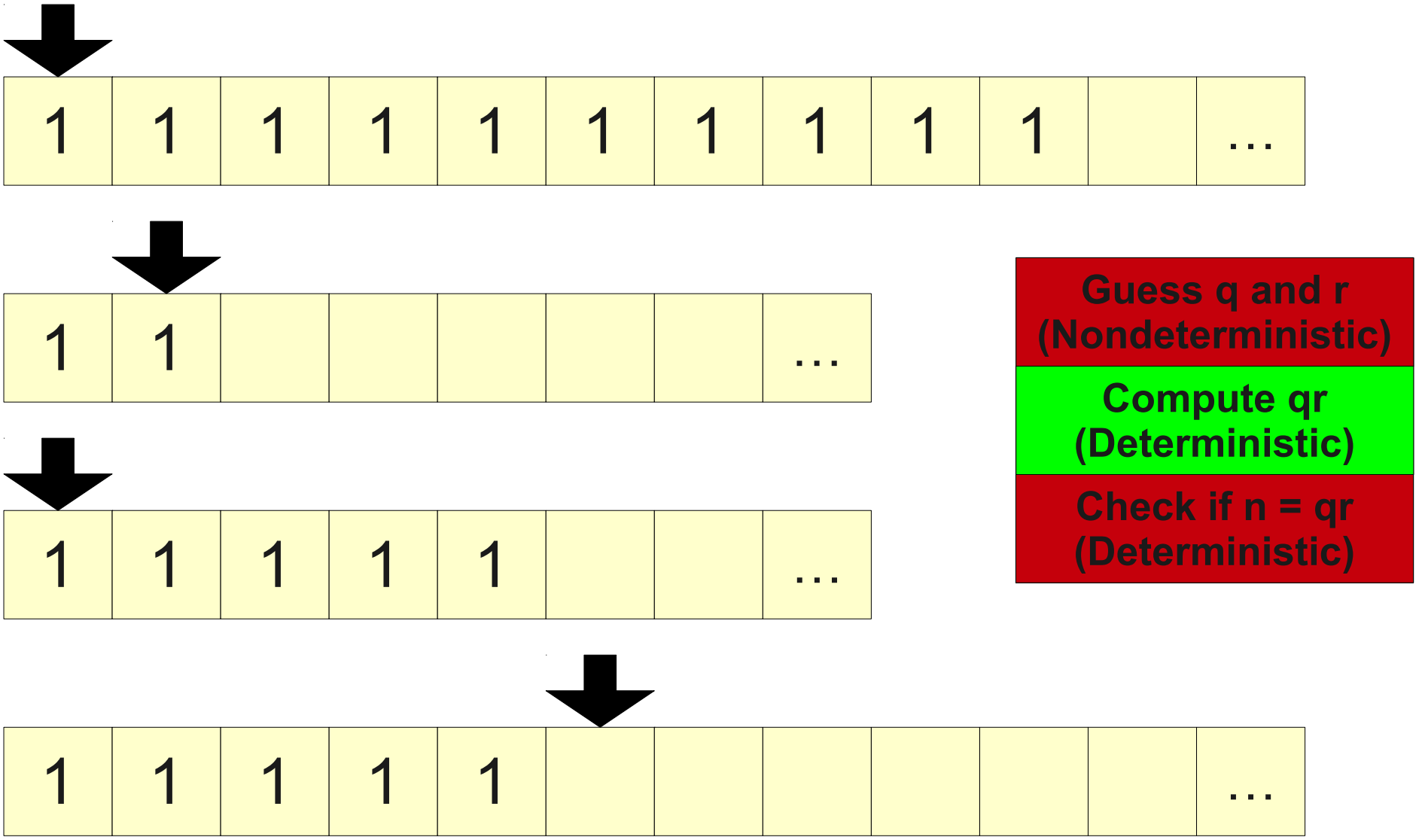


Nondeterministic Algorithms



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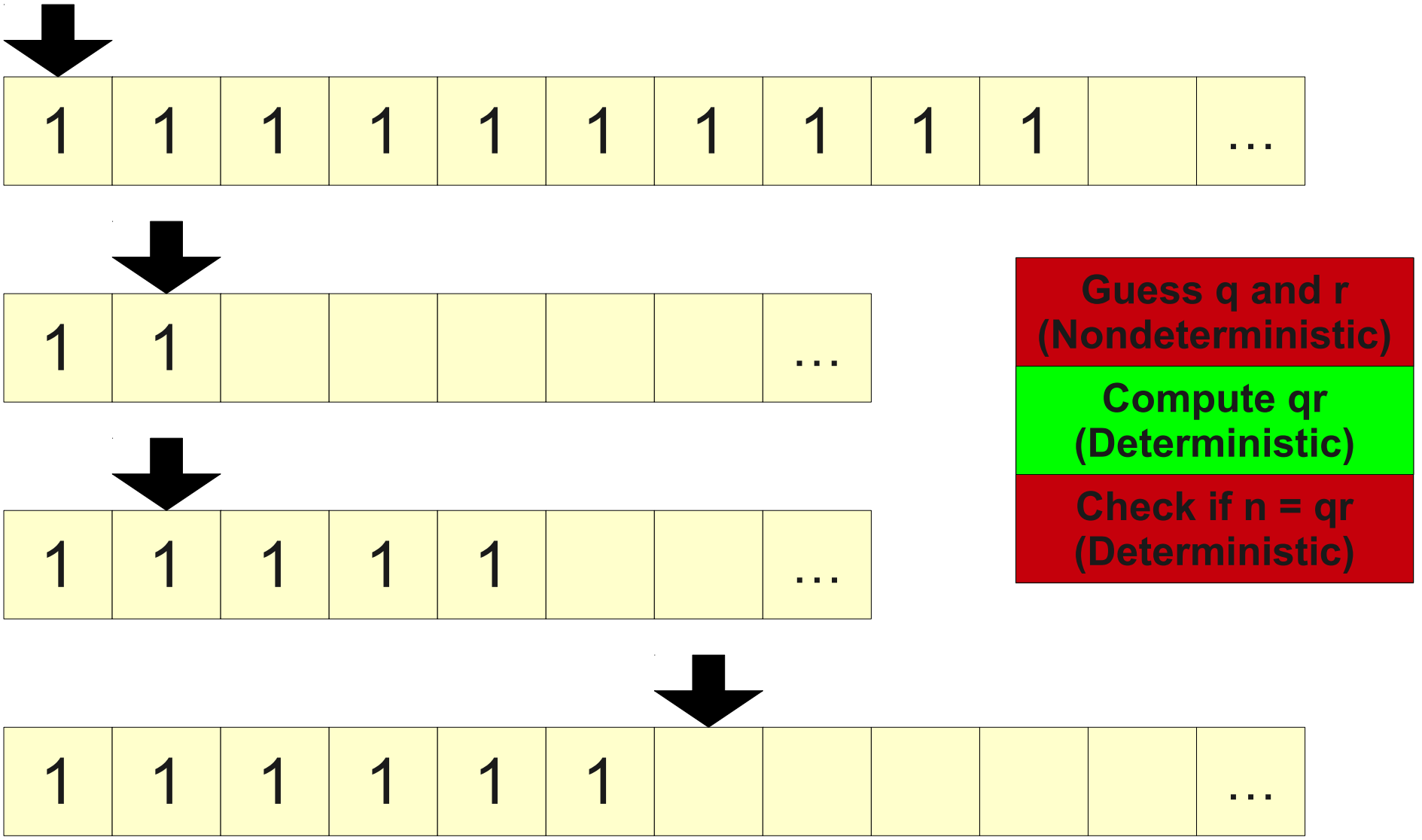

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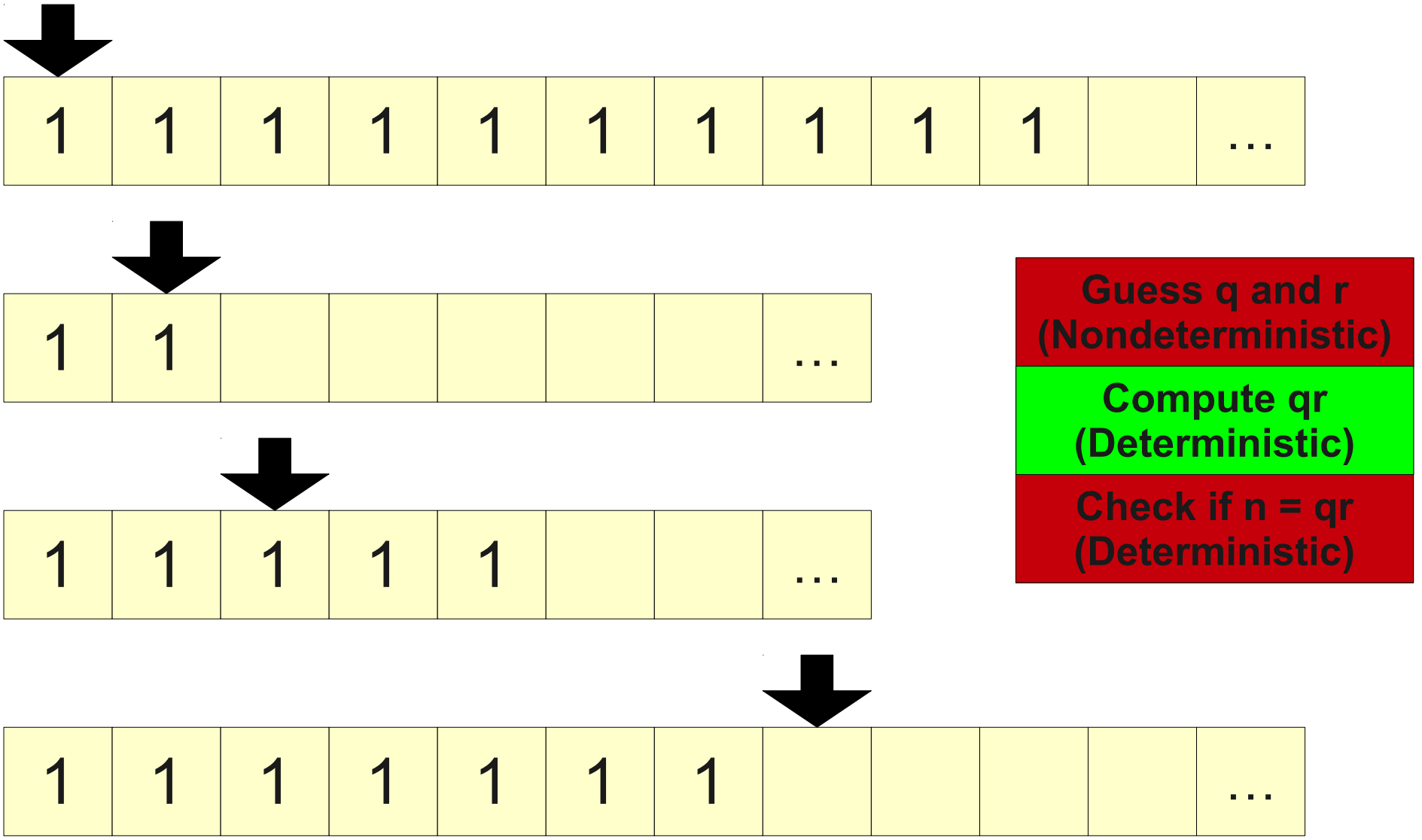

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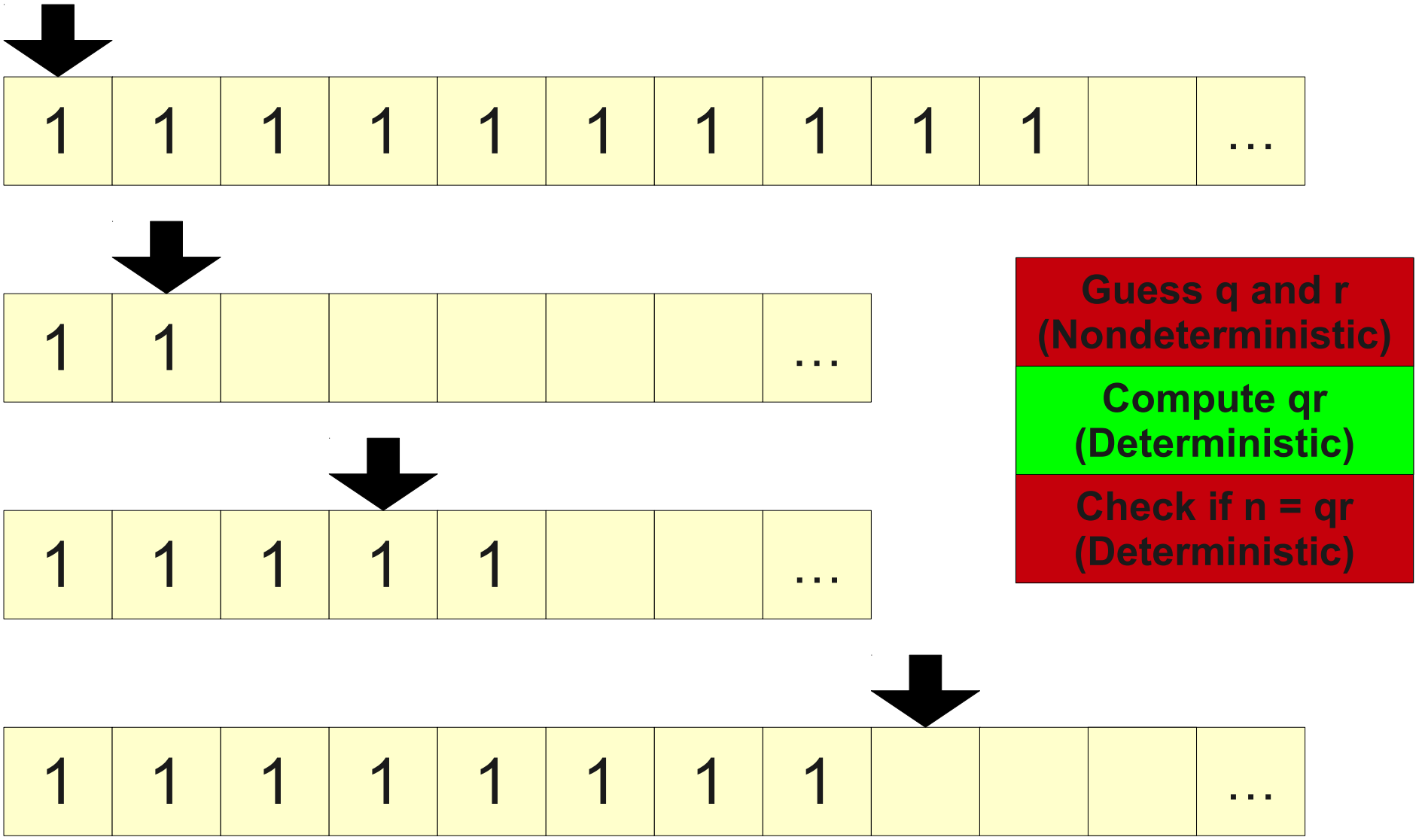

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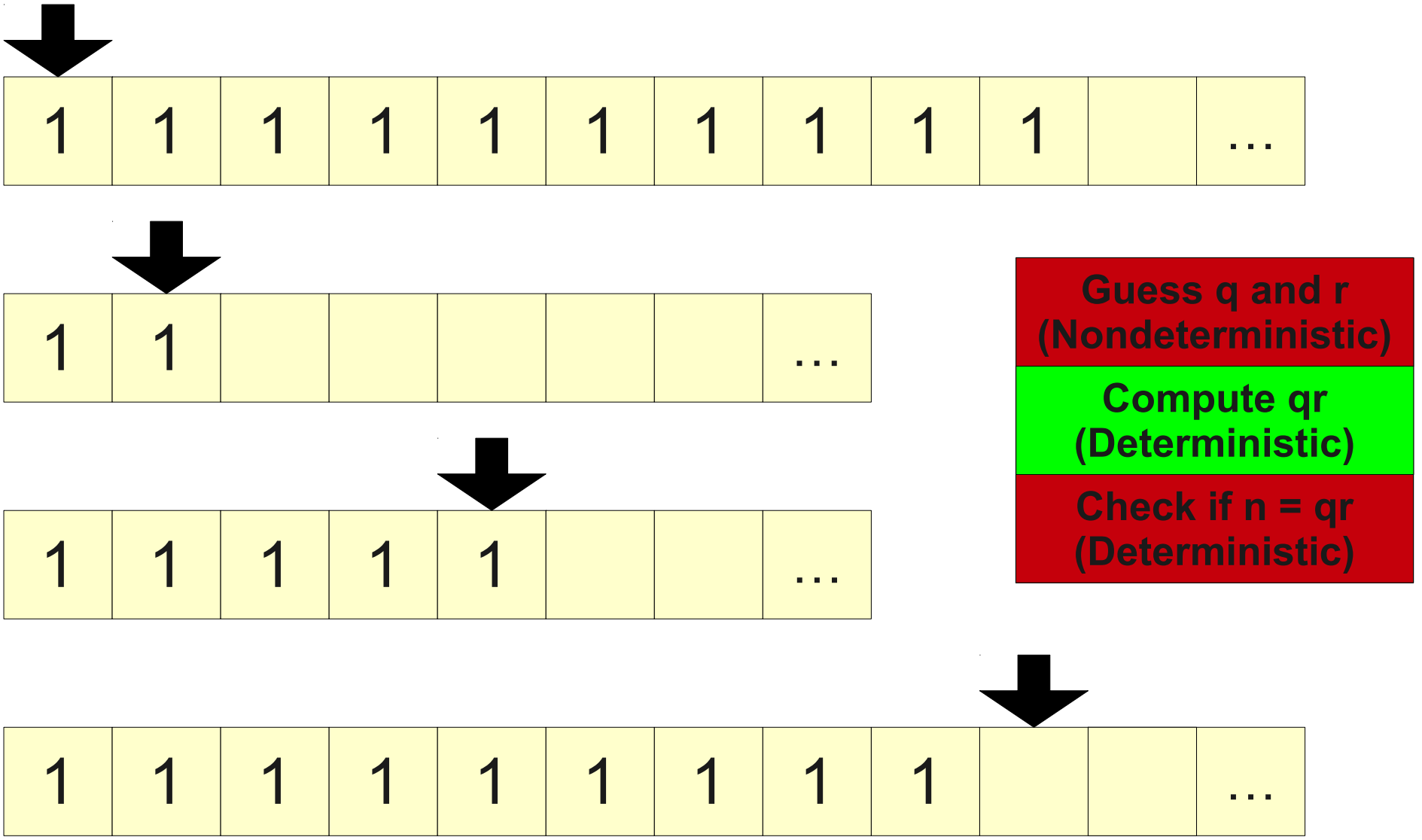

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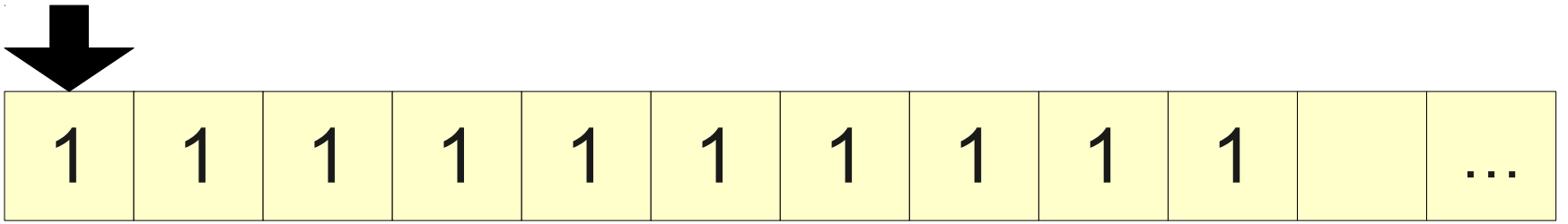

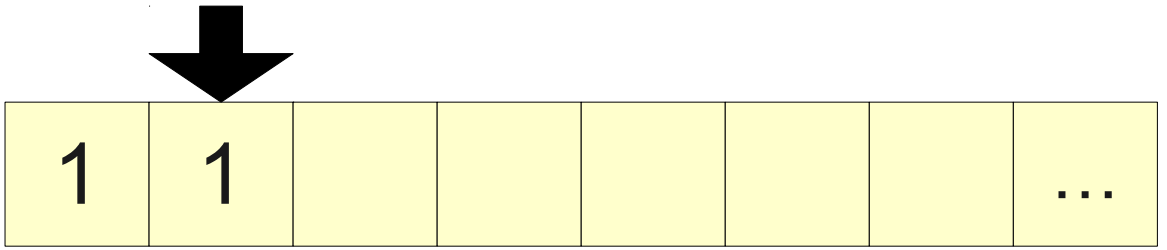

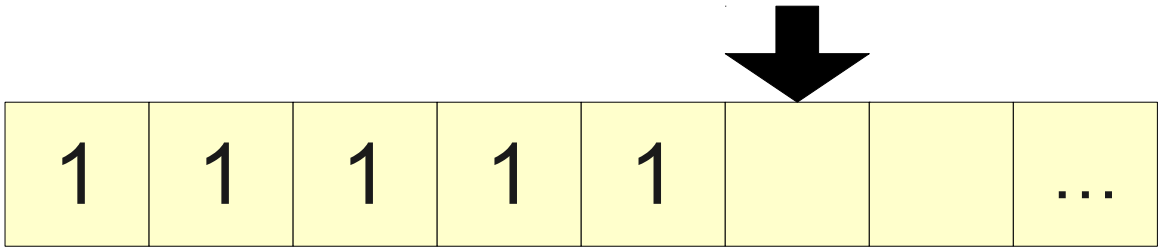

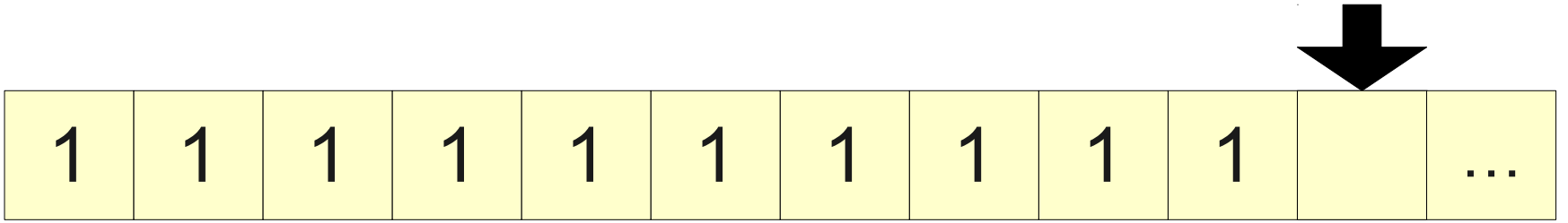

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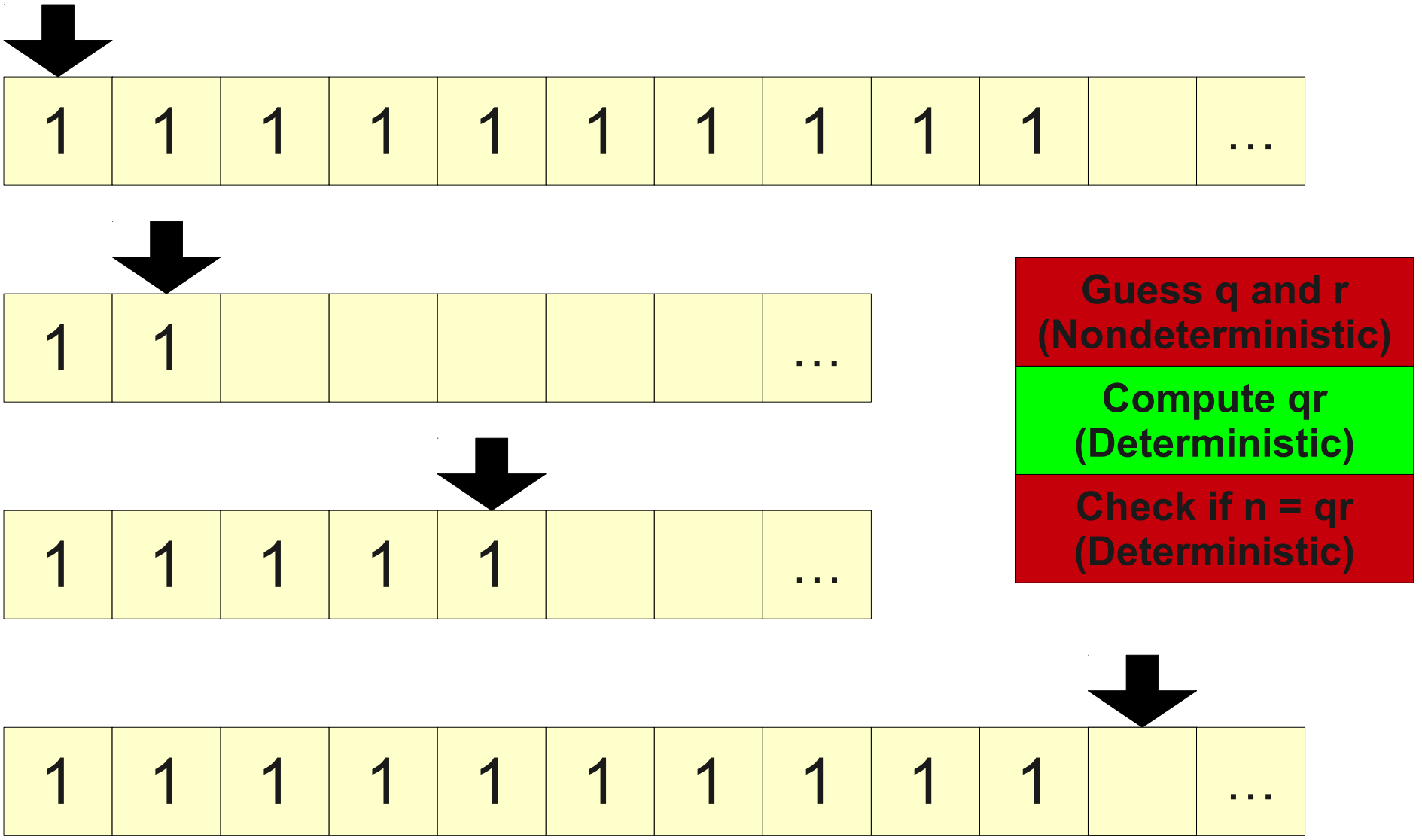

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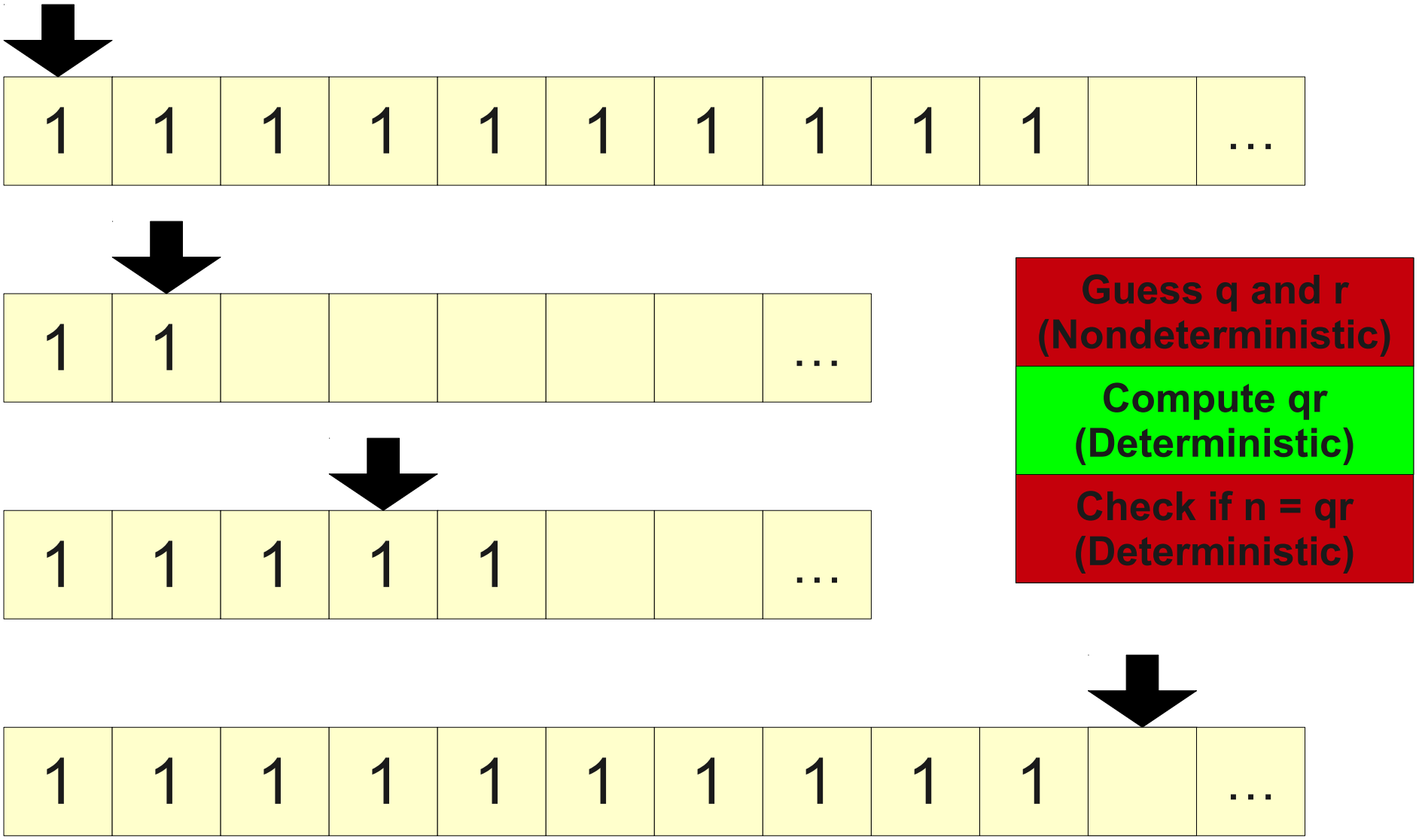

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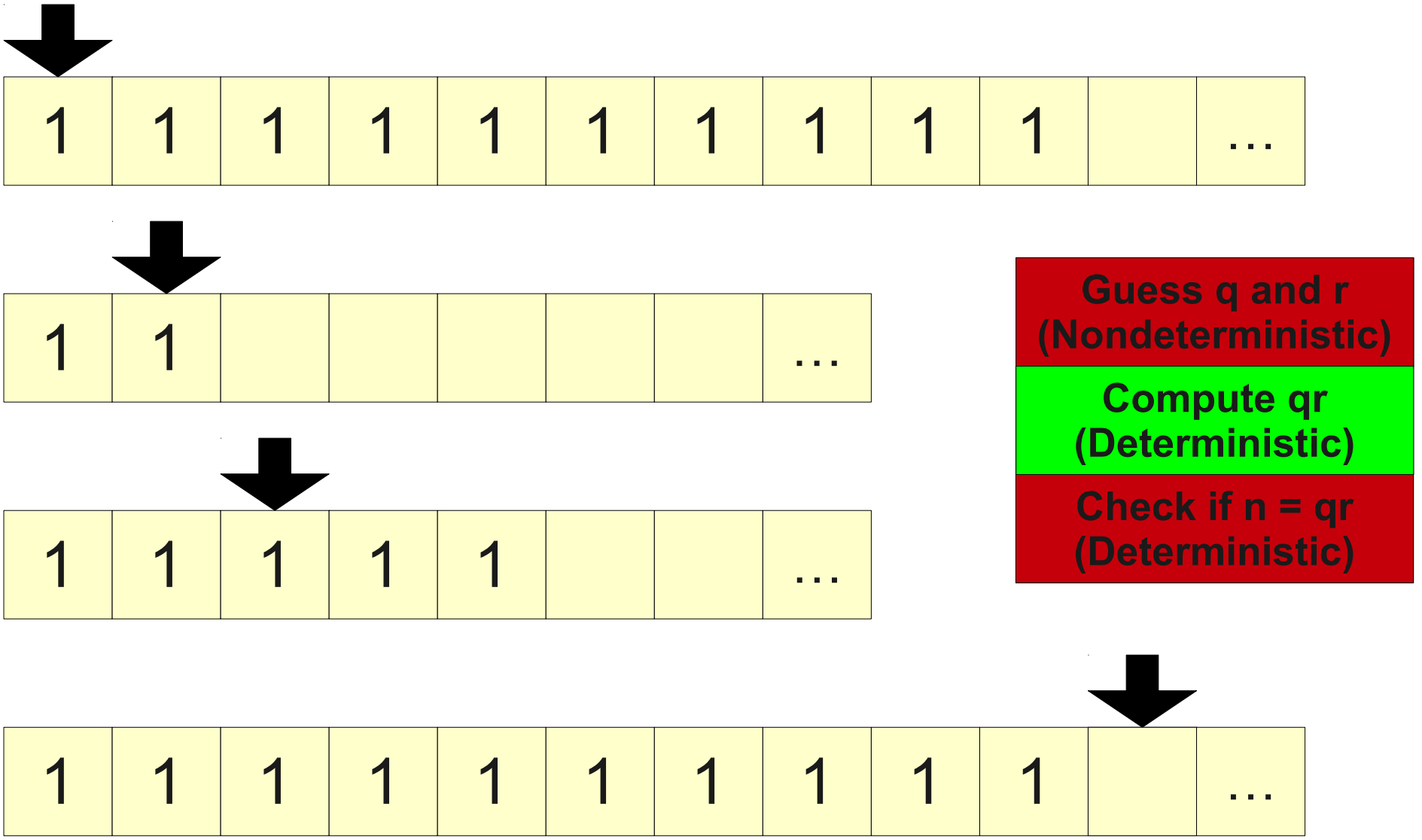

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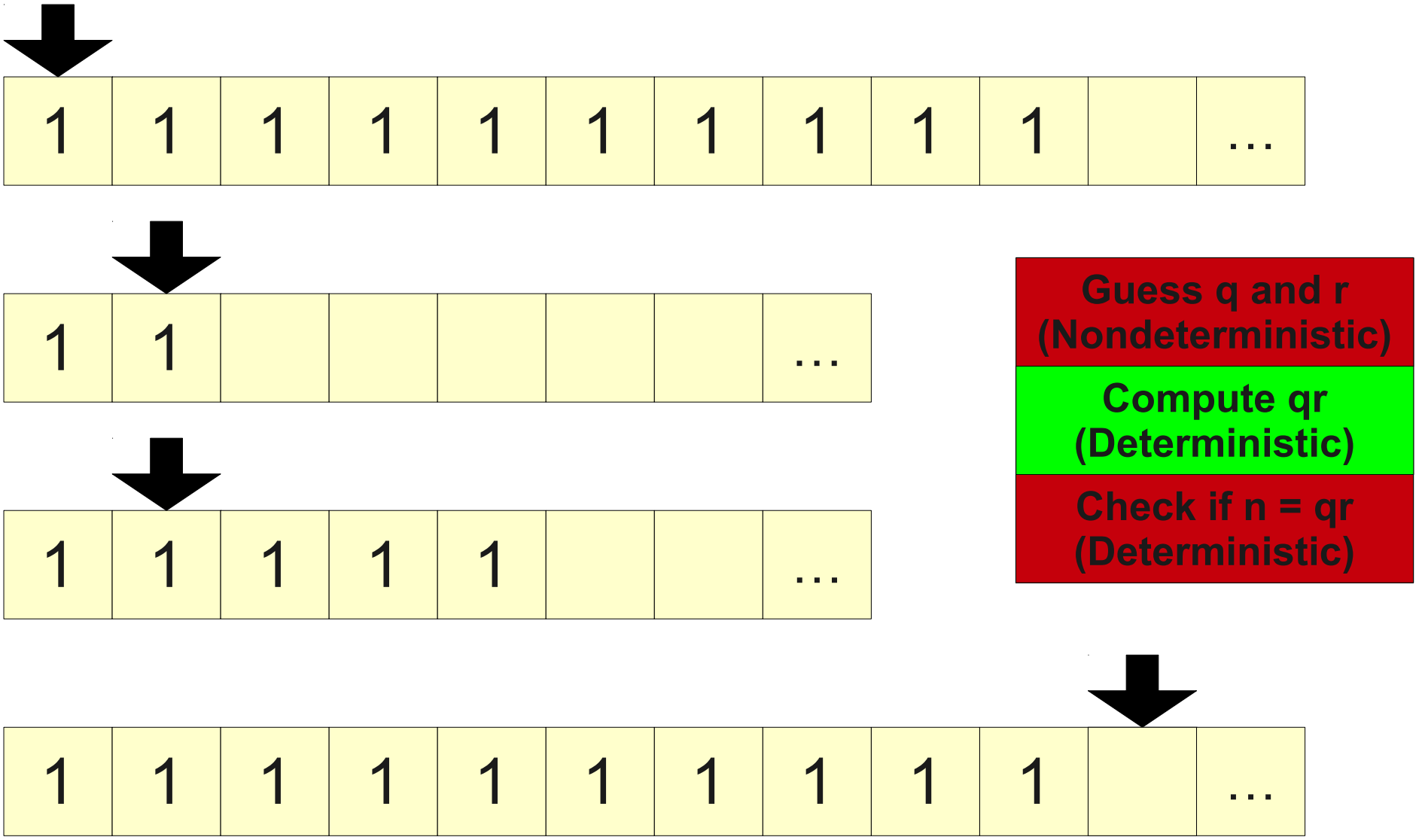

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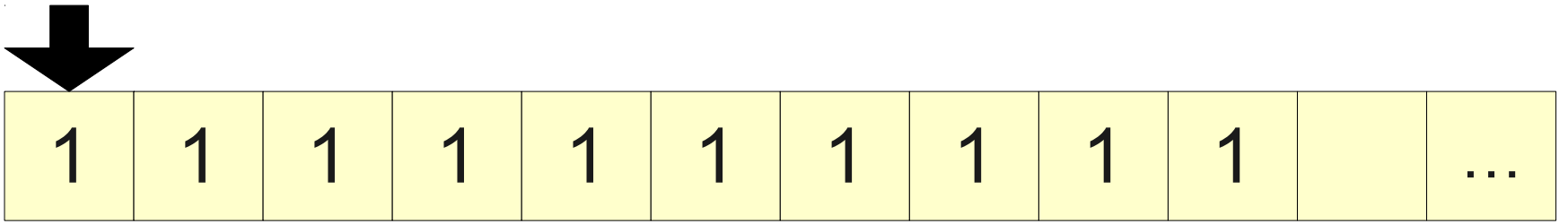

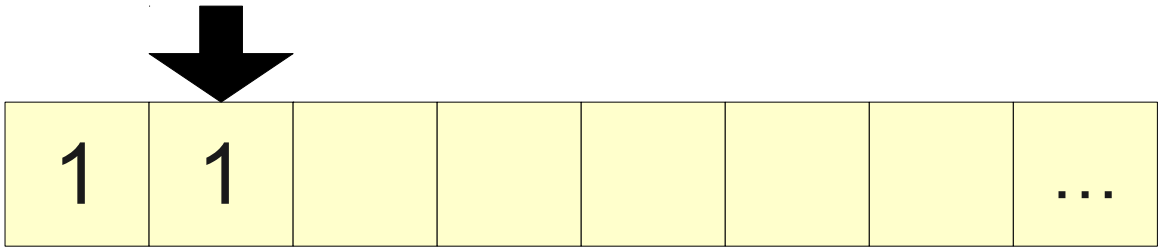

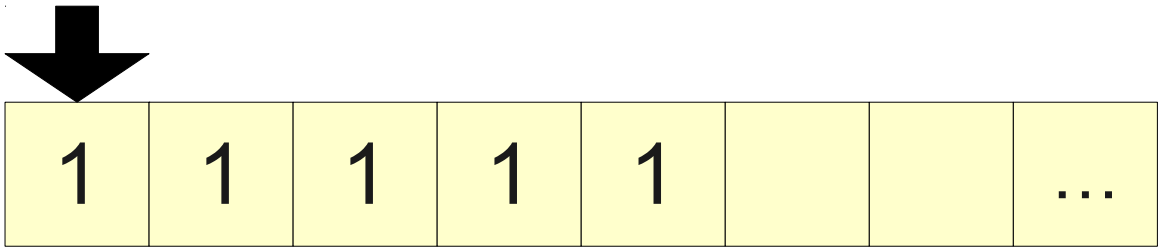

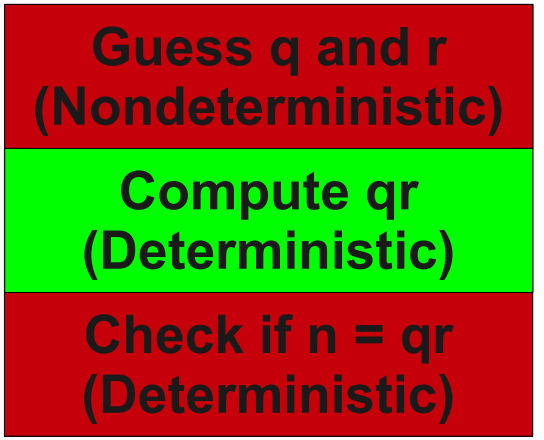

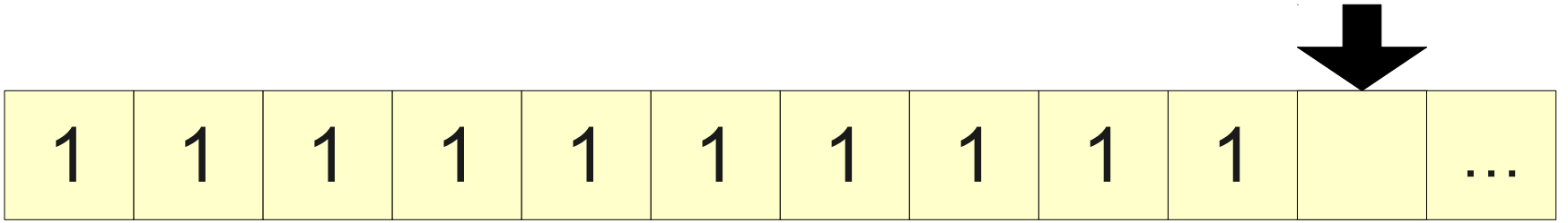

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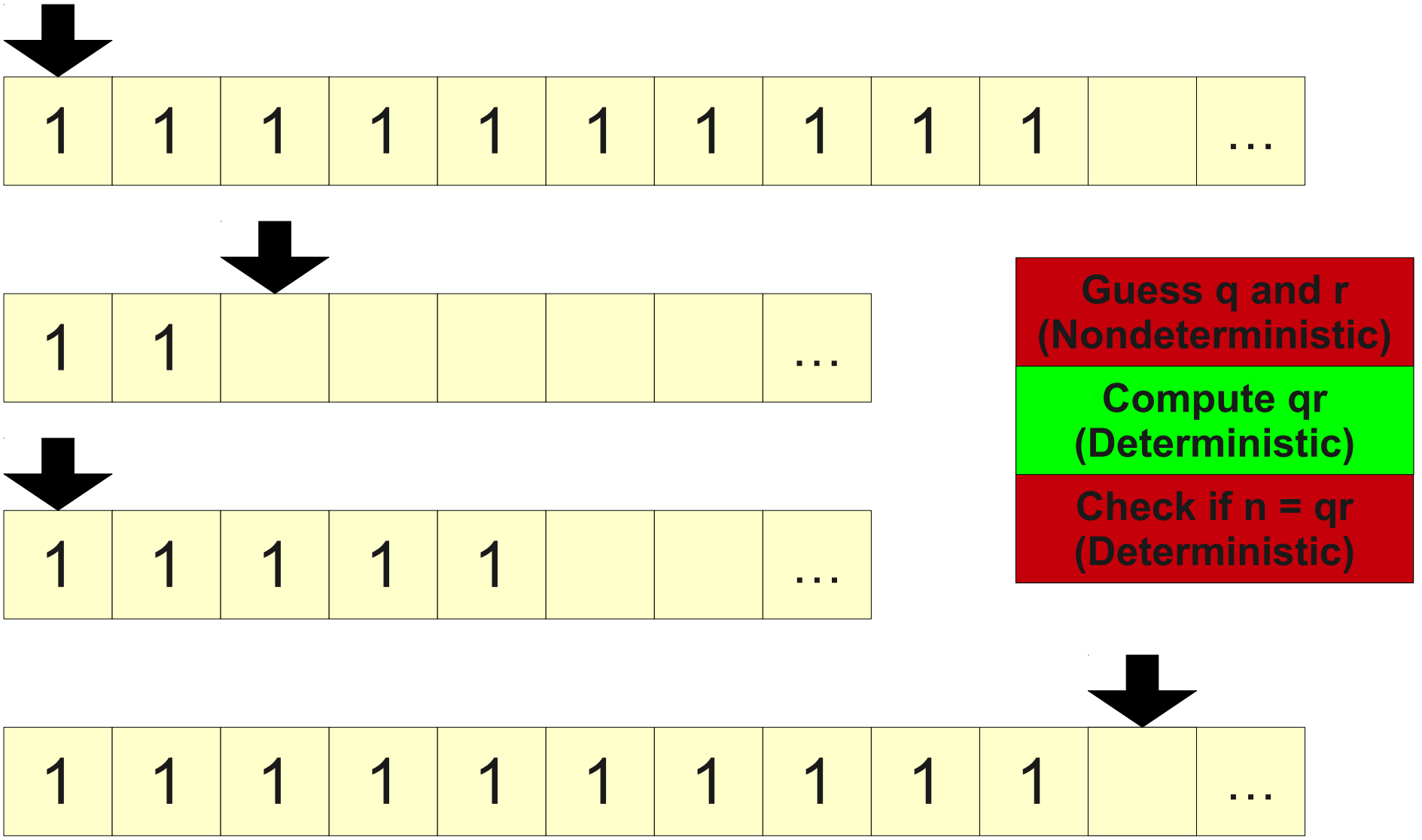

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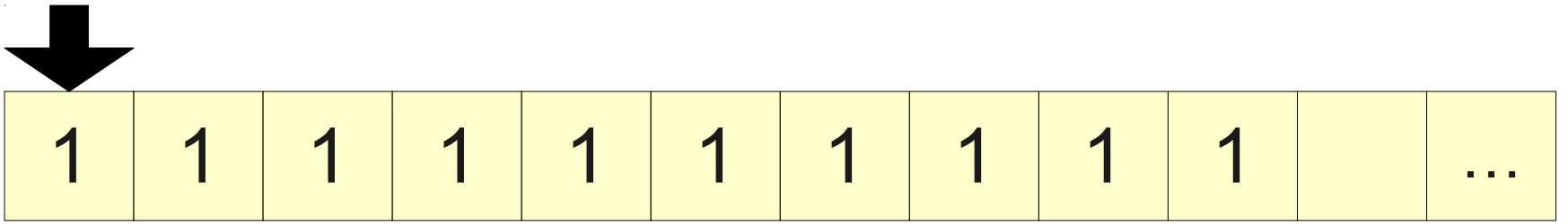

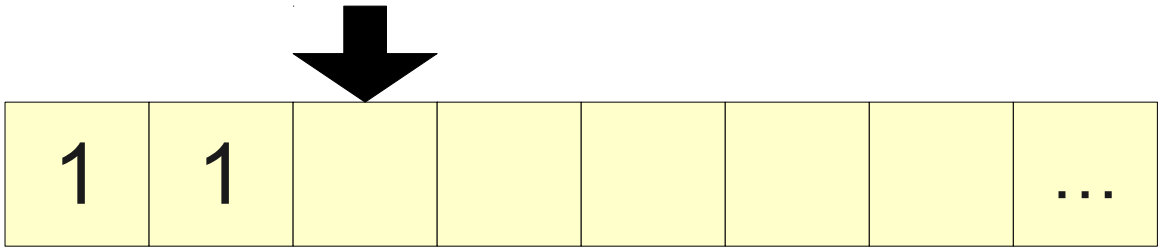

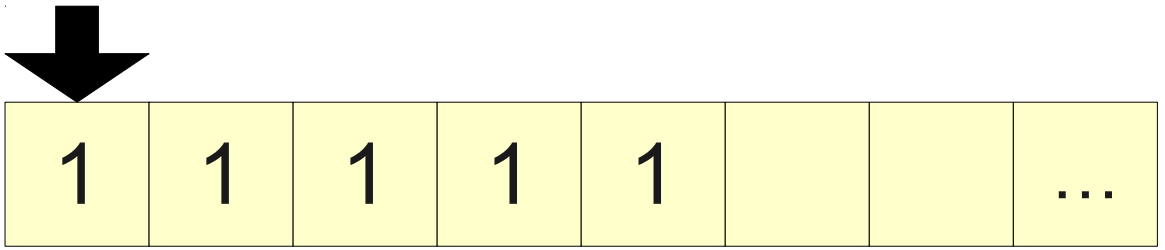

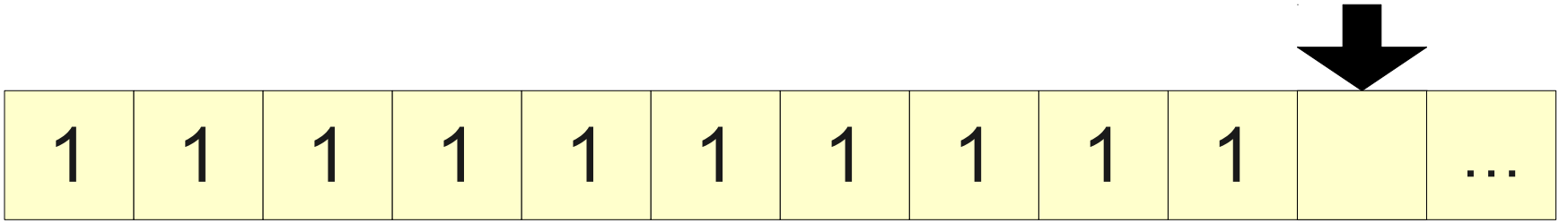

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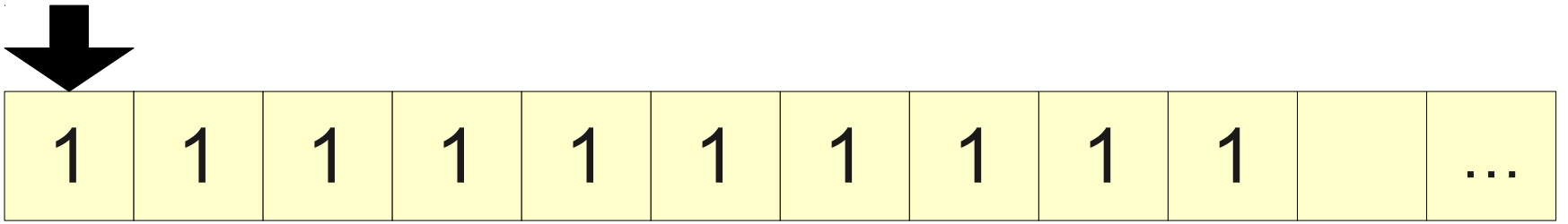

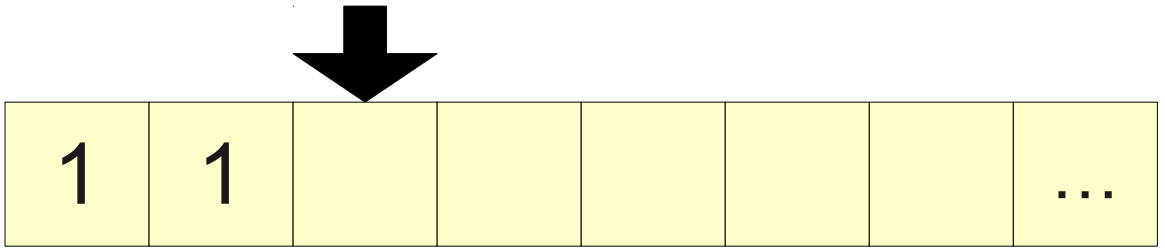

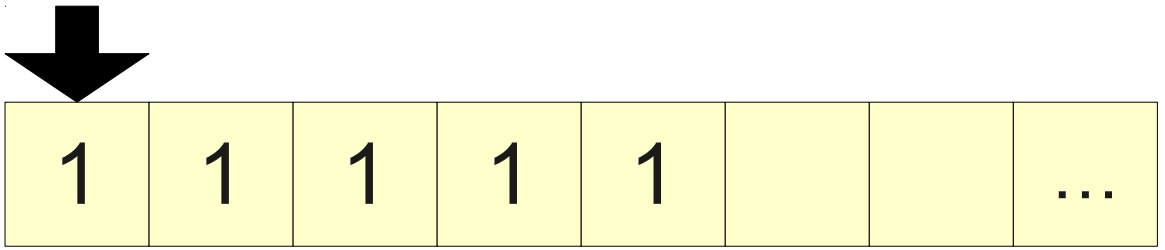

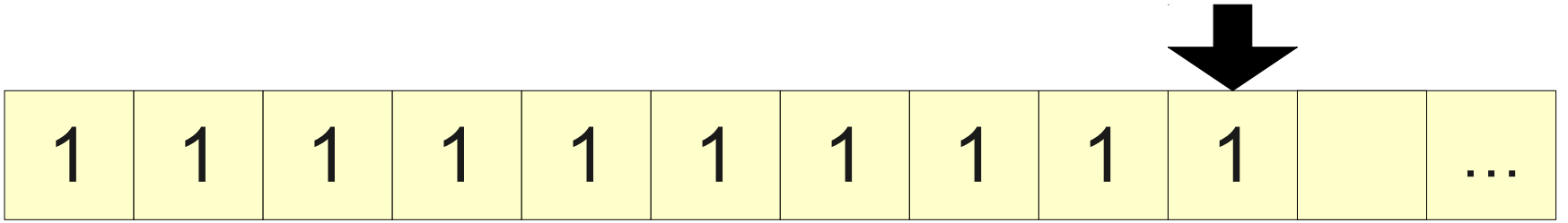

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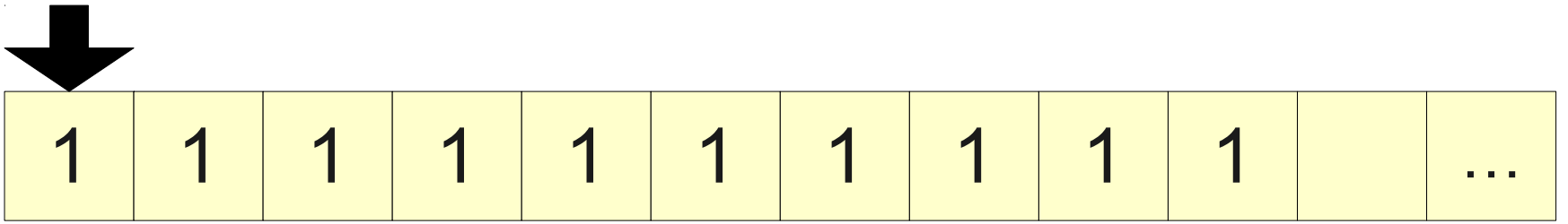

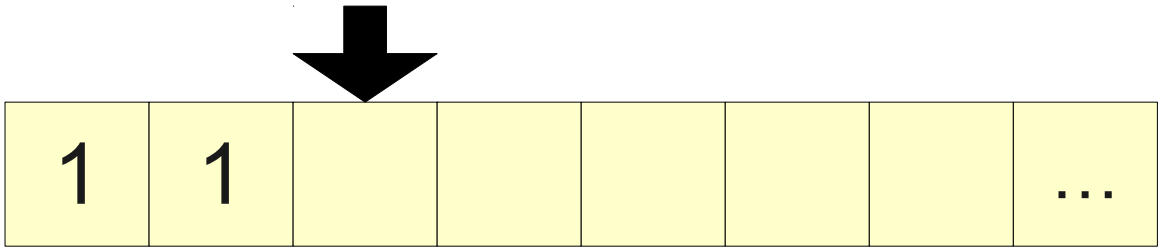

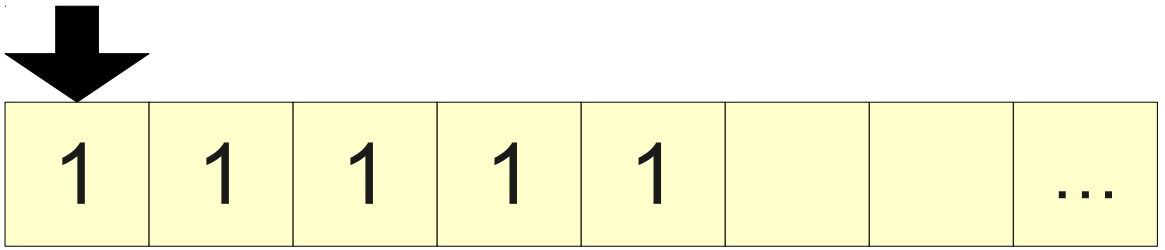

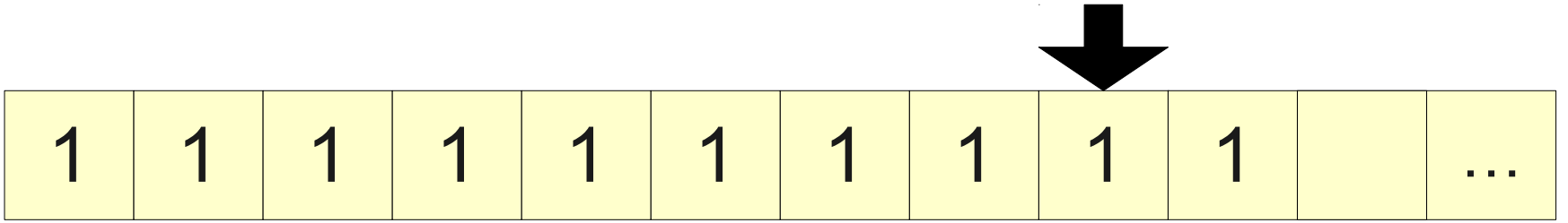

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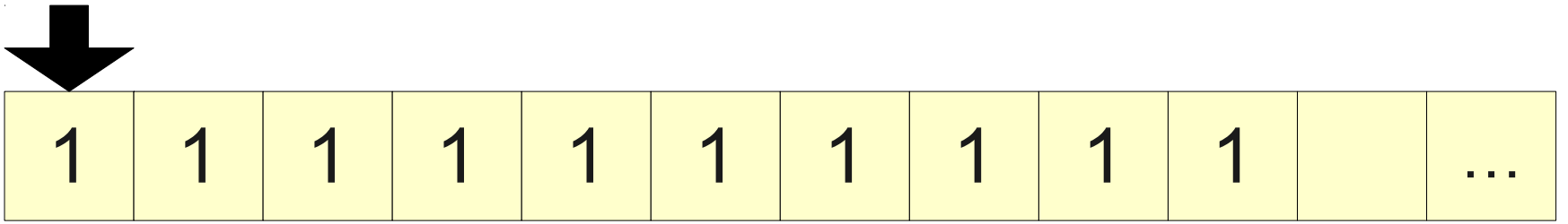

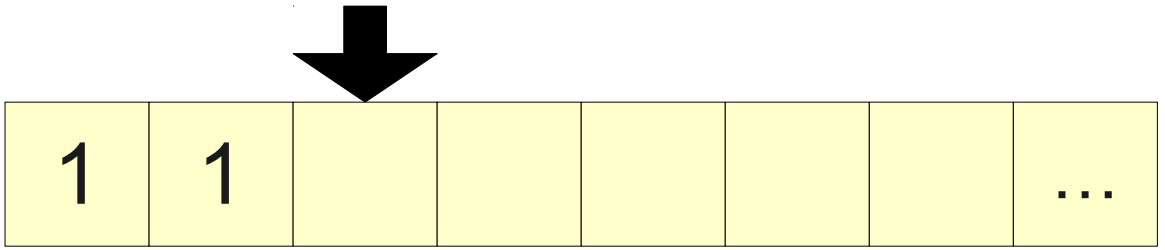

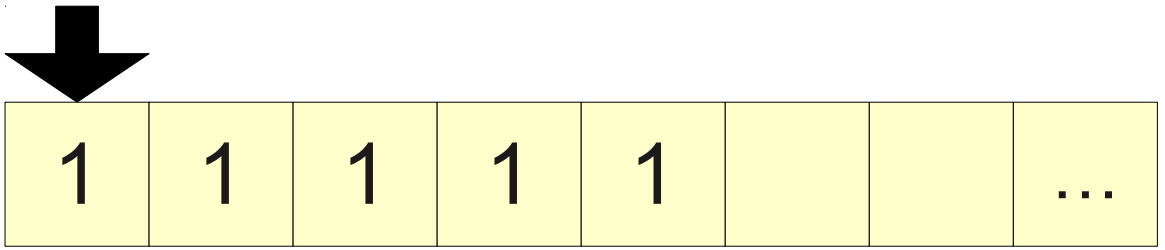

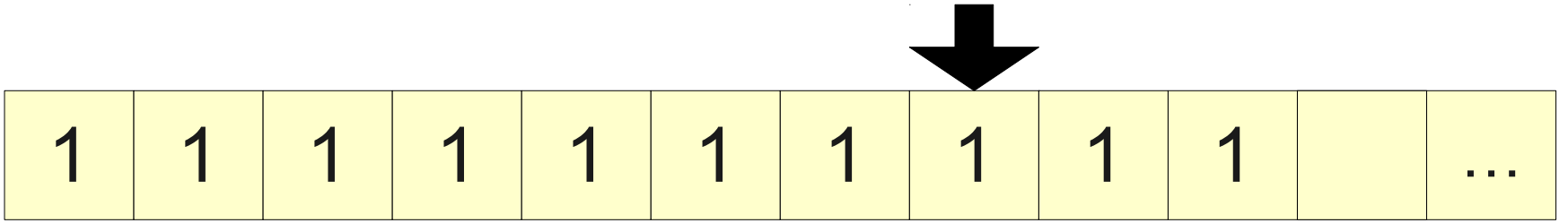

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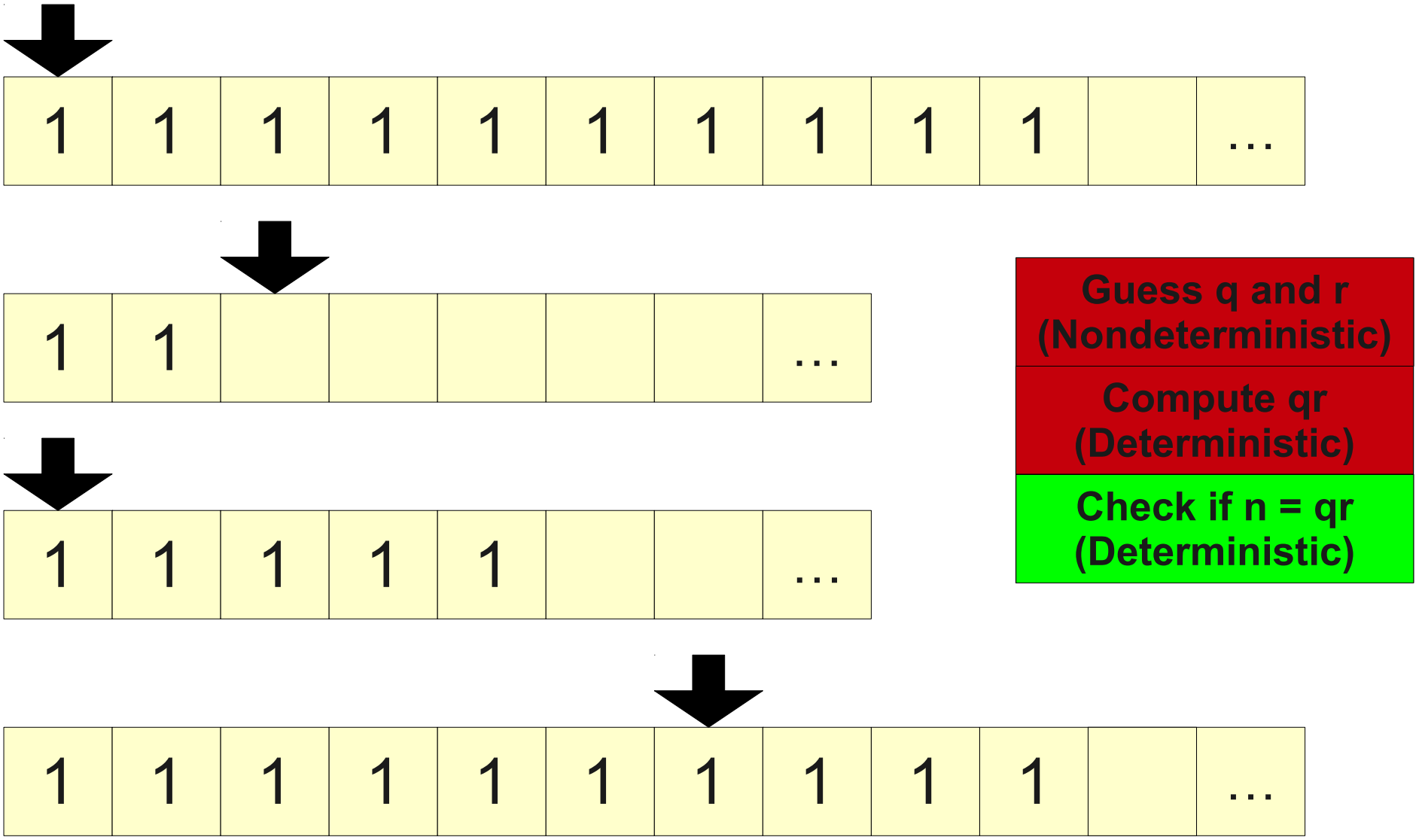

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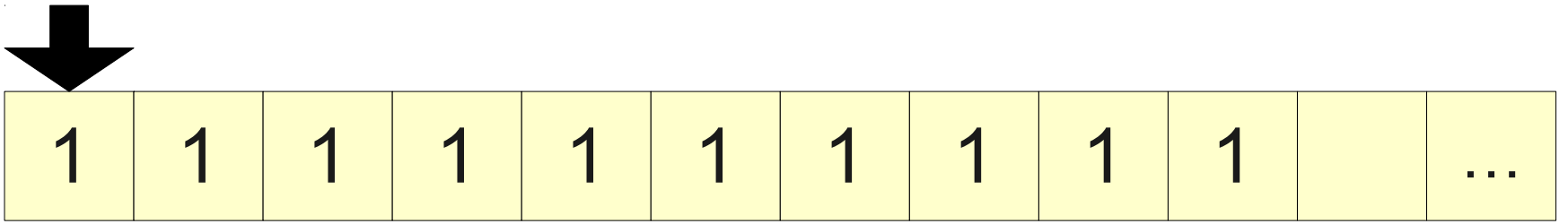

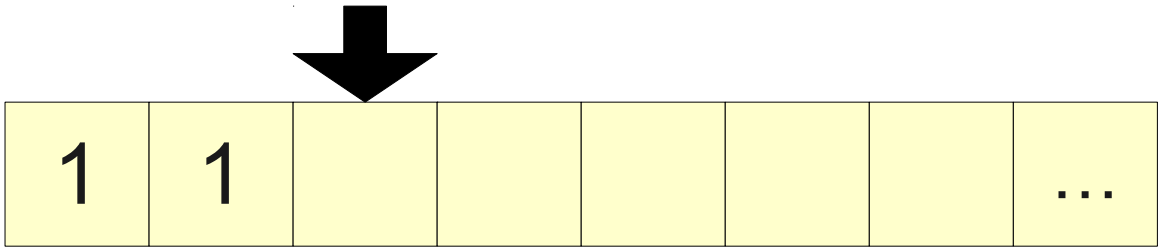

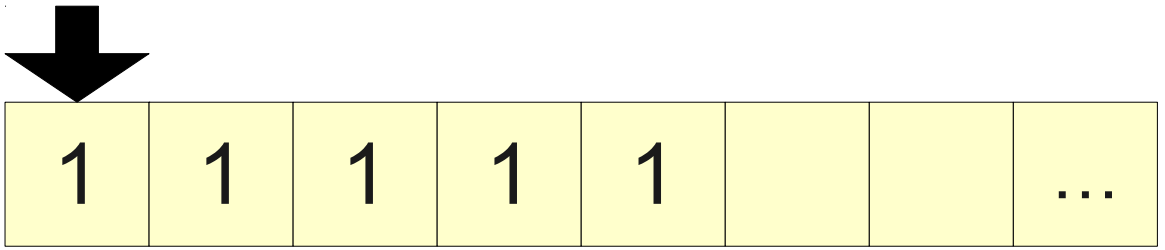

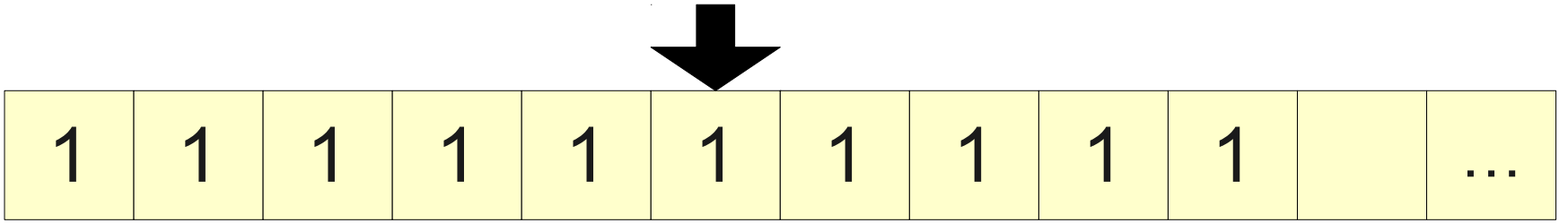

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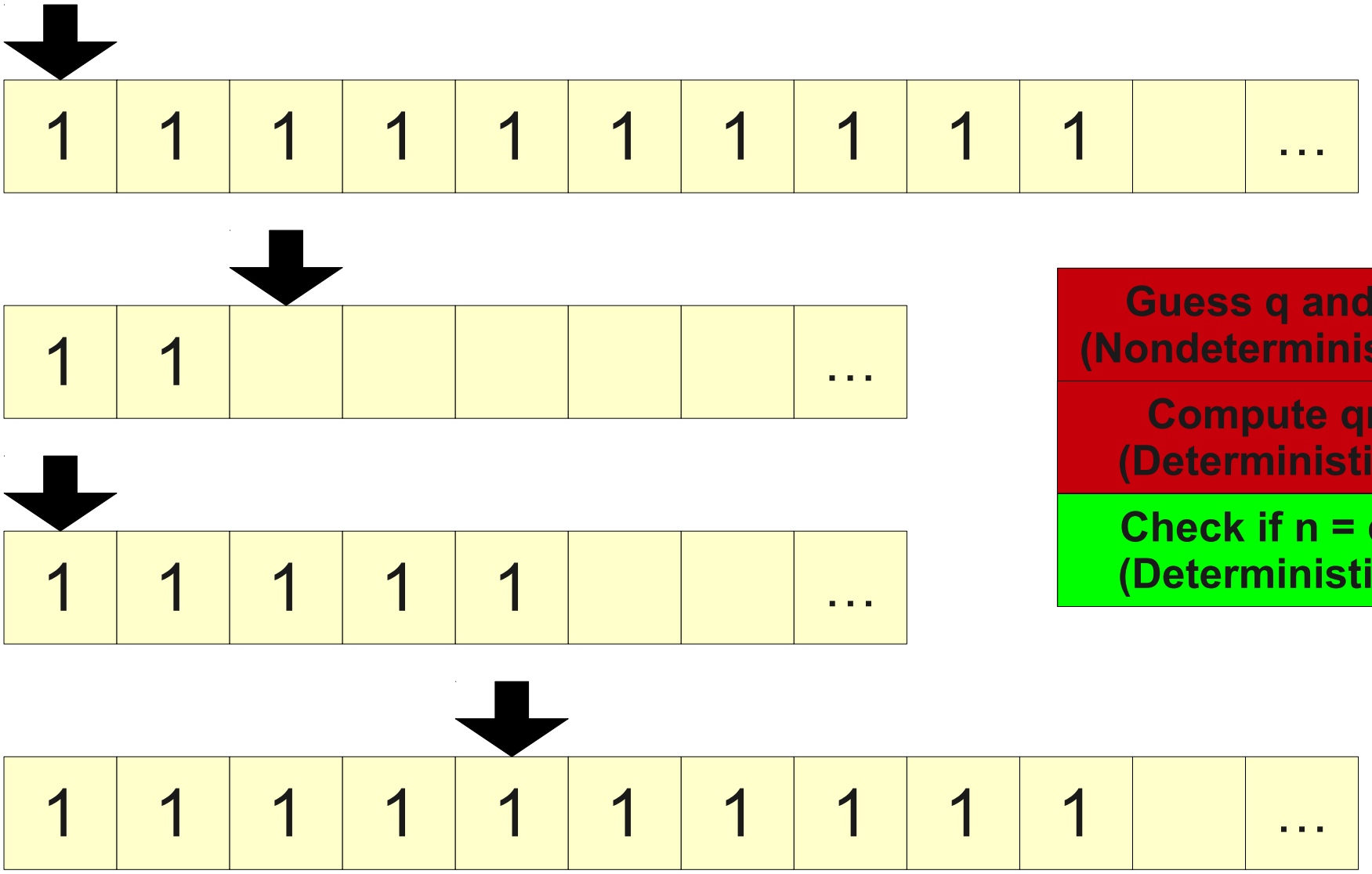

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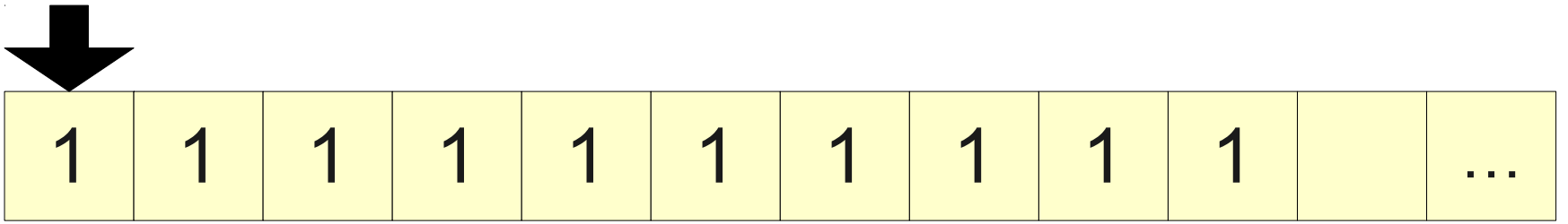

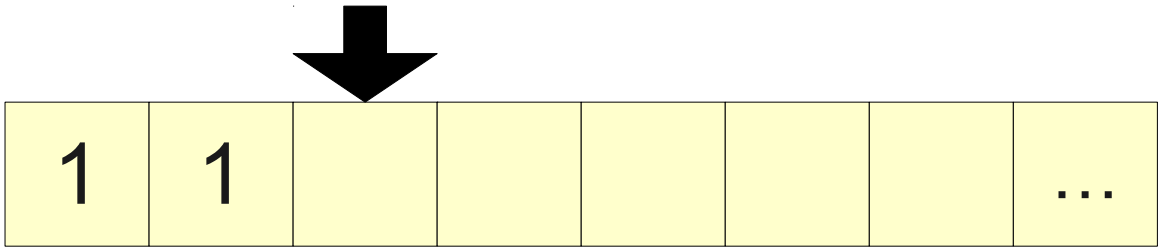

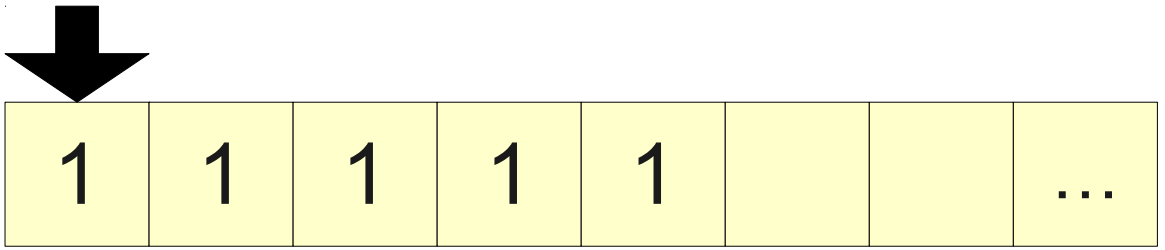

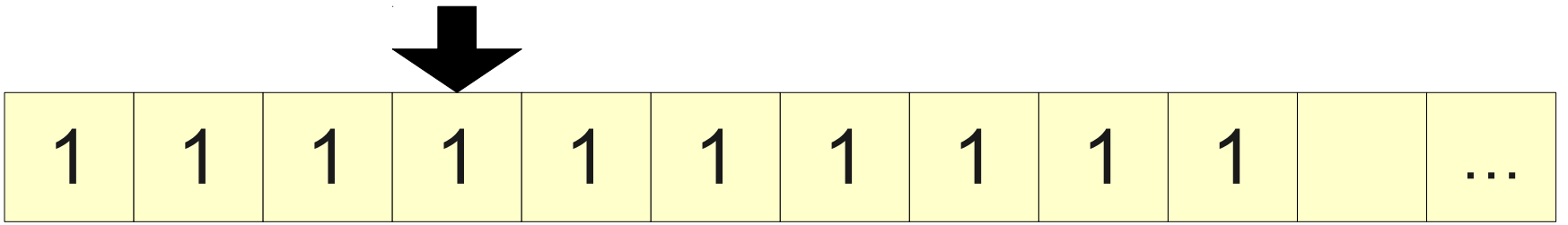

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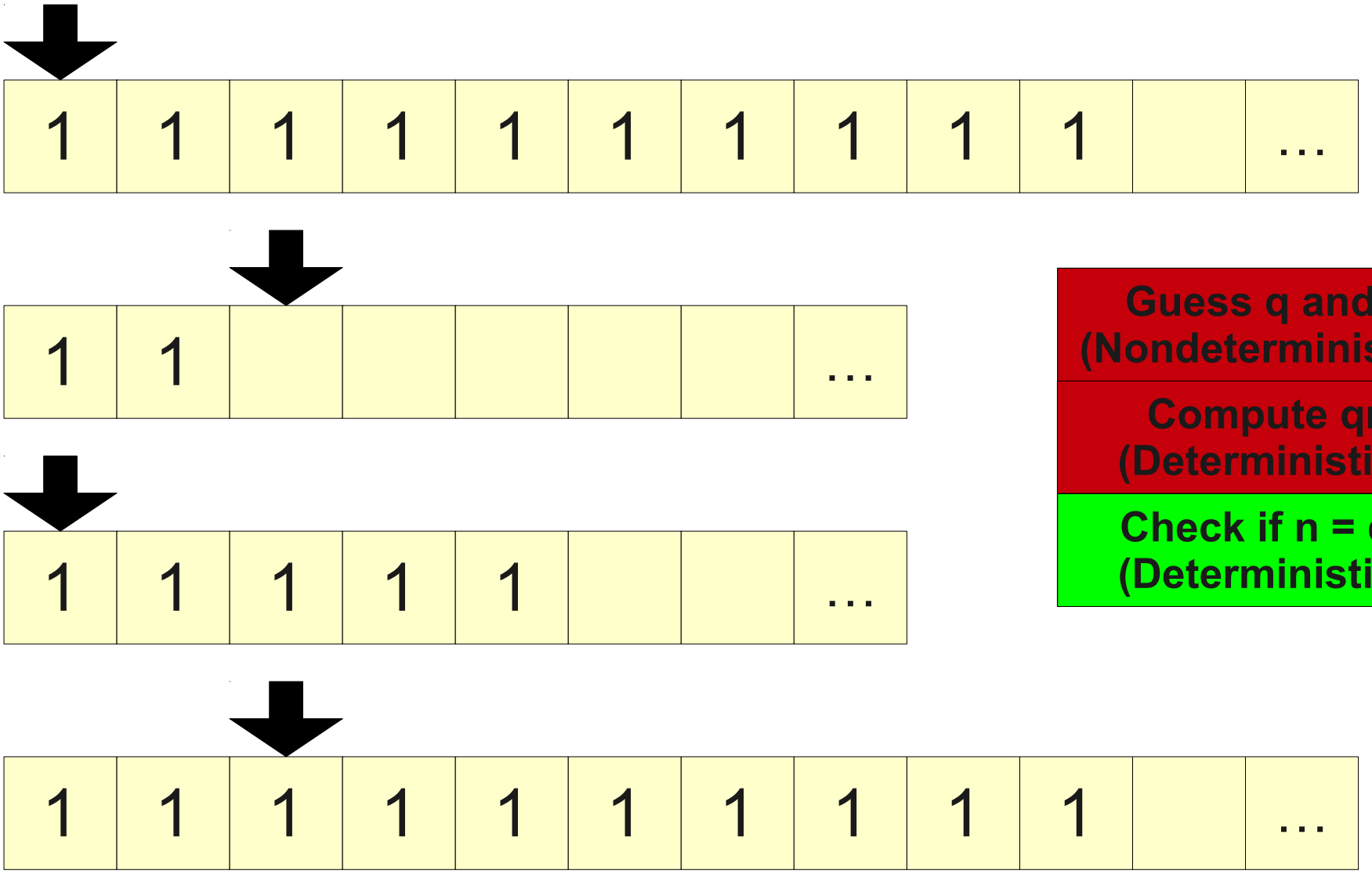

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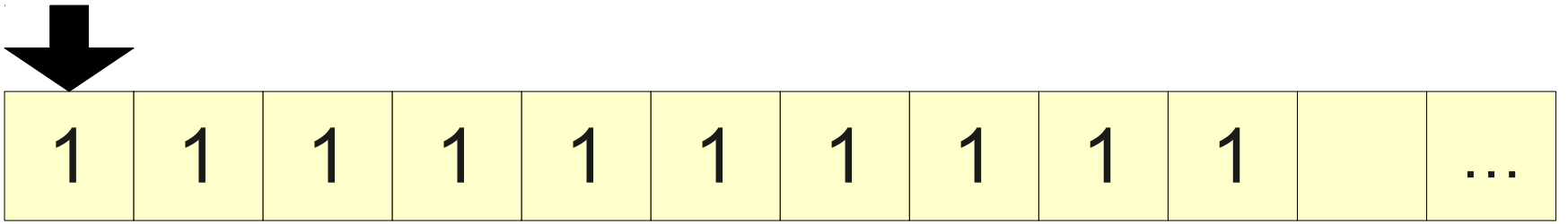

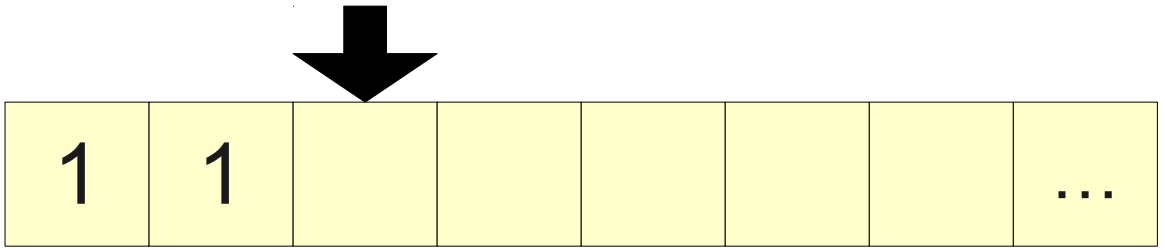

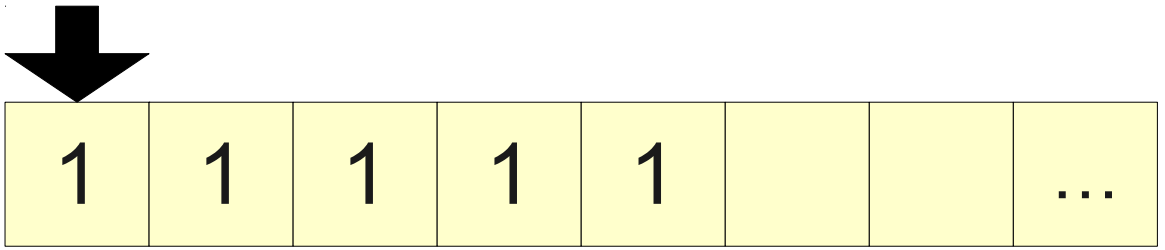

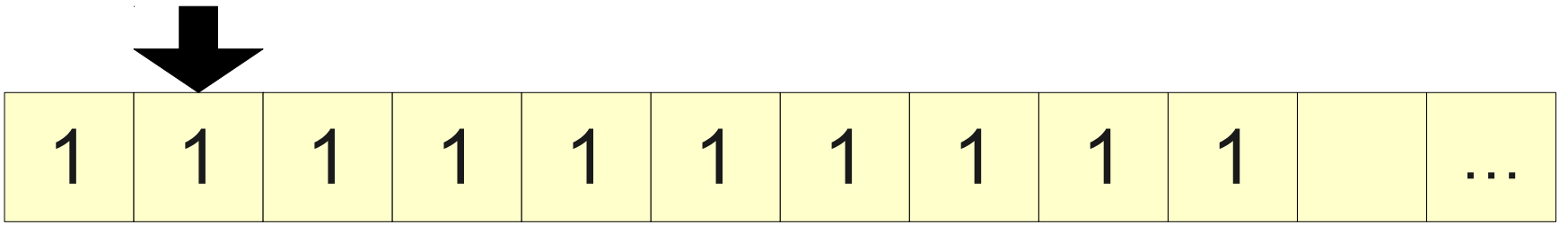

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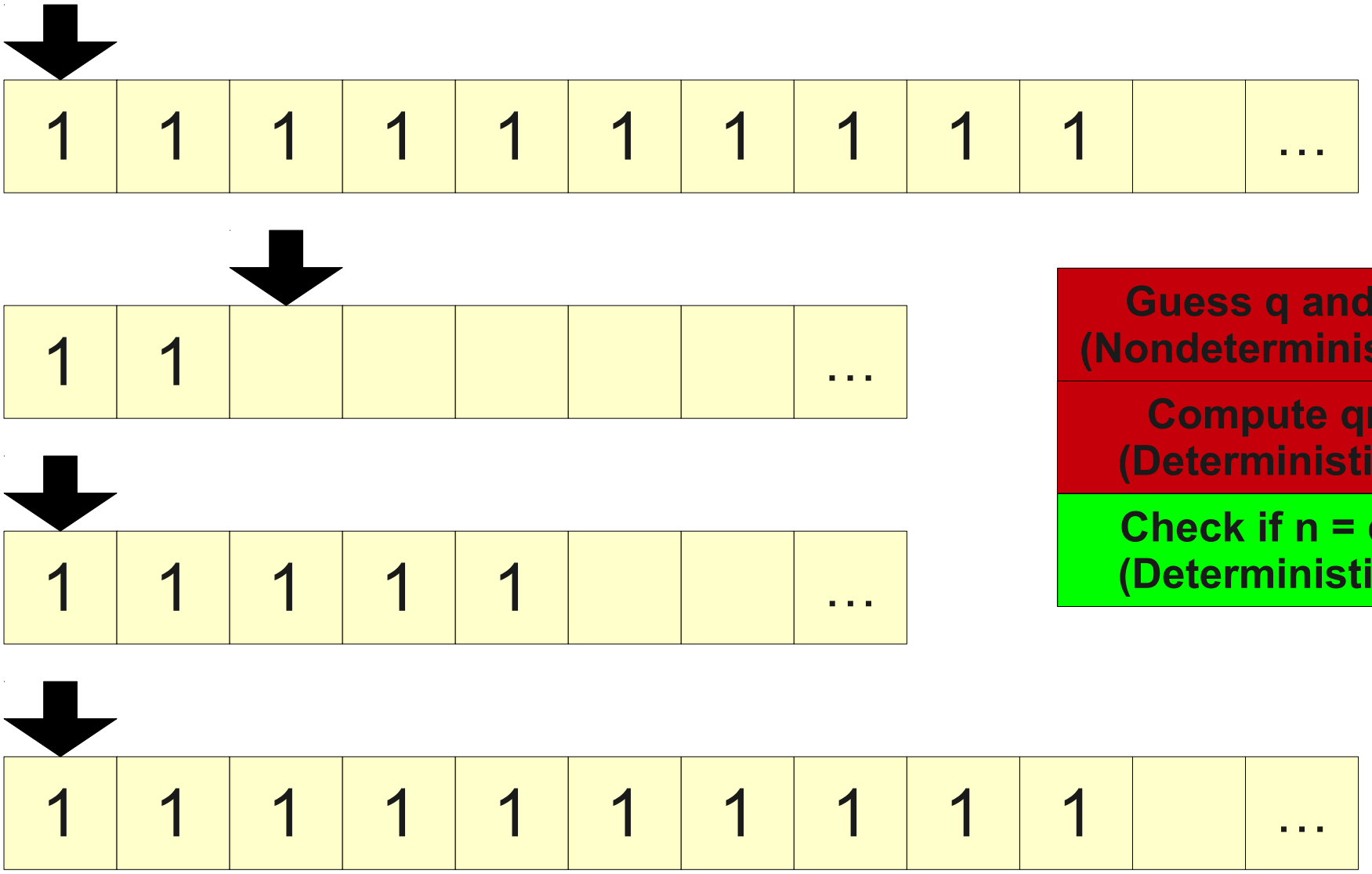

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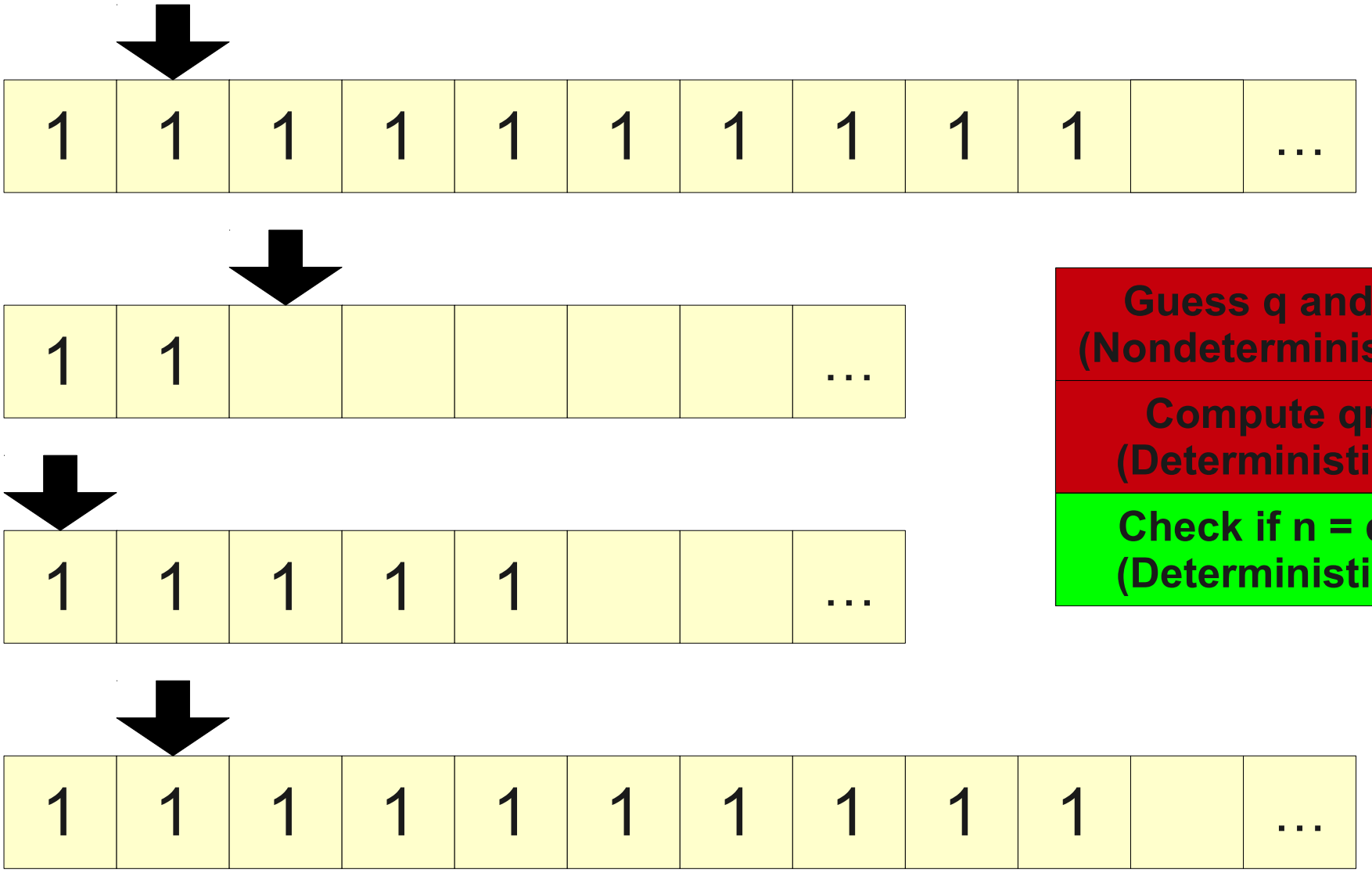

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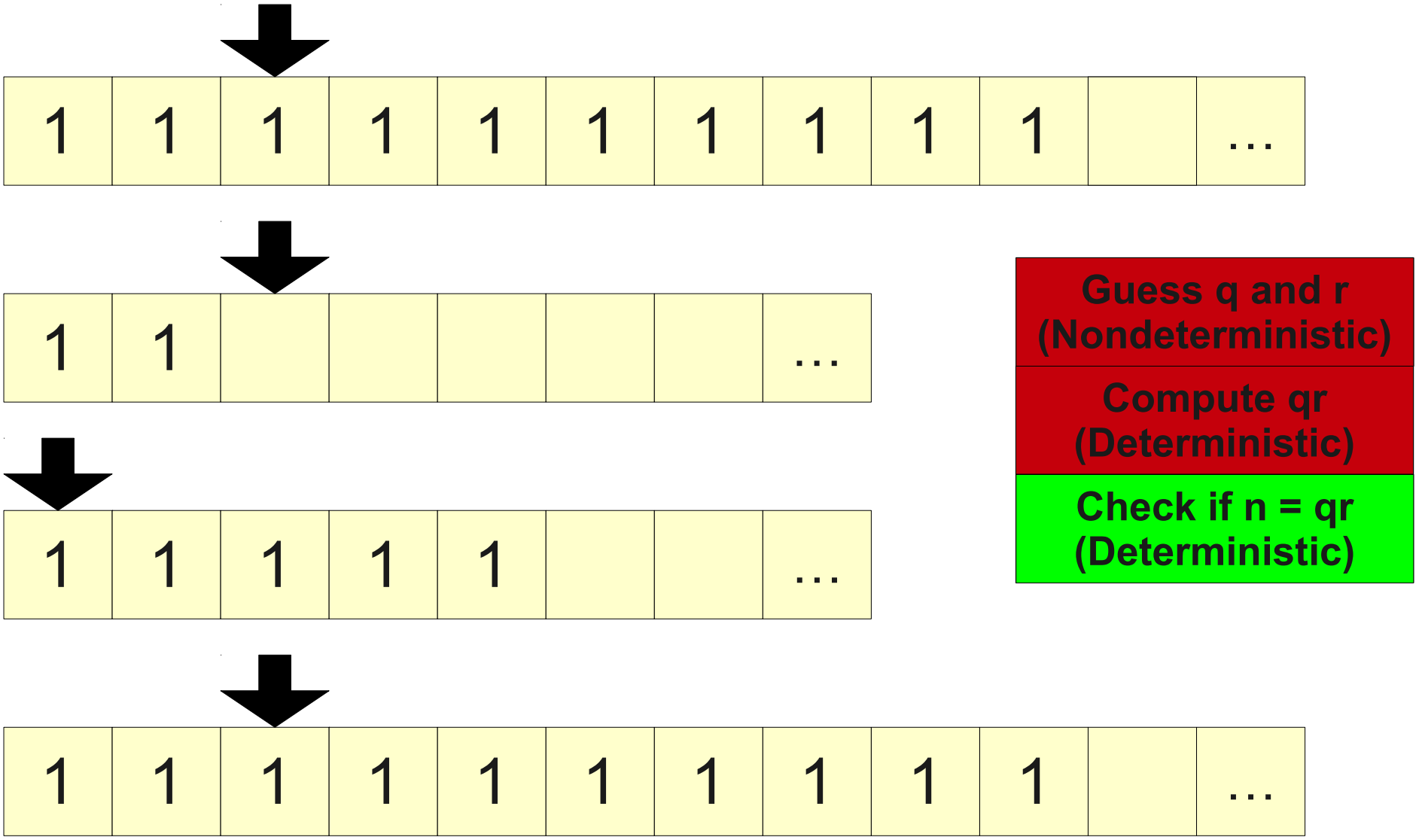

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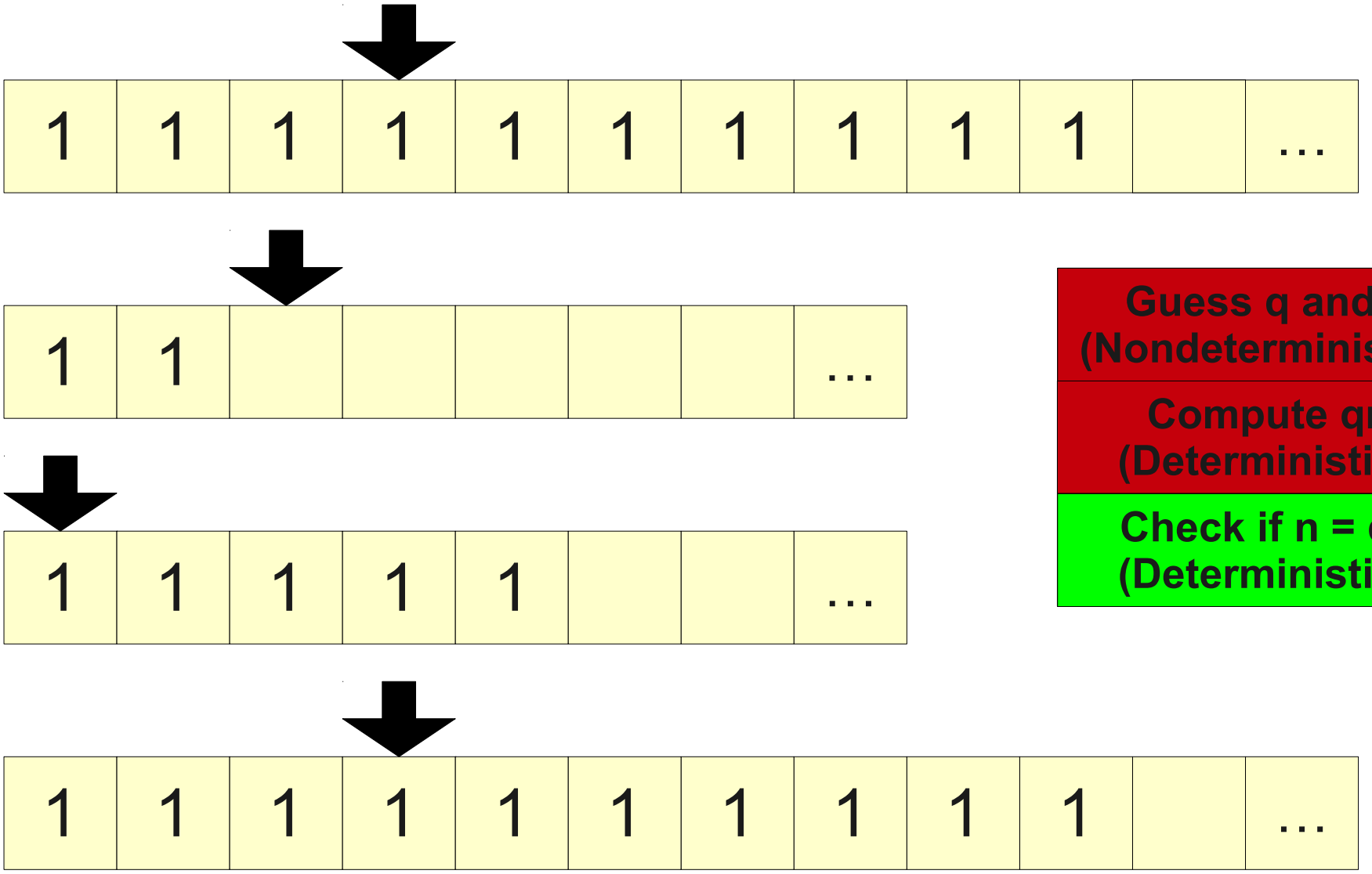

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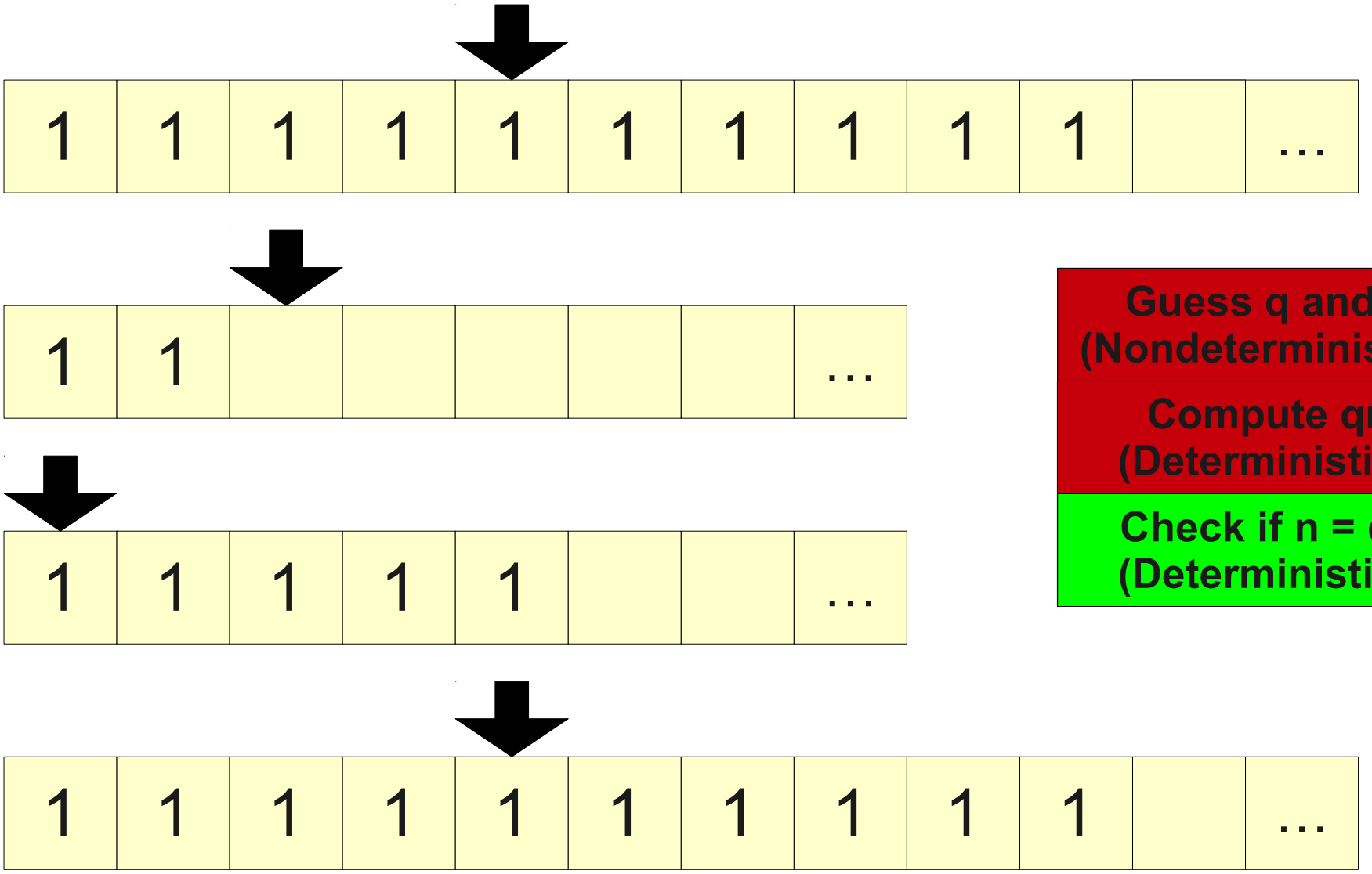

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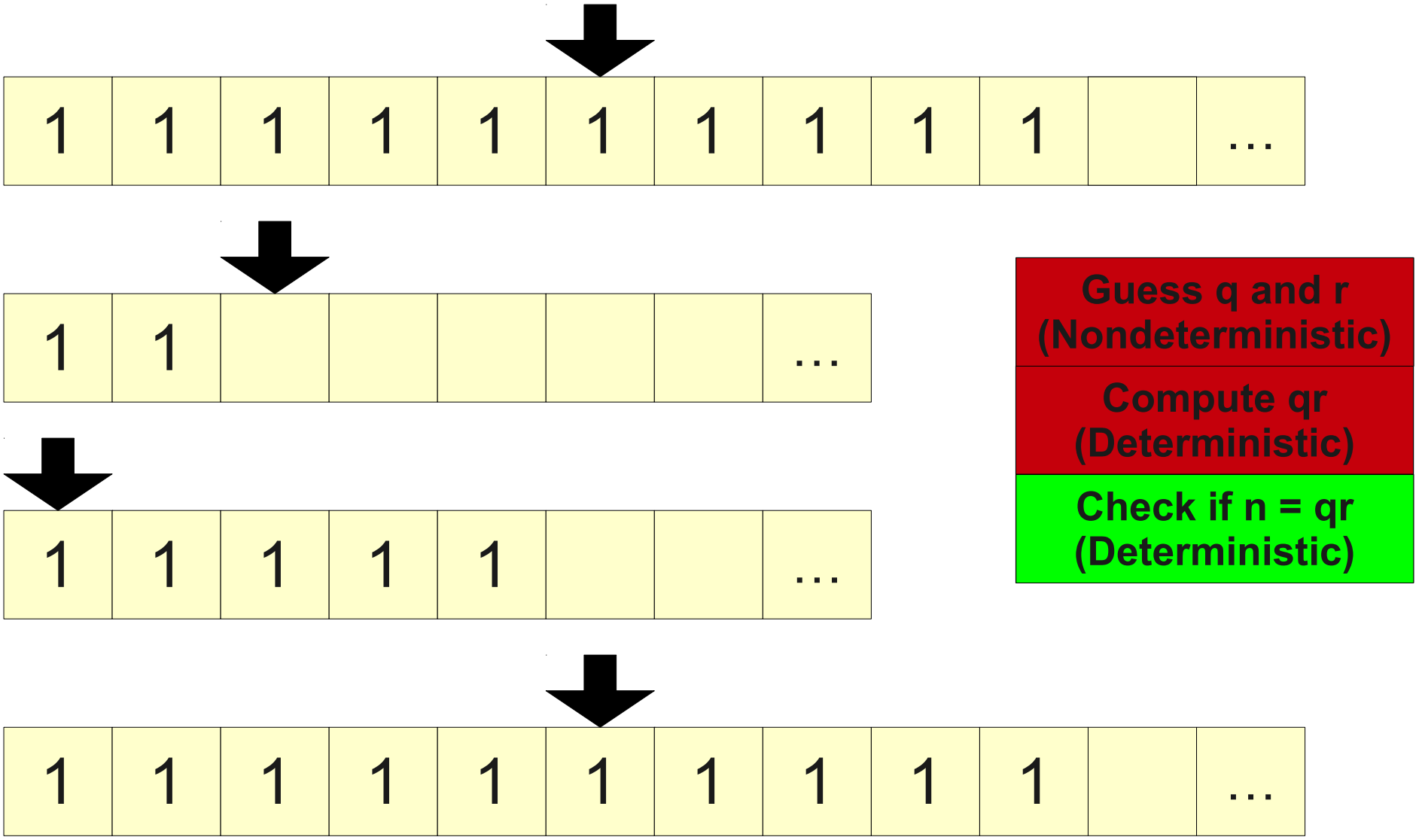

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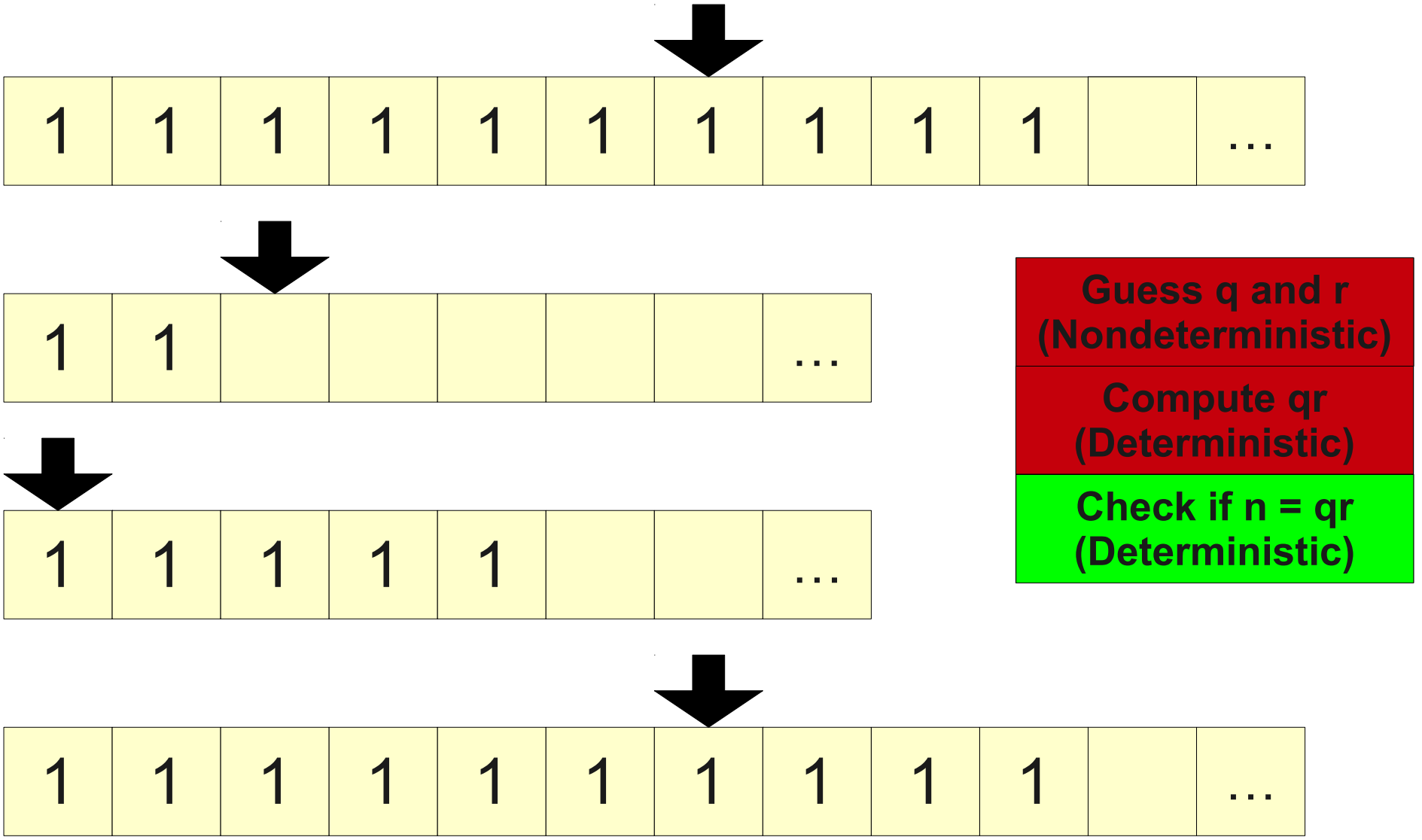
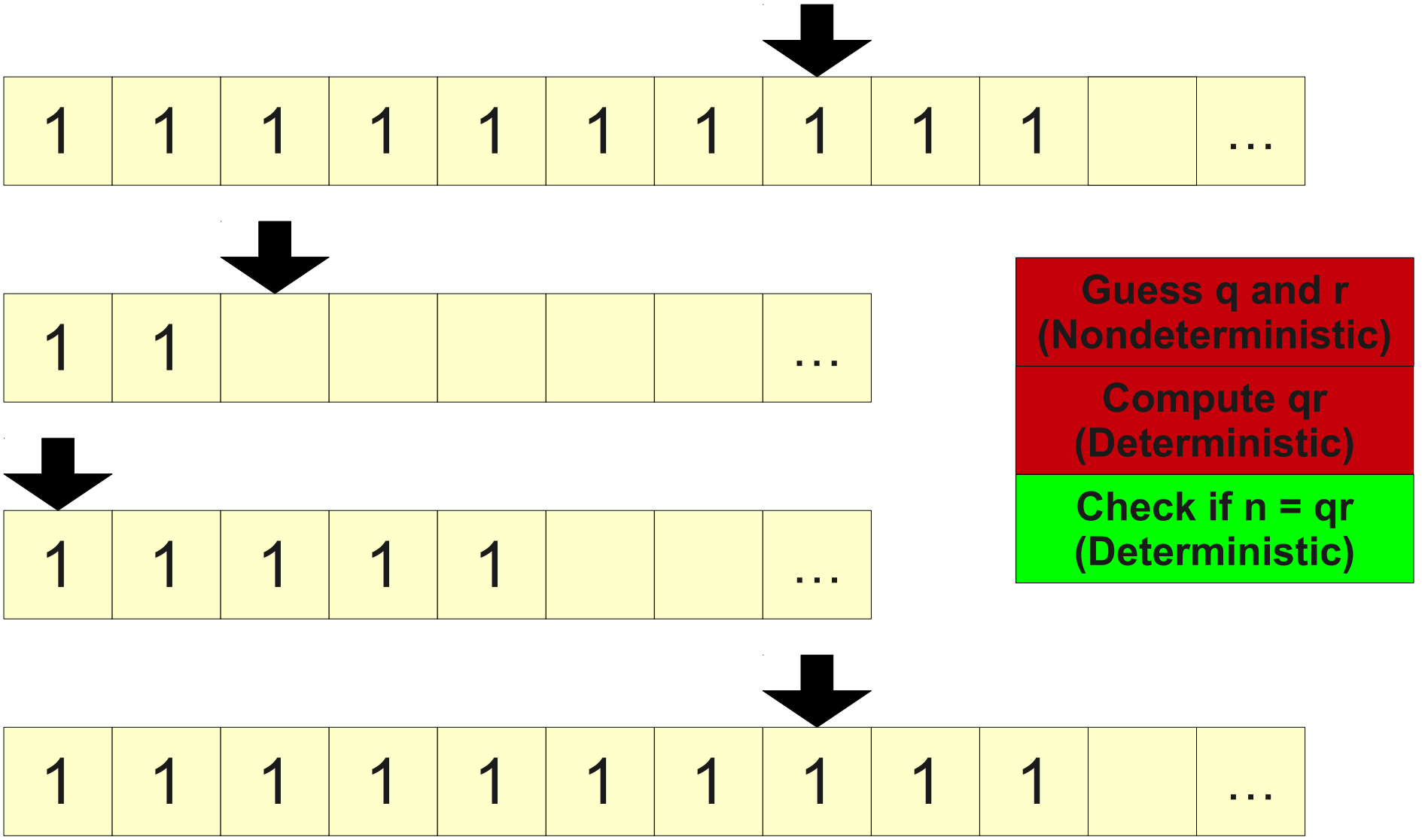


Nondeterministic Algorithms



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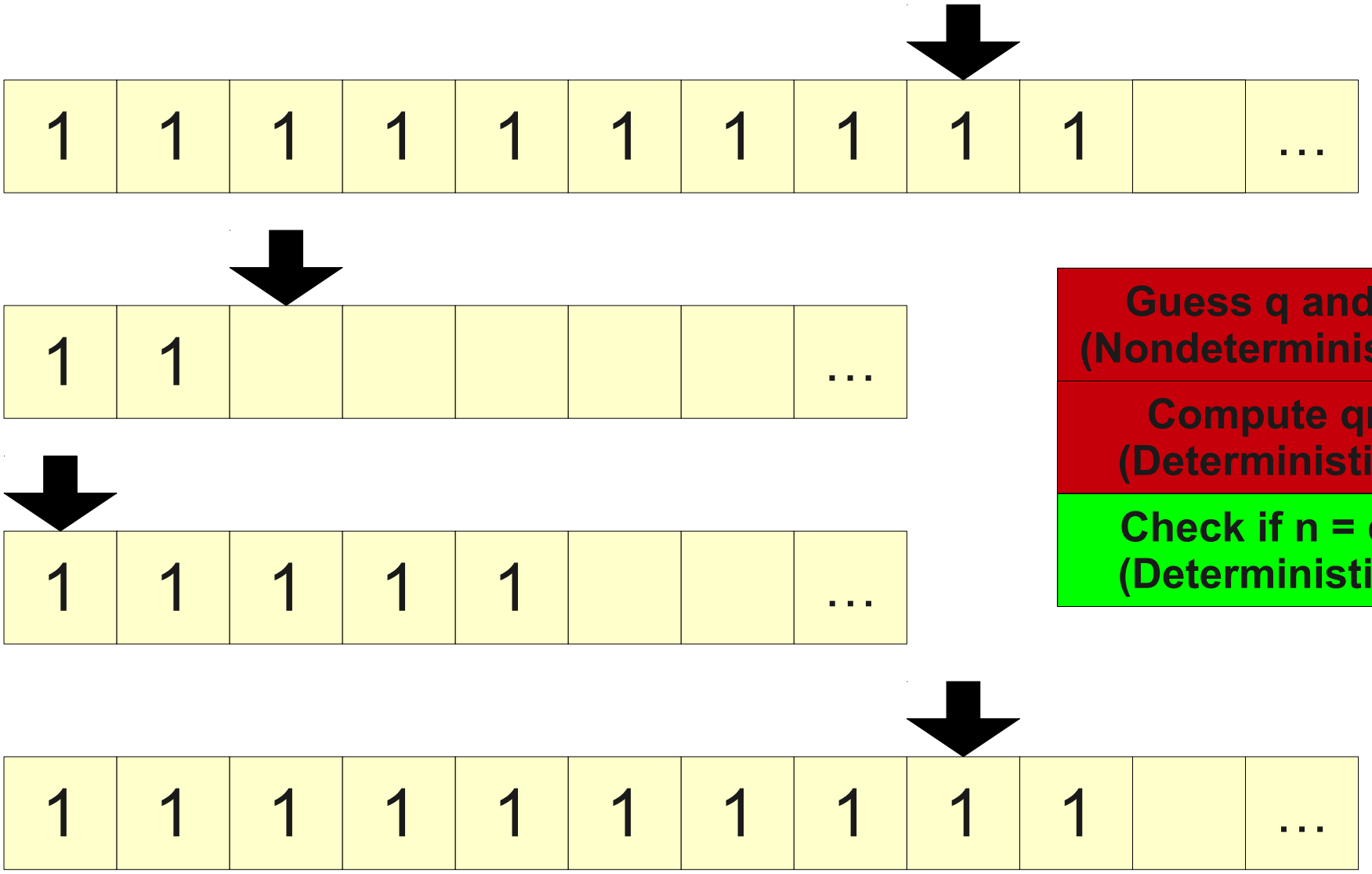

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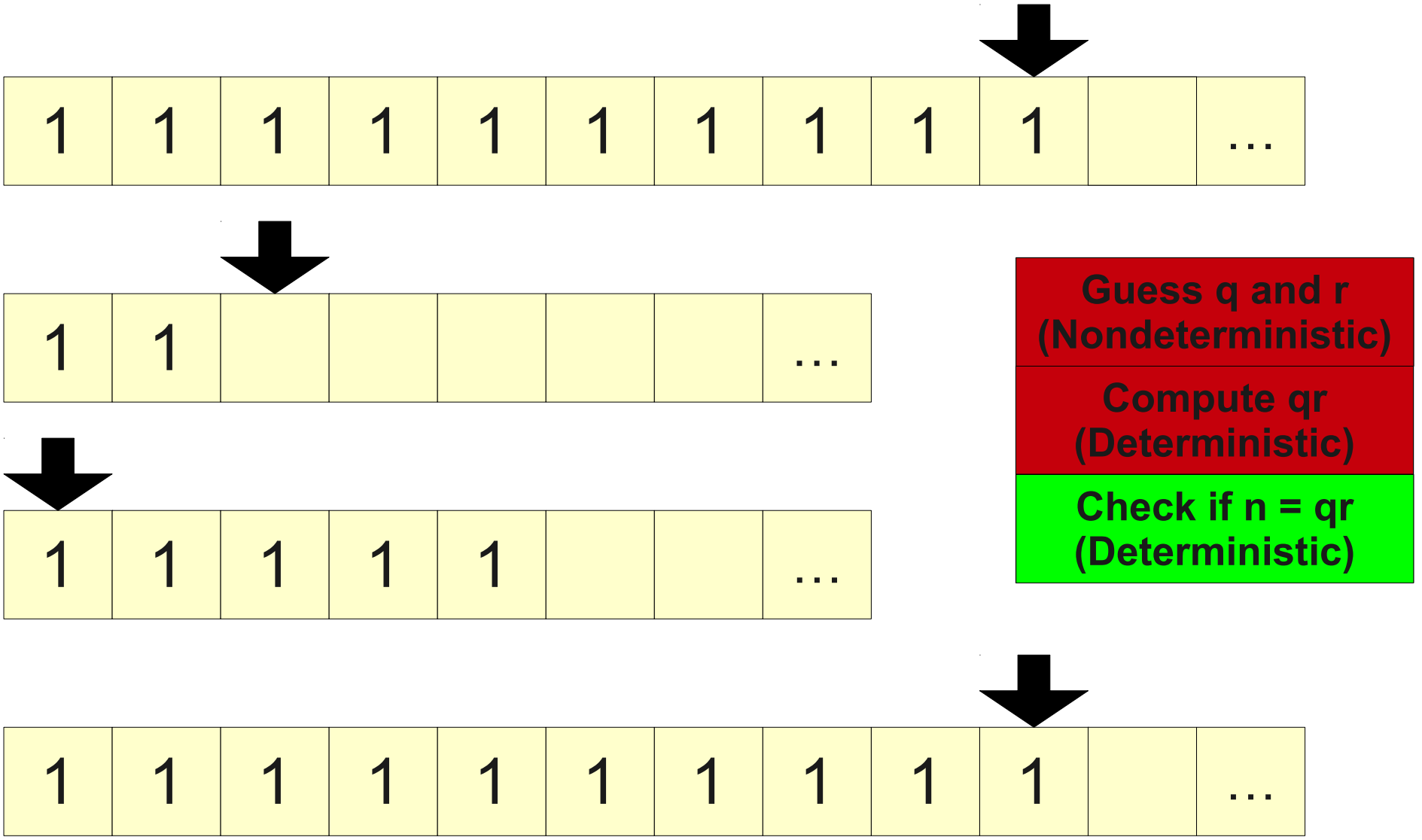

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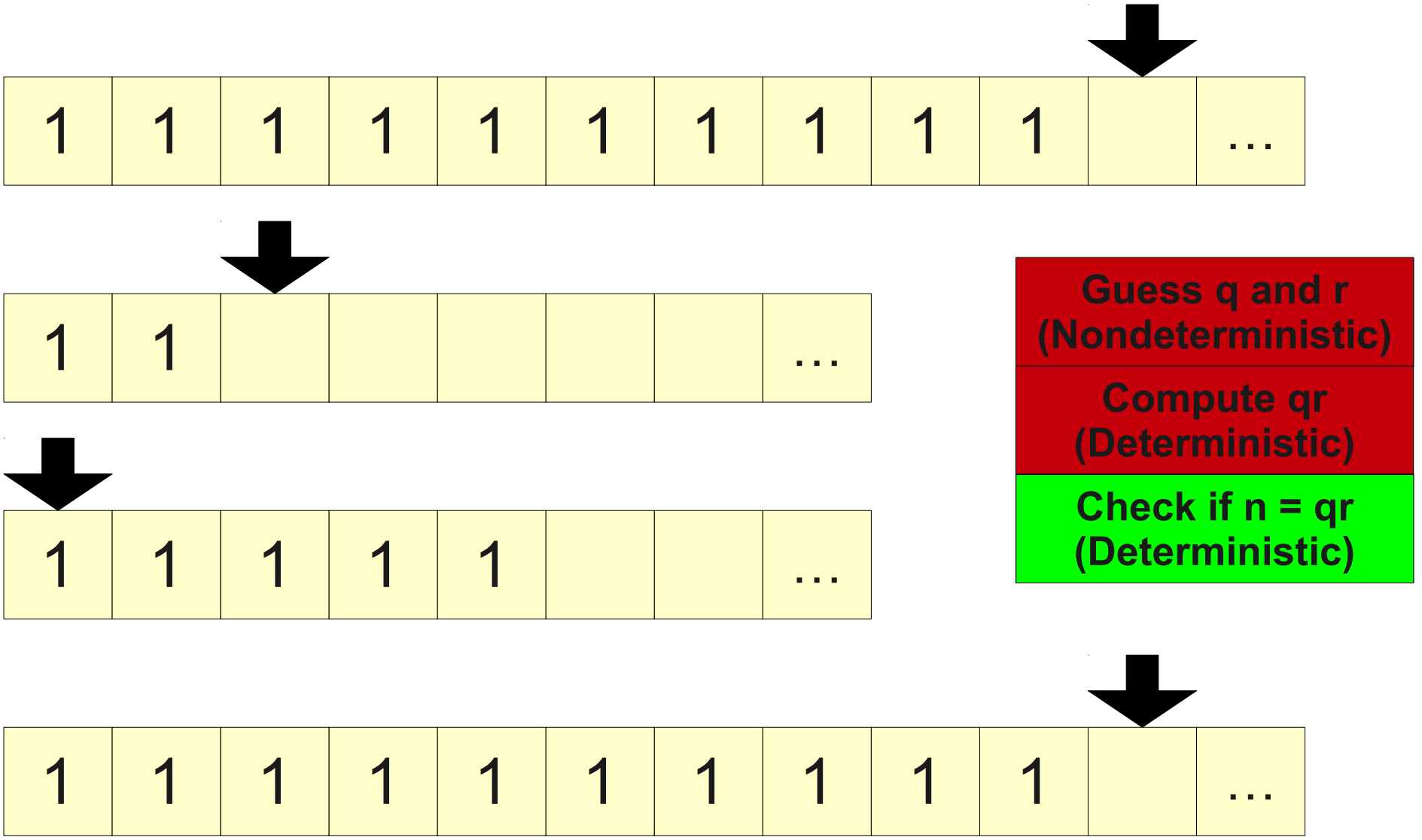


A Nondeterministic Programming Language

- Just as we can have nondeterministic TMs, we can build a nondeterministic variant of **WB**.
- Define **NDWB** to be the language **WB** (the simplest version) with the addition of nondeterministic choice.
- The line of code

$$\texttt{N : \{ } \mathit{line}_1\texttt{, } \mathit{line}_2\texttt{, } \ldots\texttt{, } \mathit{line}_m \texttt{ \}}$$

- Means "nondeterministically choose to execute one of line_1, line_2, …, line_m."

A Simple **NDWB** Program

- Suppose we want to build an NDWB program for the following language over Σ = { `0`, `1` }:

  { *w* | *w* contains `101` as a substring. }

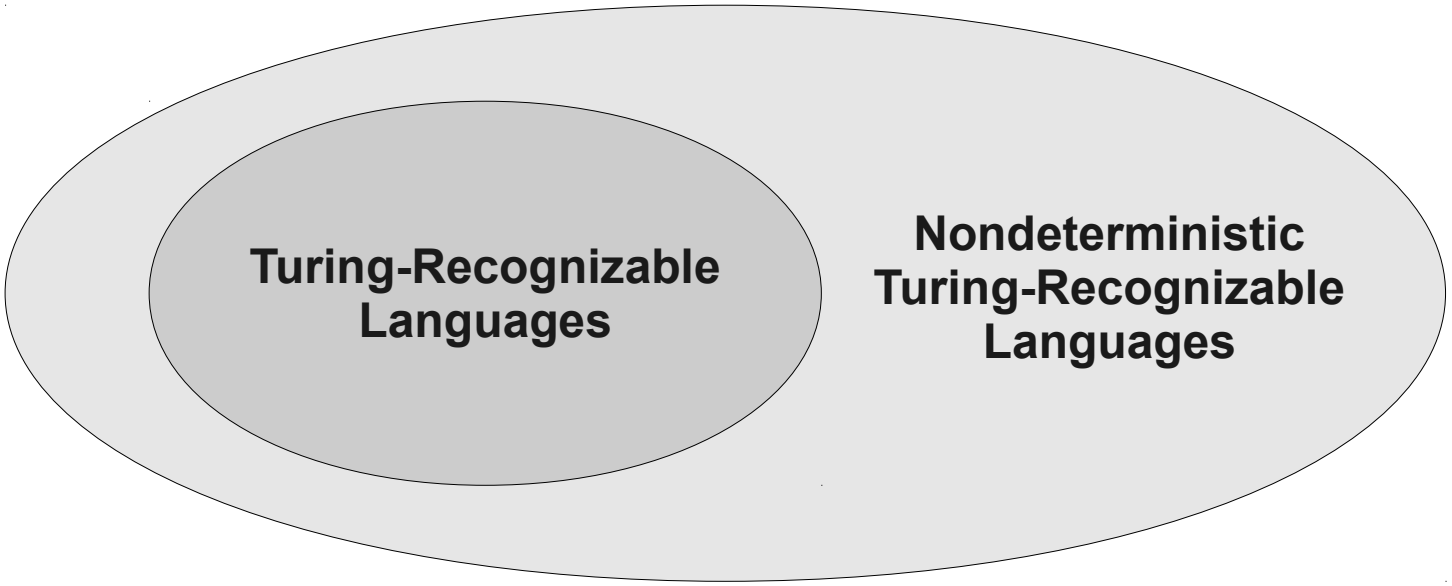
- This is not particularly hard to do deterministically (it's a regular language), but it's interesting to see how we might do this nondeterministically.

A Simple **NDWB** Program

```
// Start
 0: If reading B, go to Rej.
 1: If reading 0, go to Next.
 2: {
    If reading 1, go to Next.
    If reading 1, go to Match.
 }

// Next:
 3: Move right.
 4: Go to Start.

// Match
 8: Move right.
 9: If reading 0, go to 11
10: Reject.
11: Move right.
12: If reading 1, go to 14
13: Go to Rej.
14: Accept.

// Rej:
15: Reject.
```

The Power of Nondeterminism

- In the case of finite automata, we saw that the DFA and NFA had equivalent power.
- In the case of pushdown automata, we saw that the DPDA was strictly weaker than the NPDA.
- What is the relative power of TMs and NTMs?



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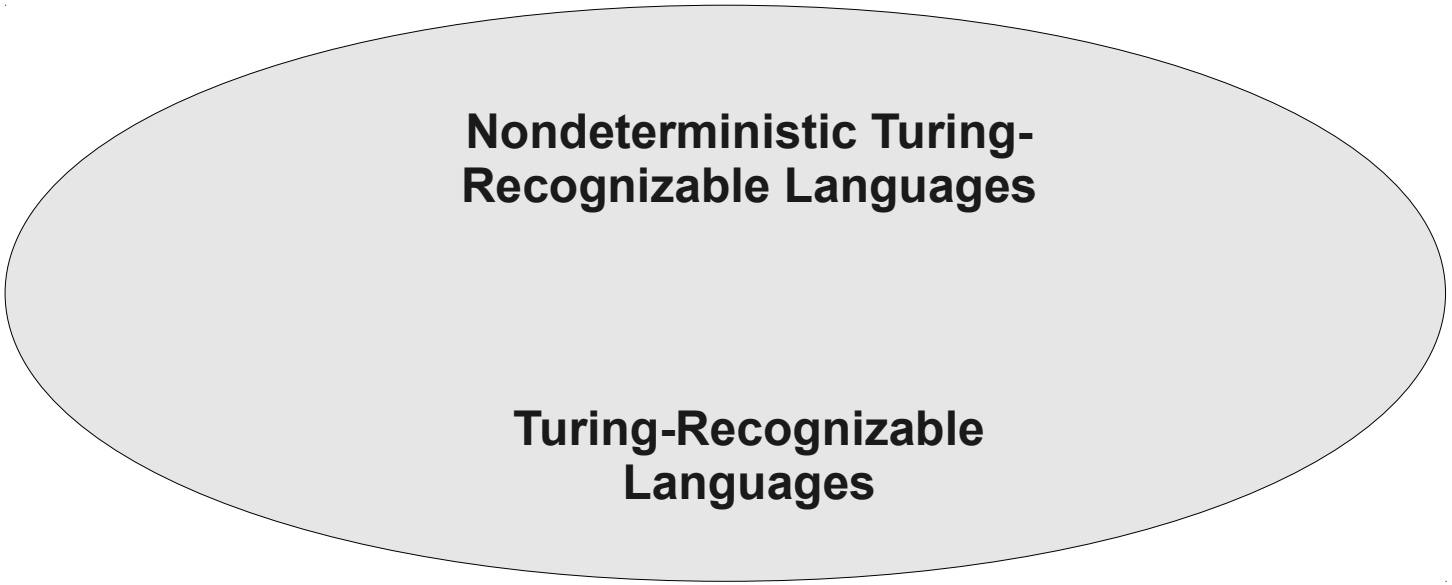
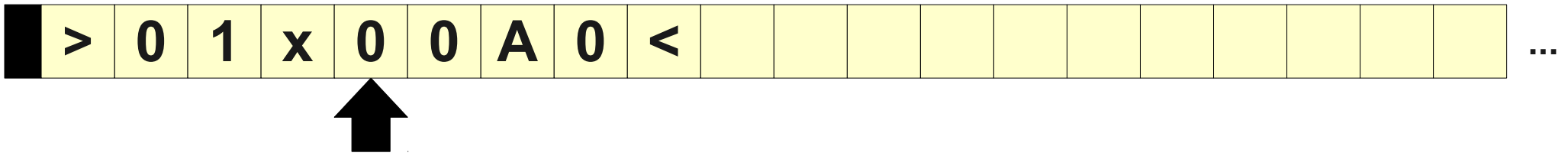


A Rather Remarkable Theorem

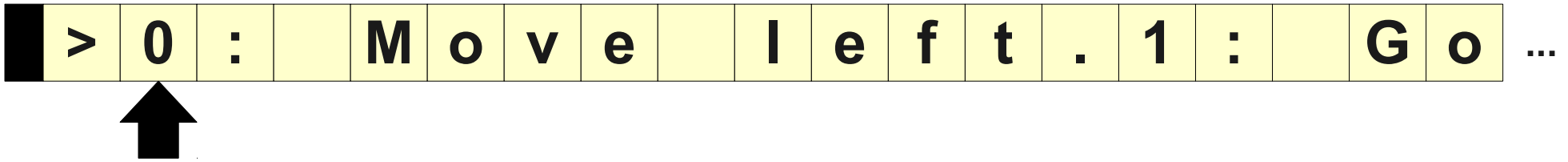
- **Theorem:** A language is recursively enumerable iff it is accepted by a nondeterministic Turing machine.
- **Theorem:** A language is recursively enumerable iff there is an **NDWB** program for it.
- How is this possible?

Sketch of the Universal **WB** Program

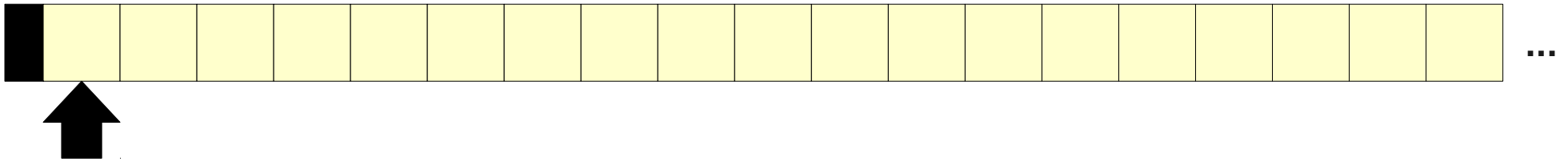
Simulated tape of the program being executed.



Program tape holding the program being executed.



Scratch tape for intermediate computation.



Variables for intermediate storage.

Instr **Letter**

Sketch of the Universal **WB** Program

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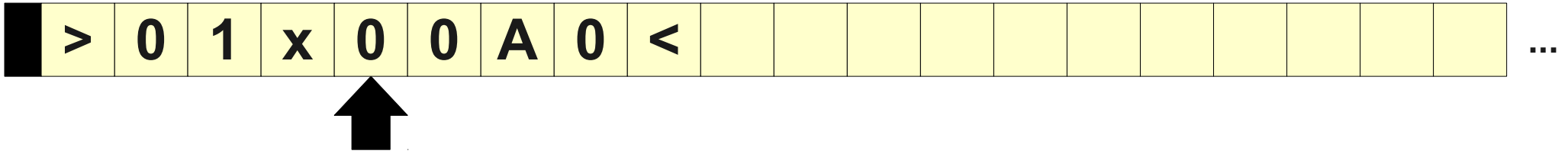

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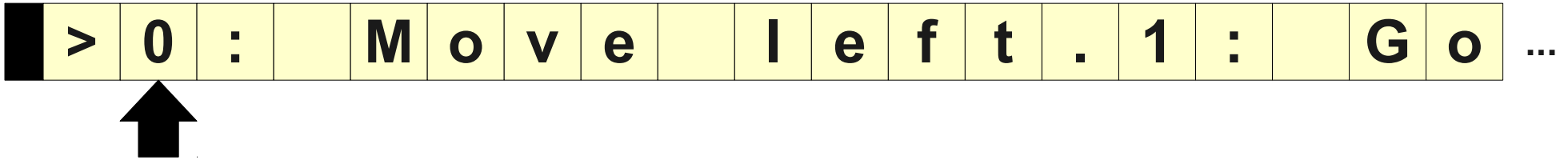


Scratch tape for intermediate computation.

Variables for intermediate storage.

Instr **Letter**

Instantaneous Descriptions

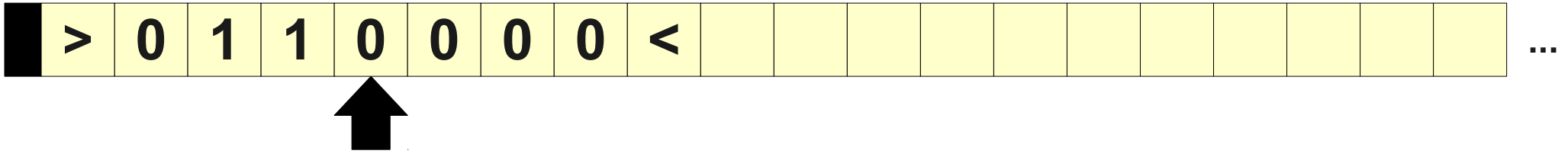
- An **instantaneous description** (or **ID**) of the execution of a program (TM, **WB** program, etc.) is a string encoding of a snapshot of that program at one instant in time.
- For Turing machines, it contains
  - The contents of the tape,
  - Where the tape head is, and
  - What state the machine is in.
- For **WB** programs, it contains
  - The contents of the tape,
  - Where the tape head is, and
  - What line of code is next to be executed.

IDs and Universal Machines

- There is a close connection between an ID and universal machines.
- The universal machines U_{TM} and U_{WB} work by repeatedly reading the ID of the machine being simulated, then executing one step of that machine.

An ID for **WB** Programs

Simulated tape of the program being executed.



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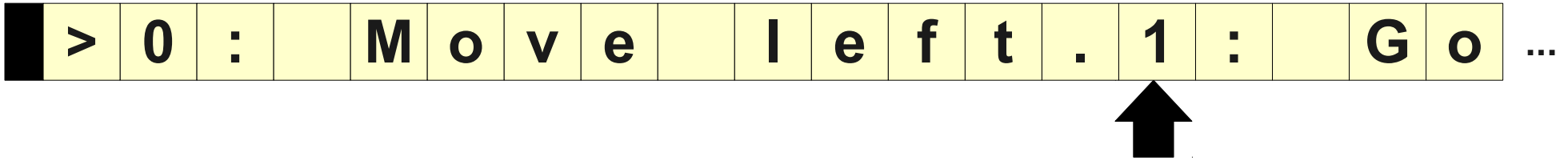
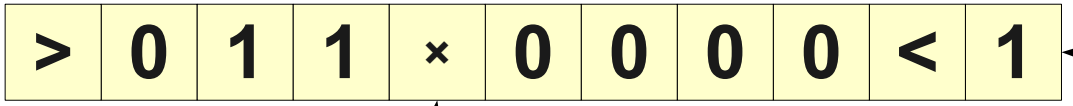




This means "the tape head is under the next symbol."

We write the line number at the end of the ID.

Manipulating IDs

- Because IDs are strings (just like machine or program encodings), we can perform all sorts of operations on them:
  - Copy them to other tapes for later use.
  - Inspect them to see the state of the machine at any instant in time.
  - Transform them to represent making changes to the program as it is running.

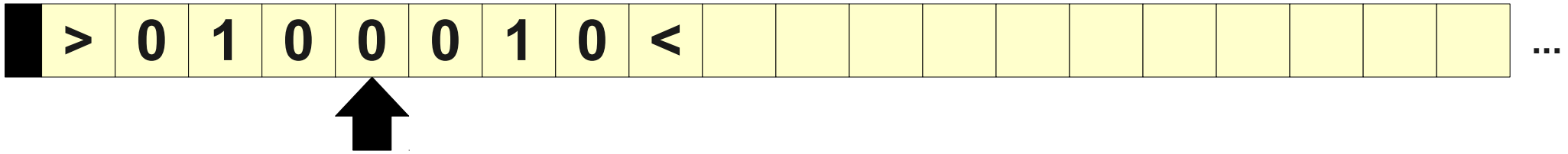
The Key Idea

- We can store IDs of all of the different branches of the NTM and execute them in a breadth-first search.
- This does a breadth-first walk of the “tree computation” interpretation of nondeterministic computation.
- If there is an accepting computation, eventually we will find it (though it might take an enormously long time to do so!)

Simulating an NTM

- Given an NTM *M*, we can simulate it with a deterministic TM as follows:
- "On input *w*:
  - Put the initial ID of *M* running on *w* into a separate tape.
  - While that tape contains IDs:
    - Copy the first ID to a work tape.
    - If this ID is in an accepting state, accept.
    - If the ID is not in a rejecting state:
      - For all possible next steps, use the universal TM to simulate the NTM making that choice, then append the resulting ID to the other tape.
  - Reject."

Simulated tape of the program being executed.



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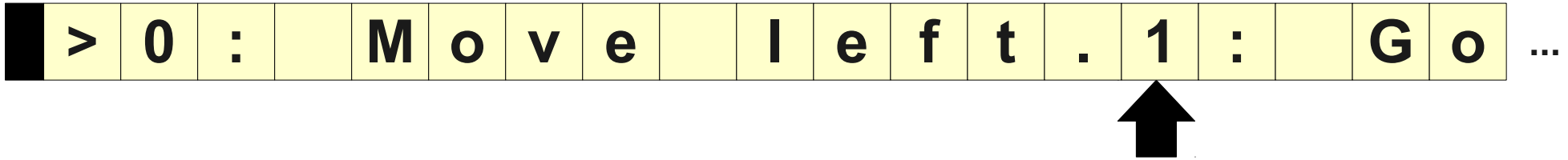
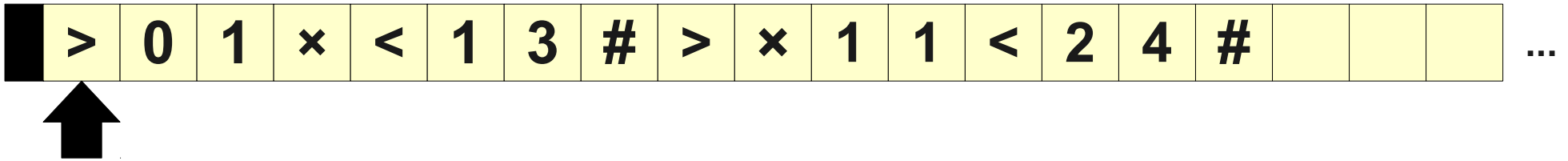
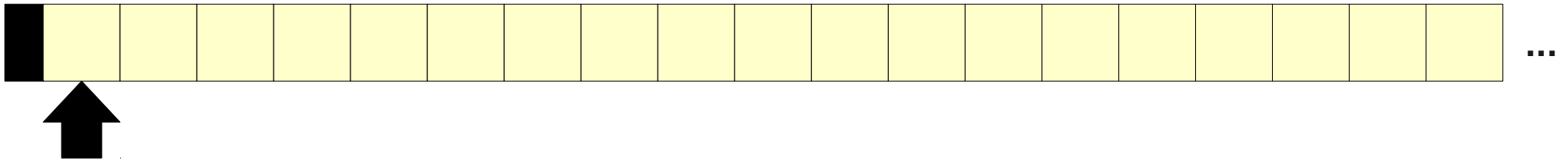


Stored IDs



Scratch tape for intermediate computation.



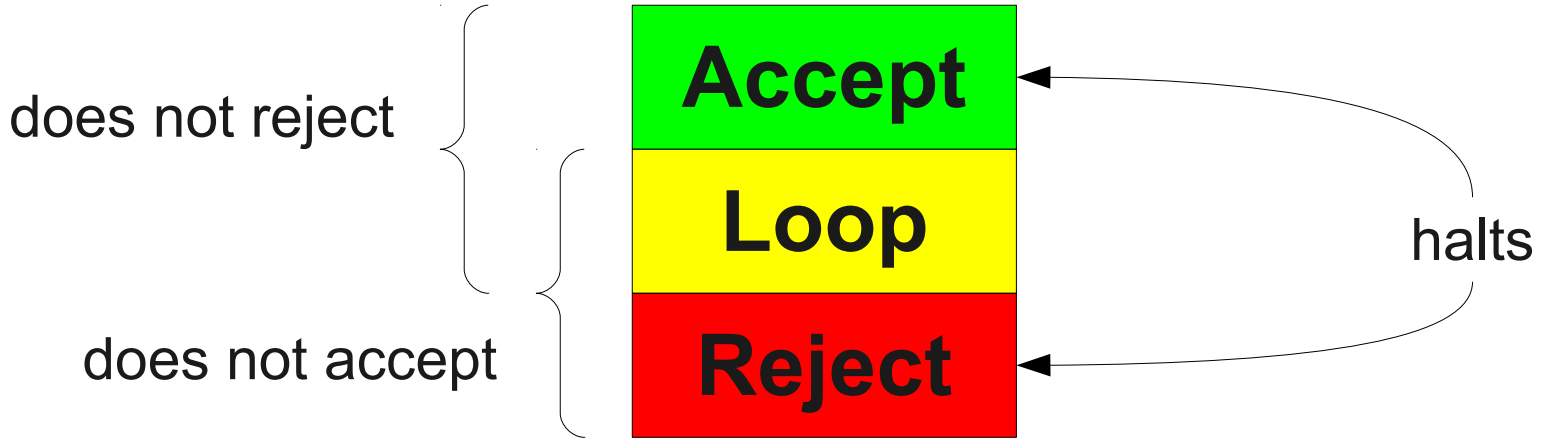
Variables for intermediate storage.

Instr **Letter**

Decidability and Recognizability

Some Important Terminology

- A TM/**WB** program **accepts** a string *w* if it enters an accept state or executes the **`Accept`** command.
- A TM/**WB** program **rejects** a string *w* if it enters a reject state or executes the **`Reject`** command.
- A TM/**WB** program **loops infinitely** on a string *w* if neither of these happens.
- A TM/**WB** program **does not accept** a string *w* if it either rejects *w* or loops infinitely on *w*.
- A TM/**WB** program **does not reject** a string *w* if it either accepts *w* or loops infinitely on *w*.
- A TM/**WB** program **halts** if it accepts or rejects.



Recall: Language of a TM

- The language of a Turing machine M, denoted $\mathcal{L}(M)$, is the set of all strings that M accepts:

$$\mathcal{L}(M) = \{ w \mid M \text{ accepts } w \}$$

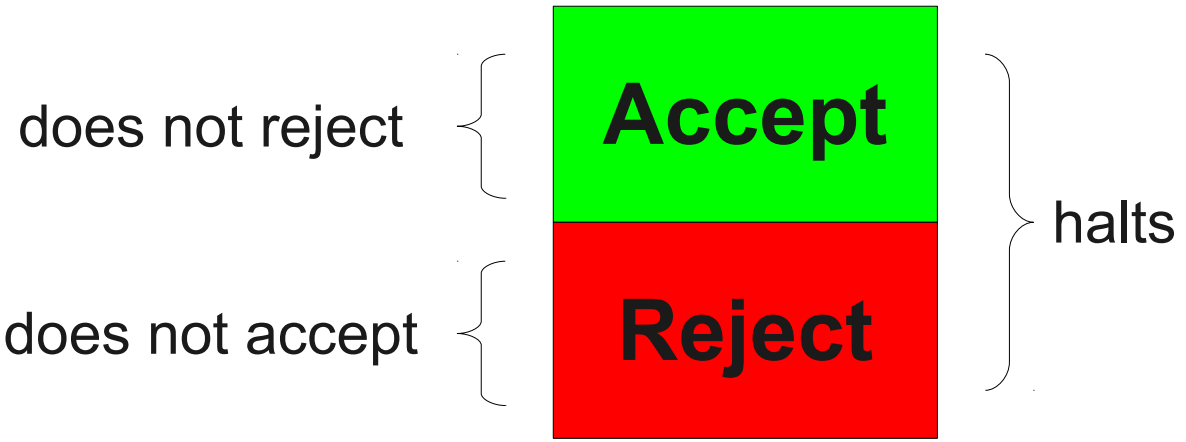
- For any $w \in \mathcal{L}(M)$, M will accept w.
- For any $w \notin \mathcal{L}(M)$, M will not accept w.
  - It might loop forever, or it might explicitly reject.
- A language is called **recognizable** or **recursively enumerable** if it is the language of some TM.

Why “Recognizable?”

- Given TM *M* with language $\mathcal{L}(M)$, running *M* on a string *w* will not necessarily tell you whether $w \in \mathcal{L}(M)$.
- If the machine is running, you can't tell whether
  - It is eventually going to halt, but just needs more time, and
  - It is never going to halt.
- However, if you know for a fact that $w \in \mathcal{L}(M)$, then the machine can confirm this.
- The machine can't *decide* whether or not $w \in \mathcal{L}(M)$, but it can *recognize* strings that are in the language.
- We sometimes call a TM for a language *L* a **recognizer** for *L*.

Deciders

- Some Turing machines always halt; they never go into an infinite loop.
- Turing machines of this sort are called **deciders**.
- For deciders, accepting is the same as not rejecting and rejecting is the same as not accepting.



Decidable Languages

- A language *L* is called **decidable** iff there is a decider *M* such that $\mathcal{L}(M) = L$.
  - These languages are also sometimes called **recursive**.
- Given a decider *M*, you *can* learn whether or not a string $w \in \mathcal{L}(M)$.
  - Run *M* on *w*.
  - Although it make take a staggeringly long time, *M* will eventually accept or reject *w*.

Decidability vs. Recognizability

- All deciders are Turing machines, but not all Turing machines are deciders.
- As a result, we know that $R \subseteq RE$.
  - R is the set of all recursive (decidable) languages; RE is the set of all recursively enumerable (recognizable) languages.
- **Enormously important question**: **Is $R = RE$?**
- This is an enormously important theoretical question. We'll explore the answer next time...

Properties of R and RE Languages

Closure under Intersection

- The R and RE languages are closed intersection.
- If $L_1 \in R$ and $L_2 \in R$, then $L_1 \cap L_2 \in$ R.
- If $L_1 \in RE$ and $L_2 \in RE$, then $L_1 \cap L_2 \in RE$.
- How would we prove this?

R is Closed Under Intersection

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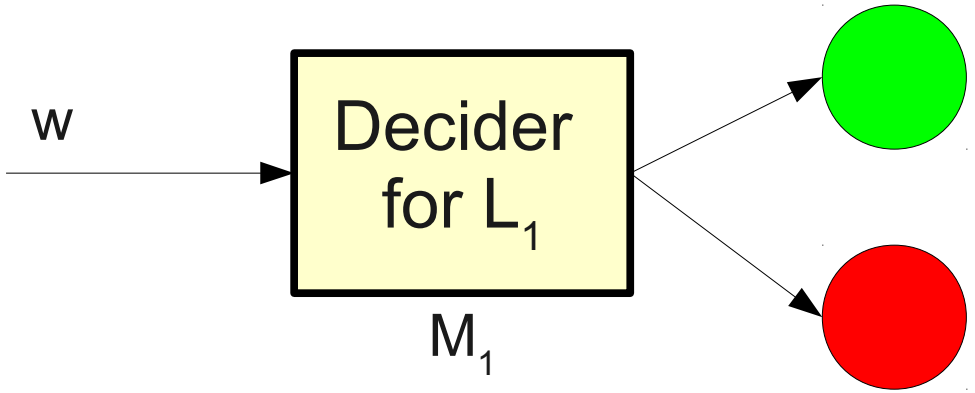

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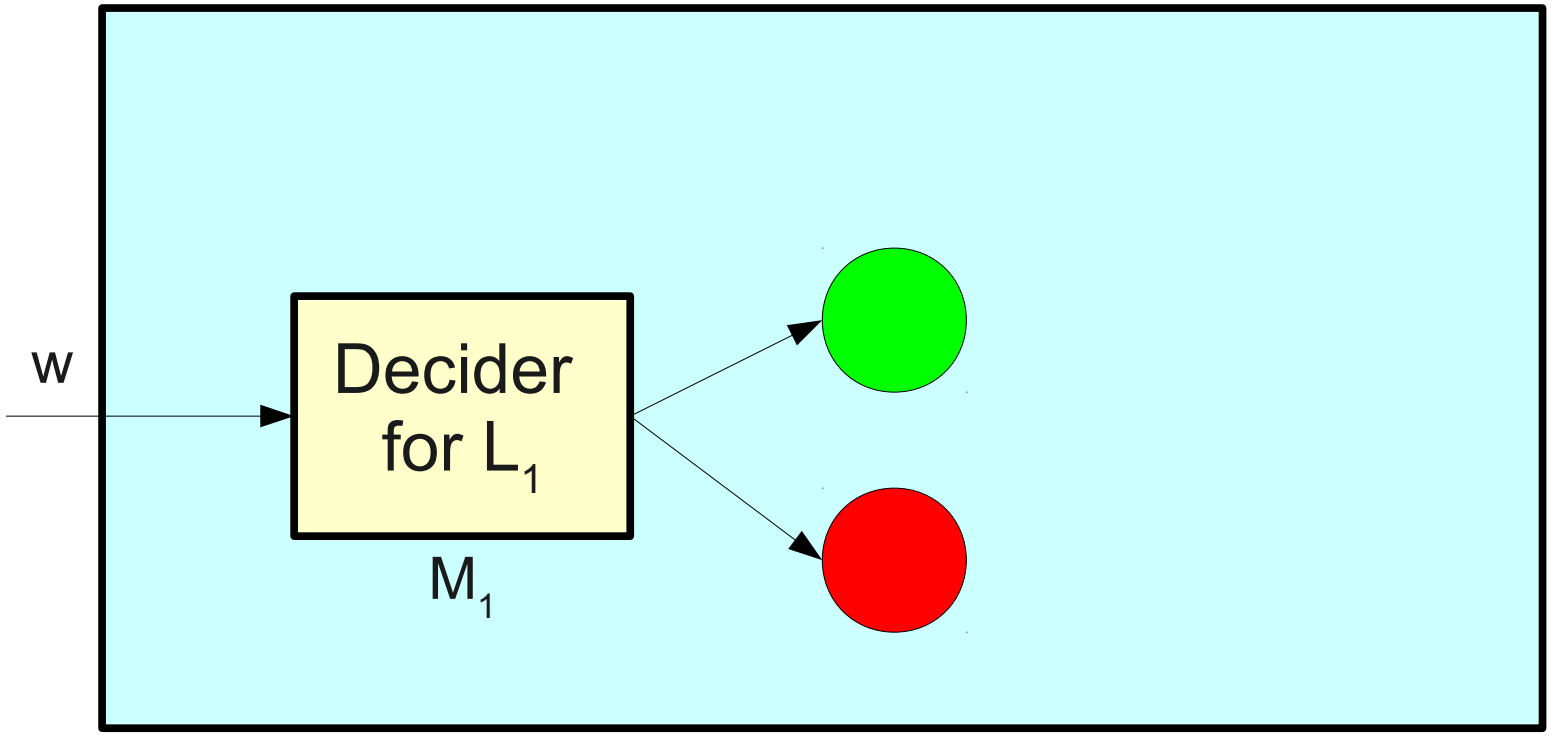

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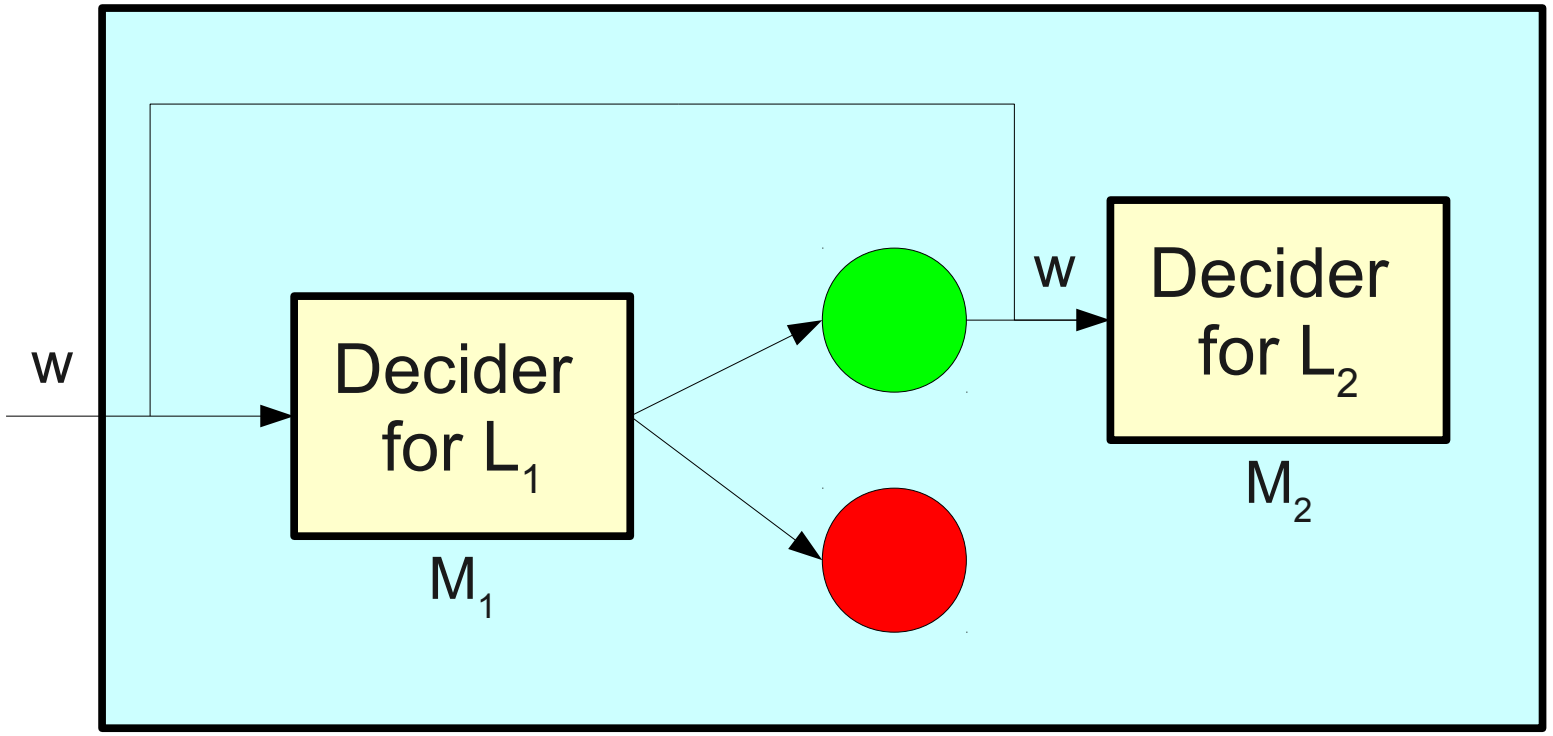

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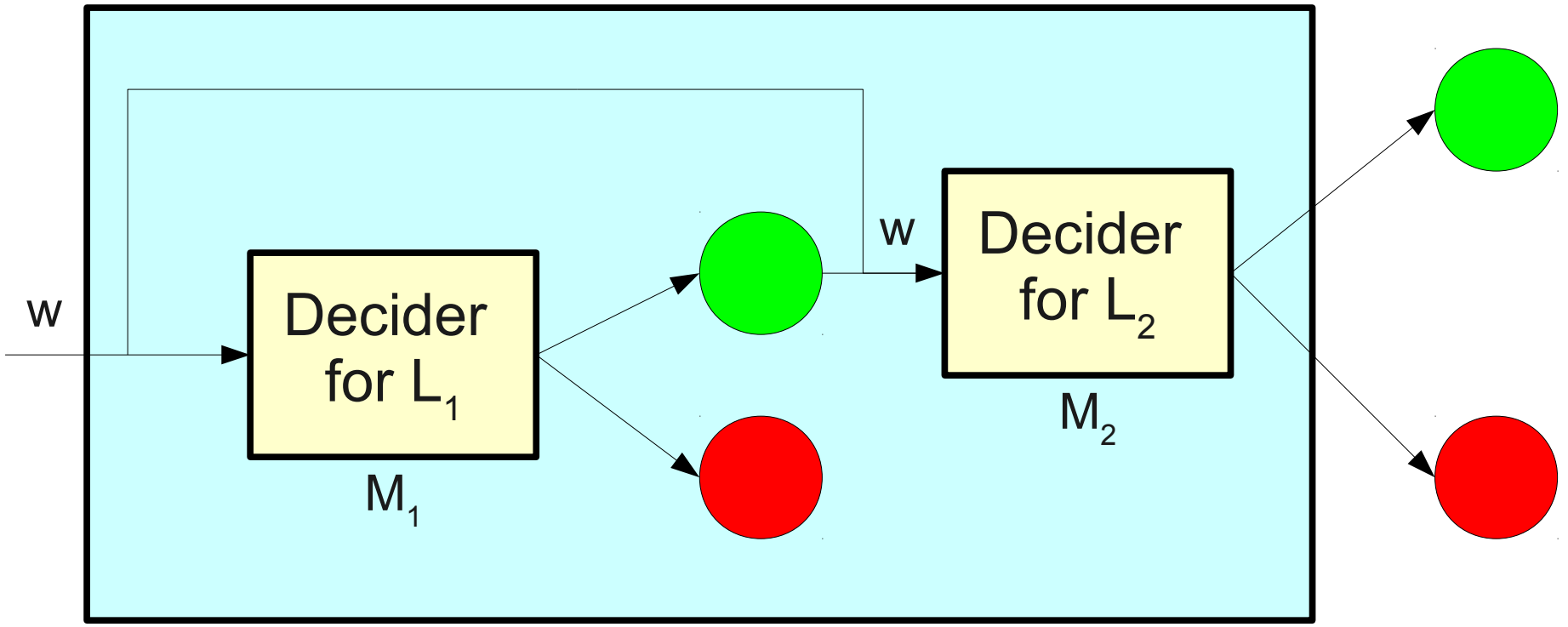

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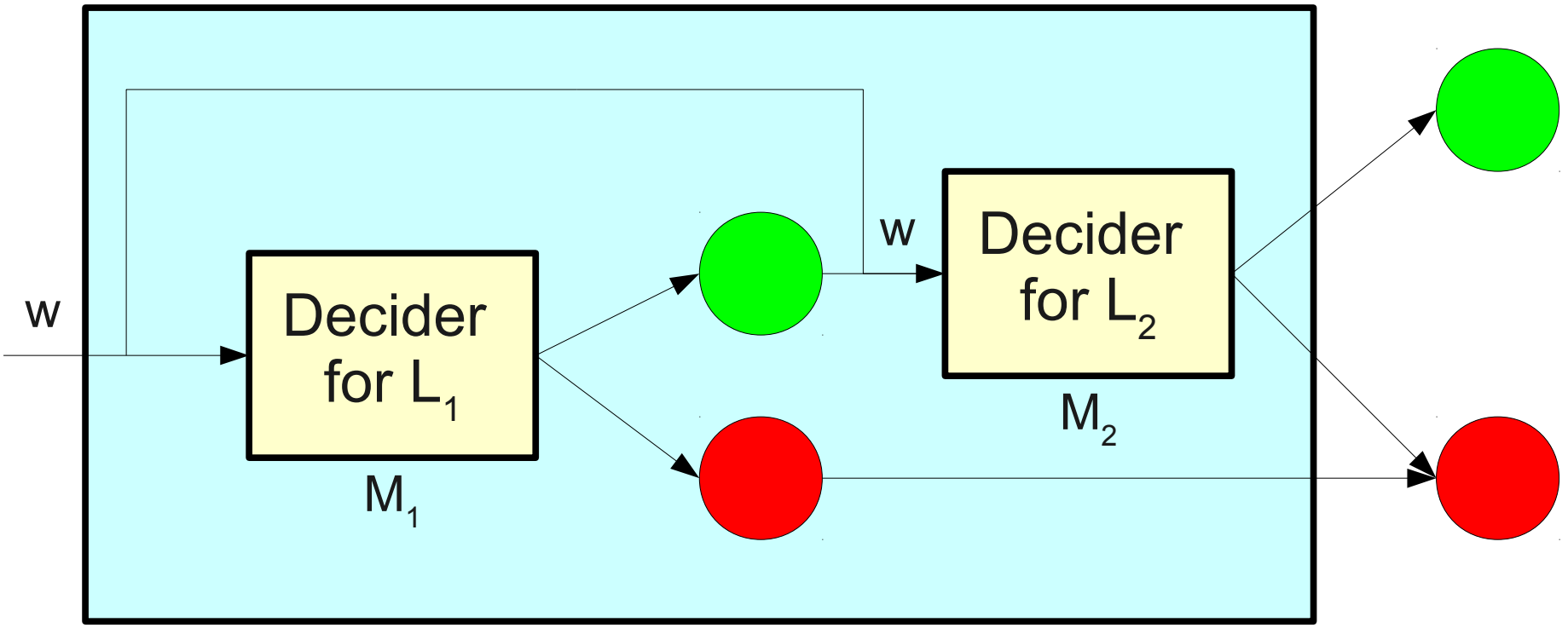

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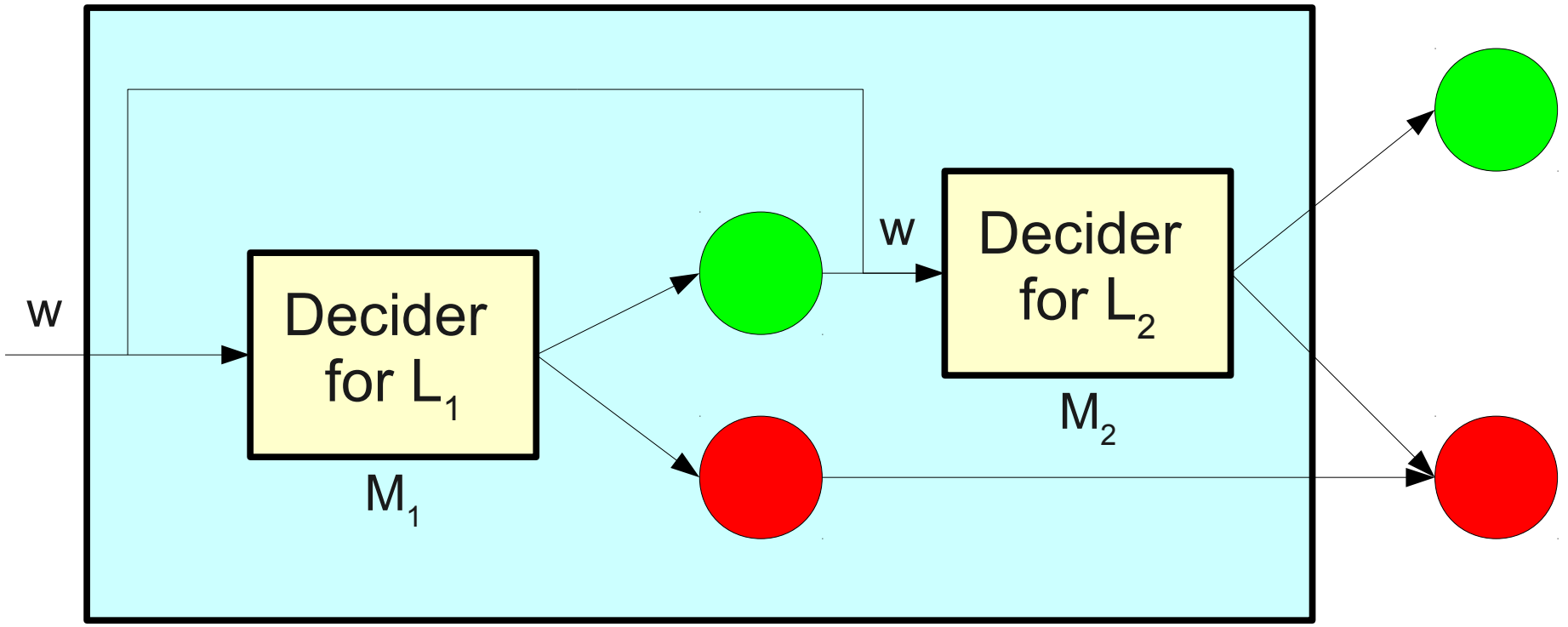
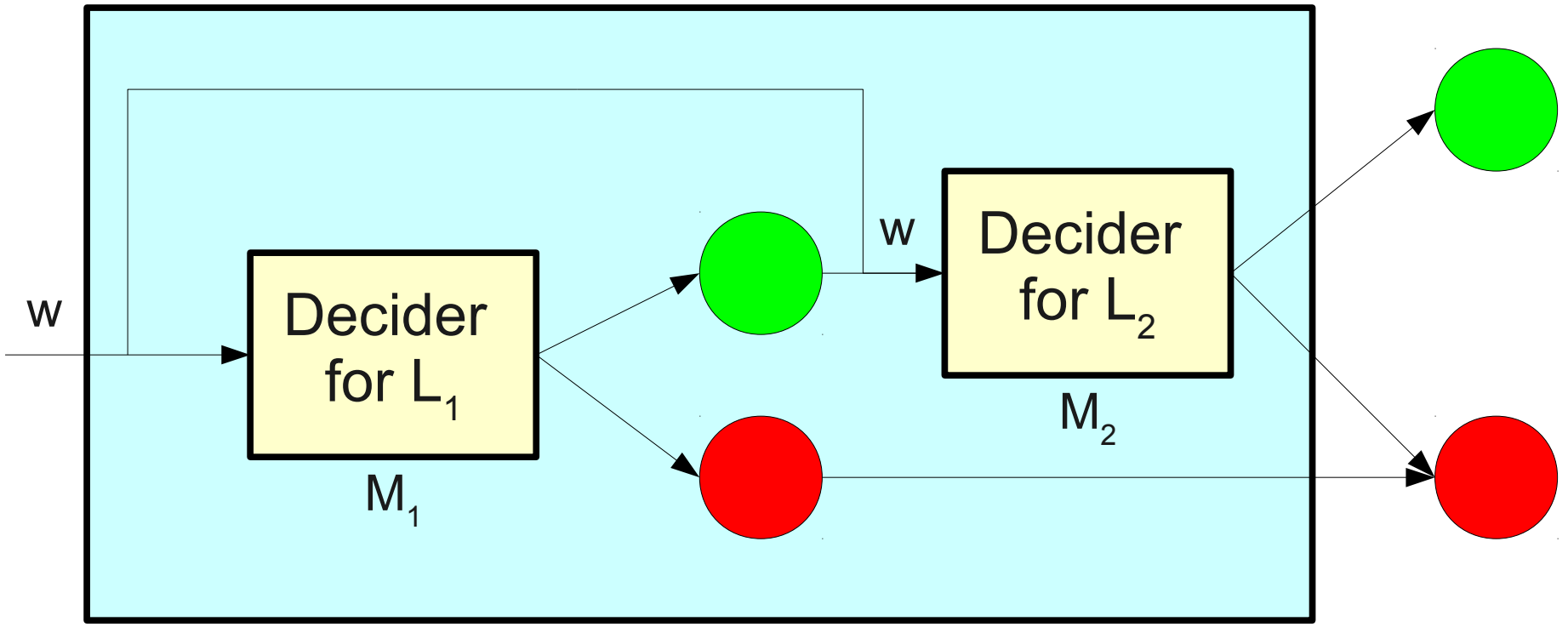


M' = “On input w:

R is Closed Under Intersection



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Run M_1 on w.

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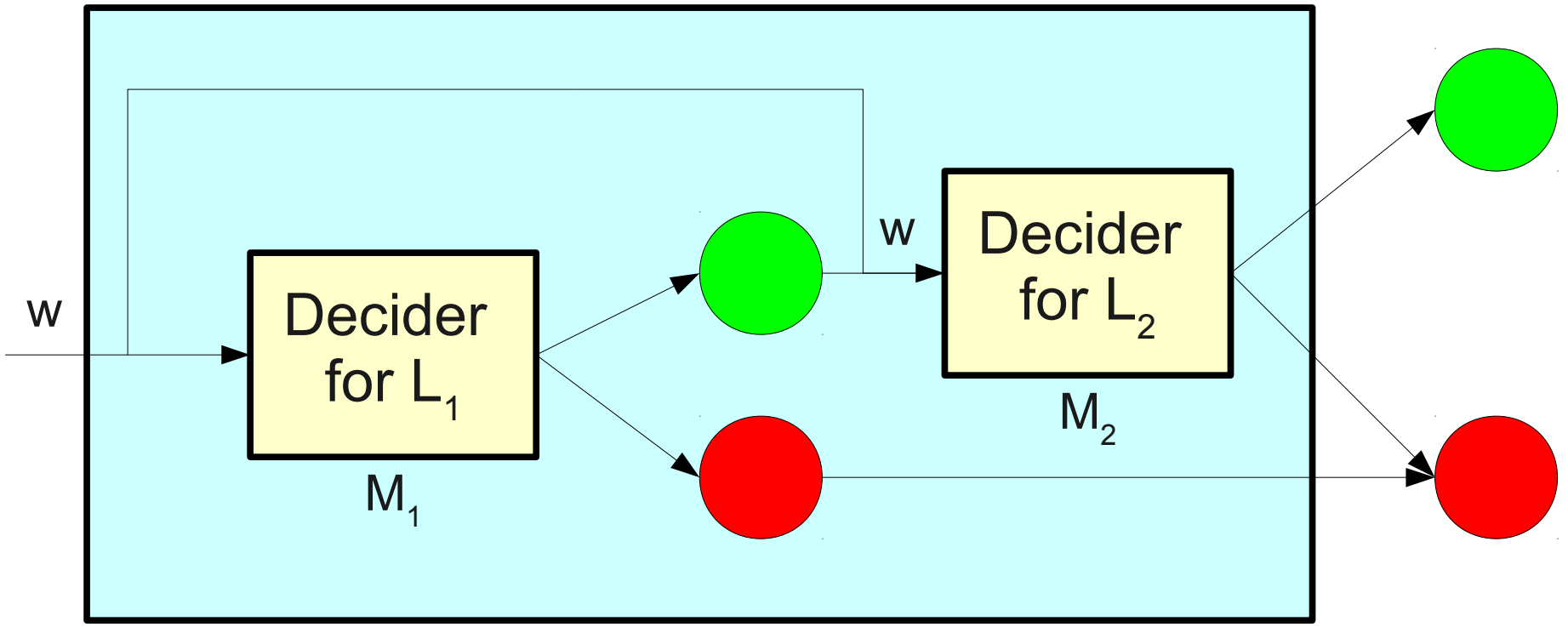

M' = “On input w:

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If M_1 rejects, reject.

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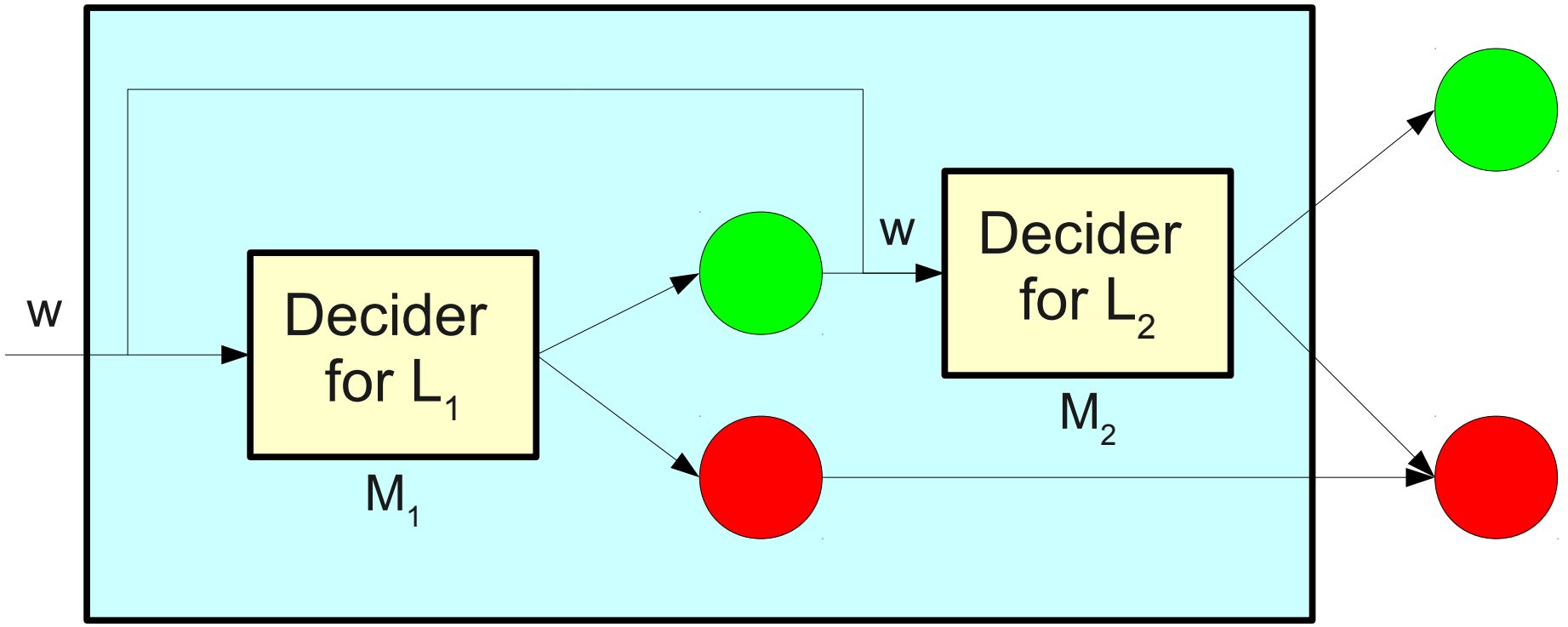

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Otherwise, run M_2 on w.

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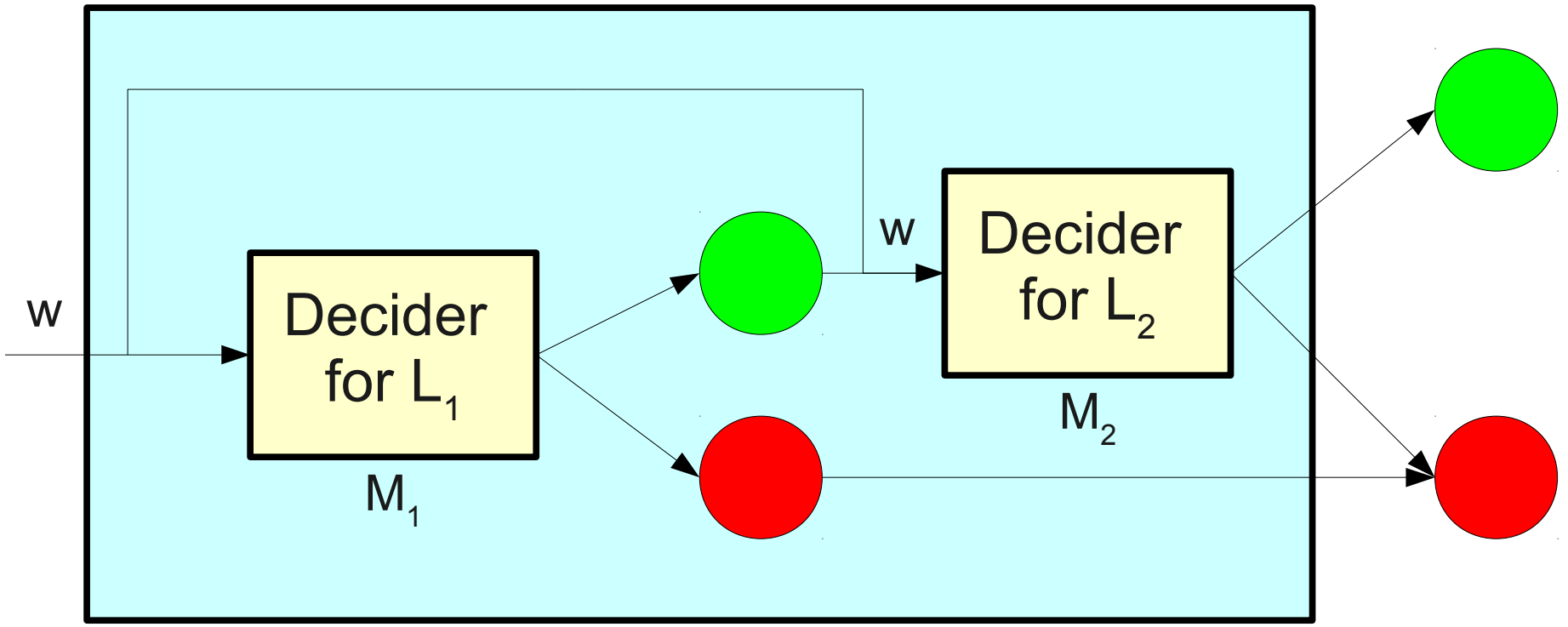

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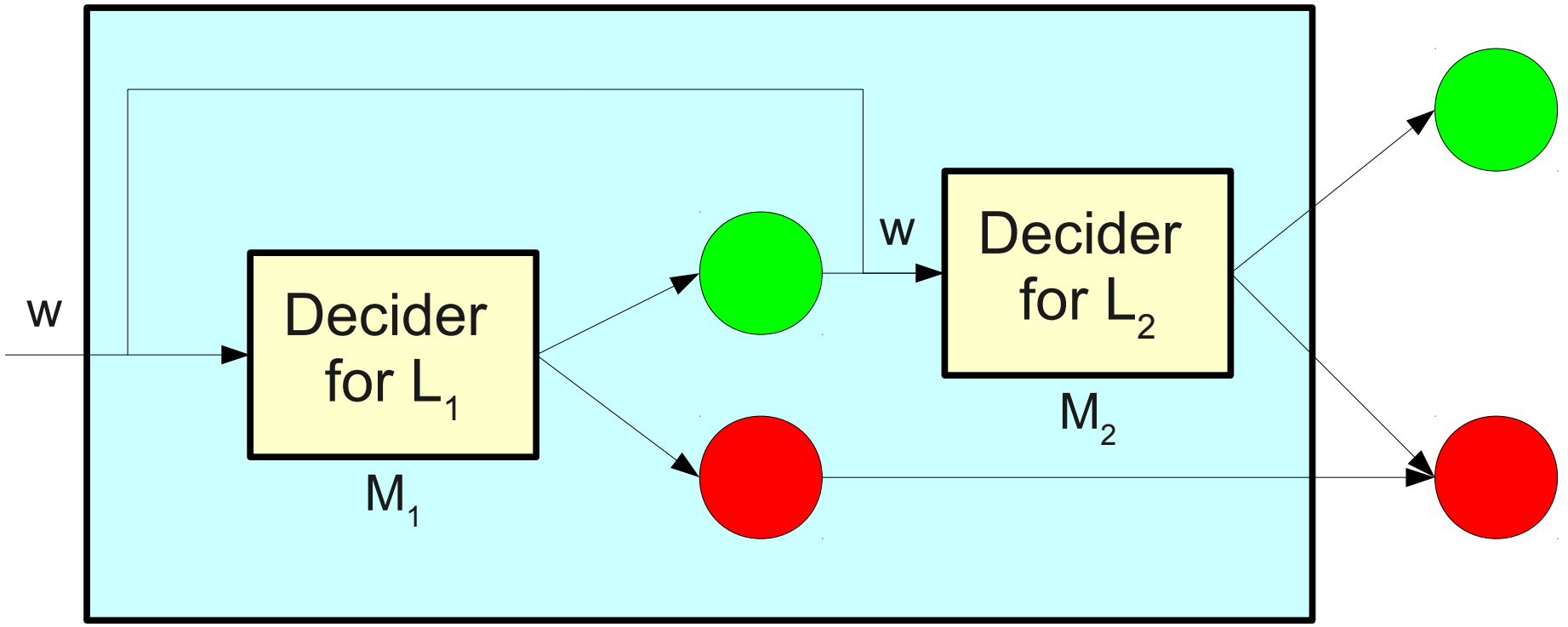

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Otherwise, run M_2 on w.

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Otherwise, reject.”

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It is critical to prove both of these properties. If we don't show that M' is a decider, then we have proven that $L_1 \cap L_2$ is RE, but not necessarily R!

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Remember that the language of a machine is the set of strings it accepts. To reason about the language of a TM, we only need to reason about what strings it accepts.

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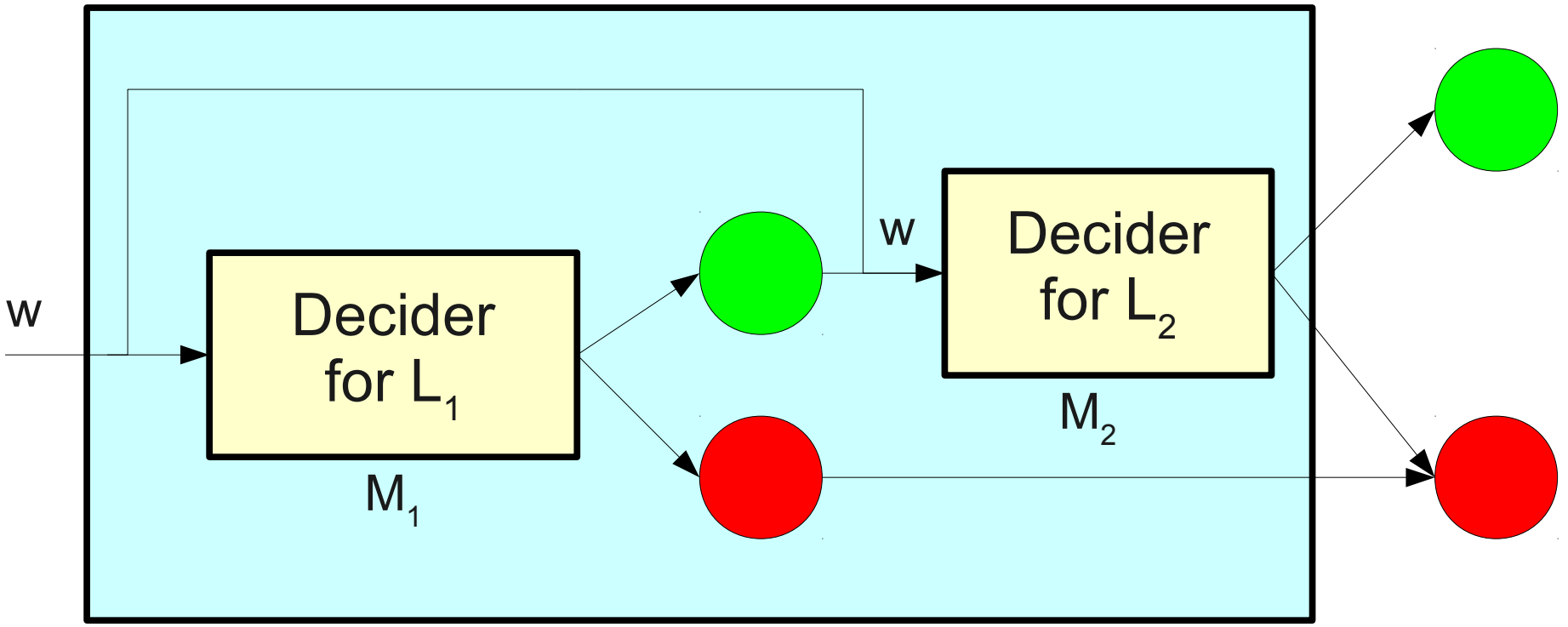

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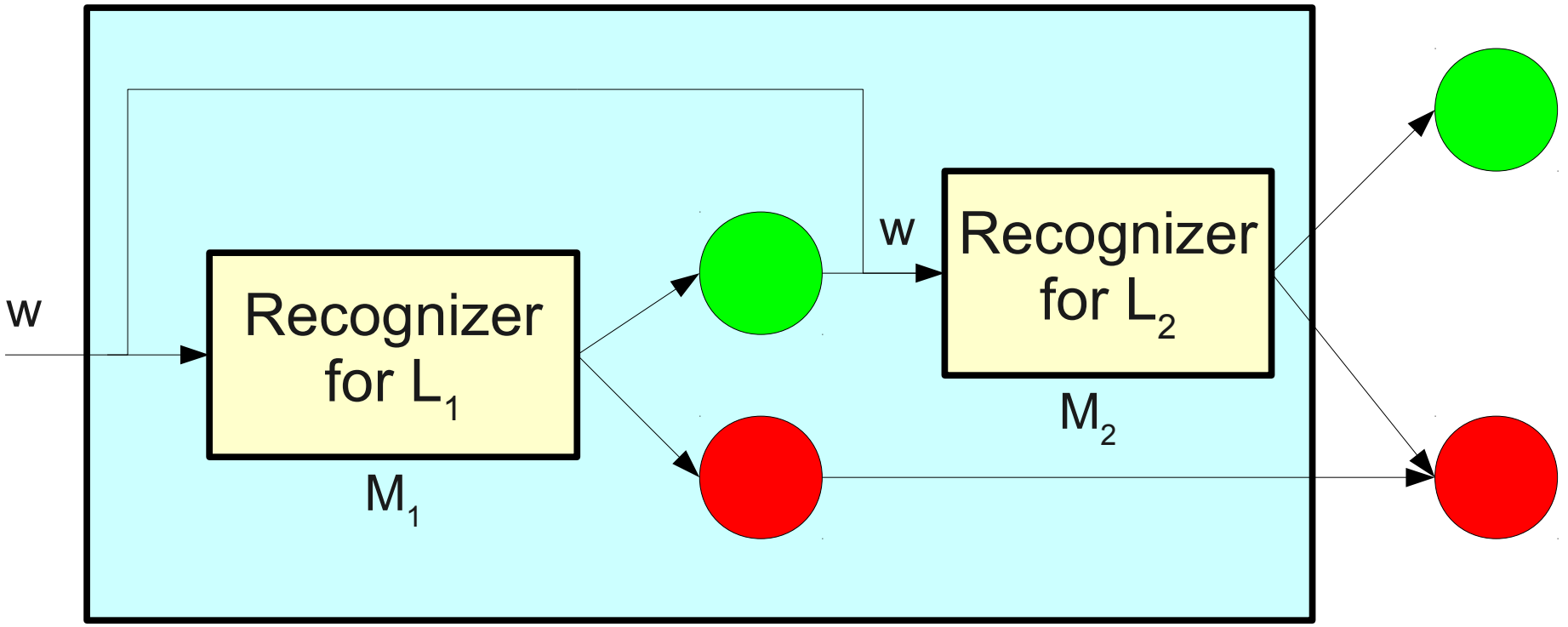

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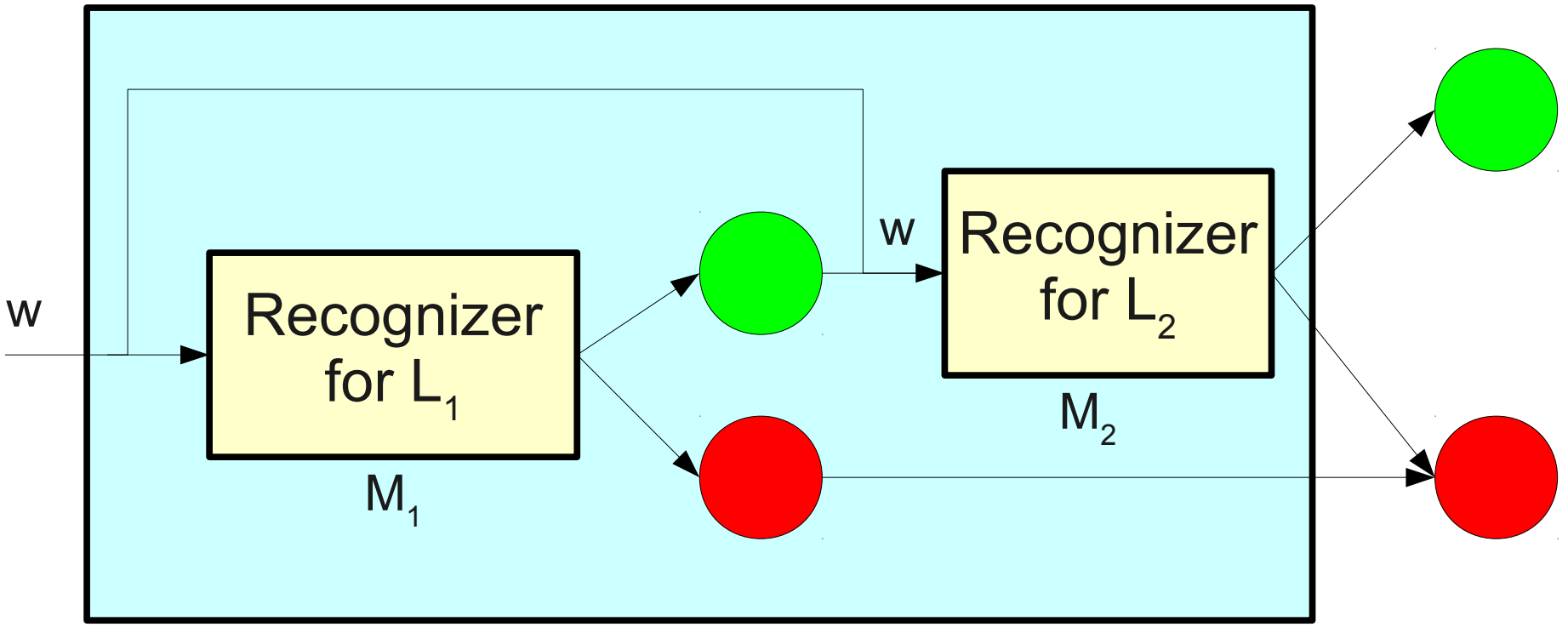

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Does this construction still work?

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Note that there is no requirement that we prove that M' halts on all inputs. All we need to do is reason about what inputs it accepts.

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We only care about what strings M' accepts, and can ignore the strings it rejects or loops infinitely on.

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- To show that R is closed under some operation:
  - Obtain deciders M_1, M_2, …, M_n for all of the original languages.
  - Construct a new TM M' based on M_1, M_2, …, M_n.
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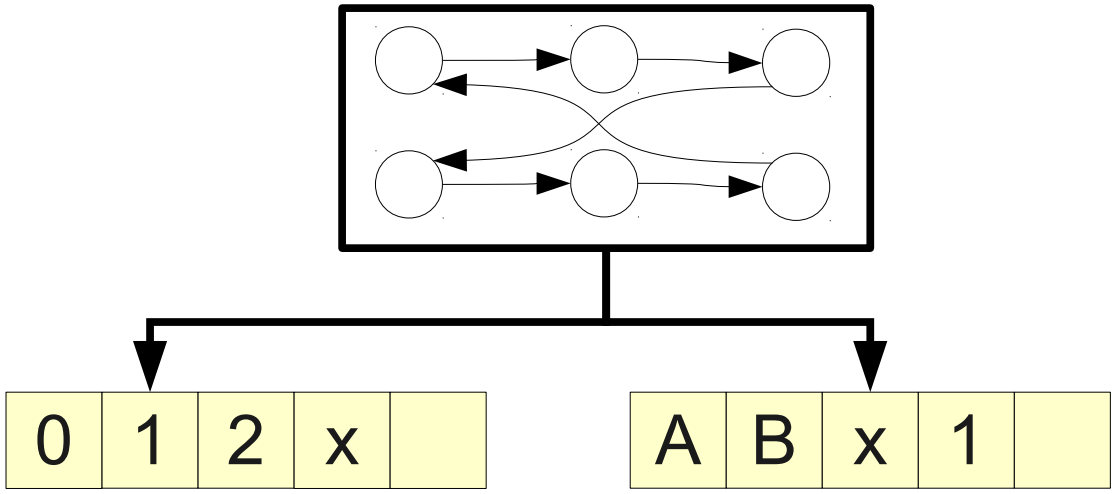

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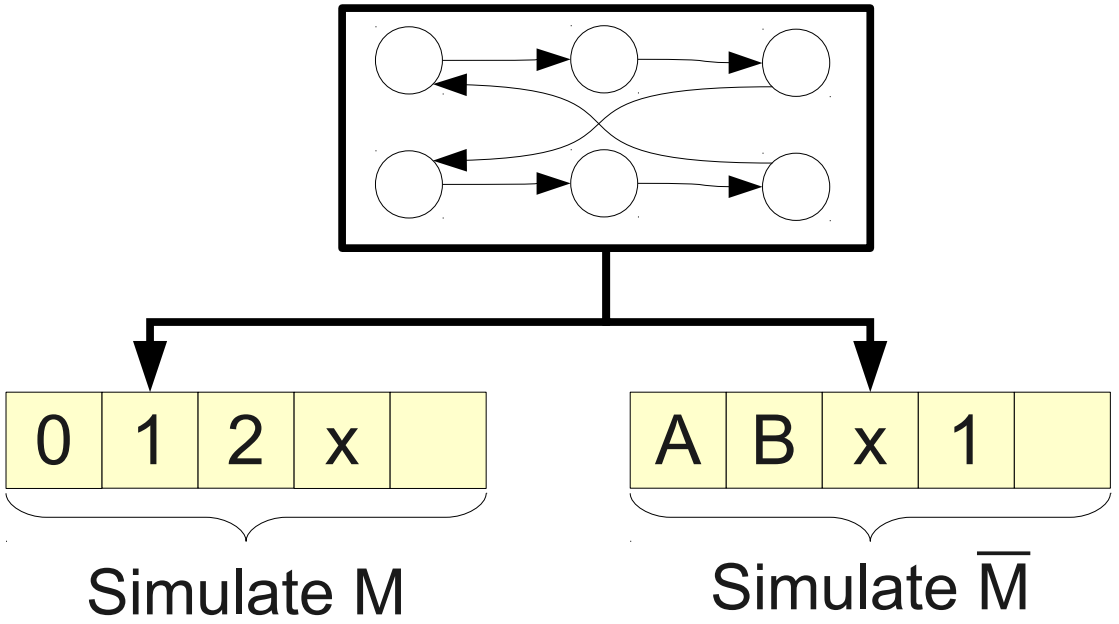

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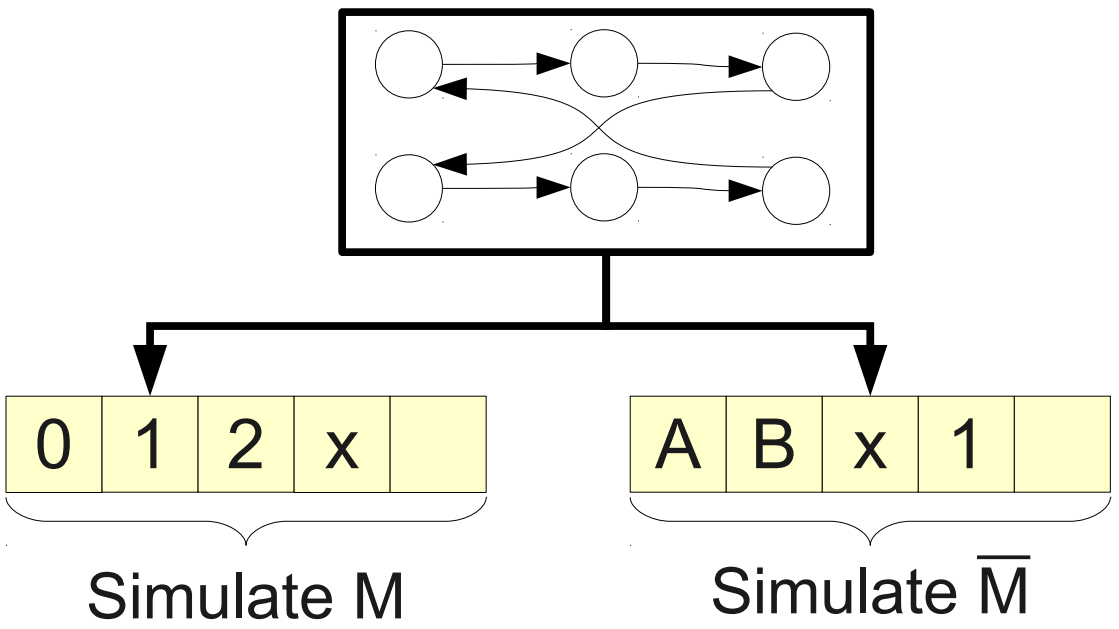

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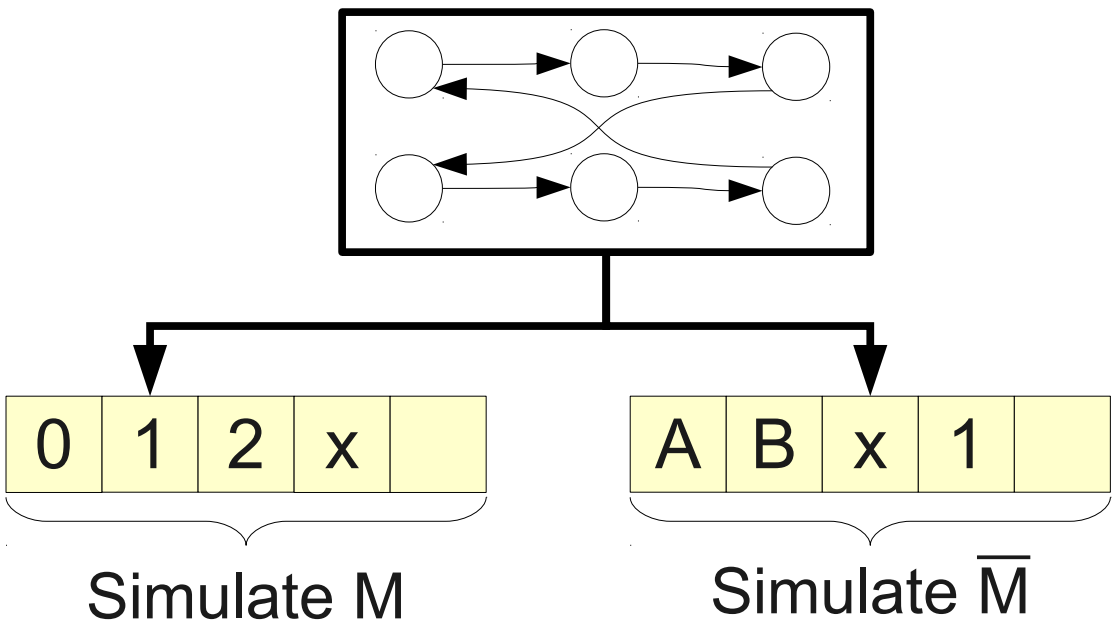

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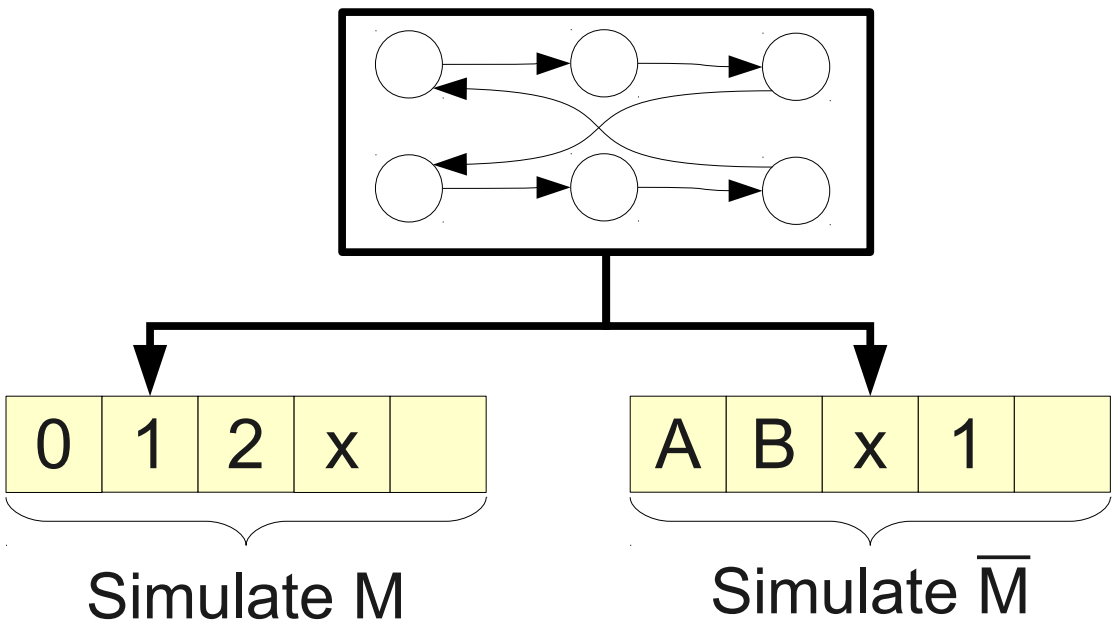

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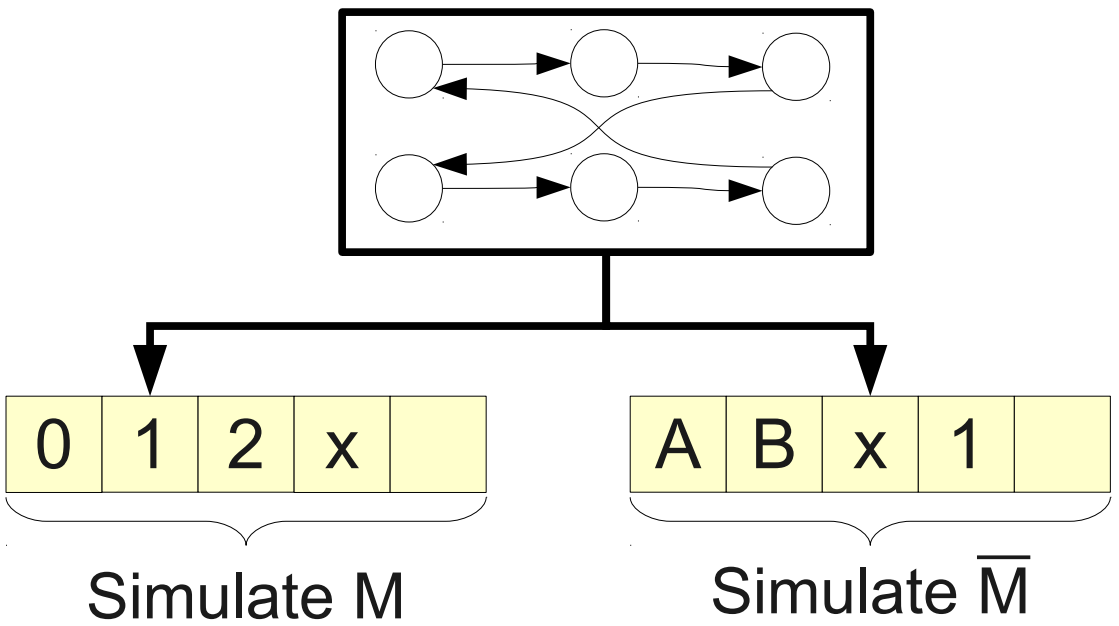

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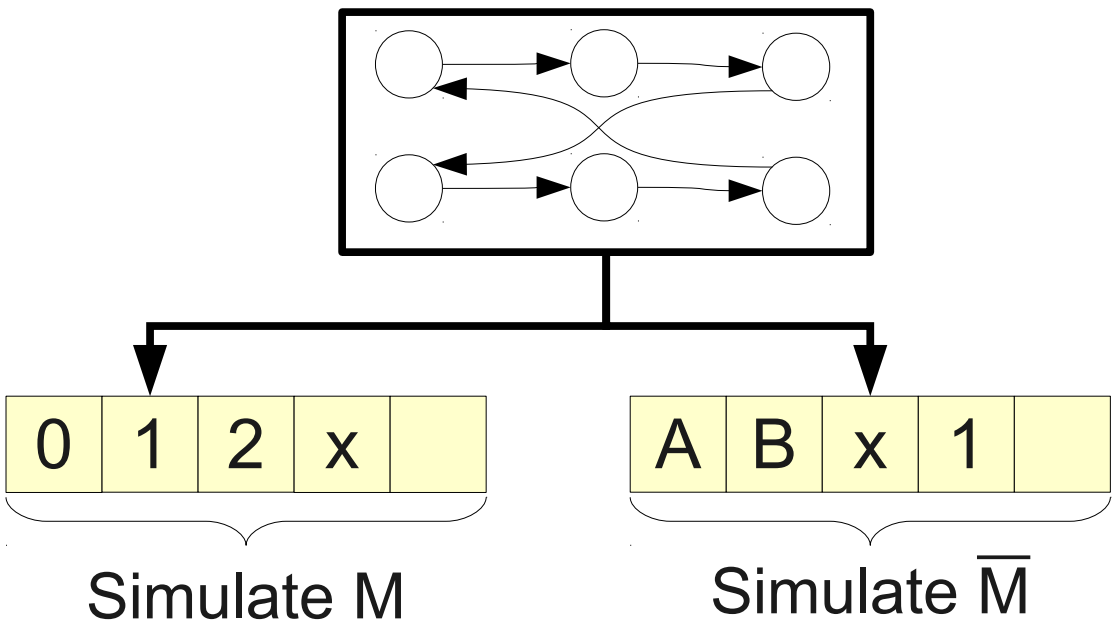

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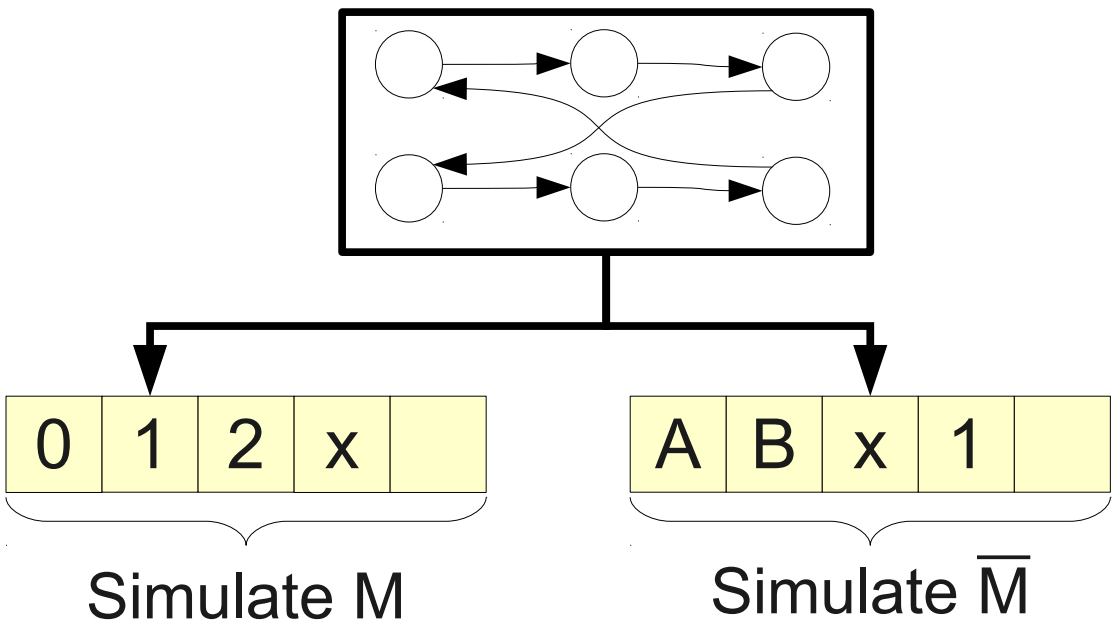

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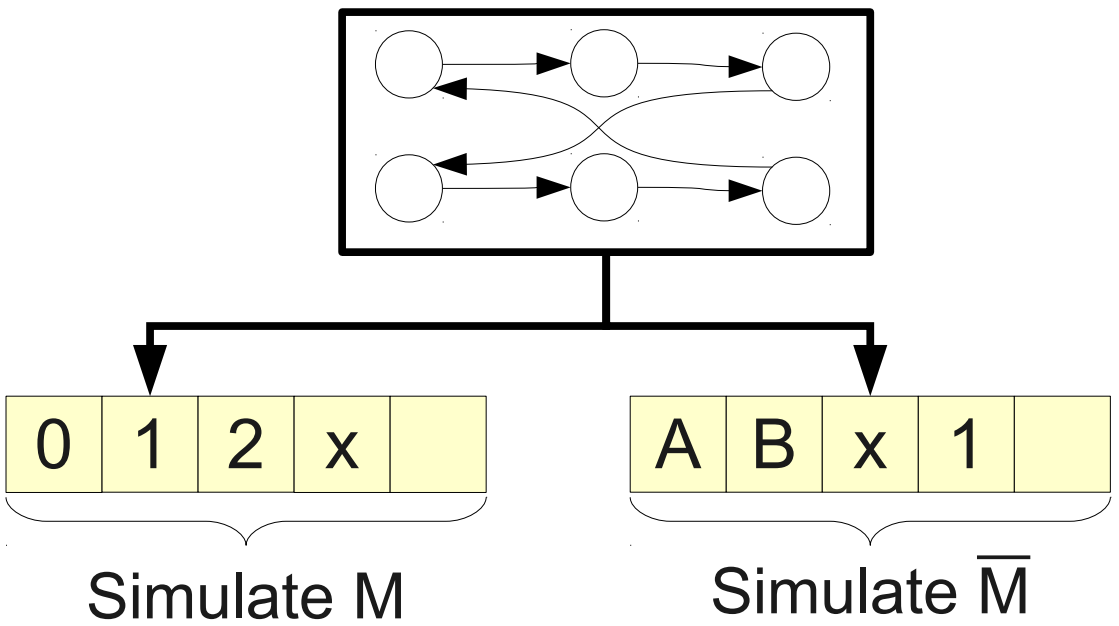

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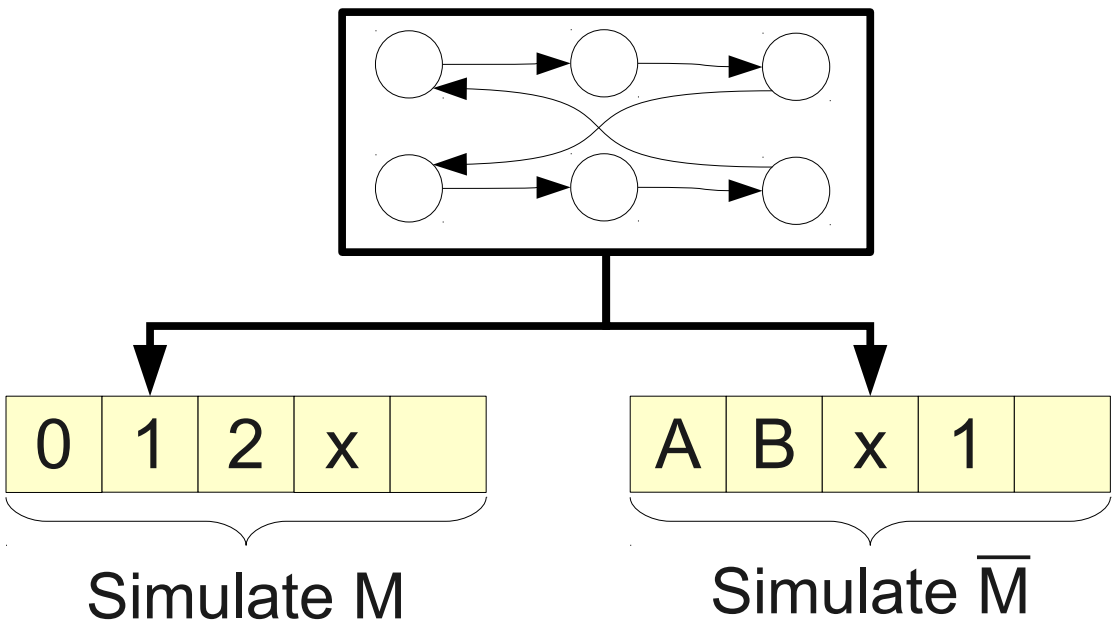

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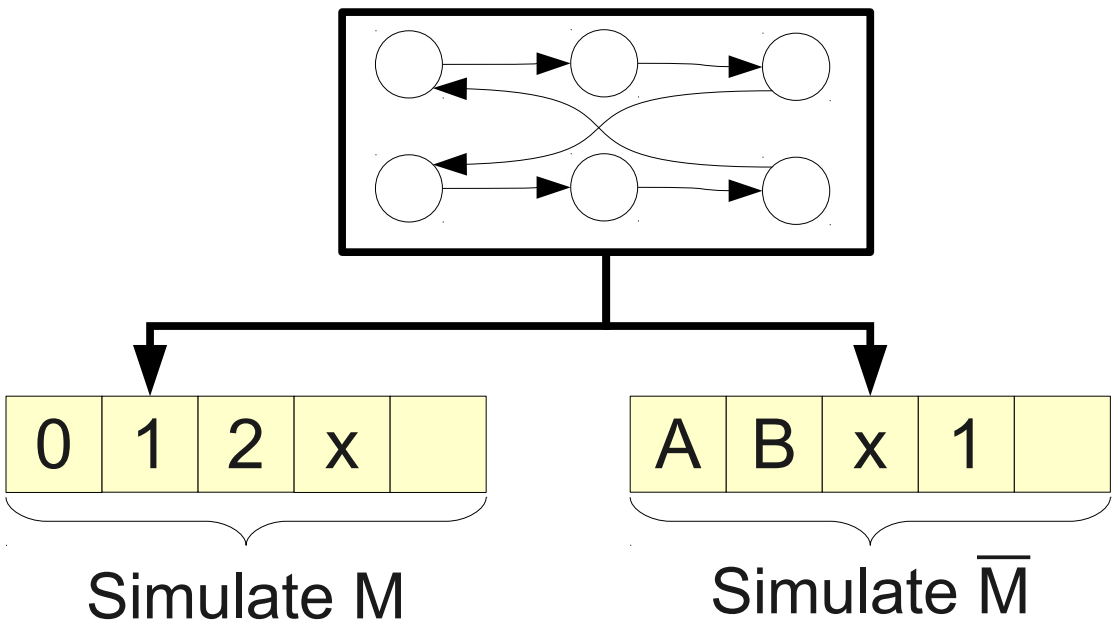

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We prove that $\mathcal{L}$(M') = L and that M' is a decider.

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We do have to consider this case - what if M would eventually accept w, but $\overline{M}$ accepts it first?

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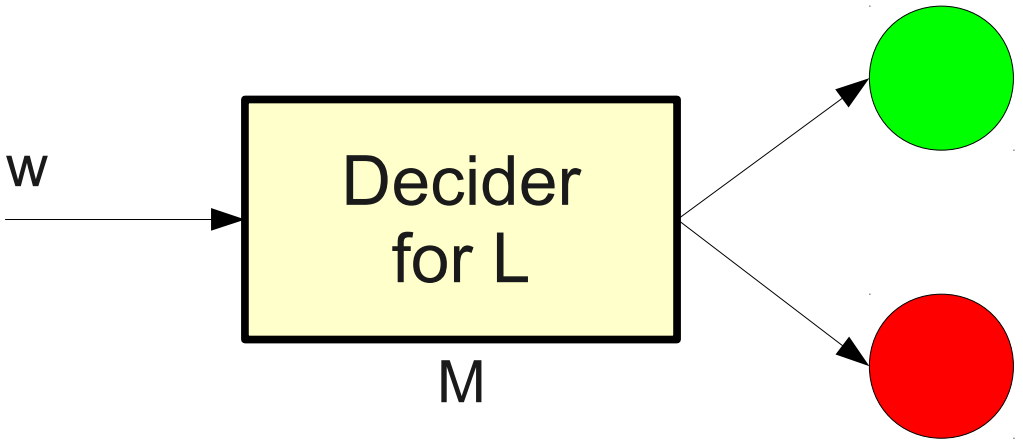

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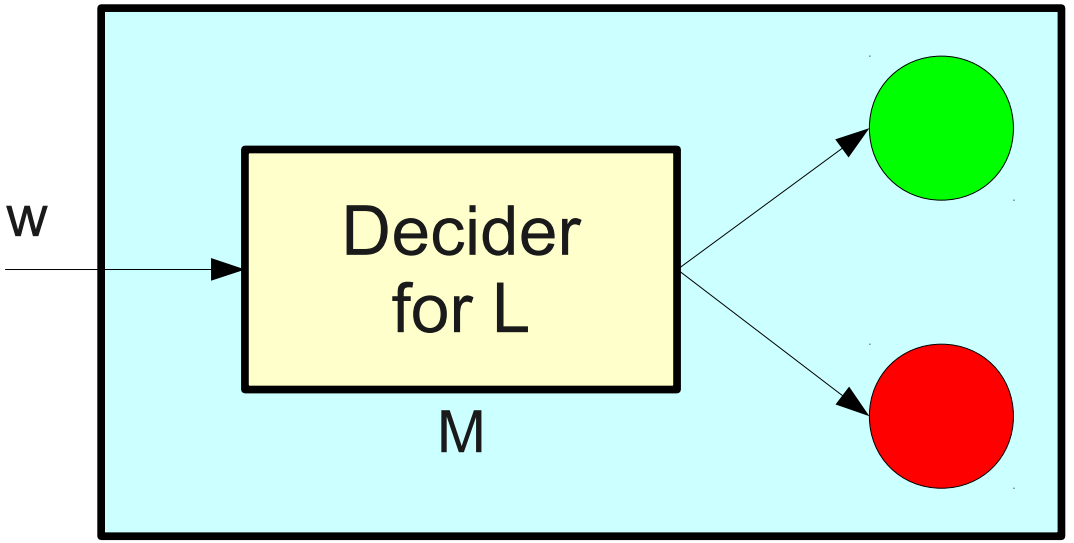

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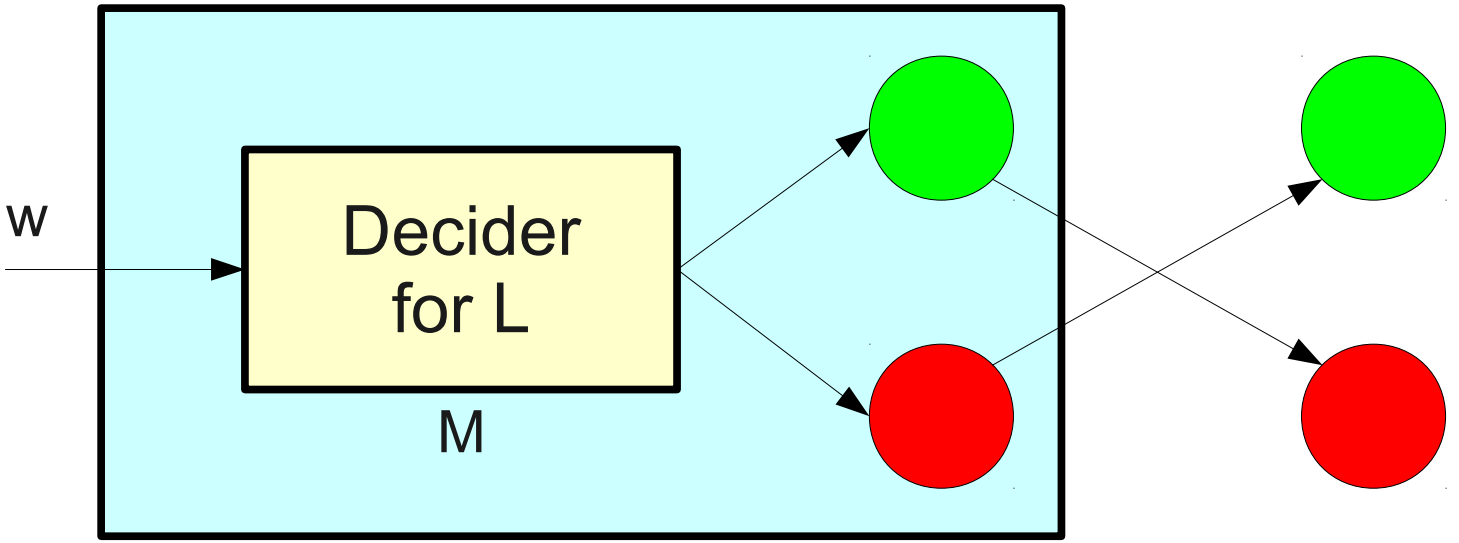

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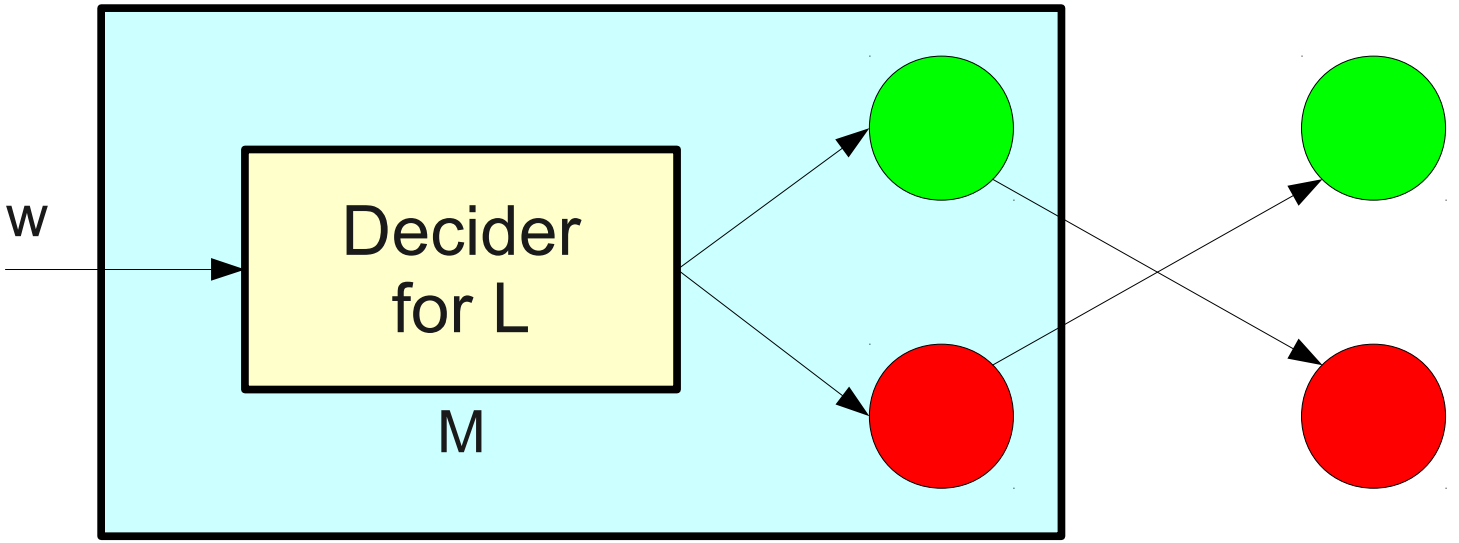

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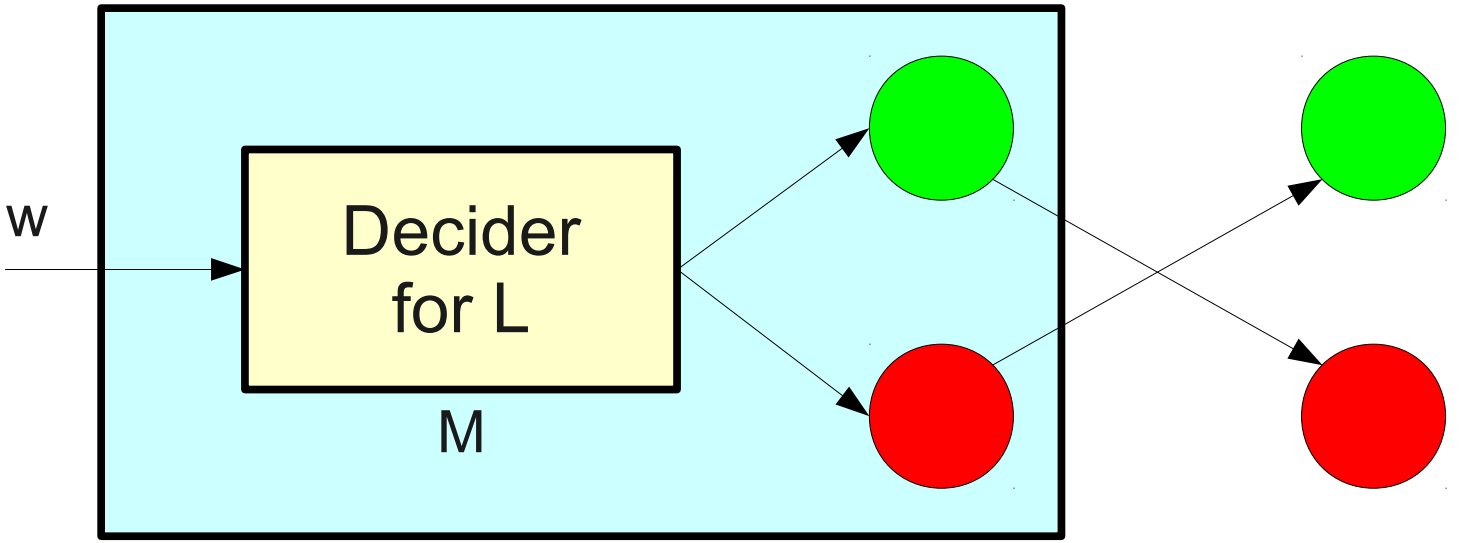

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Why doesn't this work for RE languages?

Next Time

- **Unsolvable Problems**
  - What languages are *not R* or *RE*?
  - What problems are provably impossible to solve?
  - Does *R* = *RE*?