Problem Set 9

In this final problem set of the quarter, you will explore the limits of what can be computed efficiently. What problems do we believe are computationally intractable? What do they look like? And are they purely theoretical, or might you might bump into one some day?

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

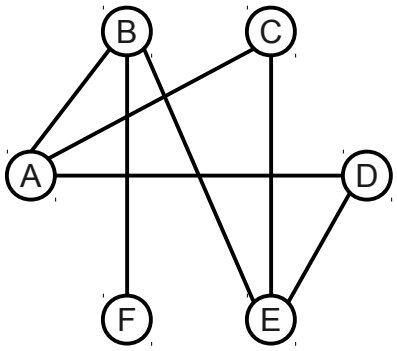
This problem set has 125 possible points. It is weighted at 5% of your total grade (note that this is slightly less than usual).

Good luck, and have fun!

Due Friday, December 7th at 2:15 PM

Problem One: The Long Path Problem (20 Points)

Given an undirected graph $G = (V, E)$, a *simple path* in a G is a path between two nodes $u, v \in V$ such that no node is repeated on the path. For example, given this graph:



A → C → E is a simple path from A to E, but A → B → E → C → A → D is not a simple path because node A is visited twice.*

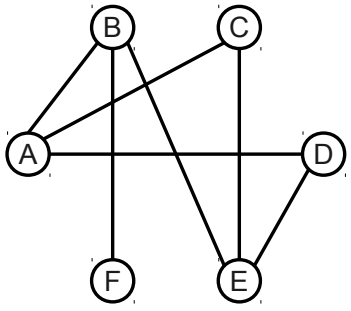
Consider the following language:

$ULONGPATH = \{ \langle G, u, v, k \rangle \mid$ G is an undirected graph,
u and v are nodes in the graph, and
there exists a simple path from u to v containing k nodes. $\}$

For example, if G is the above graph, then $\langle G, \text{D}, \text{F}, 6 \rangle \in ULONGPATH$ because there is a simple path of six nodes from D to F (namely, D→A→C→E→B→F), but $\langle G, \text{A}, \text{C}, 5 \rangle \notin ULONGPATH$ because there is no simple path of 5 nodes from A to C.

i. Prove that $ULONGPATH \in$ **NP** by designing a polynomial-time verifier for it. Prove your verifier is correct by proving there is some x such that $\langle G, u, v, k \rangle \in ULONGPATH$ iff your verifier accepts $\langle G, u, v, k, x \rangle$. However, you can just give an informal justification as to why your verifier runs in polynomial time.

In an undirected graph $G = (V, E)$, a *Hamiltonian path* is a simple path between two nodes u and v that visits every node in the graph exactly once. For example, in this graph:



The path F → B → E → C → A → D is a Hamiltonian path from F to D, but F → B → E → D is not (because it doesn't visit every node), nor is F → B → E → D → A → C → E → D (because it is not a simple path).

* Although we have defined a path in a graph as a series of edges, it is often easier to reason about the path as the series of nodes that it passes through. Throughout this problem set, we will adopt this convention.

The language *UHAMPATH* is defined as follows:

$$UHAMPATH = \{\langle G, u, v\rangle \mid G \text{ is an undirected graph and there is a Hamiltonian path from } u \text{ to } v.\}$$

UHAMPATH is known to be **NP**-complete by a fairly clever series of reductions from SAT; see Sipser, page 291 (second edition) or page 319 (third edition) for more details.

ii. Prove that $ULONGPATH \in \mathbf{NPC}$ by showing that $UHAMPATH \leq_P ULONGPATH$. Prove your reduction is correct (i.e., $\langle G, u, v\rangle \in UHAMPATH$ iff $f(\langle G, u, v,\rangle) \in ULONGPATH$), but feel free to justify informally why your reduction works in polynomial time.

Problem Two: 4-Colorability (35 Points)

If you'll recall, an undirected graph G is called 3-colorable iff there exists a way to color each the nodes in G one of three colors such that no two nodes of the same color are connected by an edge. We formalized the 3-coloring problem as a language as follows:

$$3COLOR = \{ \langle G\rangle \mid G \text{ is an undirected, 3-colorable graph } \}$$

As we saw in lecture, *3COLOR* is **NP**-complete.

An undirected graph is called 4-colorable iff there is a way to color each of the nodes in G one of four colors so that no two nodes of the same color are connected by an edge. We can formalize the 4-coloring problem as a language as follows:

$$4COLOR = \{ \langle G\rangle \mid G \text{ is an undirected, 4-colorable graph } \}$$

Note that every 3-colorable graph is also 4-colorable, but not all 4-colorable graphs are 3-colorable. In other words, 3-colorability is a stricter requirement than 4-colorability. However, it is still the case that *4COLOR* is **NP**-complete, and in this problem you will prove this result.

i. Prove that $4COLOR \in \mathbf{NP}$ by designing a polynomial-time NTM for it. Prove that your NTM is correct, then justify informally why it runs in polynomial time.

ii. Prove that *4COLOR* is **NP**-hard by proving $3COLOR \leq_P 4COLOR$. That is, show how to take an arbitrary graph G and construct (in polynomial time) a graph G' such that graph G is 3-colorable iff graph G' is 4-colorable.

For simplicity, you do not need to formally prove that your reduction is correct and runs in polynomial time. Instead, briefly answer each of the following questions about your reduction (two or three sentences apiece should be sufficient):

1. If the original graph G is 3-colorable, why is your new graph G' 4-colorable?
2. If your new graph G' is 4-colorable, why is the original graph G 3-colorable?
3. Why can your reduction be computed in polynomial time?

Problem Three: Resolving P $\stackrel{?}{=}$ NP (30 Points)

This problem explores the question

What would it take to prove whether or not P = NP?

For each statement below, decide whether the statement would definitely prove **P** = **NP**, definitely prove **P** ≠ **NP**, or would do neither. Write "**P** = **NP**," "**P** ≠ **NP**," or "neither" as your answer to each question. Please don't just write "true" or "false." No justification is necessary.

1. There is a **P** language that can be decided in polynomial time.
2. There is an **NP** language that can be decided in polynomial time.
3. There is an **NP-*complete*** language that can be decided in polynomial time.
4. There is an **NP-*hard*** language that can be decided in polynomial time.
5. There is an **NP** language that ***cannot*** be decided in polynomial time.
6. There is an **NP-*complete*** language that ***cannot*** be decided in polynomial time.
7. There is an **NP-*hard*** language that ***cannot*** be decided in polynomial time.
8. There is ***some*** **NP**-complete language that can be decided in $O(2^n)$ time.
9. There is ***no*** **NP**-complete language that can be decided in $O(2^n)$ time.
10. There is a polynomial-time ***verifier*** for every language in **NP**.
11. There is a polynomial-time ***decider*** for every language in **NP**.
12. There is a language $L \in$ **P** where $L \leq_P$ 3SAT.
13. There is a language $L \in$ **NP** where $L \leq_P$ 3SAT.
14. There is a language $L \in$ **NPC** where $L \leq_P$ 3SAT.
15. There is a language $L \in$ **P** where 3SAT $\leq_P L$.
16. There is a language $L \in$ **P** where 3SAT $\leq_M L$.
17. All languages in **P** are decidable.
18. All languages in **NP** are decidable.
19. There is a polynomial-time algorithm that correctly decides SAT for all strings of length ***at most*** 10^{100}, but that might give incorrect answers for longer strings.
20. There is a polynomial-time algorithm that correctly decides SAT for all strings of length ***at least*** 10^{100}, but that might give incorrect answers for shorter strings.

Problem Four: The Big Picture (35 Points)

We have covered a *lot* of ground in this course throughout our whirlwind tour of computability and complexity theory. This last question surveys what we have covered so far by asking you to see how everything we have covered relates.

Take a minute to review the hierarchy of languages we explored:

$$\mathbf{REG} \subset \mathbf{DCFL} \subset \mathbf{CFL} \subset \mathbf{P} \subseteq \mathbf{NP} \subset \mathbf{R} \subset \mathbf{RE} \subset \mathbf{ALL}$$

The following questions ask you to provide examples of languages at different spots within this hierarchy. In each case, you should provide an example of a language, but you don't need to formally prove that it has the properties required. Instead, describe a proof technique you could use to show that the language has the required properties. There are many correct answers to these problems, and we'll accept any of them.

i. Give an example of a regular language. How might you prove that it is regular?

ii. Give an example of a context-free language is not regular. How might you prove that it is context-free? How might you prove that it is not regular?

iii. Give an example of a language in **P** that is not context-free. How might you prove that it is in **P**? How might you prove that it is not context-free?

iv. Give an example of a language in **NP** suspected not to be in **P**. How might you prove that it is in **NP**? Why do we think that it is not contained in **P**?

v. Give an example of a language in **RE** not contained in **R**. How might you prove that it is **RE**? How might you prove that it is not contained in **R**?

vi. Give an example of a language in co-**RE** not contained in **R**. How might you prove that it is co-**RE**? How might you prove that it is not contained in **R**?

vii. Give an example of a language that is neither **RE** nor co-**RE**. How might you prove it is not contained in **RE**? How might you prove it is not contained in co-**RE**?

Problem Five: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

i. **If we should keep any one thing about this course the same in future offerings, what would it be?**

ii. **If you could change any one thing about this course, what would it be?**

iii. **What topic did you think was the most interesting? What topic did you think was the least interesting?**

Submission instructions

There are three ways to submit this assignment:

1. Hand in a physical copy of your answers at the start of class. This is probably the easiest way to submit if you are on campus.
2. Submit a physical copy of your answers in the filing cabinet in the open space near the handout hangout in the Gates building. If you haven't been there before, it's right inside the entrance labeled "Stanford Engineering Venture Fund Laboratories." There will be a clearly-labeled filing cabinet where you can submit your solutions.
3. Send an email with an electronic copy of your answers to the submission mailing list (cs103-aut1213-submissions@lists.stanford.edu) with the string "[PS9]" somewhere in the subject line. If you do submit electronically, please submit your assignment as a single PDF if at all possible. Sending multiple files makes it harder to print out and grade your submission.

Extra Credit Problem: P $\stackrel{?}{=}$ NP (Worth an A+, $1,000,000, and a Stanford Ph.D)

Prove or disprove: **P** = **NP**.