Problem Set 4

This fourth problem set explores propositional and first-order logic, along with its applications. Once you've completed it, you should have a much stronger understanding of mathematical logic and its properties.

In any question that asks for a proof, you **must** provide a rigorous mathematical proof. You cannot draw a picture or argue by intuition. If we specify that a proof must be done a certain way, you must use that particular proof technique; otherwise you may prove the result however you wish.

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

This problem set has 150 possible points. It is weighted at 7% of your total grade. The earlier questions serve as a warm-up for the later problems, so be aware that the difficulty of the problems does increase over the course of this problem set.

Good luck, and have fun!

Checkpoint due Monday, October 22 at 2:15PM

Assignment due Friday, October 26 at 2:15PM

Write your solutions to the following problems and submit them by Monday, October 22nd at the start of class. These problems will be graded based on whether or not you submit solutions, rather than the correctness of your solutions. We will try to get these problems returned to you with feedback on your proof style this Wednesday, October 24th. Submission instructions are on the last page of this problem set.

Note that this question has two parts.

Checkpoint Problem: Translating into Logic, Part I (25 Points if Submitted)

In each of the following, you will be given a list of first-order predicates and functions along with an English sentence. In each case, write a statement in first-order logic that expresses the indicated sentence. Your statement may use any first-order construct (equality, connectives, quantifiers, etc.), but you must only use the predicates, functions, and constants provided.

As an example, if you were given just the predicates *Integer(x)*, which states that *x* is an integer, and the function *Plus(x, y)*, which represents the sum of *x* and *y*, you could write the statement "there is some even integer" as

$$\exists n.\ \exists k.\ (Integer(n) \land Integer(k) \land Plus(k, k) = n)$$

since this asserts that some integer *n* is equal to 2*k* for some integer *k*. However, you could not write

$$\exists n.\ (Integer(n) \land Even(n))$$

because there is no *Even* predicate. Similarly, you could not write

$$\exists n \in \mathbb{Z}.\ \exists k \in \mathbb{Z}.\ Plus(k, k) = n$$

Because there is no ∈ predicate and no constant symbol ℤ. The point of this question is to get you to think how to express certain concepts in first-order logic given a limited set of predicates, so feel free to write any formula you'd like as long as you don't invent your own predicates, functions, or constants.

i. Given the predicate

   Fools(x, y, t), which states that *x* fools *y* at time *t*,
   Person(x), which states that *x* is a person, and
   Time(t), which states that *t* is a time,

   along with the constant *you*, write a statement in first-order logic that says "you can fool some of the people all of the time and all of the people some of the time, but not all of the people all of the time." That is, there is a time where you can fool everyone, and there is a person that you can always fool, but you cannot always fool every person.

ii. Given the predicate

   Integer(x), which states that *x* is an integer,

   the function

   Product(x, y), which yields the product of *x* and *y*,

   and the constant symbols -1, 0, 1, and 2, write a statement in first-order logic that says "$\sqrt{2}$ is irrational." For this problem, assume that the definition of a rational number means that *p* and *q* must not have any factors in common other than 1 and -1.

Problem One: The Epimenides Paradox (12 points)

In logic, we assume that every statement is either true or false. However, some statements called *logical paradoxes* break this rule and can be neither true or false. For example, the statement "this statement is false" is a paradox – if it were true, it would have to be false, and if it were false, it would have to be true. The statement is therefore a paradox – it must be either true or false, but it can be neither true nor false.

One of the earliest paradoxes is the called the *Epimenides Paradox*, which is stated as follows:

Epimenides, a Cretan, says "All Cretans always lie."

According to the ancient Greeks, this statement is a paradox because Epimenides can neither tell the truth nor lie. A sketch of the argument is as follows:

> "If Epimenides tells the truth, then all Cretans always lie. Since Epimenides is himself a Cretan, then he must be lying, which is impossible because we know that Epimenides is telling the truth. Thus it is not possible for Epimenides to be telling the truth.
>
> If, on the other hand, Epimenides is lying, then his statement is false and all Cretans never lie. Since Epimenides himself is a Cretan, then he must be telling the truth, which is impossible because we know that he was lying. Thus it is not possible for Epimenides to be lying.
>
> Thus Epimenides must be neither lying nor telling the truth – a paradox!"

However, there is a flaw in the above line of reasoning, and despite its name the Epimenides Paradox is **not** a paradox.

Identify the flaw in this reasoning. Since this is not really a paradox, Epimenides must either be lying or telling the truth. Is Epimenides lying or telling the truth? If he's telling the truth, why doesn't his statement contradict itself? If he's lying, why doesn't his statement contradict itself? Justify your answer.

Problem Two: Propositional Negations (16 points)

For each of the following propositional logic statements, find another statement that is the negation of the given statement. The statement you choose should only have ¬ directly applied to propositions. For example, to get the contradiction of the statement

$p \to q \to r$

you might use the following line of reasoning:

$\neg(p \to q \to r)$

$p \land \neg(q \to r)$

$p \land q \land \neg r$

Once you have found your negation, prove that is is correct by constructing a truth table for the negation of the original statement and showing it is equal to the truth table for your resulting statement. For the above case, we would construct truth tables for $\neg(p \to q \to r)$ and $p \land q \land \neg r$ as follows:

p	q	r	$\neg(p \to q \to r)$
F	F	F	**F** F T F T F
F	F	T	**F** F T F T T
F	T	F	**F** F T T F F
F	T	T	**F** F T T T T
T	F	F	**F** T T F T F
T	F	T	**F** T T F T T
T	T	F	**T** T F T F F
T	T	T	**F** T T T T T

p	q	r	$p \land q \land \neg r$
F	F	F	F **F** F F T F
F	F	T	F **F** F F F T
F	T	F	F **F** T T T F
F	T	T	F **F** T F F T
T	F	F	T **F** F F T F
T	F	T	T **F** F F F T
T	T	F	T **T** T T T F
T	T	T	T **F** T F F T

Since these truth tables have the same truth values, the formulas are equivalent. The truth tables above are very detailed and you don't need to provide this level of detail in yours. However, you should at least specify the truth value of each connective.

i. $p \leftrightarrow q$

ii. $r \lor (\neg p \land q)$

iii. $\neg(p \to q) \to (p \to \neg q)$

Problem Three: Sufficient Connectives (20 Points)

As we saw in lecture, some propositional connectives can be written in terms of other connectives. For example, $p \to q$ is equivalent to $\neg p \lor q$, so it's possible to rewrite all formulas in propositional logic without ever using $\to$ by replacing all instances of $\varphi \to \psi$ with $\neg\varphi \lor \psi$.

This shows that it's possible to drop down from our seven initial quantifiers to just six ($\neg$, $\land$, $\lor$, $\leftrightarrow$, $\top$, and $\bot$) without losing any expressive power – any propositional formula that can be written with the initial seven quantifiers can be rewritten using just six of them. Is it possible to drop down to an even smaller number? Could we get by with just five quantifiers? Or four? Or even fewer? In this problem, we will explore the following question:

How many logical connectives are necessary to express all propositional formulas?

It turns out that we can drop down from the six connectives $\neg$, $\land$, $\lor$, $\leftrightarrow$, $\top$, and $\bot$ to a set of just five, since the $\leftrightarrow$ operator can be replaced by expressions involving just $\neg$, $\land$, $\lor$, $\top$, and $\bot$.

i. Find a formula that is logically equivalent to $p \leftrightarrow q$ that only uses the connectives $\neg$, $\land$, $\lor$, $\top$, and $\bot$. Do not introduce any new propositional variables. Prove that your formula is logically equivalent to $p \leftrightarrow q$ by showing that its truth table is the same as the truth table for $p \leftrightarrow q$.

ii. Find a formula that is logically equivalent to $\top$ that uses only the connectives $\neg$, $\land$, and $\lor$. You may introduce other propositional variables if you would like. Prove that your formula is equivalent to $\top$ by showing that its truth table always evaluates to $\top$.

Given that you now have a new formula for $\top$, you can easily obtain one for $\bot$ by just negating the formula for $\top$. As a result, at this point you've eliminated all connectives except for $\neg$, $\land$, and $\lor$. But why stop there? Let's see just how much redundancy there is.

iii. Find a formula that is logically equivalent to $p \lor q$ that uses just $\neg$ and $\land$. Do not introduce any new propositional variables. Prove that your formula is equivalent to $p \lor q$ by showing that its truth table is the same as the truth table for $p \lor q$.

We are now down to just $\neg$ and $\wedge$, and it might seem like we can't go any further. However, it's possible to replace these two connectives with just one connective. The ***nor*** connective, denoted $p \downarrow q$, has the following truth table:

p	q	$p \downarrow q$
F	F	T
F	T	F
T	F	F
T	T	F

Notice that the truth table for $p \downarrow q$ is the same as the truth table for $\neg(p \vee q)$, hence the name "nor" (as in "neither p nor q")

iv. Find a formula logically equivalent to $\neg p$ that uses just $\downarrow$. Do not introduce any new propositional variables. Prove that your formula is equivalent to $\neg p$ by showing that its truth table is identical to the truth table for $\neg p$.

v. Find a formula logically equivalent to $p \wedge q$ that uses just $\downarrow$. Do not introduce any new propositional variables. Prove that your formula is equivalent to $p \wedge q$ by showing that its truth table is identical to the truth table for $p \wedge q$.

You have just shown that every propositional logic formula can be written purely in terms of $\downarrow$, since you can

- Eliminate $\leftrightarrow$, $\rightarrow$, $\top$, and $\bot$ by converting to $\neg$, $\wedge$, and $\vee$,
- Eliminate $\vee$ by converting to $\neg$ and $\wedge$, and
- Eliminate $\neg$ and $\wedge$ by converting to $\downarrow$.

Because $\downarrow$ is powerful enough to derive the rest of the connectives, it is sometimes referred to as a ***sole sufficient operator***.

Problem Four: Translating into Logic, Part II (28 points)

In each of the following, you will be given a list of first-order predicates and functions along with an English sentence. In each case, write a statement in first-order logic that expresses the indicated sentence. Your statement may use any first-order construct (equality, connectives, quantifiers, etc.), but you must only use the predicates, functions, and constants provided. Refer to the checkpoint problem for more detailed instructions.

i. Given the predicate

Natural(x), which states that *x* is an natural number,

the function

Product(x, y), which yields the product of *x* and *y*,

and the constants 1 and 137, write a statement in first-order logic that says "137 is prime."

ii. Given the predicates

Word(x), which states that x is a word,
Definition(x), which states that x is a definition, and
Means(x, y), which states that x means y,

write a statement in first-order logic that says "some words have exactly two meanings."

iii. Given the predicates

$x \in y$, which states that x is an element of y, and
Set(S), which states that S is a set,

write a statement in first-order logic that says "every set has a power set."

iv. Given the predicates

Lady(x), which states that x is a lady,
Glitters(x), which states that x glitters,
IsSureIsGold(x, y), which states that x is sure that y is gold,
Buying(x, y), which states that x buys y,
StairwayToHeaven(x), which states that x is a Stairway to Heaven,

write a statement in first-order logic that says "There's a lady who's sure all that glitters is gold, and she's buying a Stairway to Heaven."*

Problem Five: First-Order Negations (16 points)

Proof by contradiction can be difficult because it is often tricky to determine what the negation of the theorem is. In this problem, you'll use first-order logic to explicitly determine the negation of statements in first-order logic.

For each of the first-order logic formulas below, find a first-order logic formula that is the negation of the original statement. Your final formula must not have any negations in it, except for direct negations of predicates. For example, the negation of

$$\forall x.\ (p(x) \to \exists y.\ (q(x) \land r(y)))$$

would be found by pushing the negation in from the outside as follows:

$$\neg(\forall x.\ (p(x) \to \exists y.\ (q(x) \land r(y))))$$

$$\exists x.\ \neg(p(x) \to \exists y.\ (q(x) \land r(y)))$$

$$\exists x.\ (p(x) \land \neg\exists y.\ (q(x) \land r(y))$$

$$\exists x.\ (p(x) \land \forall y.\ \neg(q(x) \land r(y)))$$

$$\exists x.\ (p(x) \land \forall y.\ (q(x) \to \neg r(y)))$$

You must show every step of the process of pushing the negation into the formula (along the lines of what is done above), but you do not need to formally prove that your result is correct.

i. $\forall x.\ (p(x) \to \exists y.\ q(x, y))$

ii. $(\forall x.\ \forall y.\ \forall z.\ (R(x, y) \land R(y, z) \to R(x, z))) \to (\forall x.\ \forall y.\ \forall z.\ (R(y, x) \land R(z, y) \to R(z, x)))$

iii. $\forall n \in \mathbb{N}.\ (n \geq 6 \to \exists x \in \mathbb{N}.\ \exists y \in \mathbb{N}.\ 3x + 4y = n)$

* Let's face it – the lyrics to Led Zeppelin's "Stairway to Heaven" are impossible to decipher. Hopefully we can gain some insight by translating them into first-order logic!

Problem Six: SAT Solving Algorithms (28 Points)

Suppose that you are handed a "black-box" SAT-solving algorithm – that is, a device that takes in a propositional logic formula φ and returns whether or not φ is satisfiable. You don't know anything about the workings of this algorithm – perhaps it converts the formula to CNF and uses DPLL, or perhaps it constructs a truth table – but you are assured that given a formula φ it returns whether or not φ is satisfiable. Let's denote this algorithm *A*, so *A*(φ) is true iff φ is satisfiable.

This question asks what else you can do with a SAT solver.

i. Create an algorithm that uses *A* as a subroutine to determine whether φ is a tautology. Prove that your algorithm is correct. Do **not** just list all possible assignments and check each individually; leverage off of algorithm *A* to get an answer directly.

ii. Suppose that you have two propositional formulas φ and ψ. You are interested in determining whether φ ≡ ψ; that is, whether φ and ψ always have the same truth values. Create an algorithm that uses *A* as a subroutine to answer this question, and prove that your algorithm is correct. Do **not** just list all possible assignments and check each individually; leverage off of algorithm *A* to get an answer directly.

iii. Suppose that you have a propositional formula φ with *n* variables that you know is satisfiable. Create an algorithm that uses *A* as a subroutine to obtain a satisfying assignment for φ using at most *n* calls to *A*. Prove that your answer is correct.

Problem Seven: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

i. How hard did you find this problem set? How long did it take you to finish? Does that seem unreasonably difficult or time-consuming for a five-unit class?

ii. Did you attend Monday's problem session? If so, did you find it useful?

iii. How is the pace of this course so far? Too slow? Too fast? Just right?

iv. Is there anything in particular we could do better? Is there anything in particular that you think we're doing well?

Submission instructions

There are three ways to submit this assignment:

1. Hand in a physical copy of your answers at the start of class. This is probably the easiest way to submit if you are on campus.
2. Submit a physical copy of your answers in the filing cabinet in the open space near the handout hangout in the Gates building. If you haven't been there before, it's right inside the entrance labeled "Stanford Engineering Venture Fund Laboratories." There will be a clearly-labeled filing cabinet where you can submit your solutions.
3. Send an email with an electronic copy of your answers to the submission mailing list (cs103-aut1213-submissions@lists.stanford.edu) with the string "[PS4]" in the subject line.

Extra Credit Problem: Insufficient Connectives (5 Points Extra Credit)

In Problem Three, you proved that the ¬ and ∧ connectives are sufficient to express all propositional logic formulas with at least one variable.

Prove that there exists a propositional logic formula φ containing at least one variable that cannot be rewritten as a formula using just the connectives ¬ and ↔.