Indirect Proofs

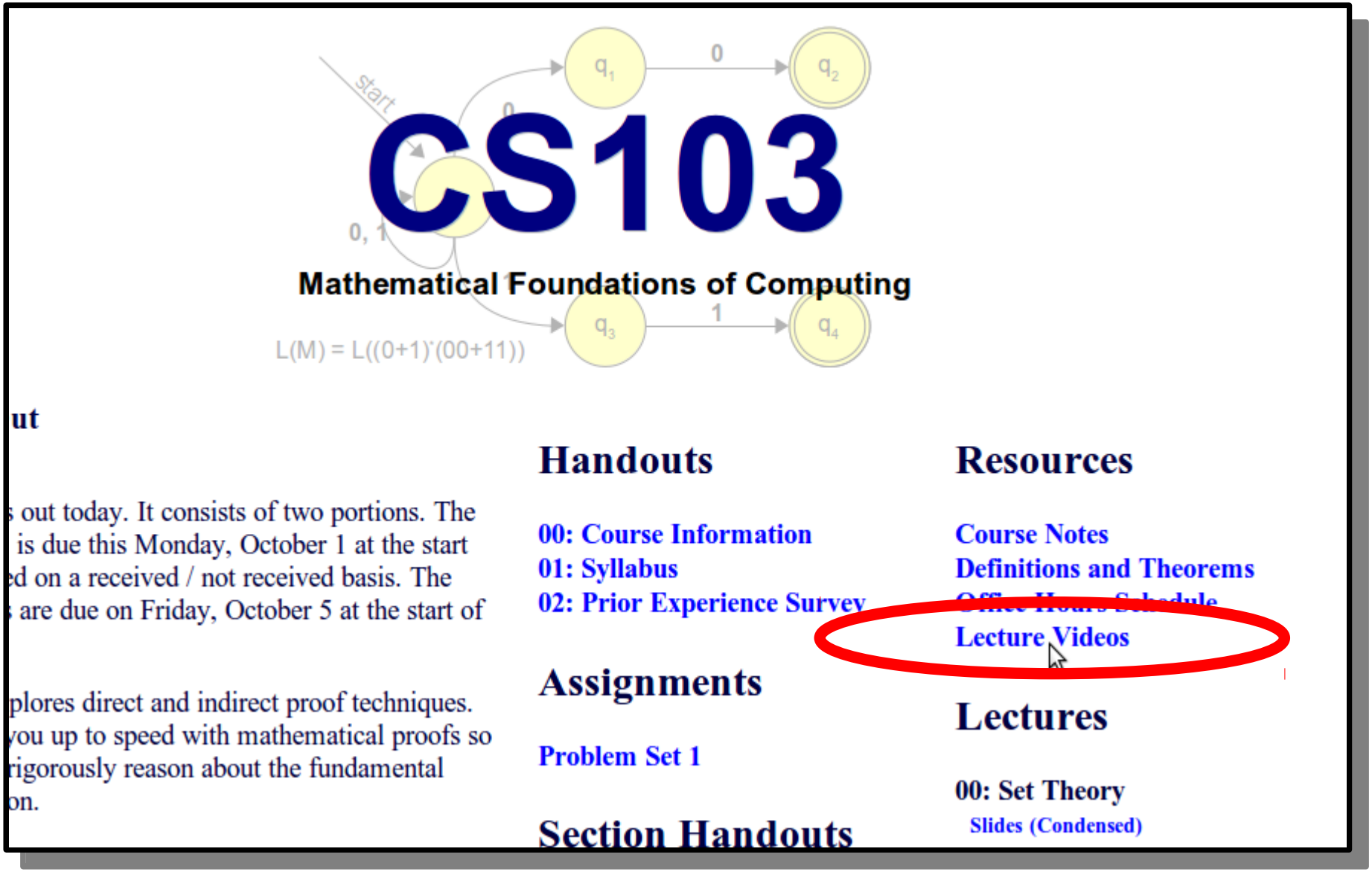
Announcements

- Problem Set 1 out.
- **Checkpoint** due Monday, October 1.
  - Graded on a “did you turn it in?” basis.
  - We will get feedback back to you with comments on your proof technique and style.
  - The more an effort you put in, the more you'll get out.
- **Remaining problems** due Friday, October 5.
  - Feel free to email us with questions!

Submitting Assignments

- You can submit assignments by
  - handing them in at the start of class,
  - dropping it off in the filing cabinet near Keith's office (details on the assignment handouts), or
  - emailing the submissions mailing list at cs103-aut1213-submissions@lists.stanford.edu and attaching your solution as a PDF.
- Late policy:
  - Three 72-hour “late days.”
  - Can use at most one per assignment.
  - No work accepted more than 72 hours after due date.

Lecture Videos



http://class.stanford.edu/cs103/Fall2012/videos

Office hours start today.

Schedule available on the course website.

Friday Four Square

- Good snacks!
- Good company!
- Good game!
- Good fun!
- **Today at 4:15 in front of Gates.**

Outline for Today

- Logical Implication
  - What does “If P, then Q” mean?
- Proof by Contradiction
  - The basic method.
  - Contradictions and implication.
  - Contradictions and quantifiers.
- Proof by Contrapositive
  - The basic method.
  - An interesting application.

Logical Implication

Implications

- An **implication** is a statement of the form

  If *P*, then *Q*.

- We write "If P, then Q" as $P \to Q$.
  - Read: "P implies Q."
- When $P \to Q$, we call P the **antecedent** and Q the **consequent**.

What does Implication Mean?

- The statement $P \to Q$ means exactly the following:

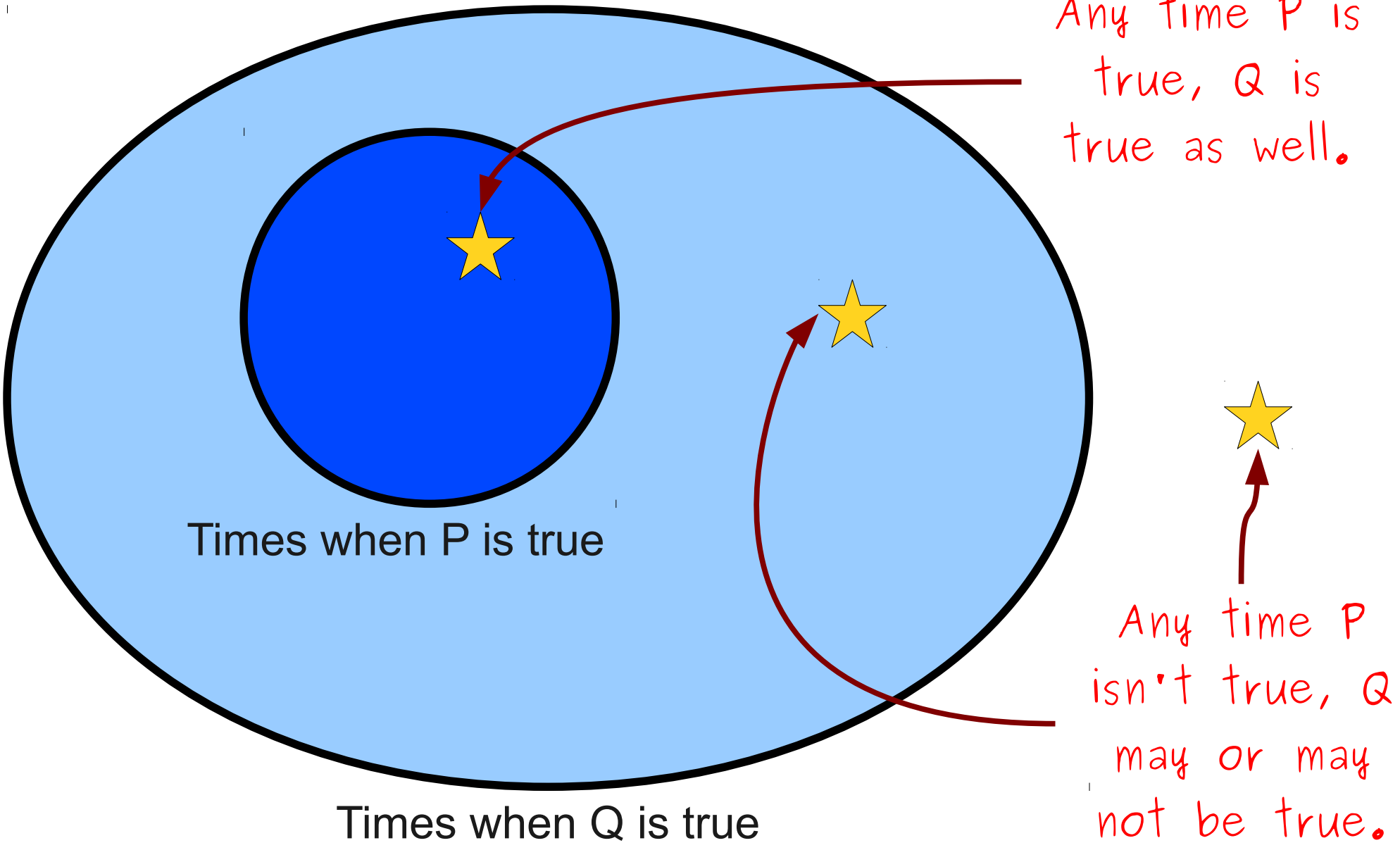
Whenever *P* is true,
Q must be true as well.

- For example:
  - n is even $\to$ n^2 is even.
  - $(A \subseteq B$ and $B \subseteq C) \to A \subseteq C$

What does Implication **Not** Mean?

- $P \to Q$ does **not** mean that whenever Q is true, P is true.
  - "If you are a Stanford student, you wear cardinal" does **not** mean that if you wear cardinal, you are a Stanford student.
- $P \to Q$ does **not** say anything about what happens if P is false.
  - "If you hit another skier, you're gonna have a bad time" doesn't mean that if you don't hit other skiers, you're gonna to have a good time.
  - **Vacuous truth:** If P is never true, then $P \to Q$ is always true.
- $P \to Q$ does **not** say anything about causality.
  - "If I want math to work, then 2 + 2 = 4" is true because any time that I want math to work, 2 + 2 = 4 already was true.
  - "If I don't want math to work, then 2 + 2 = 4" is also true, since whenever I don't want math to work, 2 + 2 = 4 is true.

Implication, Diagrammatically



Alternative Forms of Implication

- All of the following are different ways of saying $P \to Q$:

If *P*, then *Q*.

***P* implies *Q*.**

***P* only if *Q*.**

***Q* whenever *P*.**

***P* is sufficient for *Q*.**

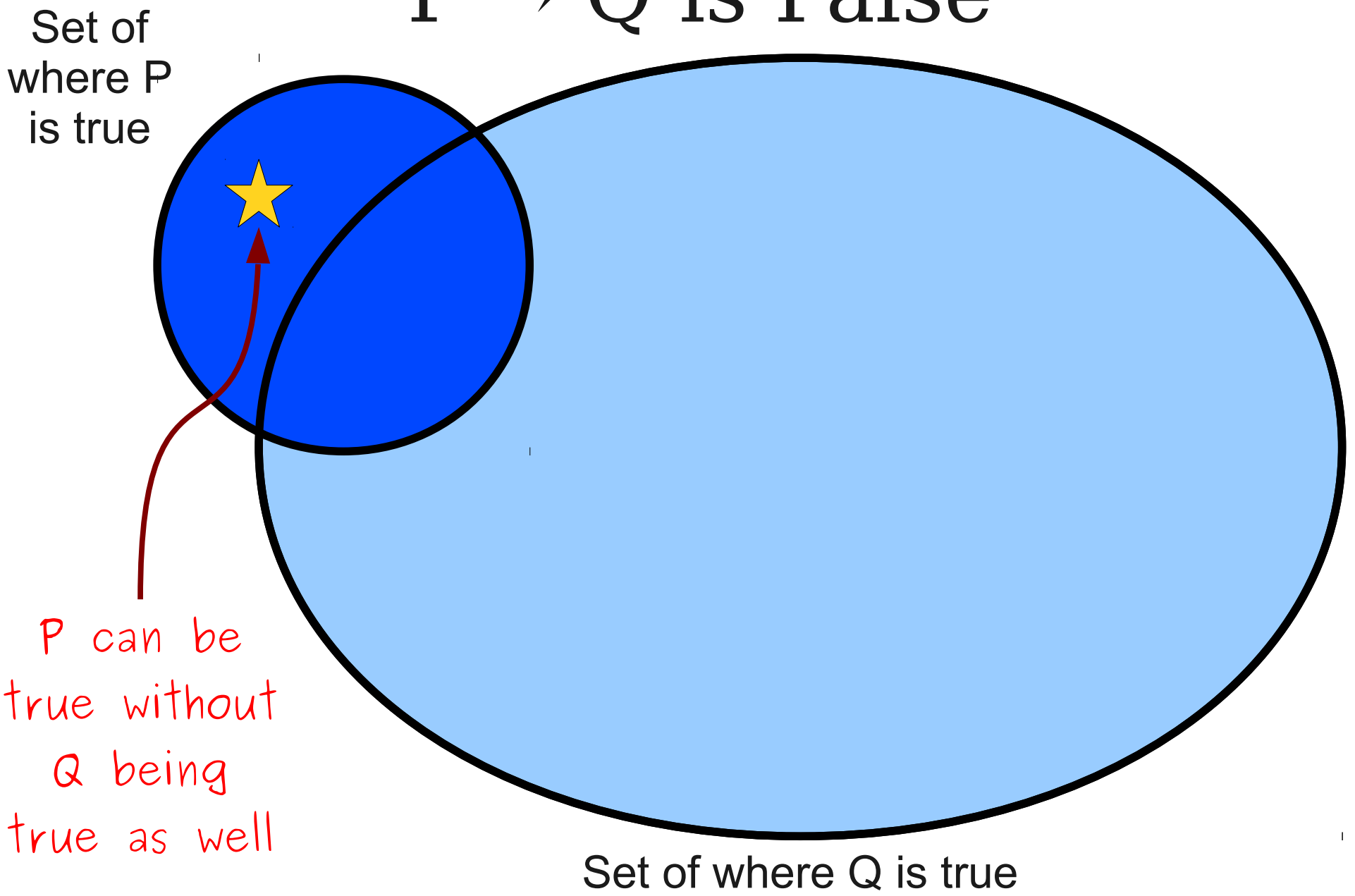
***Q* is necessary for *P*.**

- Why?

When P Does Not Imply Q

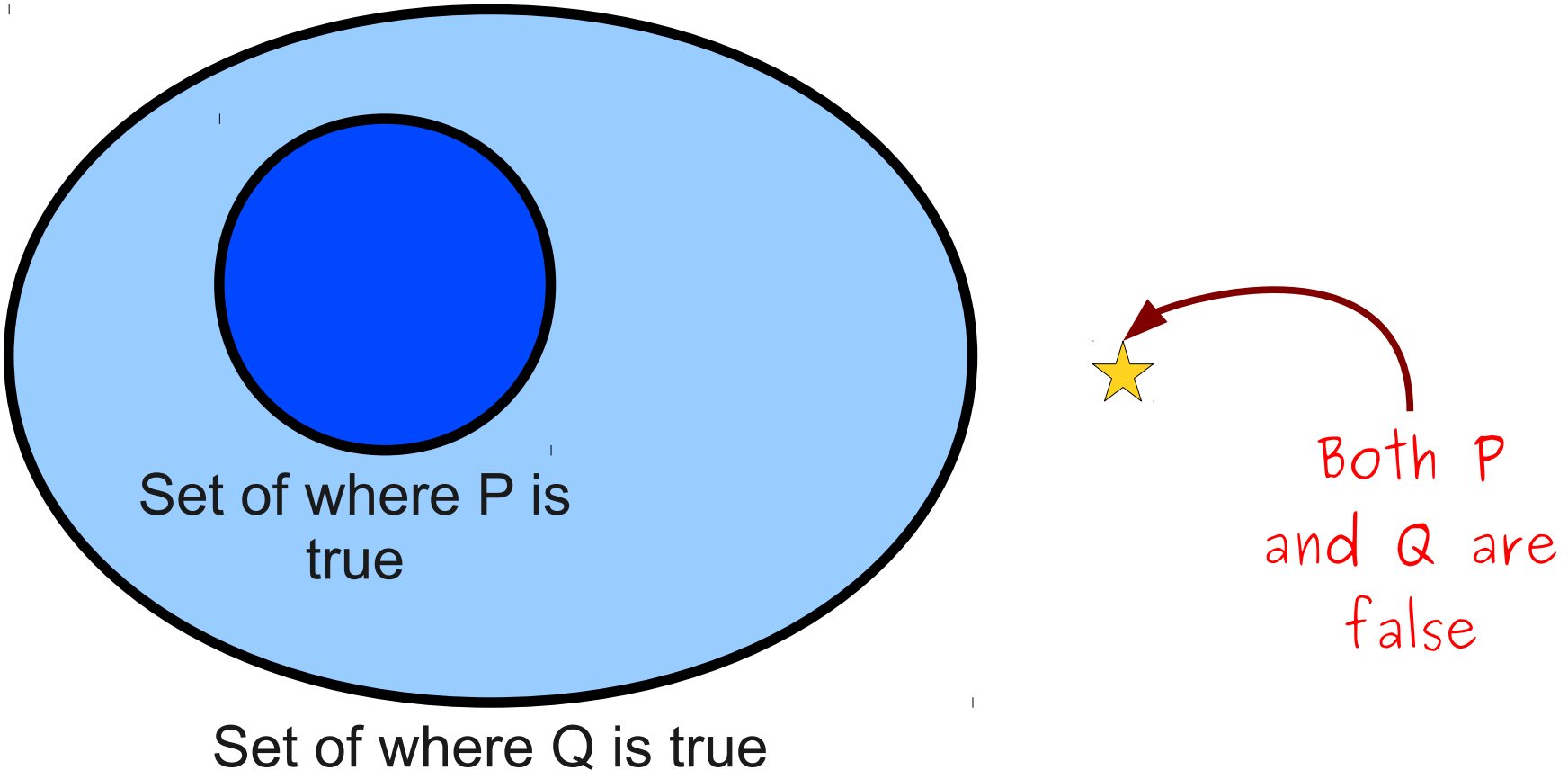
- What would it mean for $P \to Q$ to be false?
- **Answer**: There must be some way for P to be true and Q to be false.
- $P \to Q$ means "any time P is true, Q is true."
  - The only way to disprove this is to show that there is some way for P to be true and Q to be false.
- To prove that $P \to Q$ is false, find an example of where P is true and Q is false.

$P \rightarrow Q$ is False



A Common Mistake

- To show that $P \to Q$ is false, it is **not** sufficient to find a case where P is false and Q is false.



Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A **proof by contradiction** is a proof that works as follows:
  - To prove that P is true, assume that P is not true.
  - Based on the assumption that P is not true, conclude something impossible.
  - Assuming the logic is sound, the only option is that the assumption that P is not true is incorrect.
  - Conclude, therefore, that P is true.

Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that $P \to Q$ is false.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

Contradictions and Implications

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that ***P* is true and *Q* is false**.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.
Proof: By contradiction; assume n^2 is even but n is odd.

Since n is odd, $n = 2k + 1$ for some integer k.

Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Now, let $m = 2k^2 + 2k$. Then $n^2 = 2m + 1$, so by definition n^2 is odd. But this is impossible, since n^2 is even.

We have reached a contradiction, so our assumption was false. Thus if n^2 is even, n is even as well. ■

A Simple Proof by Contradiction

Theorem: If n^2 is even, then n is even.

Proof: By contradiction; assume n^2 is even but n is odd.

The three key pieces:

1. State that the proof is by contradiction.
2. State what the negation of the original statement is.
3. State you have reached a contradiction and what the contradiction entails.

You must include all three of these steps in your proofs!

We have reached a contradiction, so our assumption was false. Thus if n^2 is even, n is even as well. ■

Biconditionals

- Combined with what we saw on Wednesday, we have proven

  If n is even, n^2 is even.

  If n^2 is even, n is even.

- We sometimes write this as

  n is even **if and only if** n^2 is even.

- This is often abbreviated

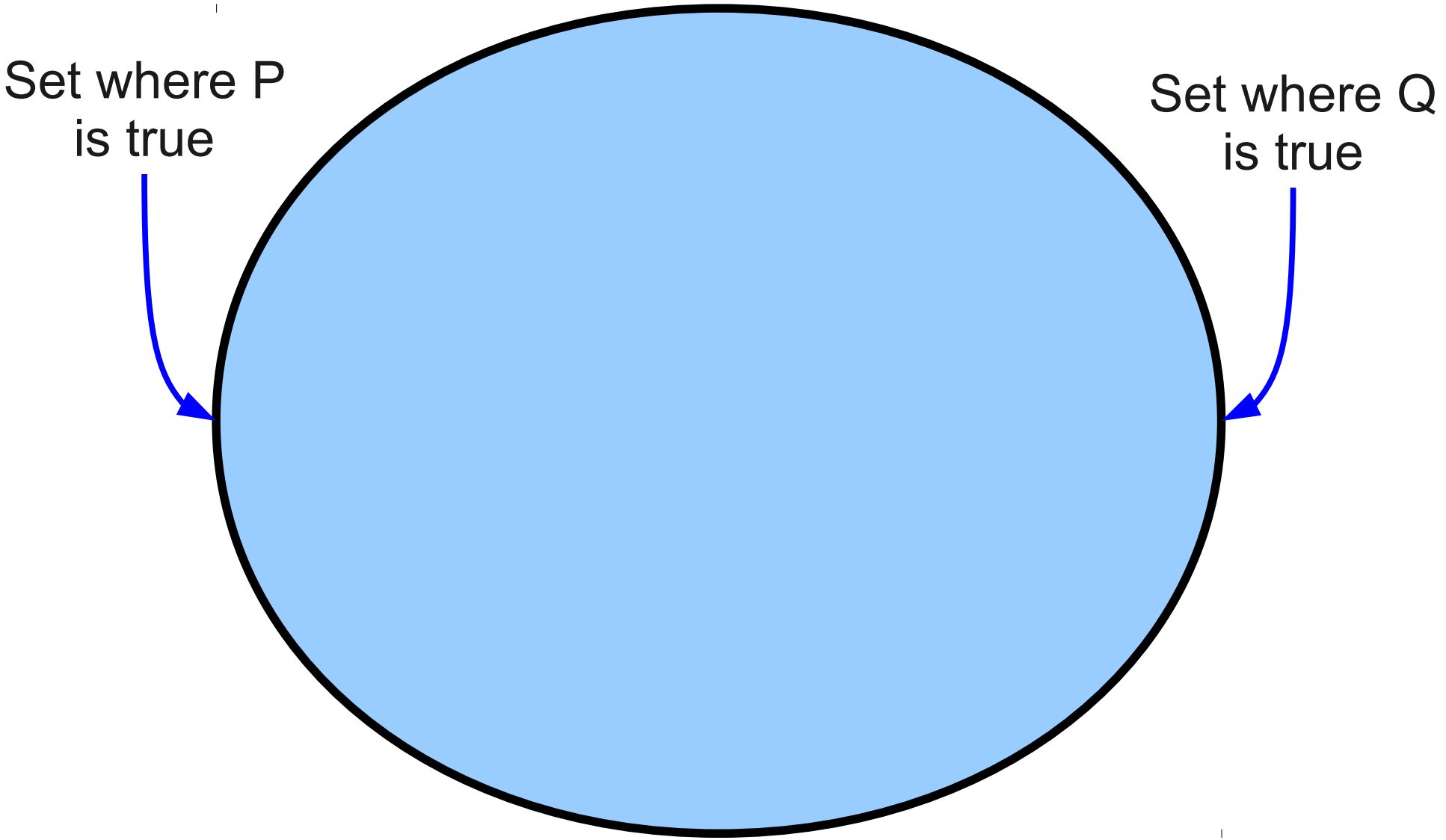
  n is even **iff** n^2 is even.

  or as

  n is even $\leftrightarrow$ n^2 is even

- This is called a **biconditional**.

$$P \leftrightarrow Q$$



Proving Biconditionals

- To prove ***P* iff *Q***, you need to prove that
  - ***P* → *Q***, and
  - ***Q* → *P***.
- You may use any proof techniques you'd like when doing so.
  - In our case, we used a direct proof and a proof by contradiction.
- **Just make sure to prove both directions of implication**!

Rational and Irrational Numbers

Rational and Irrational Numbers

- A **rational number** is a number r that can be written as

$$r = \frac{p}{q}$$

  where
  - p and q are integers,
  - $q \neq 0$, and
  - p and q have no common divisors other than ± 1.
- A number that is not rational is called **irrational**.

A Famous and Beautiful Proof

Theorem: $\sqrt{2}$ is irrational.

Proof: By contradiction; assume $\sqrt{2}$ is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{2}$, and p and q have no common divisors other than 1 and -1.

Since $p / q = \sqrt{2}$ and $q \neq 0$, we have $p = \sqrt{2}\, q$, so $p^2 = 2q^2$.

Since q^2 is an integer and $p^2 = 2q^2$, we have that p^2 is even. By our earlier result, since p^2 is even, we know p is even. Thus there is an integer k such that $p = 2k$.

Therefore, $2q^2 = p^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$.

Since k^2 is an integer and $q^2 = 2k^2$, we know q^2 is even. By our earlier result, since q^2 is even, we have that q is even. But this means that both p and q have 2 as a common divisor. This contradicts our earlier assertion that their only common divisors are 1 and -1.

We have reached a contradiction, so our assumption was incorrect. Consequently, $\sqrt{2}$ is irrational. ■

A Famous and Beautiful Proof

Theorem: √2 is irrational.

Proof: By contradiction; assume √2is rational. Then there exists integers p and q such that $q \neq 0$, $p / q = \sqrt{}$, and p and q have no common divisors other than 1 and -1.

The three key pieces:

1. State that the proof is by contradiction.
2. State what the negation of the original statement is.
3. State you have reached a contradiction and what the contradiction entails.

You must include all three of these steps in your proofs!

We have reached a contradiction, so our assumption was incorrect. Consequently, √2 is irrational. ∎

A Word of Warning

- To attempt a proof by contradiction, make sure that what you're assuming actually is the opposite of what you want to prove!
- Otherwise, your **entire proof is invalid**.

An Incorrect Proof

Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

Proof: By contradiction; assume that for any natural number n, the sum of all smaller natural numbers is equal to n. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our assumption was false and the theorem must be true. ■

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Is this really the opposite of the original statement?

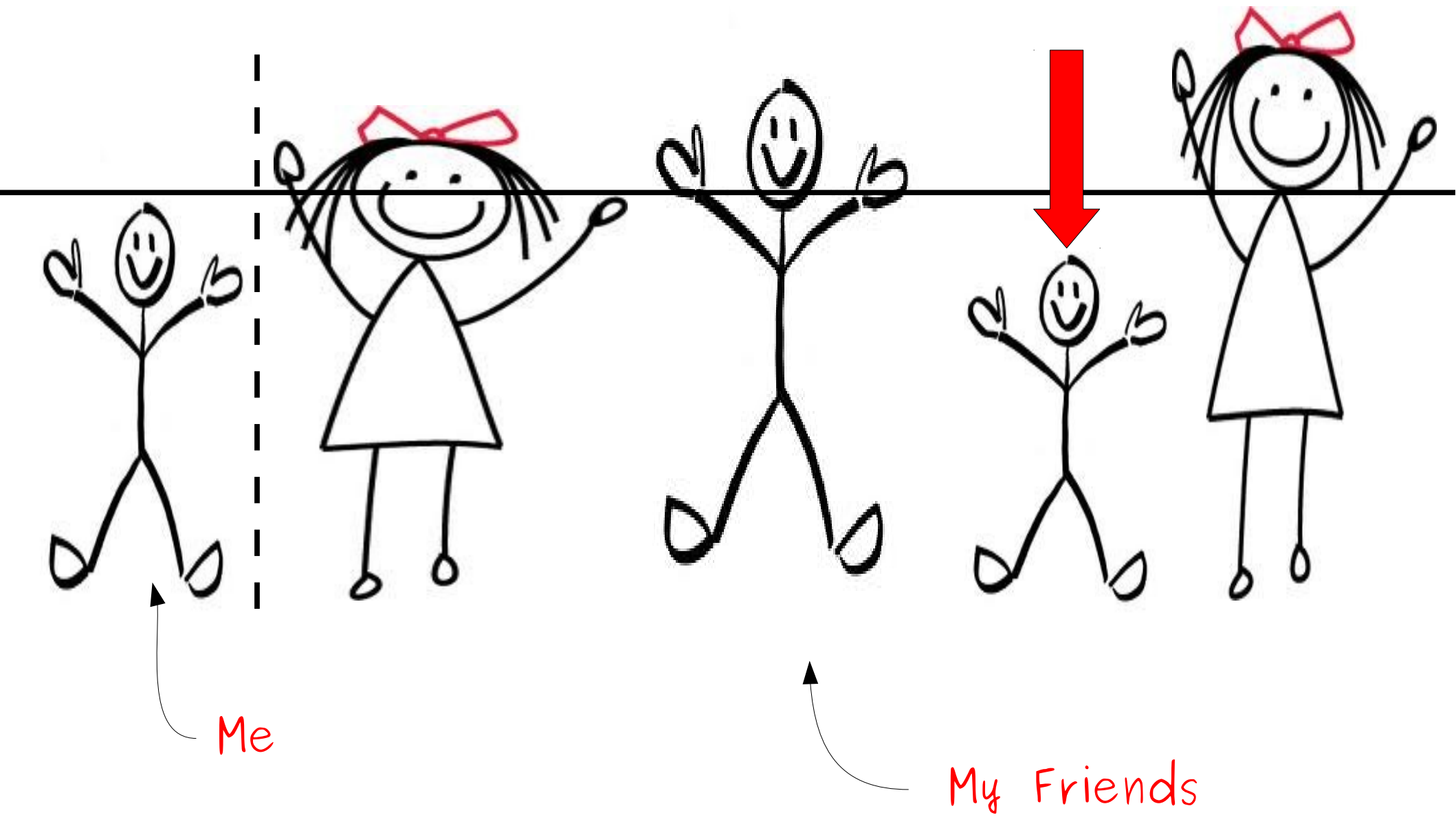
The contradiction of the universal statement

For all *x*, P(*x*) is true.

is **not**

For all *x*, P(*x*) is false.

“All My Friends Are Taller Than Me”



The contradiction of the universal statement

For all x, P(x) is true.

is the existential statement

There exists an x such that P(x) is false.

For all natural numbers n,
the sum of all natural numbers
smaller than n is not equal to n.

becomes

There exists a natural number n such that
the sum of all natural numbers
smaller than n is equal to n

An Incorrect Proof

Theorem: For any natural number n, the sum of all natural numbers less than n is not equal to n.

Proof: By contradiction; assume that for any natural number n, the sum of all smaller natural numbers is equal to n. But this is clearly false, because $5 \neq 1 + 2 + 3 + 4 = 10$. We have reached a contradiction, so our assumption was false and the theorem must be true. ■

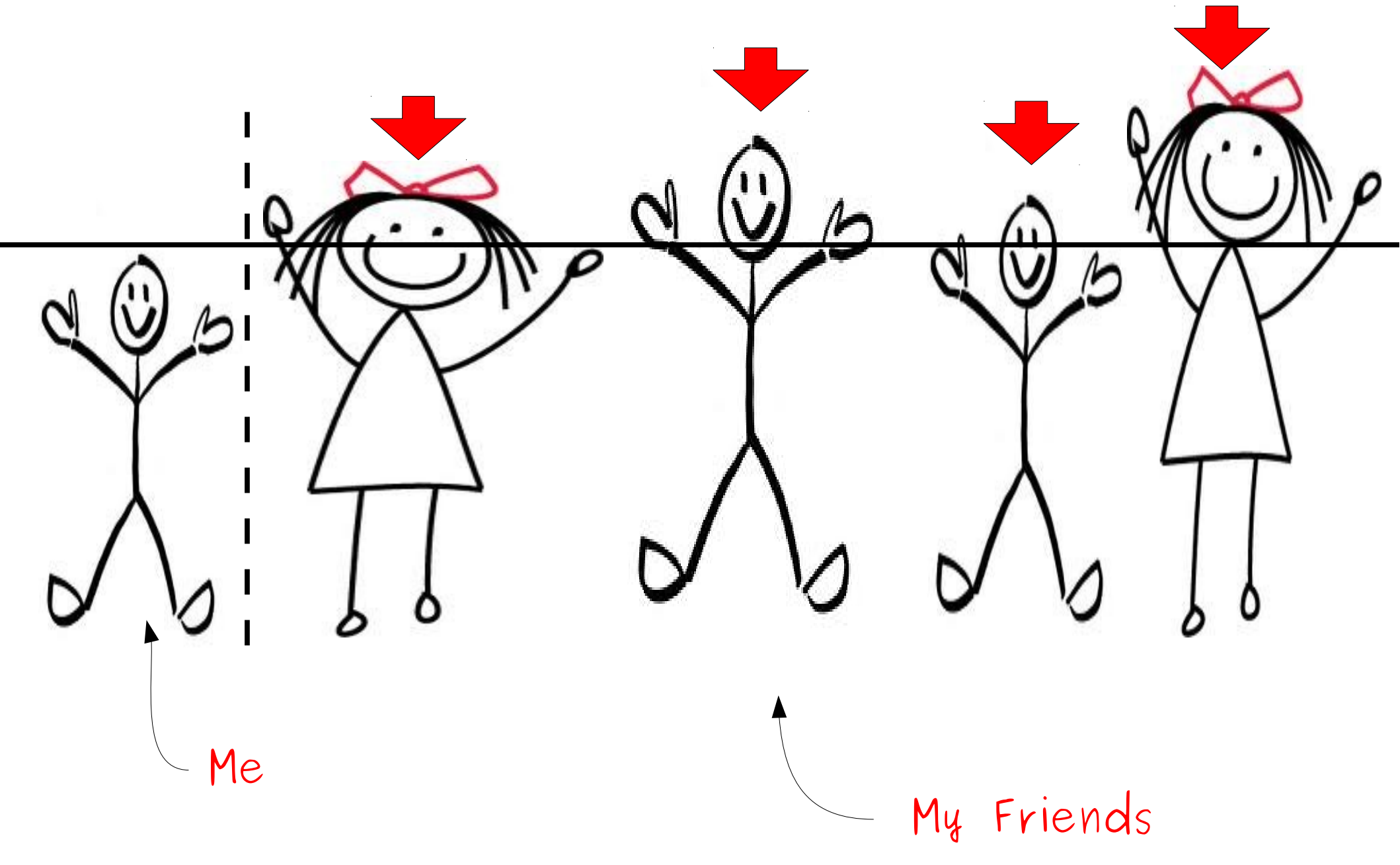
The contradiction of the existential statement

There exists an *x* such that P(*x*) is true.

is **not**

There exists an *x* such that P(*x*) is false.

“Some Friend Is Shorter Than Me”



The contradiction of the existential statement

There exists an x such that P(x) is true.

is the universal statement

For all x, P(x) is false.

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer m, $m \leq n$.

Proof: By contradiction; assume that there exists an integer n such that for every integer m, $m > n$.

Since for any m, we have that $m > n$ is true, it should be true when $m = n - 1$. Thus $n - 1 > n$. But this is impossible, since $n - 1 < n$.

We have reached a contradiction, so our assumption was incorrect. Thus there exists an integer n such that for every integer m, $m \leq n$. ■

A Terribly Flawed Proof

Theorem: There exists an integer n such that for every integer m, $m \leq n$.

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There exists an integer n such that for every integer m, $m \leq n$.

becomes

For every integer n, "for every integer m, $m \leq n$" is false.

For every integer m, $m \leq n$

becomes

There exists an integer m such that $m > n$

There exists an integer n such that
for every integer m, $m \leq n$.

becomes

For every integer n,
There exists an integer m such that
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Proof: By contradiction; assume that there exists an integer n such that for every integer m, $m > n$.

Since for any m, we have that $m > n$ is true, it should be tr… $> n$.
But this is in…

We have rea… assumption was incorrect. Thus there exists an integer n such that for every integer m, $m \leq n$. ∎

For every integer *n*,
There exists an integer *m* such that
m > n

Proof by Contrapositive

Honk **if** You Love Formal Logic

The Contrapositive

- The **contrapositive** of "If P, then Q" is the statement "If **not** Q, then **not** P."
- Example:
  - "If I stored the cat food inside, then the raccoons wouldn't have stolen my cat food."
  - Contrapositive: "If the raccoons stole my cat food, then I didn't store it inside."
- Another example:
  - "If I had been a good test subject, then I would have received cake."
  - Contrapositive: "If I didn't receive cake, then I wasn't a good test subject."

Notation

- Recall that we can write "If P, then Q" as $P \to Q$.
- Notation: We write "not P" as $\boldsymbol{\neg P}$.
- Examples:
  - "If P is false, then Q is true:" $\neg P \to Q$
  - "Q is false whenever P is false:" $\neg P \to \neg Q$
- The contrapositive of $P \to Q$ is $\boldsymbol{\neg Q \to \neg P}$.

An Important Result

Theorem: If $\neg Q \to \neg P$, then $P \to Q$.

Proof: By contradiction. Assume that $\neg Q \to \neg P$, but that $P \to Q$ is false. Since $P \to Q$ is false, it must be true that P is true and $\neg Q$ is true. Since $\neg Q$ is true and $\neg Q \to \neg P$, we know that $\neg P$ is true. But this means that we have shown P and $\neg P$, which is impossible. We have reached a contradiction, so if $\neg Q \to \neg P$, then $P \to Q$. ■

An Important Proof Strategy

To show that $P \to Q$, you may instead show that $\neg Q \to \neg P$.

This is called a **proof by contrapositive**.

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; **???**

If

n^2 is even

then

n is even

If

n is odd

then

n^2 is odd

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Since $(2k^2 + 2k)$ is an integer, n^2 is odd. ■

Theorem: If n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Since n is odd, $n = 2k + 1$ for some integ

$n^2 =$

$n^2 =$

$n^2 =$

Since

odd. ■

Notice the structure of the proof. We begin by announcing that it's a proof by contrapositive, then state the contrapositive, and finally prove it.

An Incorrect Proof

Theorem: For any sets A and B,
if $x \notin A \cap B$, then $x \notin A$.

Proof: By contrapositive; we show that
if $x \in A \cap B$, then $x \in A$.

Since $x \in A \cap B$, $x \in A$ and $x \in B$.
Consequently, $x \in A$ as required. ■

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Proof: By contrapositive; we show that if $x \in A \cap B$, then $x \in A$.

Since $x \in A \cap B$, $x \in A$ and $x \in B$. Consequently, $x \in A$ as required. ■

Common Pitfalls

To prove P → Q by contrapositive, show that

$$\neg Q \rightarrow \neg P$$

Do not show that

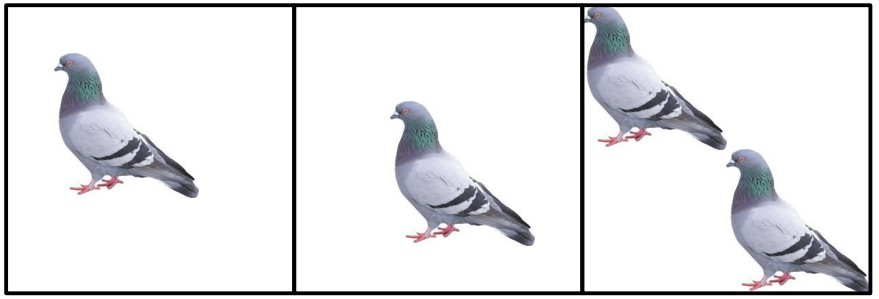
$$\neg P \rightarrow \neg Q$$

(Showing $\neg P \rightarrow \neg Q$ proves that $Q \rightarrow P$, not the other way around!)

The Pigeonhole Principle

The Pigeonhole Principle

- Suppose that you have n pigeonholes.
- Suppose that you have $m > n$ pigeons.
- If you put the pigeons into the pigeonholes, some pigeonhole will have more than one pigeon in it.



Theorem: Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects.

Proof: By contrapositive; we prove that if every bin contains at most one object, then $m \leq n$.

Let x_i be the number of objects in bin i. Since m is the number of total objects, we have that

$$m = \sum_{i=1}^{n} x_i$$

Since every bin has at most one object, $x_i \leq 1$ for all i. Thus

$$m = \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} 1 = n$$

So $m \leq n$, as required. ■

Using the Pigeonhole Principle

- The pigeonhole principle is an enormously useful lemma in many proofs.
  - If we have time, we'll spend a full lecture on it in a few weeks.
- General structure of a pigeonhole proof:
  - Find m objects to distribute into n buckets, with $m > n$.
  - Using the pigeonhole principle, conclude that some bucket has at least two objects in it.
  - Use this conclusion to show the desired result.

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.
- Each day, two people in New York City drink the same amount of water, to the thousandth of a fluid ounce.
  - No one can drink more than 50 gallons of water each day.
  - That's 6,400 fluid ounces. This gives 6,400,000 possible numbers of thousands of fluid ounces.
  - There are about 8,000,000 people in New York City proper.

Next Time

- **Proof by Induction**
  - Proofs on sums, programs, algorithms, etc.