Second Practice CS103 Midterm Exam

This midterm exam is open-book, open-note, open-computer, but closed-network. This means that if you want to have your laptop with you when you take the exam, that's perfectly fine, but you **must not** use a network connection. You should only use your computer to look at notes you've downloaded in advance. Although you may use laptops, you **must** hand-write all of your solutions on this physical copy of the exam. No electronic submissions will be considered without prior consent of the course staff.

Normally, we would provide space on the exam for you to write your answers, but in the interest of saving paper we've eliminated most whitespace from this practice exam.

You have three hours to complete this midterm. There are 180 total points, and this midterm is worth 15% of your total grade in this course. You may find it useful to read through all the questions to get a sense of what this midterm contains. As a rough sense of the difficulty of each question, there is one point on this exam per minute of testing time.

Good luck!

Question	**Points**
(1) First-Order Logic	/20
(2) Finding Flaws in Proofs	/20
(3) Graph Coloring	/40
(4) Euclid's Algorithm	/40
(5) Finite Automata	/20
(6) Circle Time!	/40
	/180

Problem 1: Translating into Logic **(20 points total)**

In each of the following, you will be given a list of first-order predicates and functions along with an English sentence. In each case, write a statement in first-order logic that expresses the indicated sentence. Your statement may use any first-order construct (equality, connectives, quantifiers, etc.), but you must only use the predicates, functions, and constants provided.

As an example, if you were given just the predicates *Integer(x)*, which states that *x* is an integer, and the function *Plus(x, y)*, which represents the sum of *x* and *y*, you could write the statement "there is some even integer" as

$$\exists n.\ \exists k.\ (Integer(n) \land Integer(k) \land Plus(k, k) = n)$$

since this asserts that some integer n is equal to $2k$ for some integer k. However, you could not write

$$\exists n.\ (Integer(n) \land Even(n))$$

because there is no *Even* predicate. Similarly, you could not write

$$\exists n \in \mathbb{Z}.\ \exists k \in \mathbb{Z}.\ Plus(k, k) = n$$

Because there is no $\in$ predicate and no constant symbol $\mathbb{Z}$. The point of this question is to get you to think how to express certain concepts in first-order logic given a limited set of predicates, so feel free to write any formula you'd like as long as you don't invent your own predicates, functions, or constants.

(i) Medical Advice **(10 Points)**

Given the predicates

Apple(x), which says that *x* is an apple,
Day(x), which says that *x* is a day,
Eats(x, y, z), which says that *x* eats *y* on day *z*, and
Near(x, y), which says that *x* is near *y*,

and the constant symbols *me* and *doctor*, write a statement in first-order logic that says "An apple a day keeps the doctor away." That is, if you eat an apple every day, then the doctor will stay away from you.

(ii) Family Trees **(10 Points)**

Given the predicates

Person(x), which says that x is a person, and
ParentOf(x, y), which says that x is the parent of y,

write a statement in first-order logic that says "no one is their own grandparent."

Problem 2: Finding Flaws in Proofs **(20 points)**

The following proofs each contain errors that let them prove results that are patently false. For each of these proofs, state what logical error the proof makes and explain why it is wrong.

(i) A Hairy Result **(10 Points)**

Theorem: For every resident of San Jose, there is some other resident of San Jose with the same number of hairs on their head as the first person.

Proof: The maximum number of hairs ever found on a human head is less than 450,000, and the population of San Jose is (as of 2012) around 900,000. By the generalized pigeonhole principle, since there are 900,000 people to distribute into 450,000 buckets, every bucket must contain at least two people. Thus for any resident of San Jose, there is some other resident of San Jose with the same number of hairs on their head. ■

(ii) Real-ity Check **(10 Points)**

Theorem: $|\mathbb{R}| = |\mathbb{Z}|$.

Proof: We exhibit a bijection $f : \mathbb{R} \to \mathbb{Z}$. Let $f(x) = x$. We prove that f is injective and surjective.

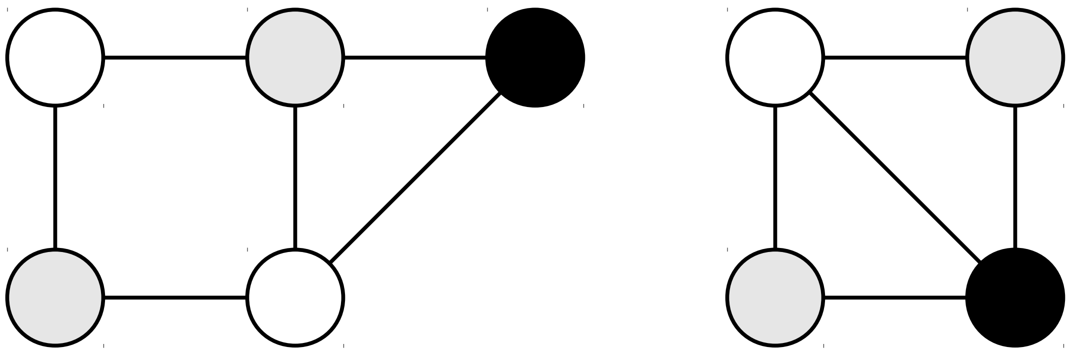
To show that f is injective, consider any x_0, x_1 such that $f(x_0) = f(x_1)$. We will prove that $x_0 = x_1$. To see this, note that by our definition of f, since $f(x_0) = f(x_1)$, we have that $x_0 = x_1$, as required. Thus f is injective.

To show that f is surjective, consider any $x \in \mathbb{Z}$. We need to show that there is an $r \in \mathbb{R}$ such that $f(r) = x$. Since $x \in \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{R}$, we know that $x \in \mathbb{R}$ as well. Thus if we take $r = x$, then $f(r) = r = x$ as required. Thus f is surjective.

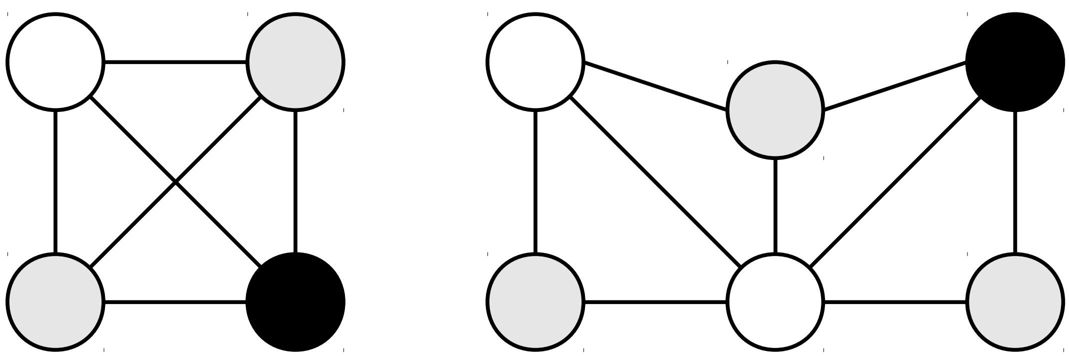
Since f is injective and surjective, it is a bijection. Thus, by definition, $|\mathbb{R}| = |\mathbb{Z}|$. ■

Problem 3: Graph Coloring **(40 points)**

Given an undirected graph G, a ***coloring*** of G is a function that associates each node in the graph with a color such that no two nodes joined by an edge have the same color. For example, the following are legal colorings:



While the following are not legal colorings:



Any graph with no self-loops (edges from a vertex to itself) with n nodes can be colored with n colors by assigning each node a unique color, though some graphs can be colored with fewer colors than this. For example, some graphs (the *bipartite graphs*) can be colored with just two colors. The minimum number of colors with which it is possible to color a graph is called the graph's *chromatic number*.

Prove that if G is an undirected graph with no self-loops where each node has at most k edges incident to it (for some natural number k), then G can be colored with $k + 1$ colors.

Problem 4: Euclid's Algorithm **(40 points total)**

For any pair of integers x and y, a number d is a *common divisor* of x and y if d divides x and d divides y. That is, there are integers m and n such that x = md and y = nd. If either $x \neq 0$ or $y \neq 0$, then the *greatest common divisor* of x and y is the largest number d that is a common divisor of x and y.

One of the oldest known algorithms is *Euclid's algorithm*, which is used to find the greatest common divisor of two integers. Euclid's algorithm is sometimes employed in RSA cryptography, which needs to search for numbers whose greater common divisor is 1. In this problem, you will explore Euclid's algorithm and will formally prove its correctness.

(i) Same Difference **(5 Points)**

Prove that if d is a common divisor of x and y, then d is a divisor of $ax + by$ for any integers a and b.

(ii) The Division Algorithm **(5 Points)**

Recall from lecture that the *division algorithm* says that for any integers x and y, with $y \neq 0$, that x can be written as $x = qy + r$ for integers q and r such that $0 \leq r < y$.

Prove that d is a common divisor of x and y iff it is a common divisor of y and r.

The result you've just proven shows that the set of divisors of x and y is the same as the set of divisors of y and r (if $y \neq 0$). This means that the greatest common divisor of x and y must be the same as the greatest common divisor of y and r. Euclid's algorithm, which dates back almost 2300 years, is based on this principle. The algorithm is defined as follows:

$$gcd(x, y) = \begin{cases} x & \text{if } y = 0 \\ gcd(y, r) & \text{otherwise} \end{cases}$$, where r is found by the division algorithm

In the remainder of this problem, you will prove that Euclid's algorithm is correct for all natural numbers x and y (except the special case where $x = 0$ and $y = 0$, where the greatest common divisor is not defined).

(iii) Proving the Base Case **(10 Points)**

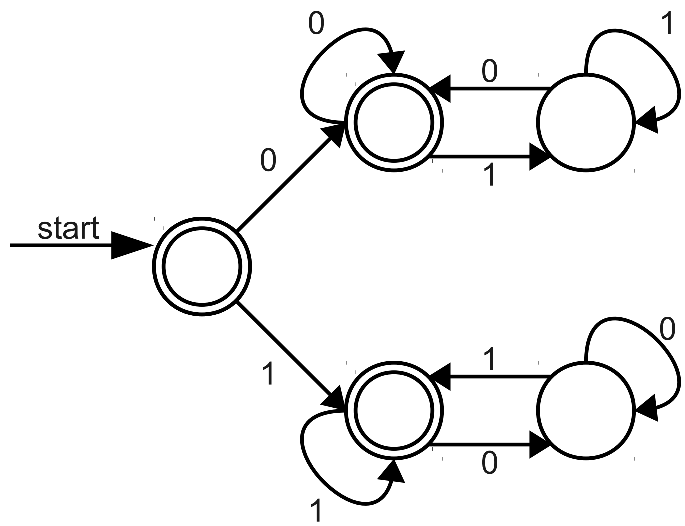
Prove that the greatest common divisor of x and 0 is x for any nonzero natural number x.

(iv) Verifying the Algorithm **(20 Points)**

Prove that *gcd(x, y)* returns the greatest common divisor of x and y, assuming that x and y are natural numbers and either $x \neq 0$ or $y \neq 0$. As a hint, try using strong induction on y.

Problem 5: Finite Automata **(20 points total)**

Below is a DFA over the alphabet {`0`, `1`}. For each of the five strings below, does the DFA accept or reject the string?



i. `01010`
ii. `000011`
iii. `10001`
iv. `1100`
v. ε

Problem 6: Circle Time! **(40 Points)**

There are 186 students enrolled in CS103 (note that $186 = 93 \times 2$). Suppose that everyone in CS103 sits in a circle. Prove that if 93 CS103 students are wearing red hats and 93 CS103 students are wearing blue hats, then someone must be sitting between two students wearing blue hats and someone must be sitting between two students wearing red hats.