Indirect Proofs

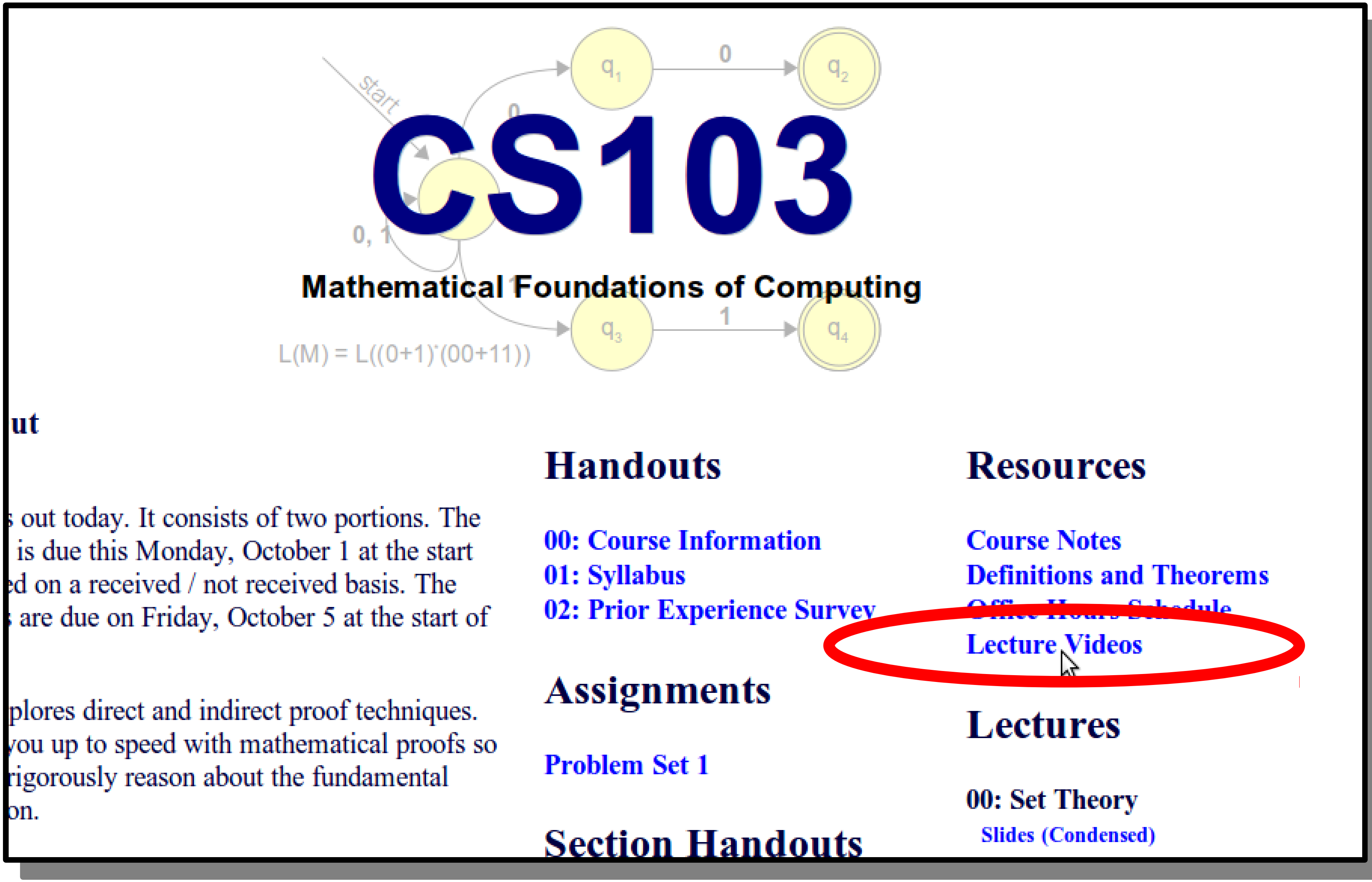
Announcements

- Problem Set 1 out.
- **Checkpoint** due Monday, October 1.
  - Graded on a “did you turn it in?” basis.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more an effort you put in, the more you'll get out.
- **Remaining problems** due Friday, October 5.
  - Feel free to email us with questions!

Submitting Assignments

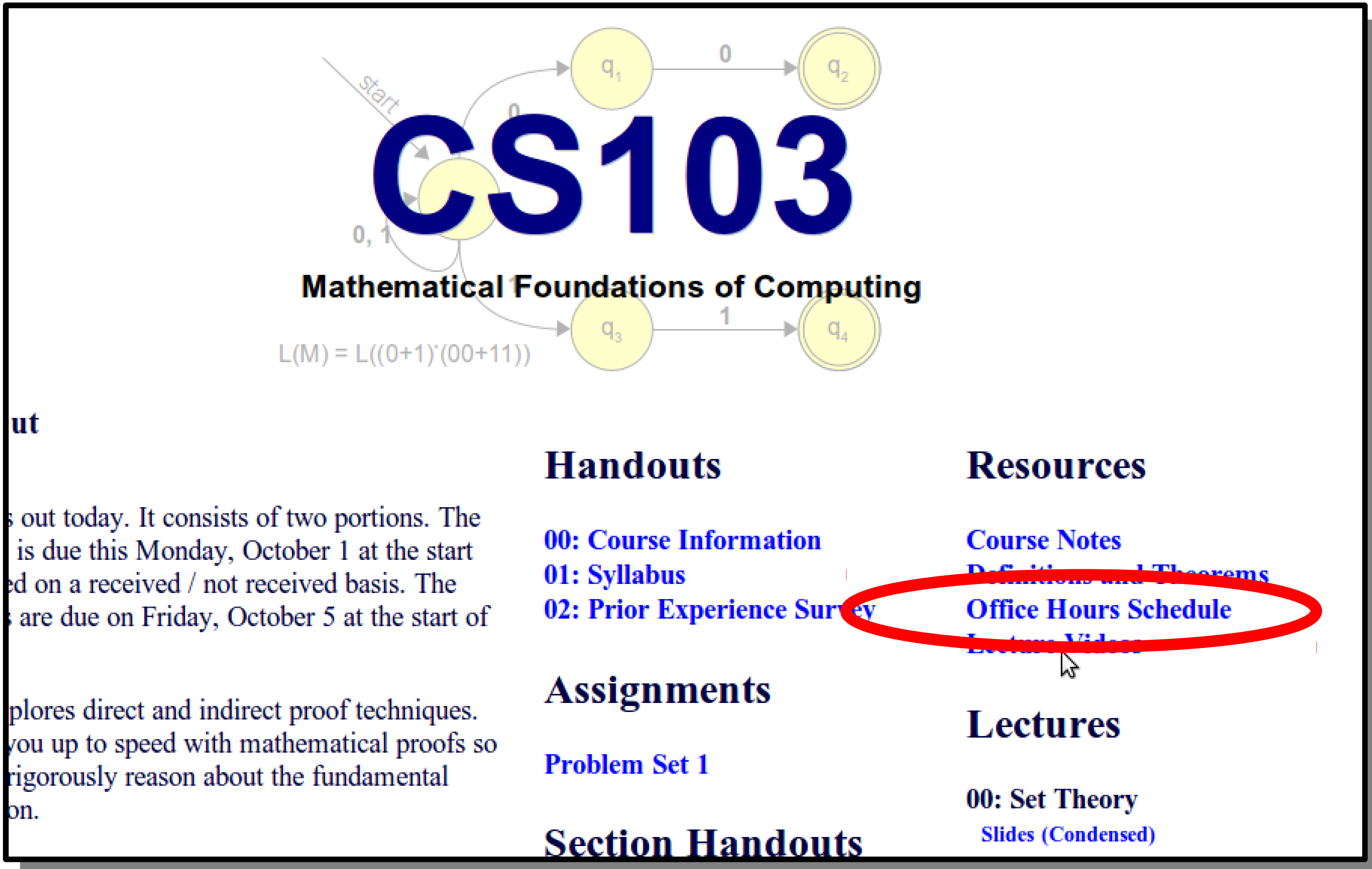
- You can submit assignments by
  - handing them in at the start of class,
  - dropping it off in the filing cabinet near Keith's office (details on the assignment handouts), or
  - emailing the submissions mailing list at cs103-aut1213-submissions@lists.stanford.edu and attaching your solution as a PDF.
- Late policy:
  - Three 72-hour "late days."
  - Can use at most one per assignment.
  - No work accepted more than 72 hours after due date.

Lecture Videos



http://class.stanford.edu/cs103/Fall2012/videos

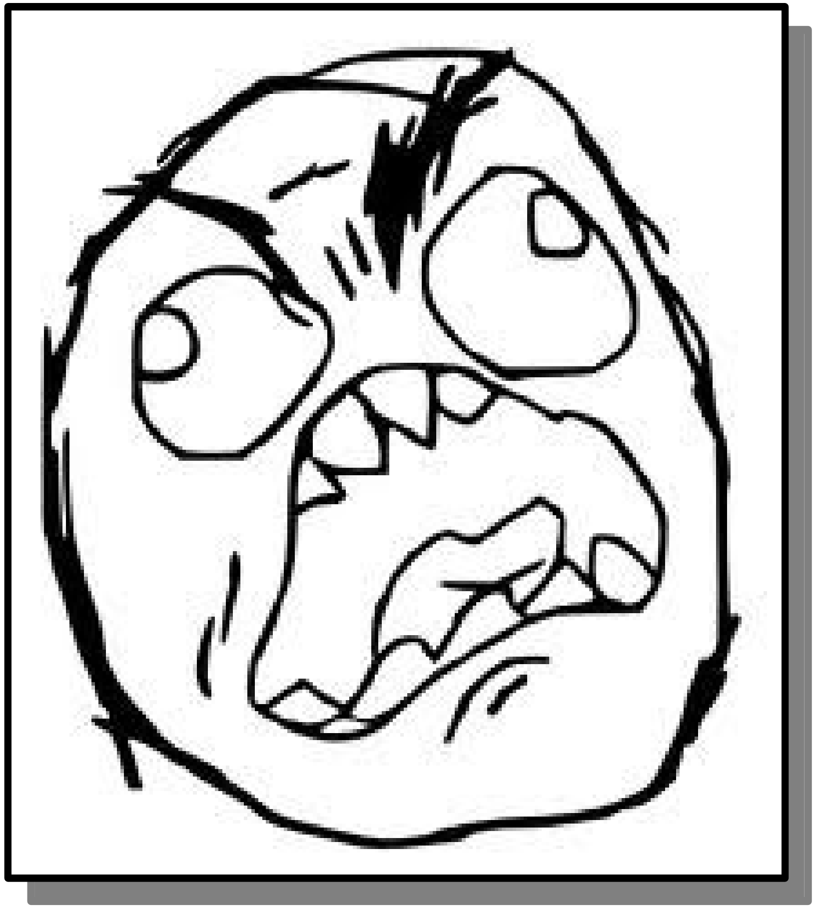
Lecture Videos



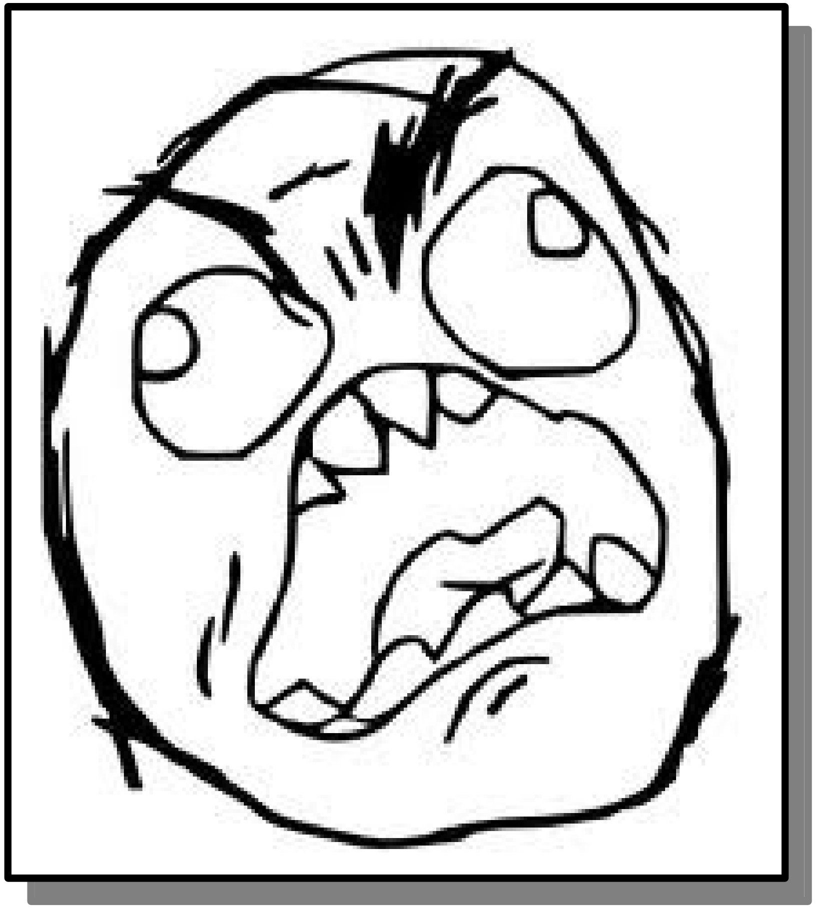
http://class.stanford.edu/cs103/Fall2012/videos

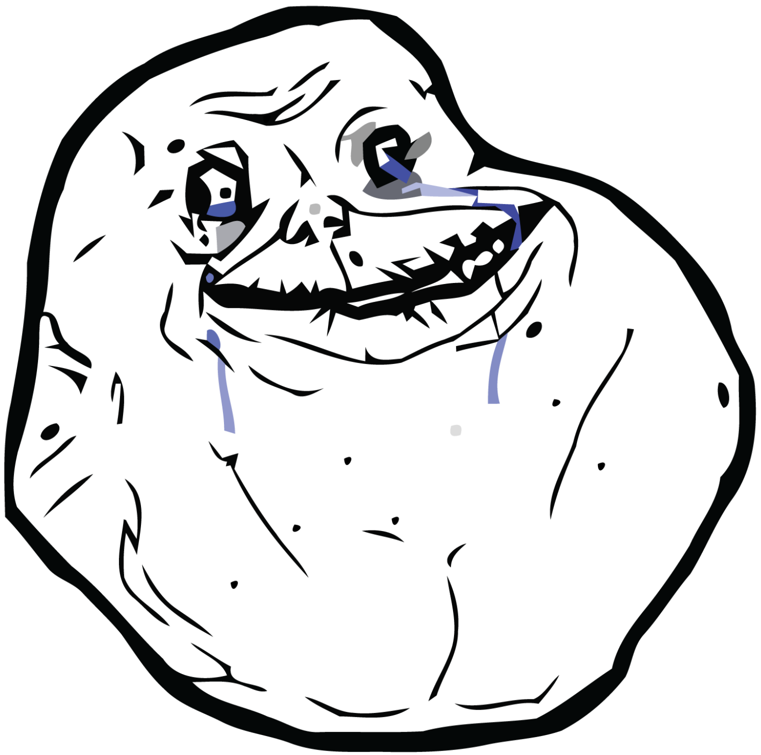
Office Hours

Office Hours

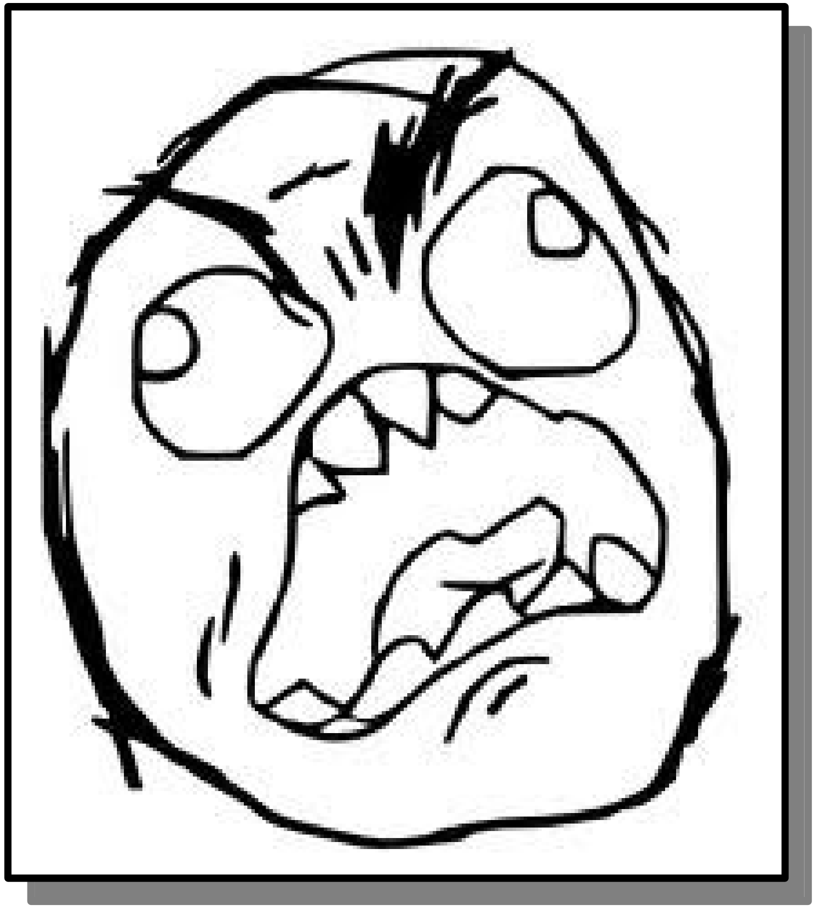


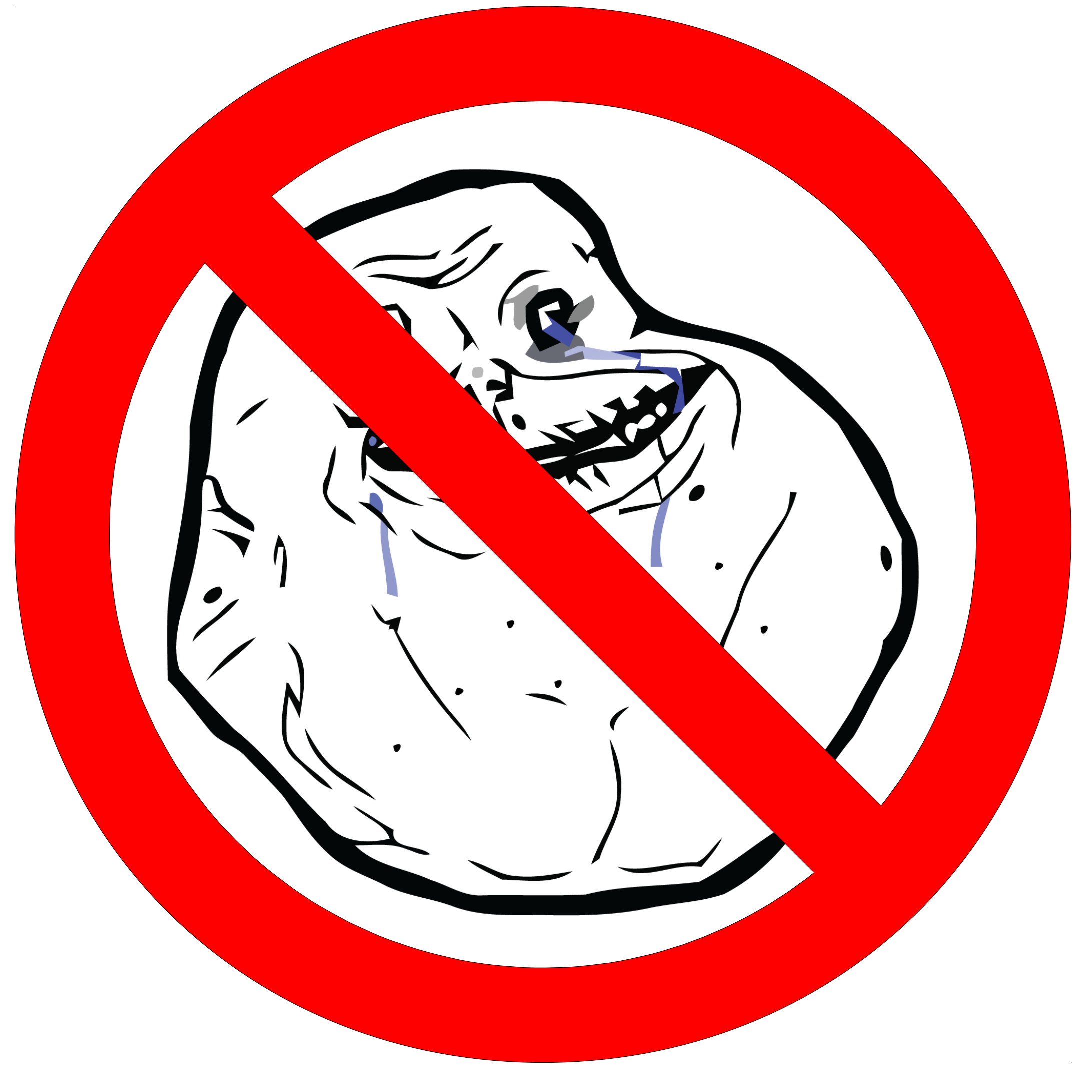
Office Hours





Office Hours





Office Hours

Office hours start today.

Schedule available on the course website.

Friday Four Square

Don't be this guy!

- Good snacks!
- Good company!
- Good game!
- Good fun!
- **Today at 4:15 in front of Gates.**

Outline for Today

- Logical Implication
  - What does "If P, then Q" mean?
- Proof by Contradiction
  - The basic method.
  - Contradictions and implication.
  - Contradictions and quantifiers.
- Proof by Contrapositive
  - The basic method.
  - An interesting application.

Logical Implication

Implications

- An **implication** is a statement of the form

  If $\boldsymbol{P}$, then $\boldsymbol{Q}$.

- We write "If P, then Q" as $\boldsymbol{P \to Q}$.
  - Read: "P implies Q."
- When $P \to Q$, we call P the **antecedent** and Q the **consequent**.

What does Implication Mean?

- The statement $P \to Q$ means exactly the following:

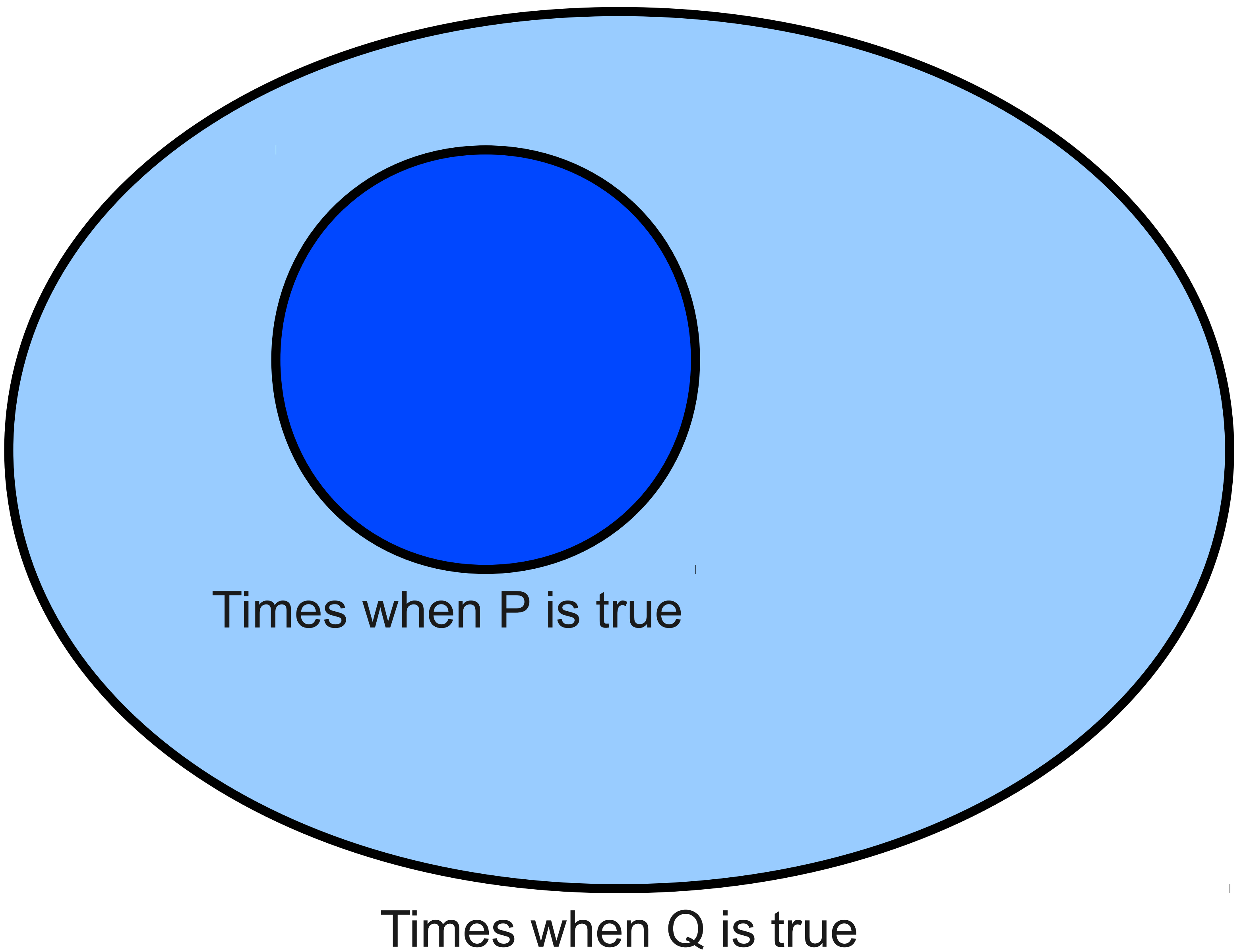
Whenever P is true,
Q must be true as well.

- For example:
  - n is even $\to n^2$ is even.
  - $(A \subseteq B$ and $B \subseteq C) \to A \subseteq C$

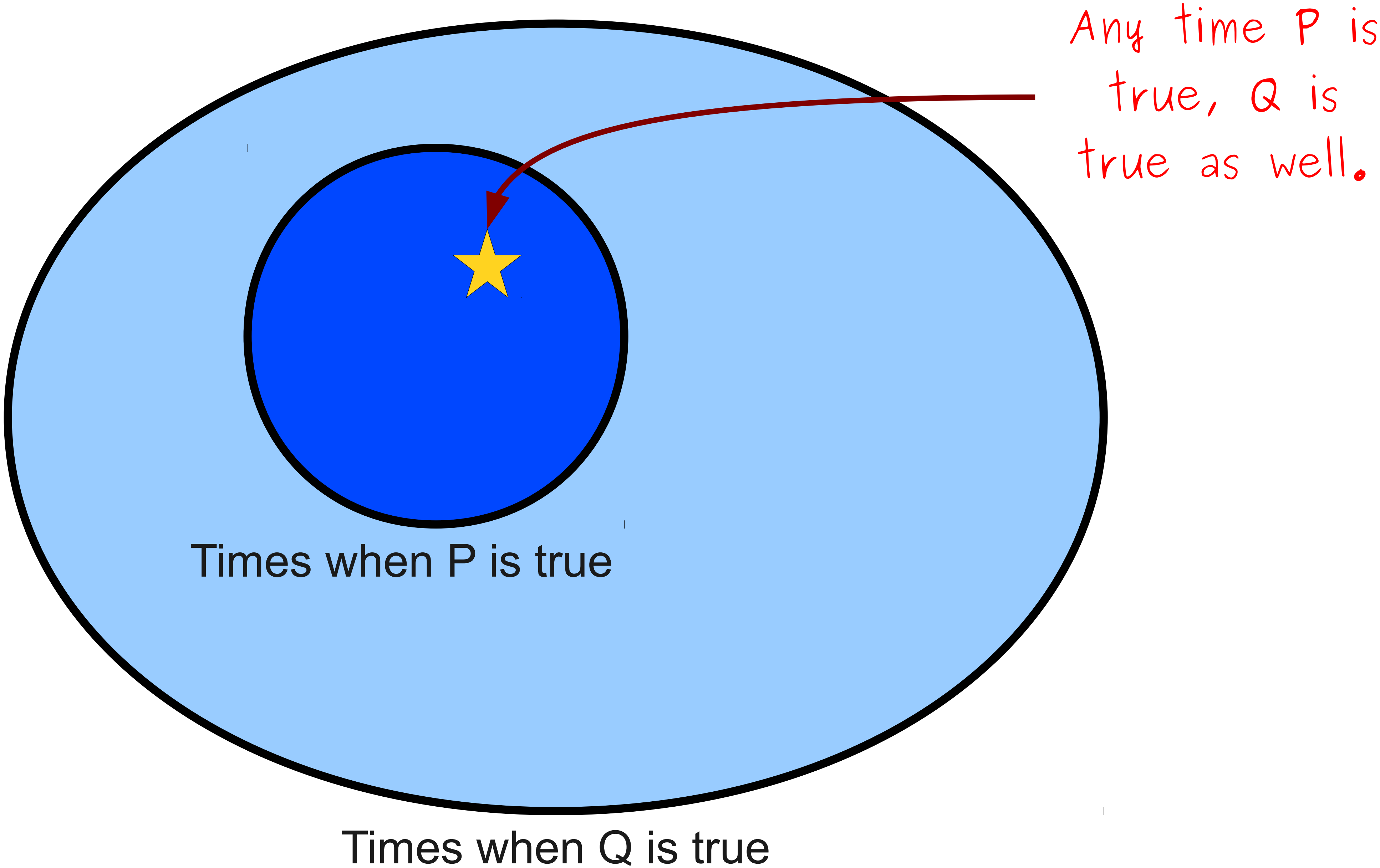
What does Implication **Not** Mean?

- $P \to Q$ does **not** mean that whenever Q is true, P is true.
  - "If you are a Stanford student, you wear cardinal" does **not** mean that if you wear cardinal, you are a Stanford student.
- $P \to Q$ does **not** say anything about what happens if P is false.
  - "If you hit another skier, you're gonna have a bad time" doesn't mean that if you don't hit other skiers, you're gonna to have a good time.
  - **Vacuous truth:** If P is never true, then $P \to Q$ is always true.
- $P \to Q$ does **not** say anything about causality.
  - "If I want math to work, then $2 + 2 = 4$" is true because any time that I want math to work, $2 + 2 = 4$ already was true.
  - "If I don't want math to work, then $2 + 2 = 4$" is also true, since whenever I don't want math to work, $2 + 2 = 4$ is true.

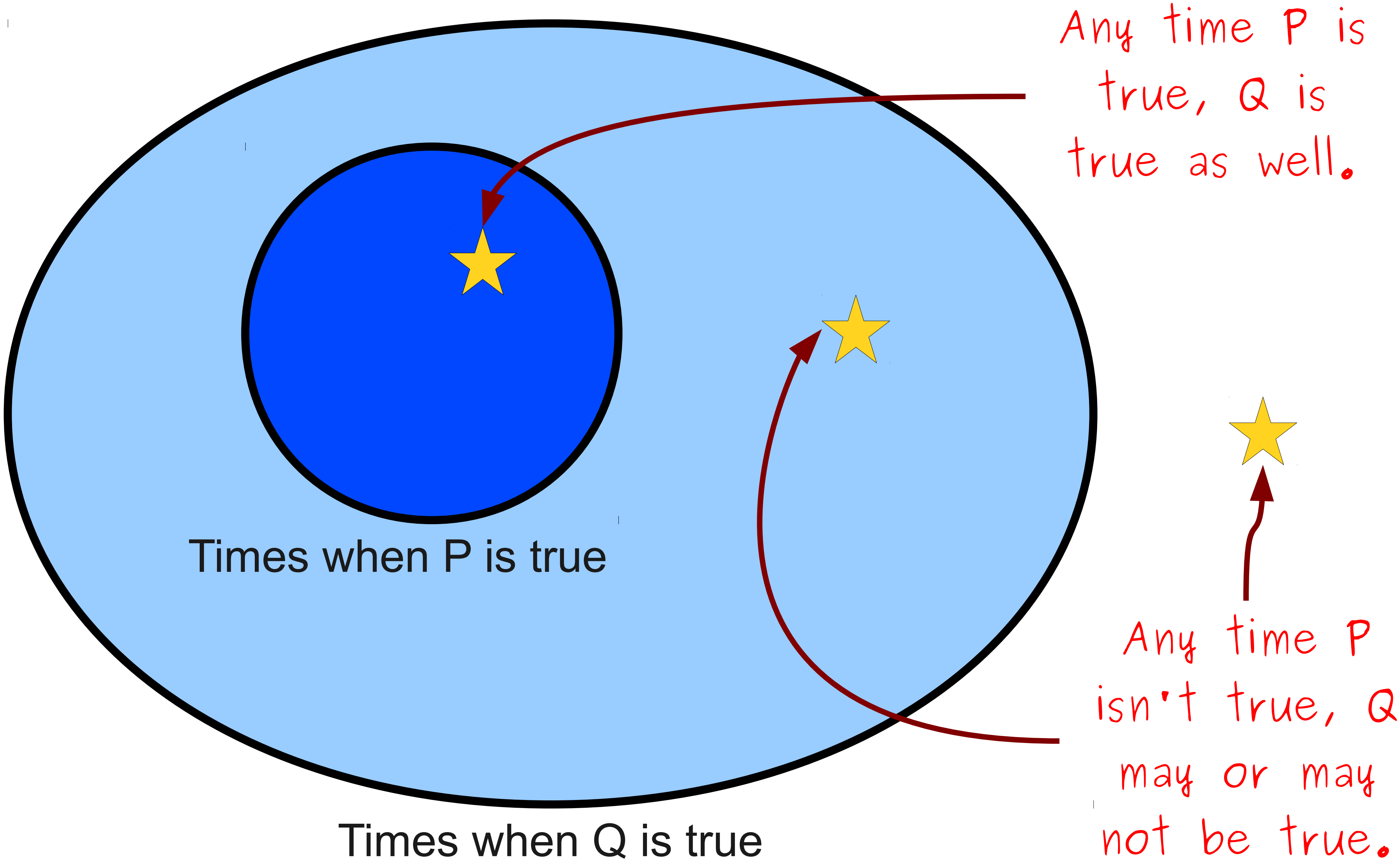
Implication, Diagrammatically



Implication, Diagrammatically



Implication, Diagrammatically



Alternative Forms of Implication

- All of the following are different ways of saying $P \to Q$:

If P, then Q.

P implies Q.

P only if Q.

Q whenever P.

P is sufficient for Q.

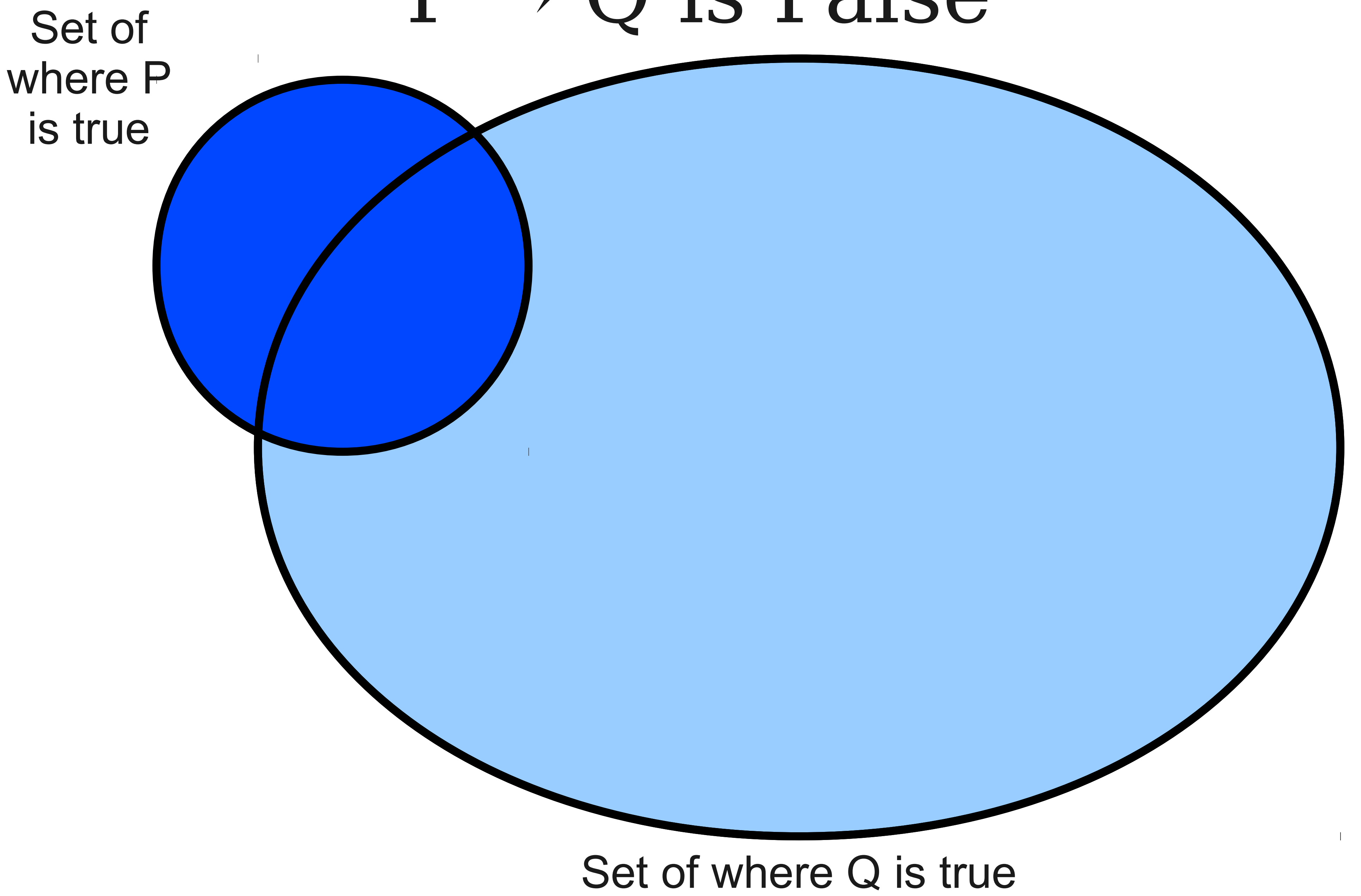
Q is necessary for P.

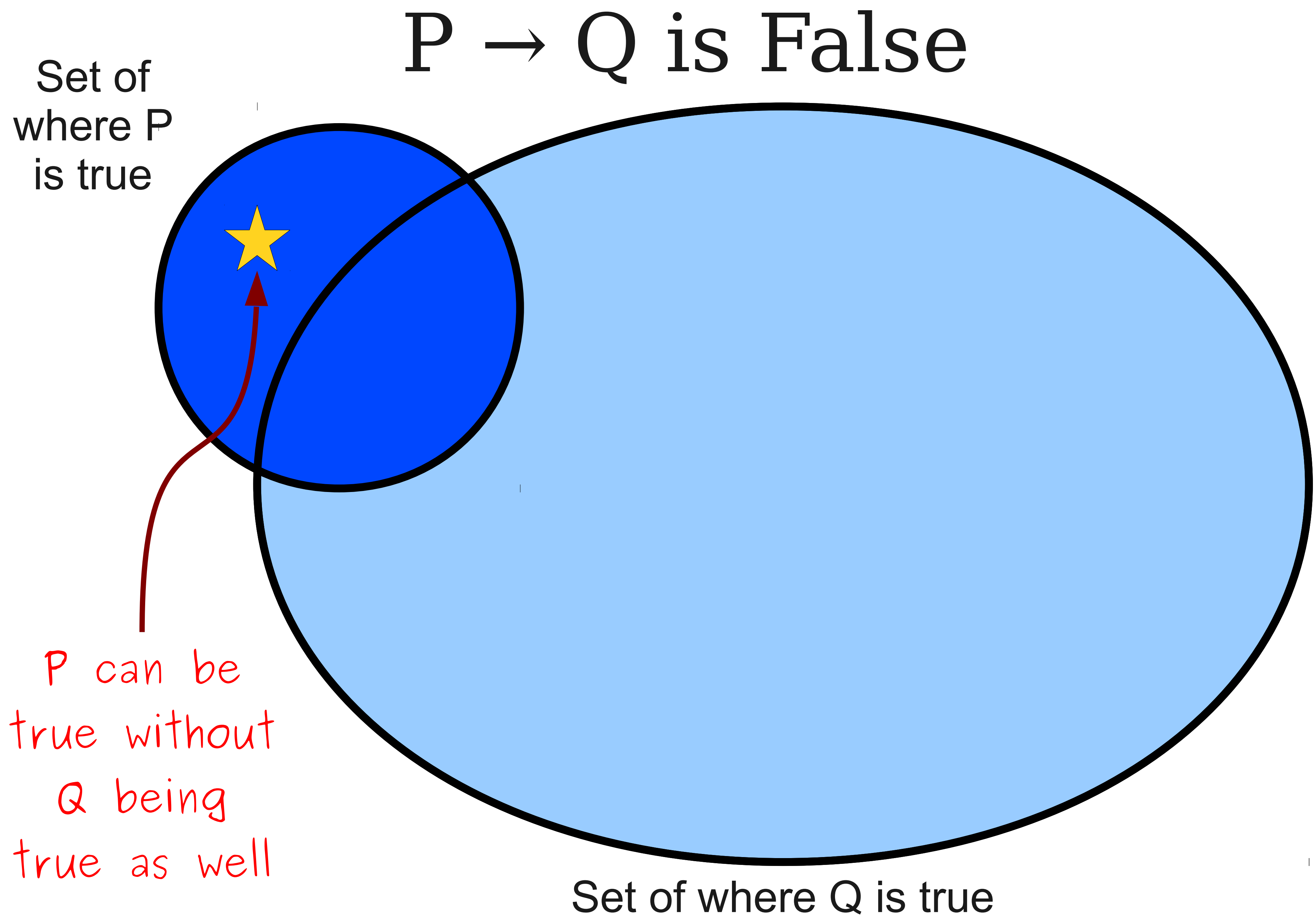
- Why?

When P Does Not Imply Q

- What would it mean for $P \to Q$ to be false?
- **Answer**: There must be some way for P to be true and Q to be false.
- $P \to Q$ means "any time P is true, Q is true."
  - The only way to disprove this is to show that there is some way for P to be true and Q to be false.
- To prove that $P \to Q$ is false, find an example of where P is true and Q is false.

P → Q is False





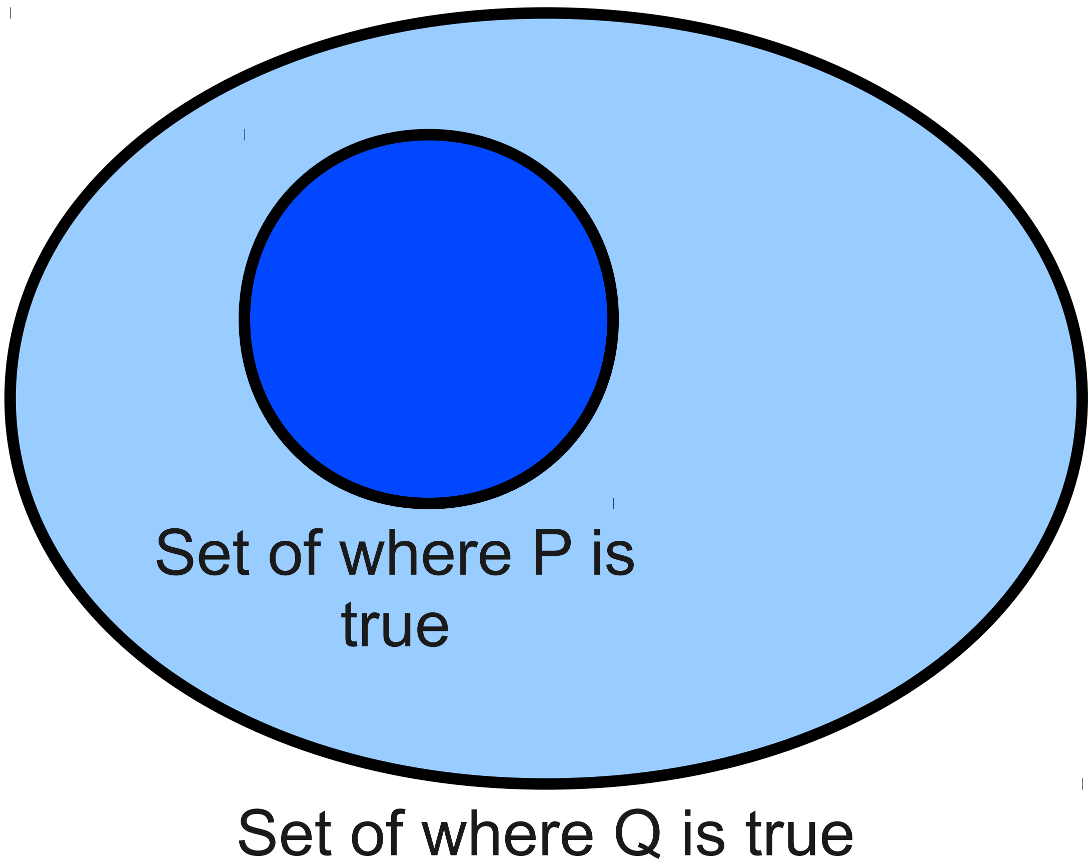
P → Q is False
Set of where P is true
P can be true without Q being true as well
Set of where Q is true

A Common Mistake

- To show that $P \to Q$ is false, it is **not** sufficient to find a case where P is false and Q is false.

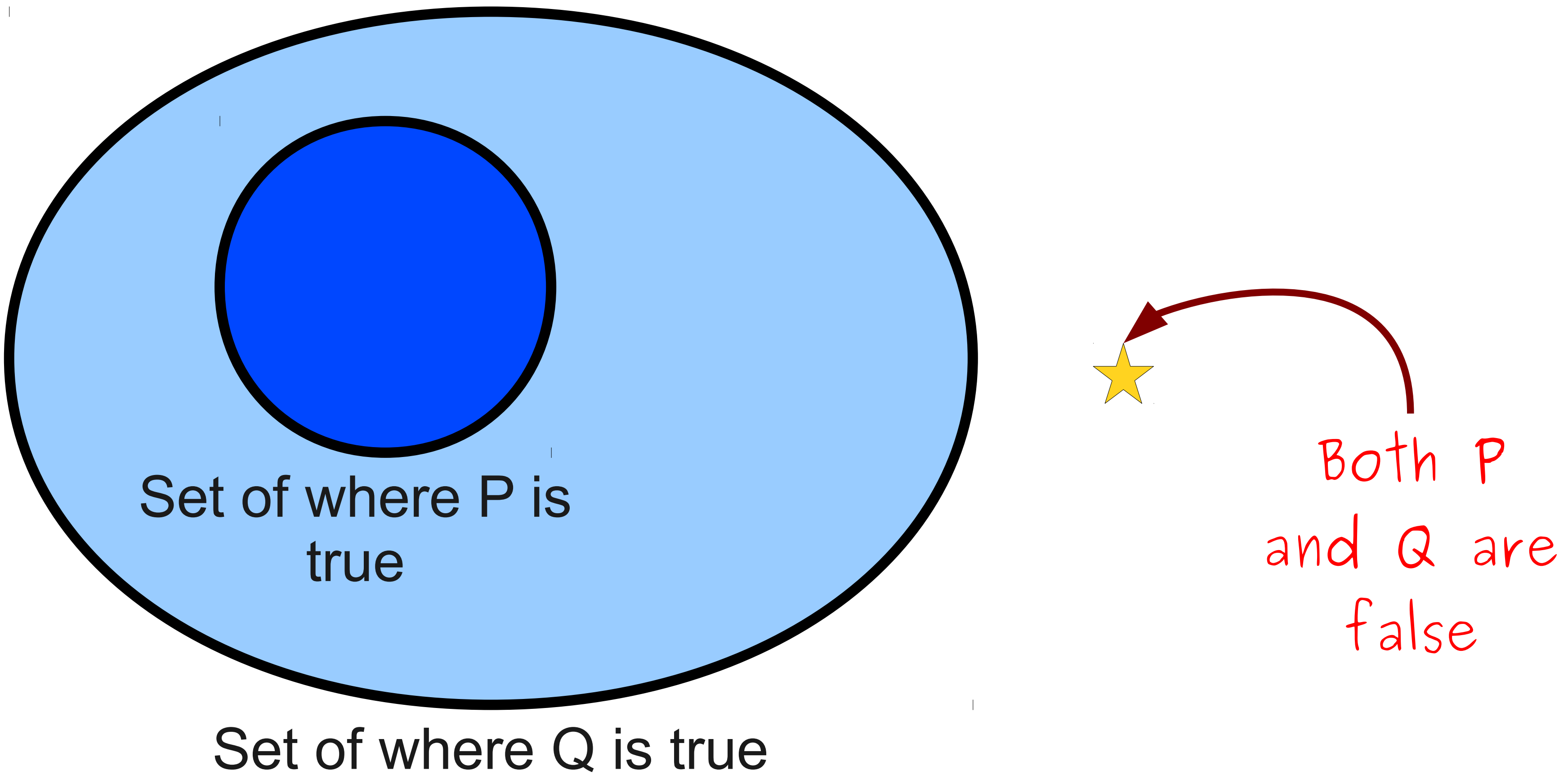
A Common Mistake

- To show that $P \to Q$ is false, it is **not** sufficient to find a case where P is false and Q is false.



A Common Mistake

- To show that $P \to Q$ is false, it is **not** sufficient to find a case where P is false and Q is false.



Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A **proof by contradiction** is a proof that works as follows:
  - To prove that P is true, assume that P is not true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only option is that the assumption that P is not true is incorrect.
  - Conclude, therefore, that P is true.

Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that $P \to Q$ is false.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that $P \to Q$ is false.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

What does this mean?

Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that ***P* is true and *Q* is false**.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction;

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; **???**

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2$

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is impossible, since n^2 is even.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is impossible, since n^2 is even.

We have reached a contradiction, so our assumption was false.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is impossible, since n^2 is even.

We have reached a contradiction, so our assumption was false. Thus if n^2 is even, n is even as well.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is impossible, since n^2 is even.

We have reached a contradiction, so our assumption was false. Thus if n^2 is even, n is even as well. ■

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is impossible, since n^2 is even.

We have reached a contradiction, so our assumption was false. Thus if n^2 is even, n is even as well. ■

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.

Proof: By contradiction; assume n^2 is even but n is odd.

The three key pieces:

1. State that the proof is by contradiction.
2. State what the negation of the original statement is.
3. State you have reached a contradiction and what the contradiction entails.

You must include all three of these steps in your proofs!

1.

We have reached a contradiction, so our assumption was false. Thus if n^2 is even, n is even as well. ■

Biconditionals

- Combined with what we saw on Wednesday, we have proven

$$\text{If } n \text{ is even, } n^2 \text{ is even.}$$

$$\text{If } n^2 \text{ is even, } n \text{ is even.}$$

- We sometimes write this as

n is even **if and only if** n^2 is even.

- This is often abbreviated

n is even **iff** n^2 is even.

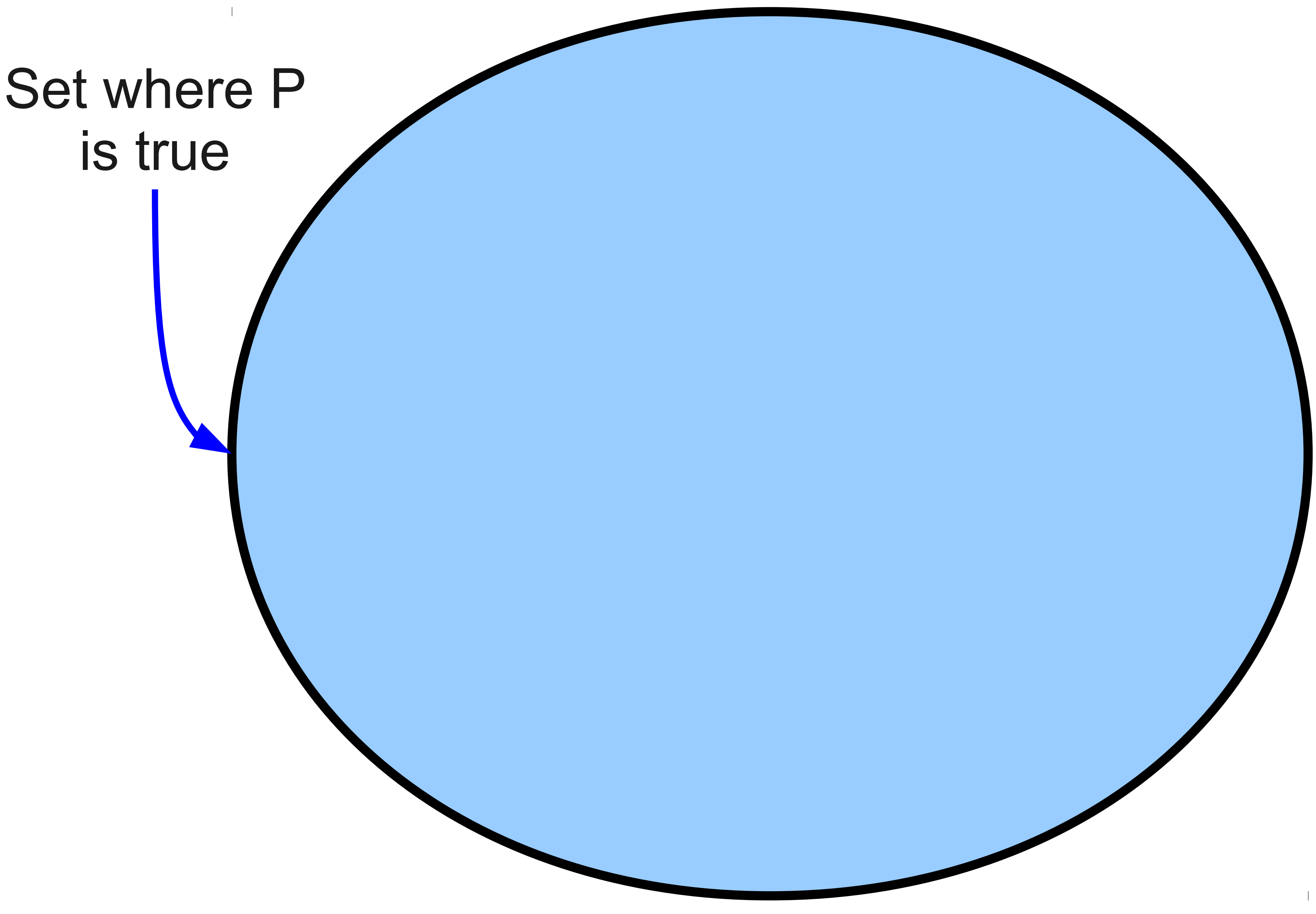
or as

$$n \text{ is even} \leftrightarrow n^2 \text{ is even}$$

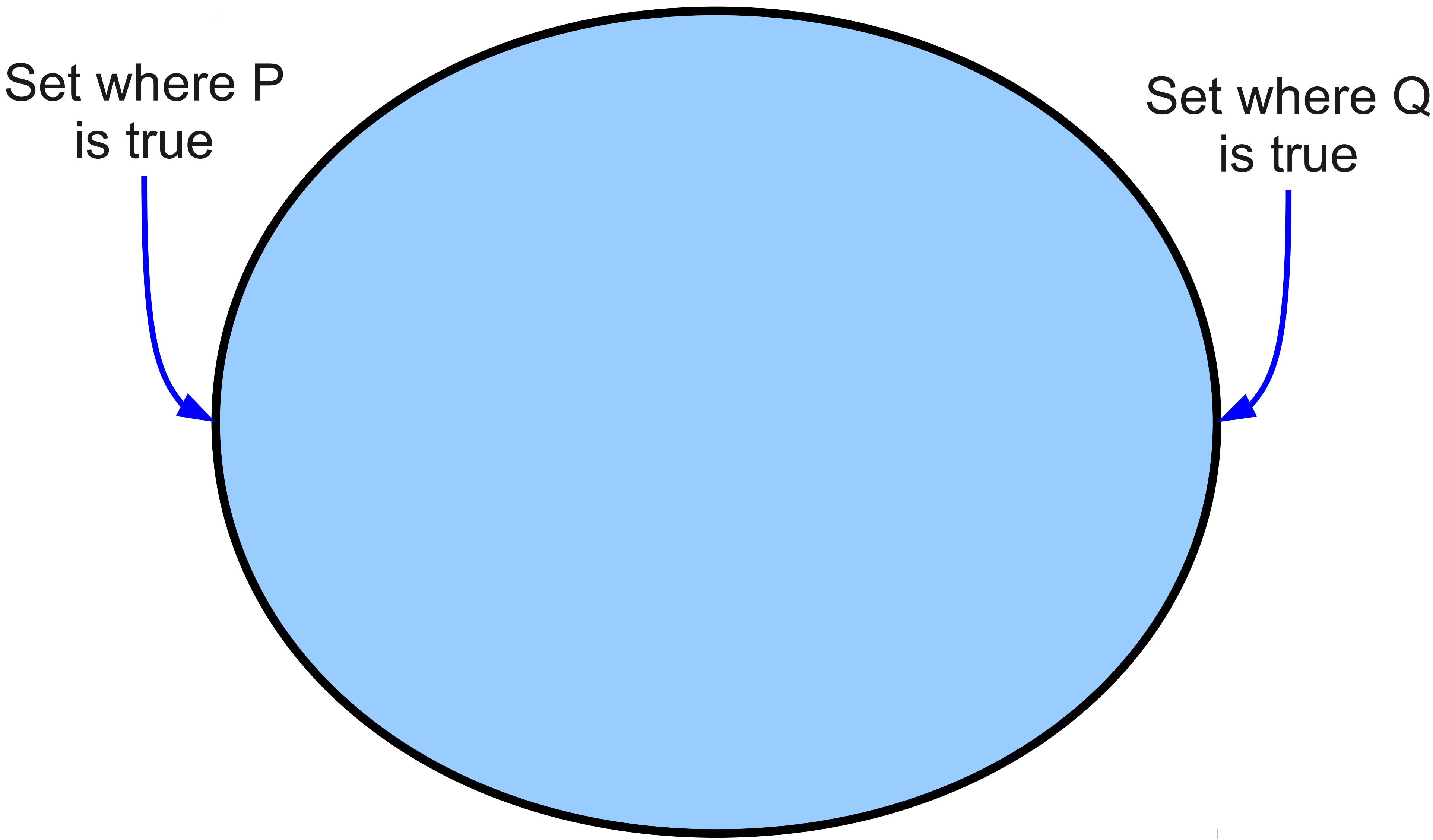
- This is called a **biconditional**.

$$P \leftrightarrow Q$$

$P \leftrightarrow Q$



$P \leftrightarrow Q$



Proving Biconditionals

- To prove ***P* iff *Q***, you need to prove that
  - $\boldsymbol{P \to Q}$, and
  - $\boldsymbol{Q \to P}$.
- You may use any proof techniques you'd like when doing so.
  - In our case, we used a direct proof and a proof by contradiction.
- **Just make sure to prove both directions of implication**!

Rational and Irrational Numbers

Rational and Irrational Numbers

- A **rational number** is a number r that can be written as

$$r = \frac{p}{q}$$

  where

  - p and q are integers,
  - $q \neq 0$, and
  - p and q have no common divisors other than ± 1.
- A number that is not rational is called **irrational**.

Rational and Irrational Numbers

A **rational number** is a number r that can be written as

$$r = \frac{p}{q}$$

where

- p and q are integers,
- $q \neq 0$, and
- p and q have no common divisors other than ± 1.

A number that is not rational is called **irrational**.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.
Proof: By contradiction;

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.
Proof: By contradiction; **???**

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.
Proof: By contradiction; assume $\sqrt{2}$ is rational.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\,q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\,q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\,q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2$

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\,q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2$

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\,q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\,q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor. This contradicts our earlier assertion that their only common divisors are 1 and -1.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor. This contradicts our earlier assertion that their only common divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\,q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor. This contradicts our earlier assertion that their only common divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect. Consequently, $\sqrt{2}$ is irrational.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor. This contradicts our earlier assertion that their only common divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect. Consequently, $\sqrt{2}$ is irrational. ■

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{\ }$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor. This contradicts our earlier assertion that their only common divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect. Consequently, $\sqrt{2}$ is irrational. ∎

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.
Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{}$, and p and q have no common divisors other than 1 and -1.

The three key pieces:

1. State that the proof is by contradiction.
2. State what the negation of the original statement is.
3. State you have reached a contradiction and what the contradiction entails.

You must include all three of these steps in your proofs!

divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect. Consequently, $\sqrt{2}$ is irrational. ■

A Word of Warning

- To attempt a proof by contradiction, make sure that what you're assuming actually is the opposite of what you want to prove!
- Otherwise, your **entire proof is invalid**.

An Incorrect Proof

Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

An Incorrect Proof

Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

Proof: By contradiction; assume that for any natural number n, the sum of all smaller natural numbers is equal to n. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our assumption was false and the theorem must be true. ■

An Incorrect Proof

Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

Proof: By contradiction; assume that for any natural number n, the sum of all smaller natural numbers is equal to n. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our assumption was false and the theorem must be true. ■

An Incorrect Proof

Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

Proof: By contradiction; assume that for any natural number n, the sum of all smaller natural numbers is equal to n. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4$ … eached a contradiction, so o… as false and the theorem must be t…

Is this really the opposite of the original statement?

The contradiction of the universal statement

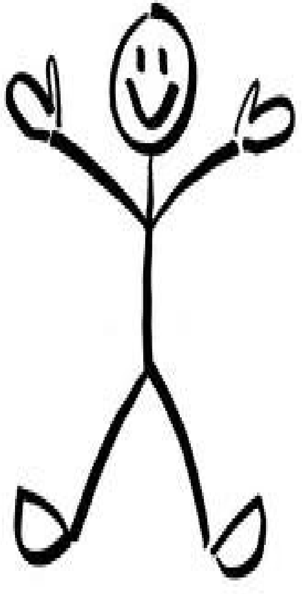
For all *x*, P(*x*) is true.

is **<u>not</u>**

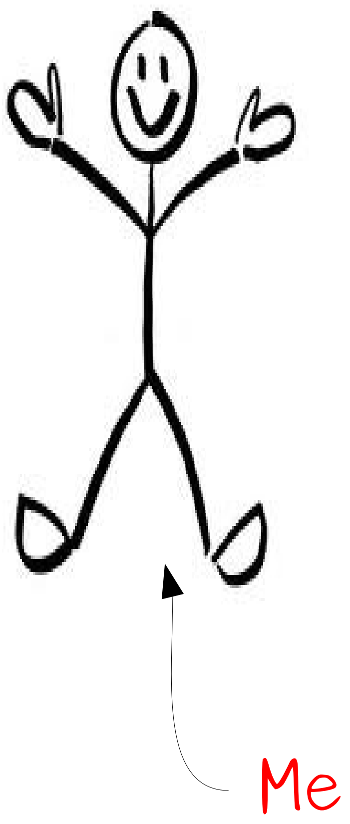
For all *x*, P(*x*) is false.

“All My Friends Are Taller Than Me”

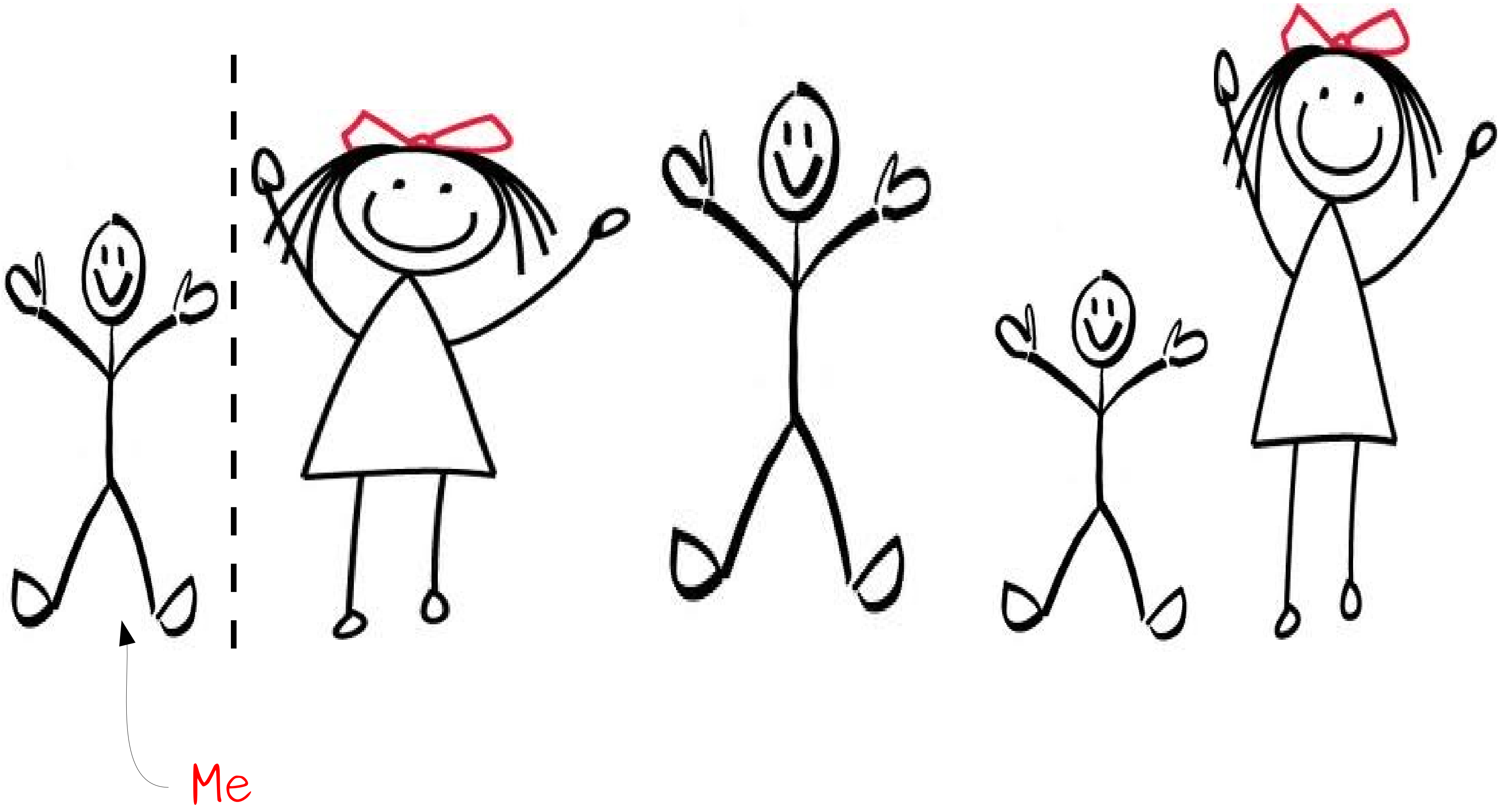
“All My Friends Are Taller Than Me”



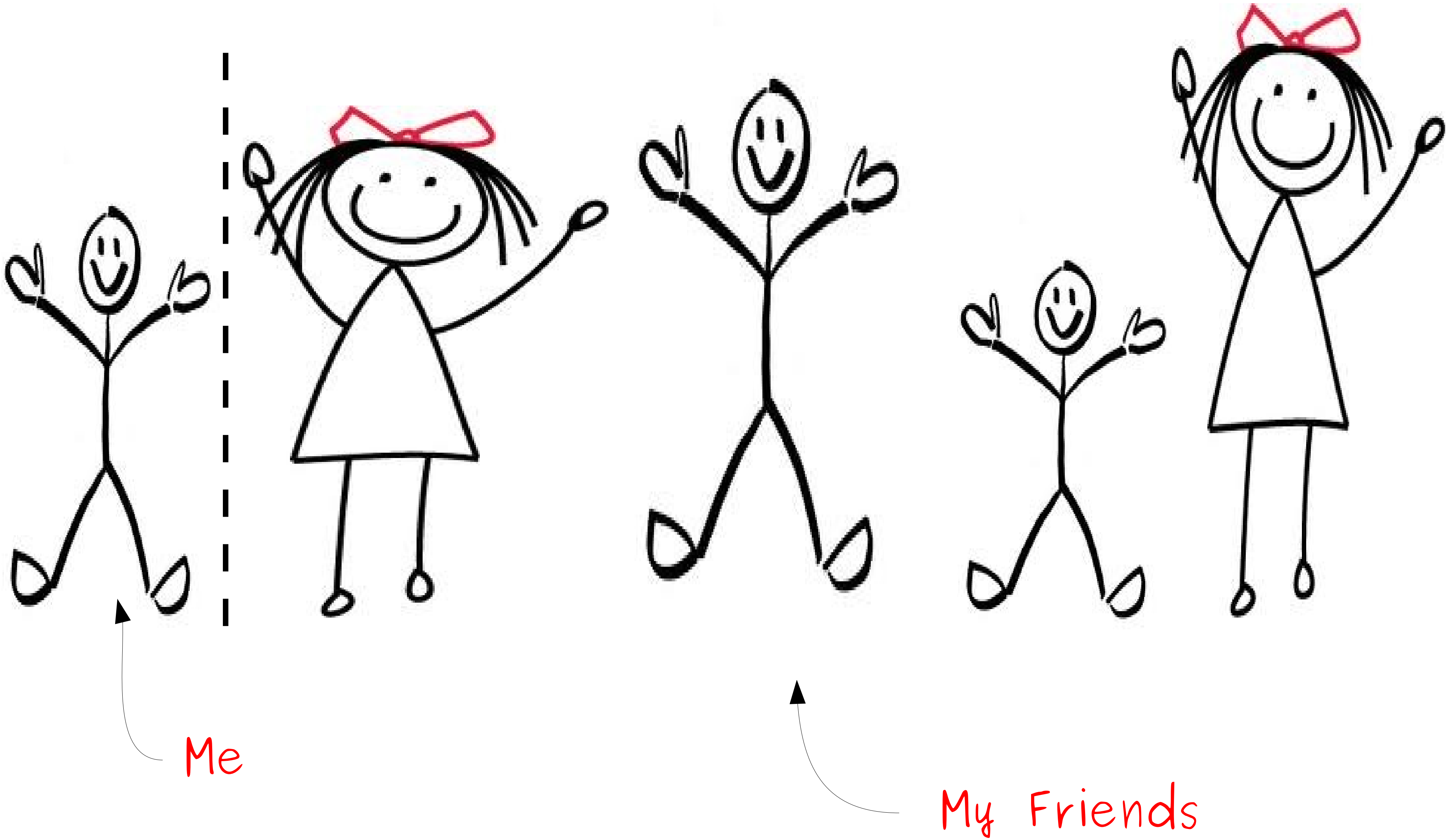
“All My Friends Are Taller Than Me”



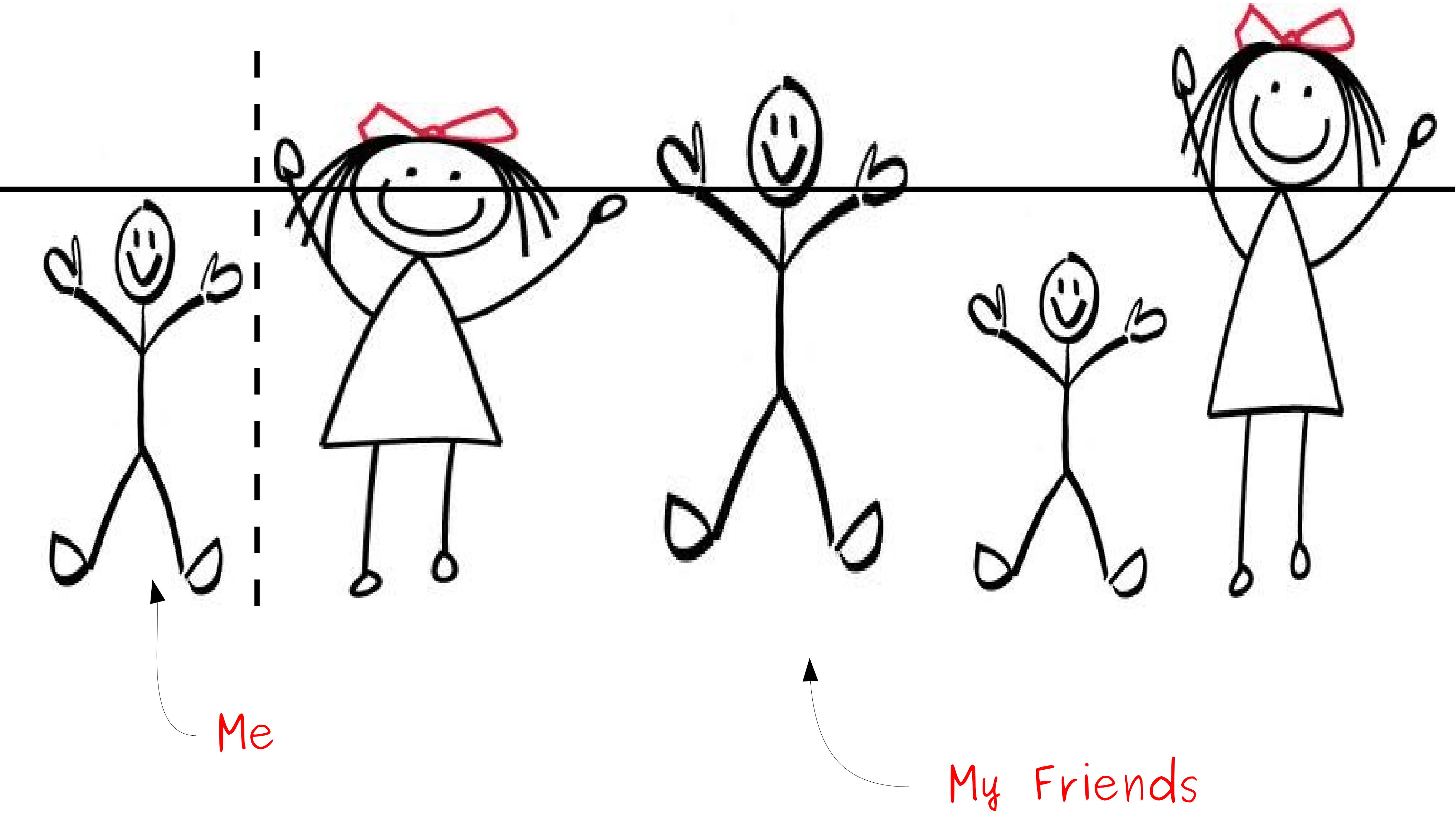
“All My Friends Are Taller Than Me”



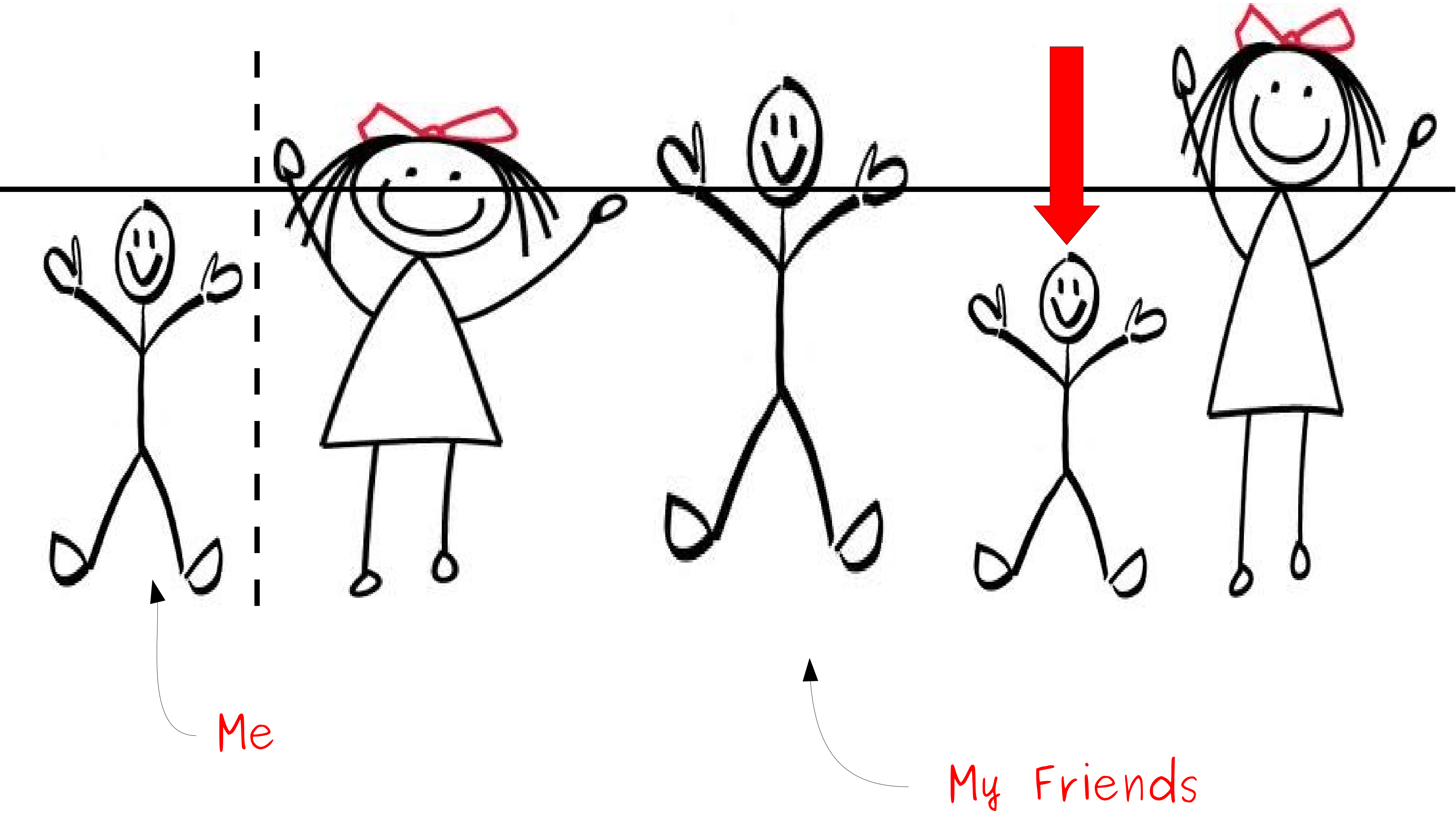
"All My Friends Are Taller Than Me"



“All My Friends Are Taller Than Me”



“All My Friends Are Taller Than Me”



The contradiction of the universal statement

For all x, P(x) is true.

is the existential statement

There exists an x such that P(x) is false.

The contradiction of the universal statement

For all *x*, P(*x*) is true.

is the existential statement

There exists an *x* such that P(*x*) is false.

For all natural numbers n,
the sum of all natural numbers
smaller than n is not equal to n.

For all natural numbers *n*,
the sum of all natural numbers
smaller than *n* is not equal to *n*.

For all natural numbers n,
the sum of all natural numbers
smaller than n is not equal to n.

becomes

For all natural numbers *n*,
the sum of all natural numbers
smaller than *n* is not equal to *n*.

becomes

There exists a natural number *n* such that
“the sum of all natural numbers
smaller than *n* is not equal to *n*”
is false.

For all natural numbers *n*,
the sum of all natural numbers
smaller than *n* is not equal to *n*.

becomes

There exists a natural number *n* such that
the sum of all natural numbers
smaller than *n* is equal to *n*

An Incorrect Proof

Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

Proof: By contradiction; assume that for any natural number n, the sum of all smaller natural numbers is equal to n. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our assumption was false and the theorem must be true. ■

An Incorrect Proof

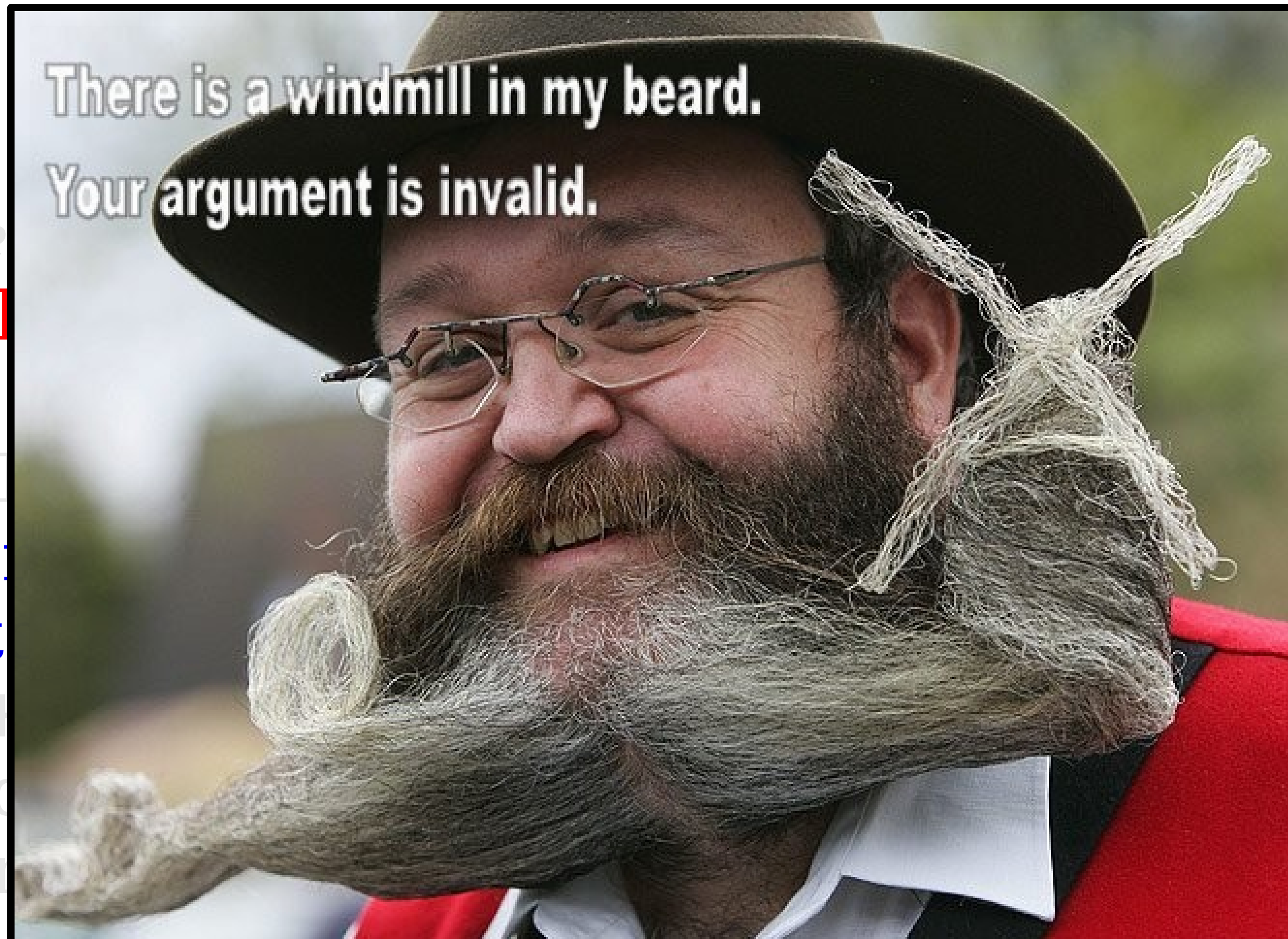
Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

Proof: By contradiction; assume that for any natural number n, the sum of all smaller natural numbers is equal to n. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our assumption was false and the theorem must be true. ■

An Incorrect Proof

Theore of all
natural

Proof: I natural
numbe bers is
equal t
$5 \neq 1 +$
contrac d the
theore



The contradiction of the existential statement

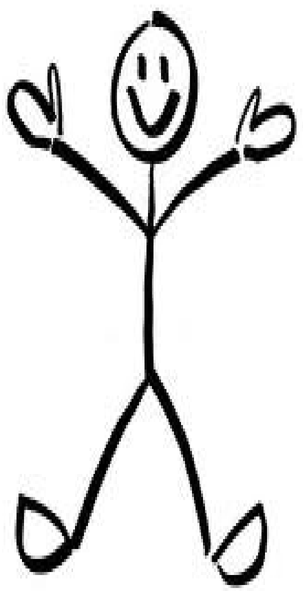
There exists an *x* such that P(*x*) is true.

is <u>**not**</u>

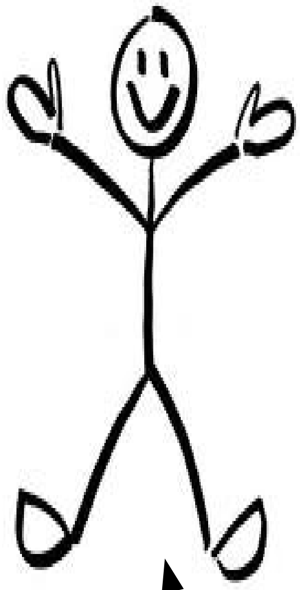
There exists an *x* such that P(*x*) is false.

“Some Friend Is Shorter Than Me”

“Some Friend Is Shorter Than Me”

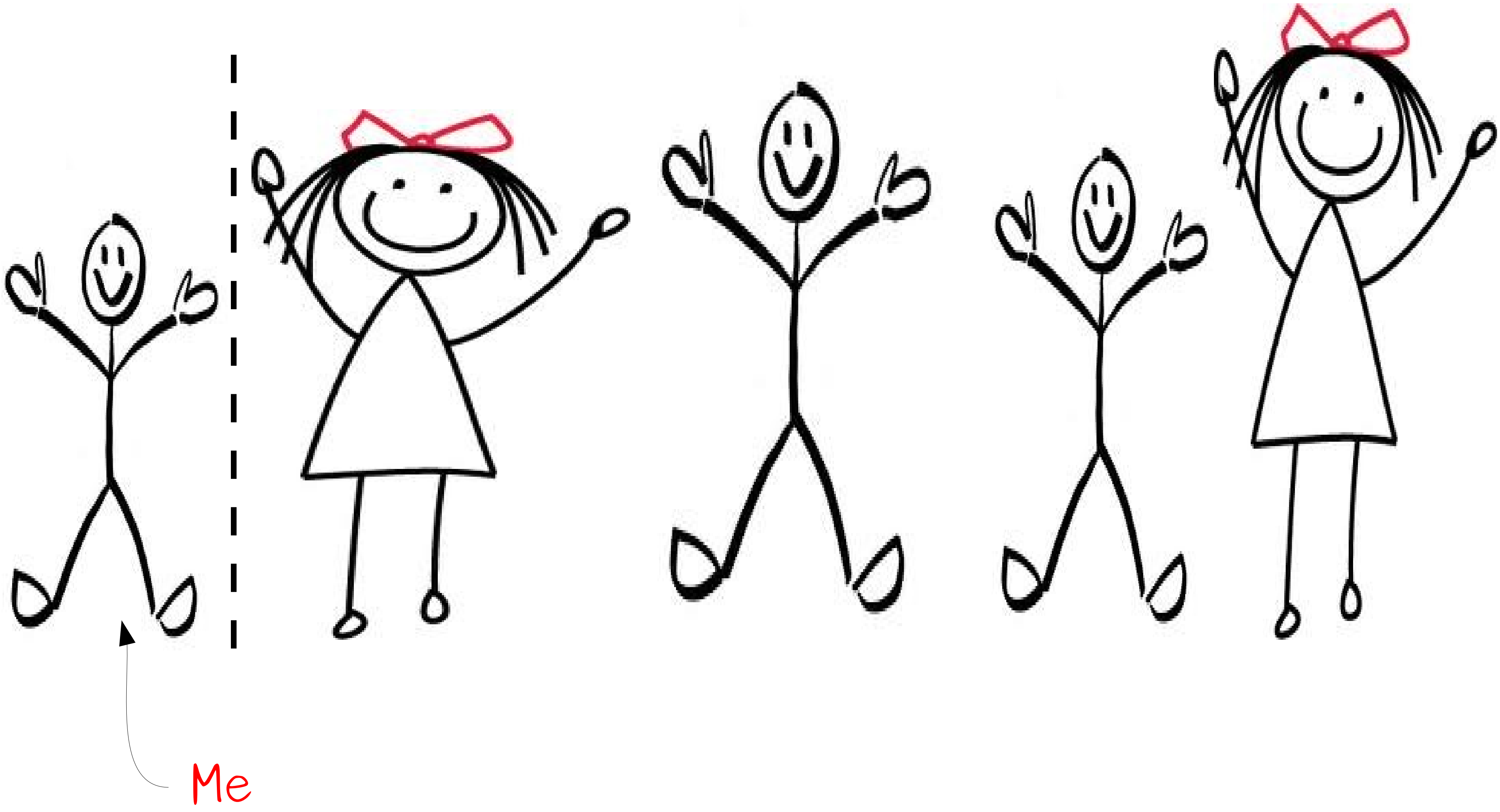


“Some Friend Is Shorter Than Me”

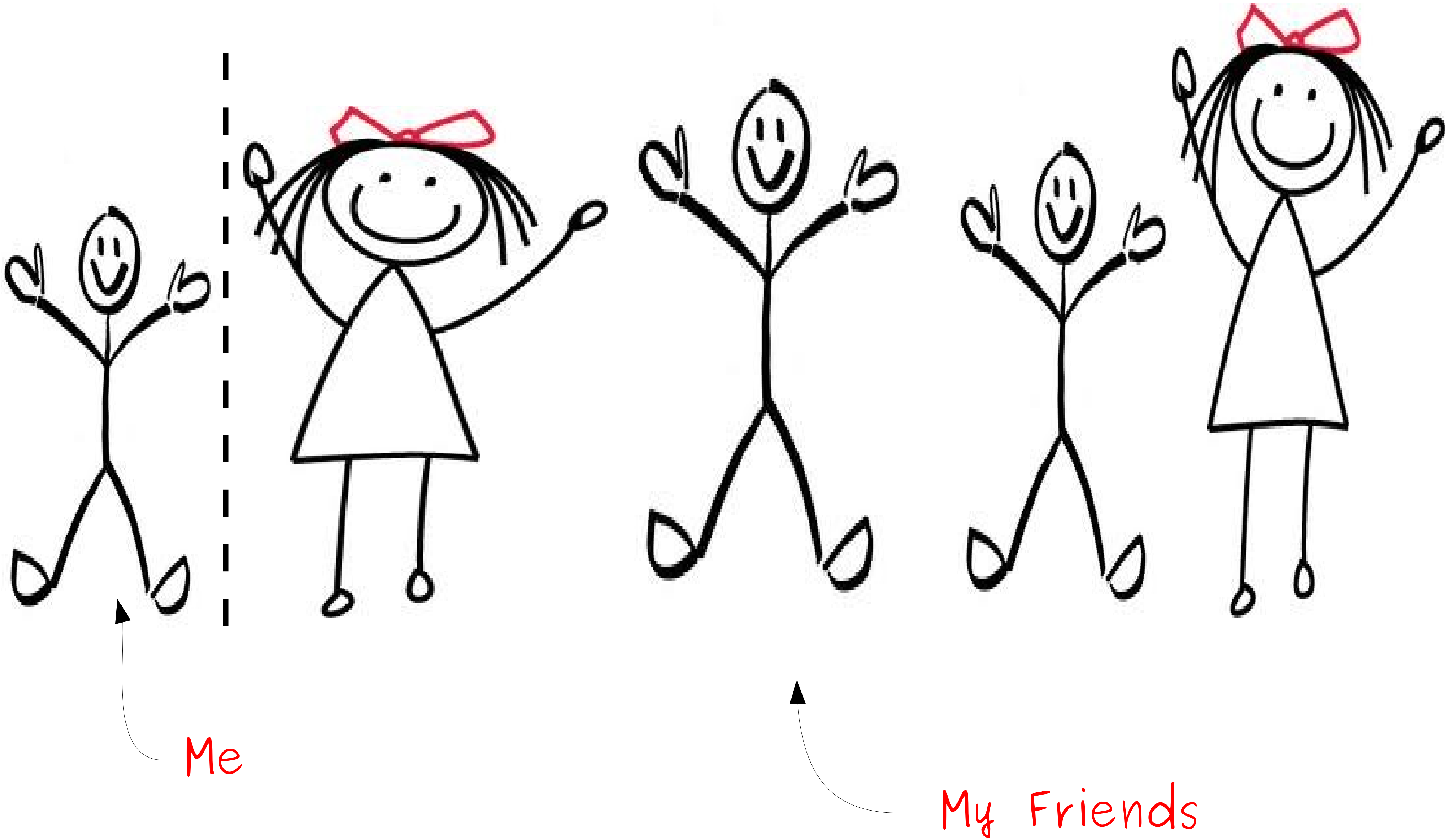


Me

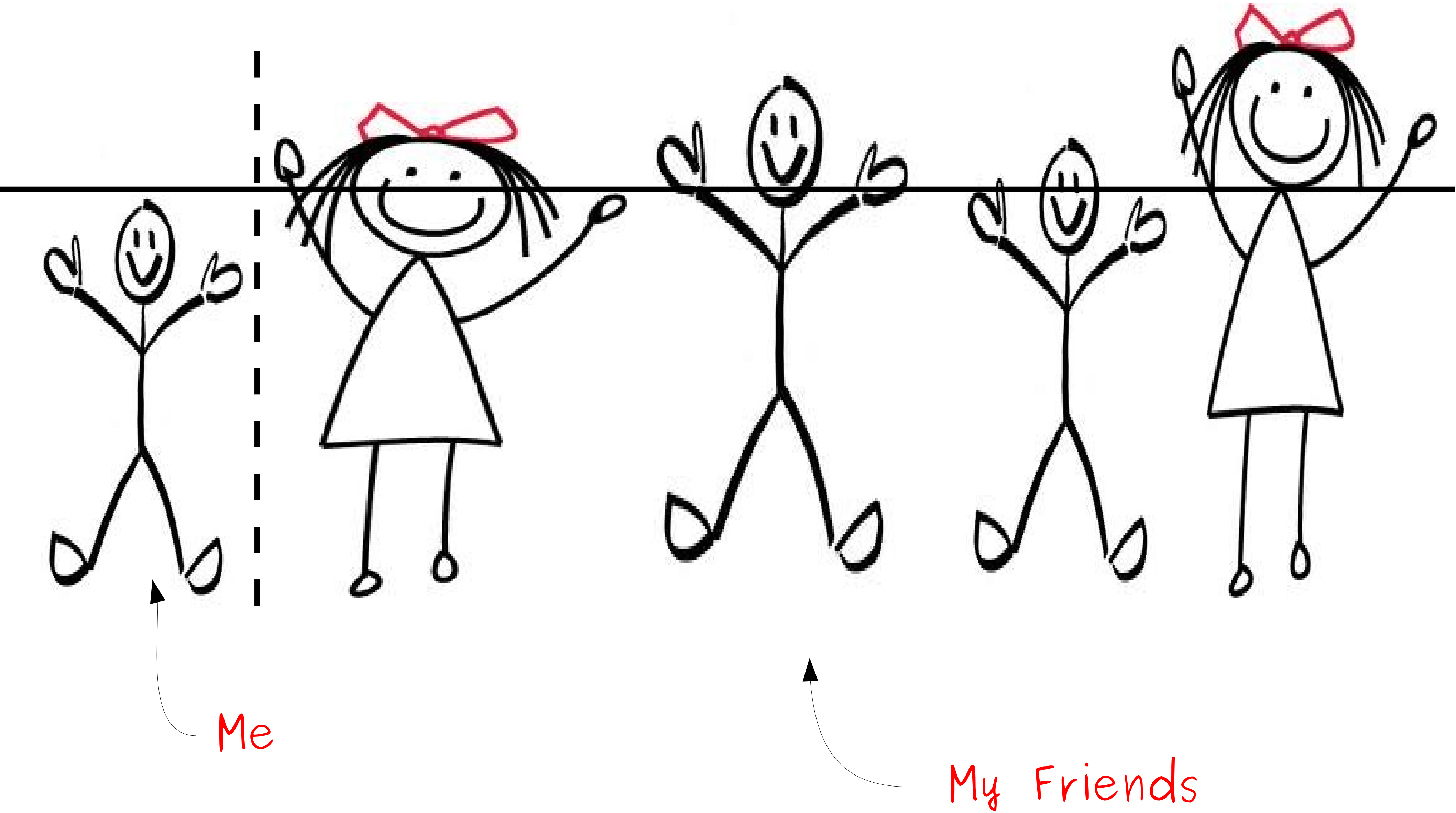
“Some Friend Is Shorter Than Me”



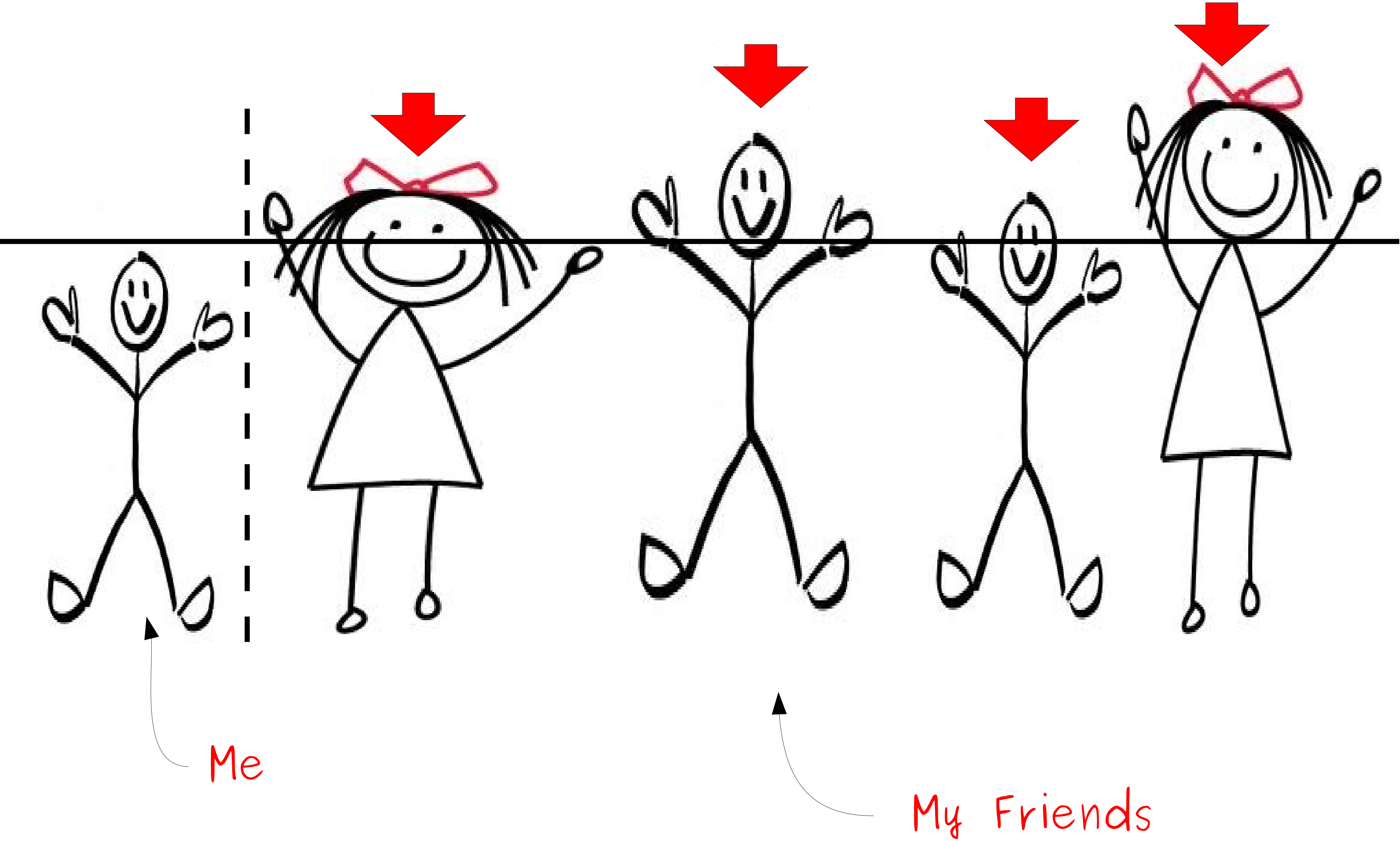
“Some Friend Is Shorter Than Me”



“Some Friend Is Shorter Than Me”



“Some Friend Is Shorter Than Me”



The contradiction of the existential statement

There exists an *x* such that P(*x*) is true.

is the universal statement

For all *x*, P(*x*) is false.

The contradiction of the existential statement

There exists an *x* such that P(*x*) is true.

is the universal statement

For all *x*, P(*x*) is false.

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer m, $m \leq n$.

Proof: By contradiction; assume that there exists an integer n such that for every integer m, $m > n$.

Since for any m, we have that $m > n$ is true, it should be true when $m = n - 1$. Thus $n - 1 > n$. But this is impossible, since $n - 1 < n$.

We have reached a contradiction, so our assumption was incorrect. Thus there exists an integer n such that for every integer m, $m \leq n$. ■

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer m, $m \leq n$.

Proof: By contradiction; assume that there exists an integer n such that for every integer m, $m > n$.

Since for any m, we have that $m > n$ is true, it should be true when $m = n - 1$. Thus $n - 1 > n$. But this is impossible, since $n - 1 < n$.

We have reached a contradiction, so our assumption was incorrect. Thus there exists an integer n such that for every integer m, $m \leq n$. ■

There exists an integer n such that for every integer m, $m \leq n$.

There exists an integer *n* such that for every integer *m*, *m* ≤ *n*.

becomes

There exists an integer n such that for every integer m, $m \leq n$.

becomes

For every integer n, "for every integer m, $m \leq n$" is false.

There exists an integer *n* such that for every integer *m*, *m* ≤ *n*.

becomes

For every integer *n*,
"for every integer *m*, *m* ≤ *n*" is false.

This statement is itself a universal statement! So let's use our existing techniques to find its negation.

There exists an integer n such that
for every integer m, $m \leq n$.

becomes

For every integer n,
"for every integer m, $m \leq n$" is false.

There exists an integer n such that
for every integer m, $m \leq n$.

becomes

For every integer n,
"for every integer m, $m \leq n$" is false.

For every integer m,
$m \leq n$

There exists an integer n such that for every integer m, $m \leq n$.

becomes

For every integer n, "for every integer m, $m \leq n$" is false.

For every integer m,

$m \leq n$

There exists an integer *n* such that
for every integer *m*, $m \leq n$.

becomes

For every integer *n*,
"for every integer *m*, $m \leq n$" is false.

For every integer *m*,
$m \leq n$

becomes

There exists an integer *m* such that
"$m \leq n$" is false.

There exists an integer *n* such that
for every integer *m*, *m* ≤ *n*.

becomes

For every integer *n*,
"for every integer *m*, *m* ≤ *n*" is false.

For every integer *m*,
$m \leq n$

becomes

There exists an integer *m* such that
m > n

There exists an integer *n* such that for every integer *m*, *m* ≤ *n*.

becomes

For every integer *n*, "for every integer *m*, *m* ≤ *n*" is false.

For every integer *m*, *m* ≤ *n*

becomes

There exists an integer *m* such that
m > n

There exists an integer *n* such that for every integer *m*, *m* ≤ *n*.

becomes

For every integer *n*,
"for every integer *m*, *m* ≤ *n*" is false.

Now that we have the negation, let's go replace it.

For every integer *m*,
m* ≤ *n

becomes

There exists an integer *m* such that
m > n

There exists an integer n such that
for every integer m, $m \leq n$.

becomes

For every integer n,
There exists an integer m such that
m > n

For every integer m,
$m \leq n$

becomes

There exists an integer m such that
m > n

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer m, $m \leq n$.

Proof: By contradiction; assume that there exists an integer n such that for every integer m, $m > n$.

Since for any m, we have that $m > n$ is true, it should be true when $m = n - 1$. Thus $n - 1 > n$. But this is impossible, since $n - 1 < n$.

We have reached a contradiction, so our assumption was incorrect. Thus there exists an integer n such that for every integer m, $m \leq n$. ■

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer m, $m \leq n$.

Proof: By contradiction; assume that there exists an integer n such that for every integer m, $m > n$.

Since for any m, we have that $m > n$ is true, it should be true when $m = n - 1$. Thus $n - 1 > n$. But this is impossible, since $n - 1 < n$.

We have reached a contradiction, so our assumption was incorrect. Thus there exists an integer n such that for every integer m, $m \leq n$. ■

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer m, $m \leq n$.

Proof: By contradiction; assume that there exists an integer n such that for every integer m, $m > n$.

Since for any m, we have that $m > n$ is true, it should be tr… $> n$. But this is in…

For every integer *n*,
There exists an integer *m* such that
m > n

We have rea… assumption was incorrect. Thus there exists an integer n such that for every integer m, $m \leq n$. ■

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer

Proof: By cont t there exists an integer n s ger m, $m > n$.

Since for any n is true, it should be tr $> n$. But this is in

T ch that

We have rea assumption w re exists an integer n such r m, $m \leq n$. ■

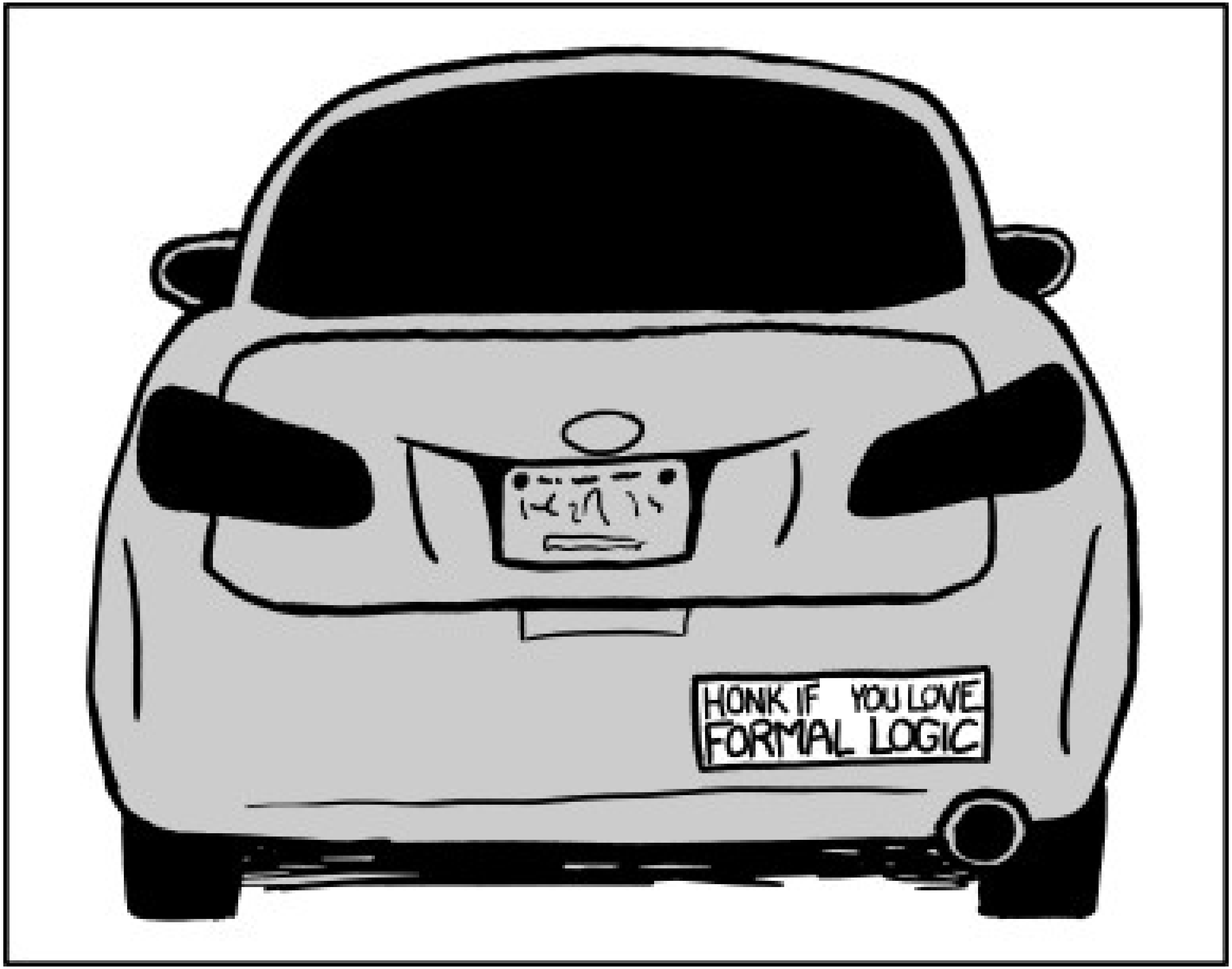
The Story So Far

For CS106B Students

- I will be holding a recap session M/W/F from 4:15 – 4:30 in my office (Gates 178) to recap the last fifteen minutes of lecture.
- Feel free to stop on by!

Proof by Contrapositive

Honk **if** You Love Formal Logic



Honk **if** You Love Formal Logic

The Contrapositive

- The **contrapositive** of “If P, then Q” is the statement “If **not** Q, then **not** P.”
- Example:
  - “If I stored the cat food inside, then the raccoons wouldn't have stolen my cat food.”
  - Contrapositive: “If the raccoons stole my cat food, then I didn't store it inside.”
- Another example:
  - “If I had been a good test subject, then I would have received cake.”
  - Contrapositive: “If I didn't receive cake, then I wasn't a good test subject.”

Notation

- Recall that we can write "If P, then Q" as $P \to Q$.
- Notation: We write "not P" as $\boldsymbol{\neg P}$.
- Examples:
  - "If P is false, then Q is true:" $\neg P \to Q$
  - "Q is false whenever P is false:" $\neg P \to \neg Q$
- The contrapositive of $P \to Q$ is $\boldsymbol{\neg Q \to \neg P}$.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

An Important Result

Theorem: If $\neg Q \rightarrow \neg P$, then $P \rightarrow Q$.
Proof: By contradiction.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.
Proof: By contradiction. ???

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.
Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false.

An Important Result

Theorem: If $\neg Q \rightarrow \neg P$, then $P \rightarrow Q$.

Proof: By contradiction. Assume that $\neg Q \rightarrow \neg P$, but that $P \rightarrow Q$ is false. Since $P \rightarrow Q$ is false, it must be true that P is true and Q is false.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.
Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and Q is false.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and $\neg Q$ is true.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.
Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and $\neg Q$ is true.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and $\neg Q$ is true. Since $\neg Q$ is true and $\neg Q \to \neg P$, we know that $\neg P$ is true.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and $\neg Q$ is true. Since $\neg Q$ is true and $\neg Q \to \neg P$, we know that $\neg P$ is true. But this means that we have shown P and $\neg P$, which is impossible.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and $\neg Q$ is true. Since $\neg Q$ is true and $\neg Q \to \neg P$, we know that $\neg P$ is true. But this means that we have shown P and $\neg P$, which is impossible. We have reached a contradiction, so if $\neg Q \to \neg P$, then $P \to Q$.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and $\neg Q$ is true. Since $\neg Q$ is true and $\neg Q \to \neg P$, we know that $\neg P$ is true. But this means that we have shown P and $\neg P$, which is impossible. We have reached a contradiction, so if $\neg Q \to \neg P$, then $P \to Q$. ■

An Important Proof Strategy

To show that $P \rightarrow Q$, you may instead show that $\neg Q \rightarrow \neg P$.

This is called a **proof by contrapositive**.

Theorem: If n^2 is even, then n is even.

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive;

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; **???**

If

n^2 is even

then

n is even

If

n^2 is even

then

n is even

If

If

n^2 is even

then

n is even

If

n is odd

If

n^2 is even

then

n is even

If

n is odd

then

If

n^2 is even

then

n is even

If

n is odd

then

n^2 is odd

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; **???**

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$n^2 = (2k + 1)^2$$

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$
\begin{aligned}
n^2 &= (2k + 1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1.
\end{aligned}
$$

Since $(2k^2 + 2k)$ is an integer, n^2 is odd.

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Since $(2k^2 + 2k)$ is an integer, n^2 is odd. ■

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$n^2 = (2k + 1)^2$$
$$n^2 = 4k^2 + 4k + 1$$
$$n^2 = 2(2k^2 + 2k) + 1.$$

Since $(2k^2 + 2k)$ is an integer, n^2 is odd. ■

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integ

$n^2 =$

$n^2 =$

$n^2 =$

Since s odd. ■

Notice the structure of the proof. We begin by announcing that it's a proof by contrapositive, then state the contrapositive, and finally prove it.

An Incorrect Proof

Theorem: For any sets A and B, if $x \notin A \cap B$, then $x \notin A$.

An Incorrect Proof

Theorem: For any sets A and B, if $x \notin A \cap B$, then $x \notin A$.

Proof: By contrapositive; we show that if $x \in A \cap B$, then $x \in A$.

Since $x \in A \cap B$, $x \in A$ and $x \in B$. Consequently, $x \in A$ as required. ■

An Incorrect Proof

Theorem: For any sets A and B,
if $x \notin A \cap B$, then $x \notin A$.

Proof: By contrapositive; we show that
if $x \in A \cap B$, then $x \in A$.

Since $x \in A \cap B$, $x \in A$ and $x \in B$.
Consequently, $x \in A$ as required. ■

An Incorrect Proof

Theorem:

Proof: ow that

nd $x \in B$.

equired. ■

Common Pitfalls

To prove P → Q by contrapositive, show that

$$\boldsymbol{\neg Q \to \neg P}$$

Do not show that

$$\boldsymbol{\neg P \to \neg Q}$$

Common Pitfalls

To prove $P \to Q$ by contrapositive, show that

$$\neg Q \to \neg P$$

Do not show that

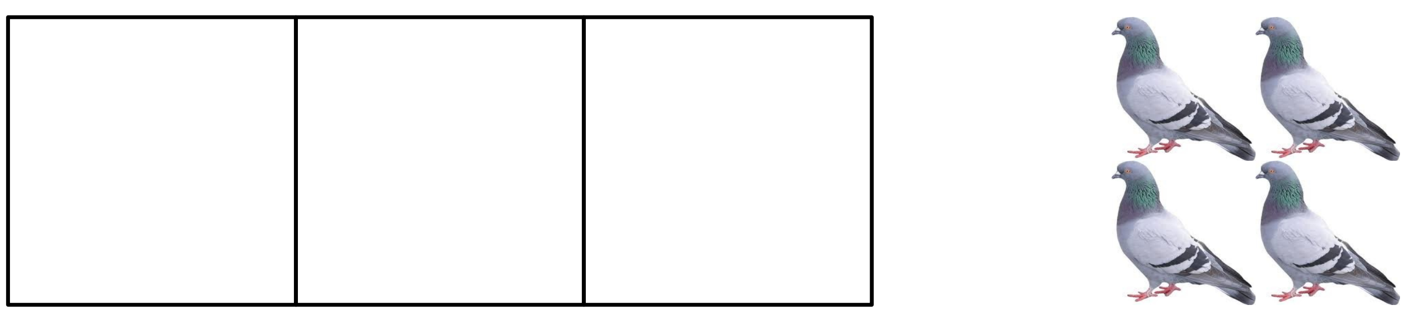
$$\neg P \to \neg Q$$

(Showing $\neg P \to \neg Q$ proves that $Q \to P$, not the other way around!)

The Pigeonhole Principle

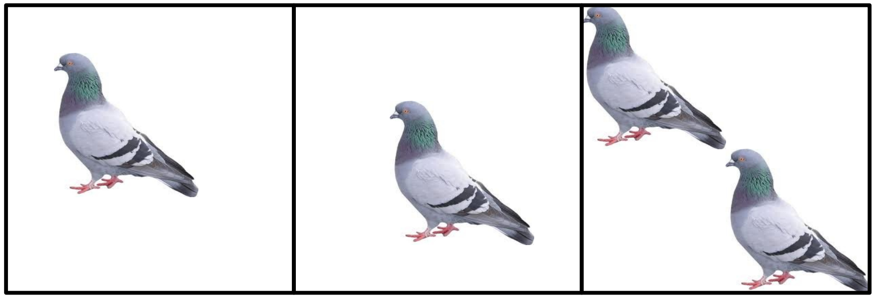
The Pigeonhole Principle

- Suppose that you have n pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



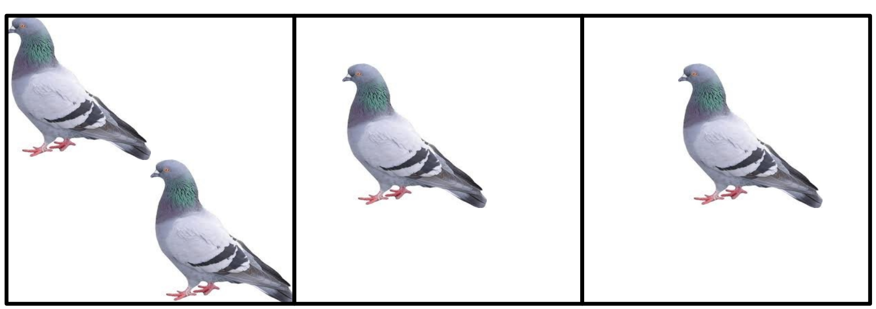
The Pigeonhole Principle

- Suppose that you have n pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



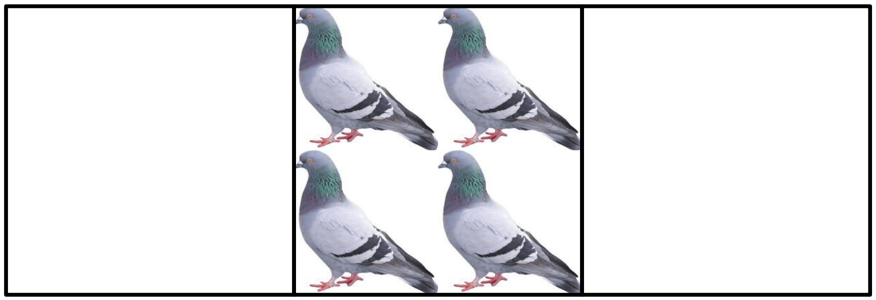
The Pigeonhole Principle

- Suppose that you have *n* pigeonholes.
- Suppose that you have *m* > *n* pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



The Pigeonhole Principle

- Suppose that you have n pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive;

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; **???**

If

$m > n$

then

there is some bin containing at least two objects

If

$m > n$

then

there is some bin containing at least two objects

If

$m > n$

then

there is some bin containing at least two objects

If

If

$m > n$

then

there is some bin containing at least two objects

If

"there is some bin containing at least two objects" is false

If

$m > n$

then

there is some bin containing at least two objects

If

every bin does not contain at least two objects

If

$m > n$

then

there is some bin containing at least two objects

If

every bin contains at most one object

If

$m > n$

then

there is some bin containing at least two objects

If

every bin contains at most one object

then

If

$m > n$

then

there is some bin containing at least two objects

If

every bin contains at most one object

then

$m \leq n$

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i.

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Since every bin has at most one object, $x_i \leq 1$ for all i.

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Since every bin has at most one object, $x_i \leq 1$ for all i. Thus

$$m = \sum_{i=1}^{n} x_i$$

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Since every bin has at most one object, $x_i \leq 1$ for all i. Thus

$$m = \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} 1$$

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Since every bin has at most one object, $x_i \leq 1$ for all i. Thus

$$m = \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} 1 = n$$

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Since every bin has at most one object, $x_i \leq 1$ for all i. Thus

$$m = \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} 1 = n$$

So $m \leq n$, as required.

Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Since every bin has at most one object, $x_i \leq 1$ for all i. Thus

$$m = \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} 1 = n$$

So $m \leq n$, as required. ■

Using the Pigeonhole Principle

- The pigeonhole principle is an enormously useful lemma in many proofs.
  - If we have time, we'll spend a full lecture on it in a few weeks.
- General structure of a pigeonhole proof:
  - Find m objects to distribute into n buckets, with $m > n$.
  - Using the pigeonhole principle, conclude that some bucket has at least two objects in it.
  - Use this conclusion to show the desired result.

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,000 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

Next Time

- **Proof by Induction**
  - Proofs on sums, programs, algorithms, etc.