Mathematical Induction

A Note to CS106B Students

- Since CS106B and CS103 overlap, I'll be repeating the last 15 minutes of lecture every M/W/F from 4:15ish to 4:30ish in my office (Gates 178).
- Stop by if you're interested!

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

$P(0)$ is true

If it starts …

… and it keeps going …

and

For any $n \in \mathbb{N}$, we have $P(n) \to P(n + 1)$

Then

… then it's always true.

For any $n \in \mathbb{N}$, $P(n)$ is true.

Another Example of Induction

Human Dominoes

- Everyone (except that last guy) eventually fell over.
- Why is that?
  - Someone fell over.
  - Once someone fell over, the next person fell over as well.

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

$\boldsymbol{P(0)}$ **is true**

and

For any $\boldsymbol{n \in \mathbb{N}}$**, we have** $\boldsymbol{P(n) \to P(n+1)}$

Then

For any $\boldsymbol{n \in \mathbb{N}}$**,** $\boldsymbol{P(n)}$ **is true.**

Induction, Intuitively

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

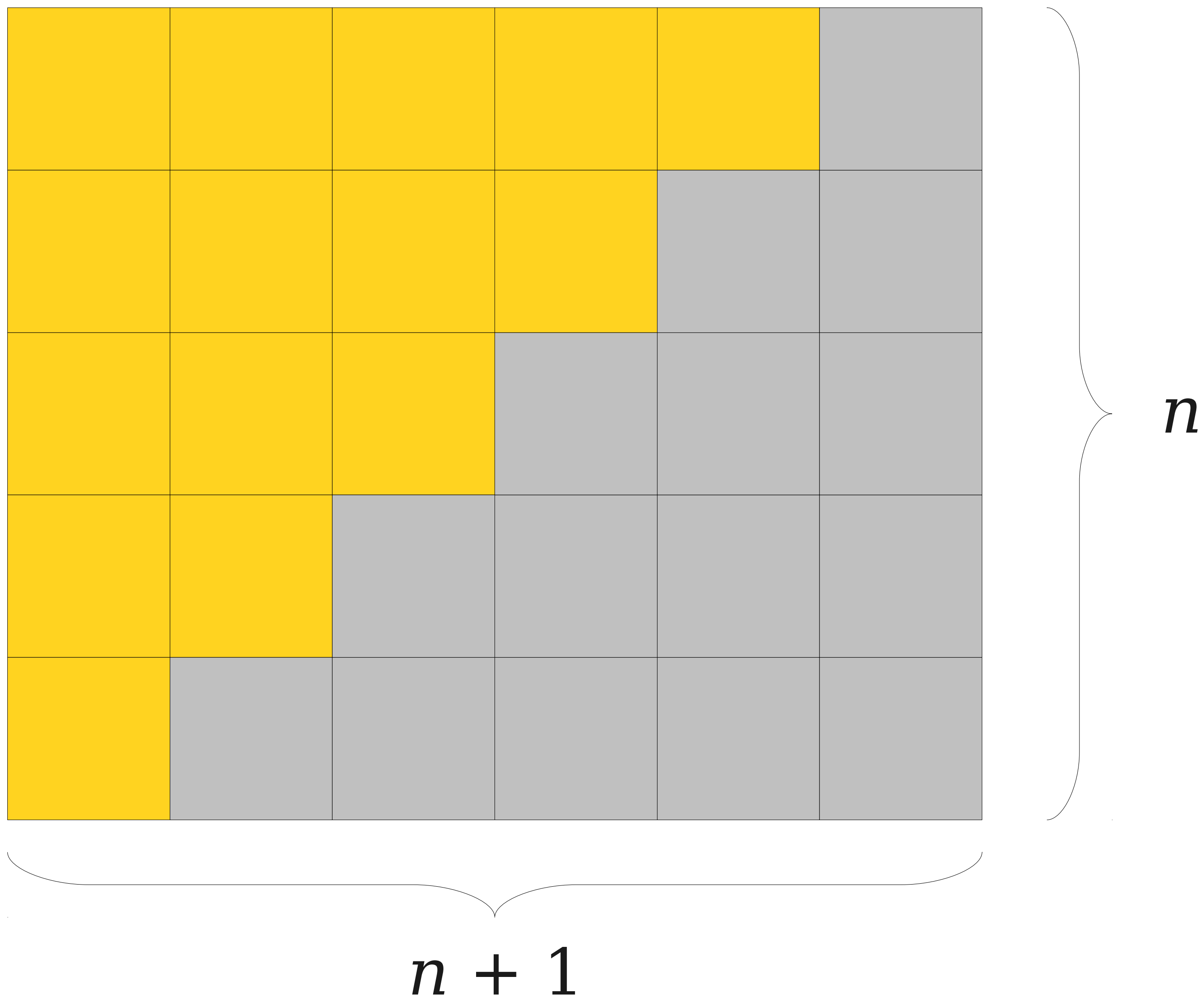
Proof by Induction

- Suppose that you want to prove that some property $P(n)$ holds of all natural numbers. To do so:
  - Prove that $P(0)$ is true.
    - This is called the **basis** or the **base case**.
  - Prove that for all $n \in \mathbb{N}$, that if $P(n)$ is true, then $P(n + 1)$ is true as well.
    - This is called the **inductive step**.
    - $P(n)$ is called the **inductive hypothesis**.
  - Conclude by induction that $P(n)$ holds for all n.

Some Sums

$1 = 1$
$1 + 2 = 3$
$1 + 2 + 3 = 6$
$1 + 2 + 3 + 4 = 10$
$1 + 2 + 3 + 4 + 5 = 15$

$$1 + 2 + \ldots + (n - 1) + n = n(n + 1) / 2$$



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Some Sums

$1 = 1$ **$= 1(1 + 1) / 2$**

$1 + 2 = 3$ **$= 2(2 + 1) / 2$**

$1 + 2 + 3 = 6$ **$= 3(3 + 1) / 2$**

$1 + 2 + 3 + 4 = 10$ **$= 4(4 + 1) / 2$**

$1 + 2 + 3 + 4 + 5 = 15$ **$= 5(5 + 1) / 2$**

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Just as in a proof by contradiction or contrapositive, we should mention this proof is by induction.

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Now, we state what property $P(n)$ we are going to prove holds for all $n \in \mathbb{N}$.

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Proof: By induction. Let $P(n)$ be "the sum of the first n positive natural numbers is $n(n + 1) / 2$." We show that $P(n)$ is true for all $n \in \mathbb{N}$.

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The first step of an inductive proof is to show P(0). We explicitly state what P(0) is, then try to prove it. We can prove P(0) using any proof technique we'd like.

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For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $1 + 2 + \ldots + n = n(n + 1) / 2$.

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For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $1 + 2 + \ldots + n = n(n + 1) / 2$.

The goal of this step is to prove

"For any $n \in \mathbb{N}$, if $P(n)$, then $P(n + 1)$"

To do this, we'll choose an arbitrary n, assume that $P(n)$ holds, then try to prove $P(n + 1)$.

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Here, we're explicitly stating $P(n + 1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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$$1 + \ldots + n + (n + 1)$$

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Consider the sum of the . This
is the sum of the first n . By
the inductive hypothesis

We're assuming that $P(n)$ is true, so we can replace this sum with the value **$n(n + 1) / 2$**.

$$1 + \ldots + n + (n+1)$$

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Thus $P(n + 1)$ is true, completing the induction.

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Thus $P(n + 1)$ is true, completing the induction. ■

Structuring a Proof by Induction

- State that your proof works by induction.
- State your choice of $P(n)$.
- Prove the base case:
  - State what $P(0)$ is, then prove it using any technique you'd like.
- Prove the inductive step:
  - State that for some arbitrary $n \in \mathbb{N}$ that you're assuming $P(n)$ and mention what $P(n)$ is.
  - State that you are trying to prove $P(n + 1)$ and what $P(n + 1)$ means.
  - Prove $P(n + 1)$ using any technique you'd like.
- This is very rigorous, so as we gain more familiarity with induction we will start being less formal in our proofs.

Notation: Summations

- Instead of writing 1 + 2 + 3 + … + n, we write

Sum from i = 1 to n

$$\sum_{i=1}^{n} i$$

of i

Summation Examples

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

$$\sum_{i=0}^{2} \left(i^2 - i\right) = \left(0^2 - 0\right) + \left(1^2 - 1\right) + \left(2^2 - 2\right) = 2$$

The Empty Sum

- A sum of no numbers is called the **empty sum** and is defined to be zero.
- Examples:

$$\sum_{i=1}^{0} 2^i = 0 \qquad \sum_{i=137}^{42} i^i = 0 \qquad \sum_{i=0}^{-1} i = 0$$

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Since the empty sum is defined to be 0, this claim is true.

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$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We need to show that $P(n + 1)$ holds, meaning that

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

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To see this, note that

$$\sum_{i=1}^{n+1} i$$

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Here, we're "peeling off" the last term of the sum. Many inductive proofs on sums will use this trick.

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Sums of Powers of Two

(empty sum) = 0

$2^0 = 1 = 1$

$2^0 + 2^1 = 1 + 2 = 3$

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Sums of Powers of Two

(empty sum) = 0 **= 2^0 - 1**
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$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

A Quick Aside

- This result helps explain the range of numbers that can be stored in an **`int`**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. We'll see one in a week.

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A Brief Interlude for Announcements

Problem Session Tonight

- Problem Session tonight, 7:00 – 7:50PM in 380-380X
- Purely optional, but should be a lot of fun!
- We'll try to get it recorded and posted online as soon as possible.

Back to our regularly scheduled programming…

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math

How Not To Induct

An Incorrect Proof

Theorem: For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$

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$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + n + 1 = \frac{1}{2}\left(n + \frac{1}{2}\right)^2 + n + 1 = \frac{\left(n + \frac{1}{2}\right)^2}{2} + \frac{2(n+1)}{2} = \frac{\left(n + \frac{1}{2}\right)^2 + 2(n+1)}{2}$$

$$= \frac{n^2 + n + \frac{1}{4} + 2\text{n} + 2}{2} = \frac{n^2 + 3\text{n} + \frac{9}{4}}{2} = \frac{\left(n + \frac{3}{2}\right)^2}{2}$$

So $P(n + 1)$ holds, completing the induction. ■

An Incorrect Proof

Theorem: For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$

Proof: By induction. Let $P(n)$ be defined as $P(n) \equiv \sum_{i=1}^{n}$

Where did we prove the base case?

Now, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds,

$$\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$$

We want to show that $P(n + 1)$ is true, which means that we want to show

$$\sum_{i=1}^{n+1} i = \frac{1}{2}\left(n + 1 + \frac{1}{2}\right)^2 = \frac{1}{2}\left(n + \frac{3}{2}\right)^2$$

To see this, note that

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + n + 1 = \frac{1}{2}\left(n + \frac{1}{2}\right)^2 + n + 1 = \frac{\left(n + \frac{1}{2}\right)^2}{2} + \frac{2(n+1)}{2} = \frac{\left(n + \frac{1}{2}\right)^2 + 2(n+1)}{2}$$

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An Incorrect Proof

$\equiv \sum_{i=1}^{n}$

olds,

Where did we prove the base case?

means that we want to show

$\frac{3}{2})^2$

$\frac{2(n+1)}{2} = \frac{(n+\frac{1}{2})^2 + 2(n+1)}{2}$

When proving $P(n)$ is true
for all $n \in \mathbb{N}$ by induction,

make sure to show the base case!

Otherwise, your argument is invalid!

The Counterfeit Coin Problem, Take Two

Problem Statement

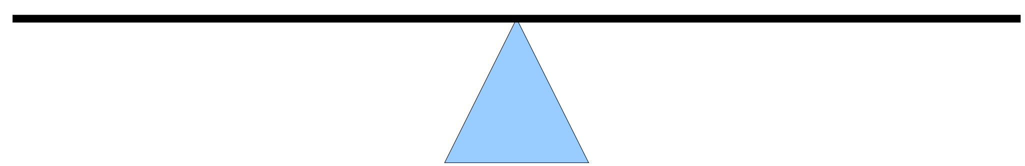
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin

1

2

3



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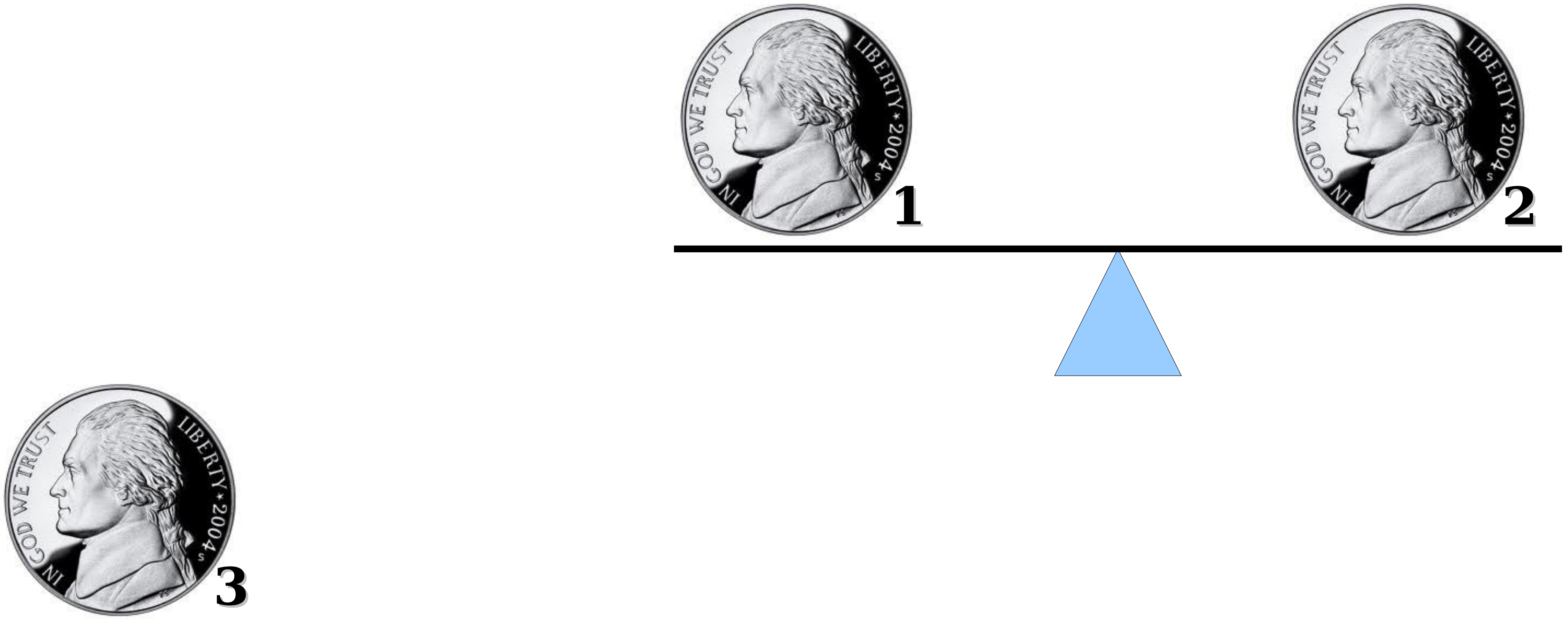

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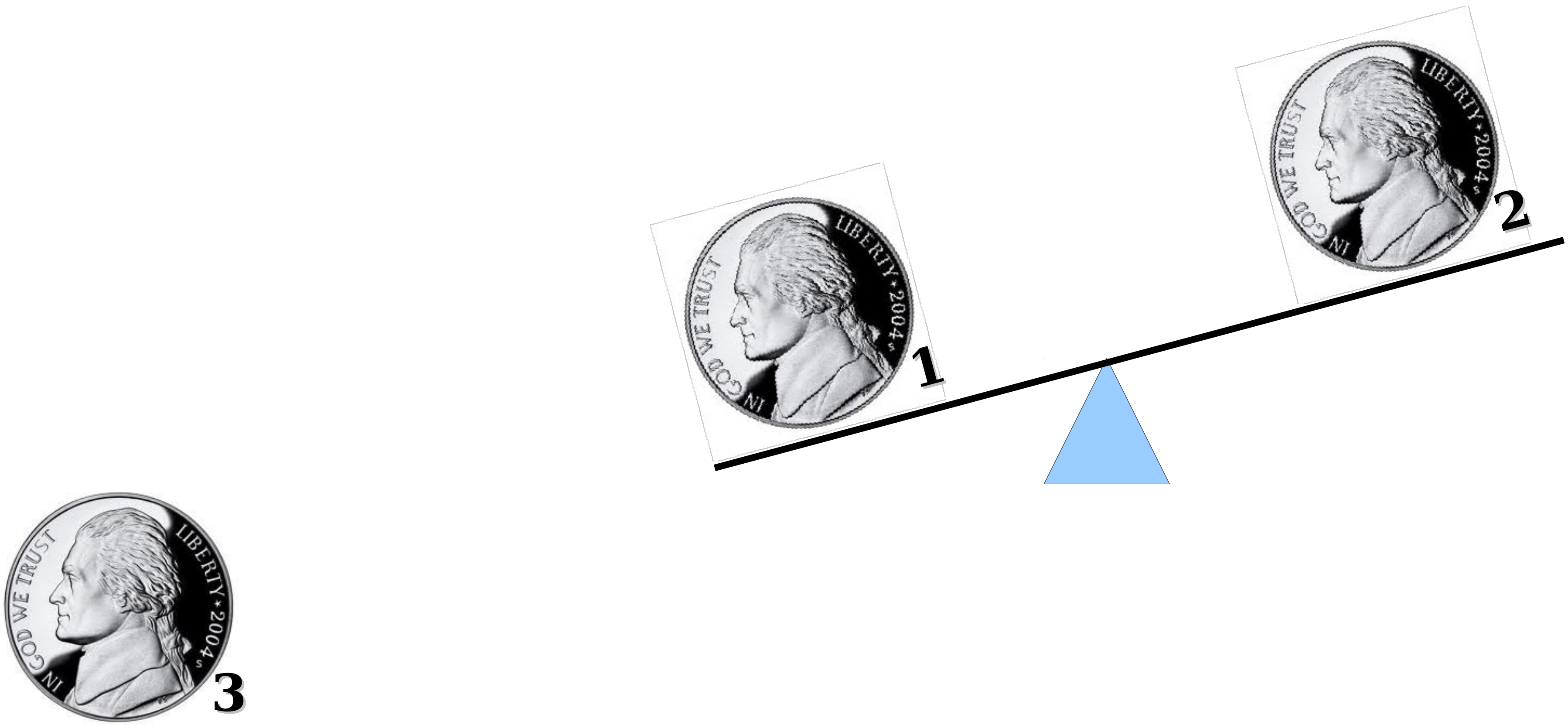

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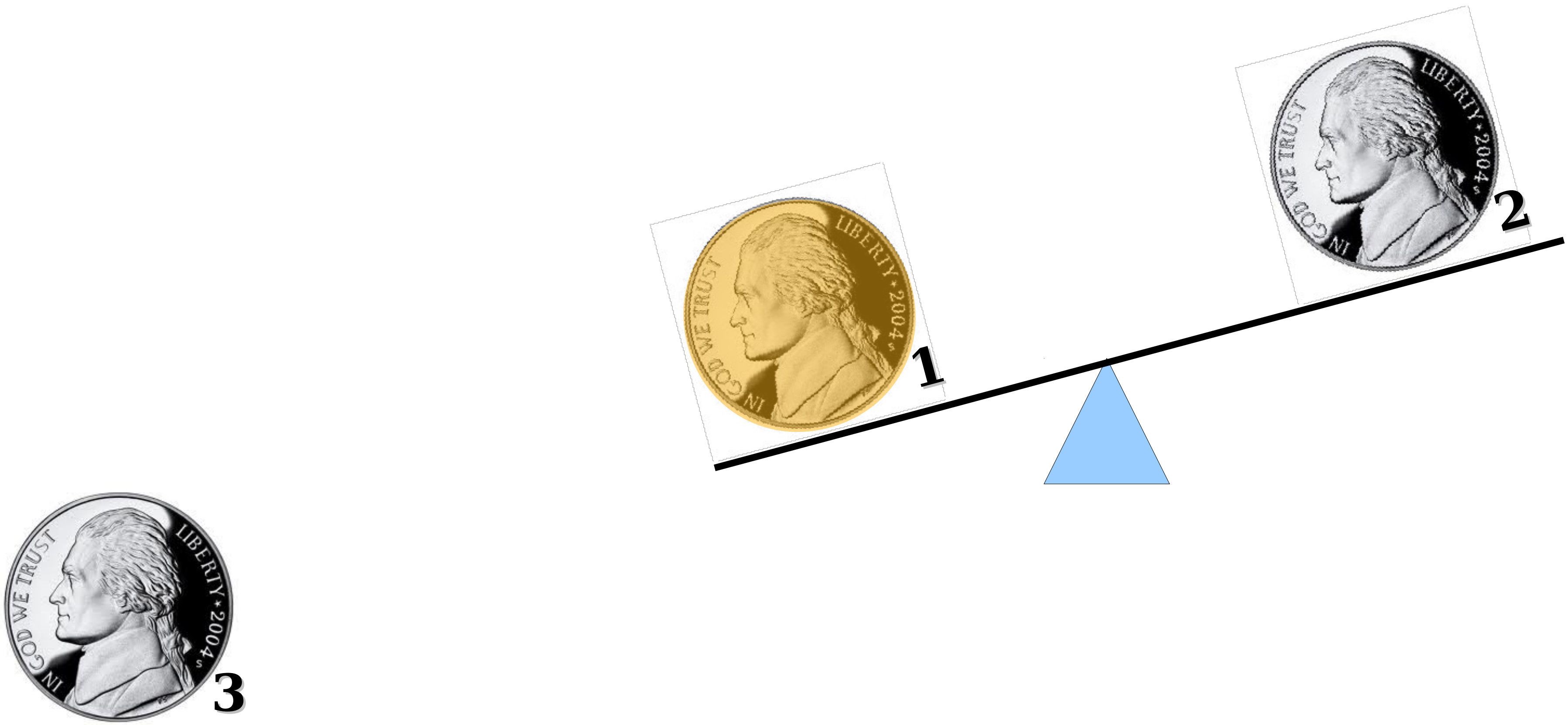

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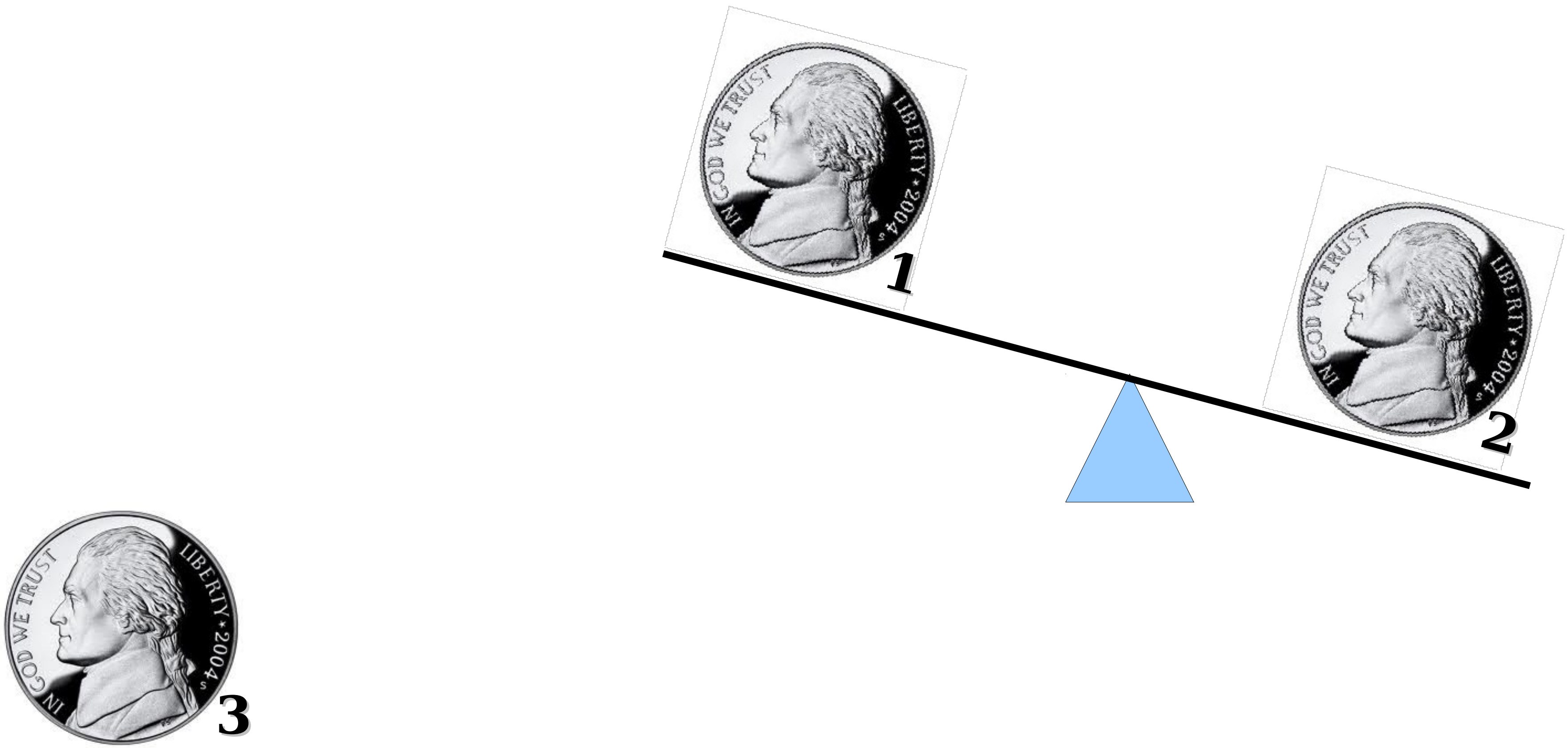

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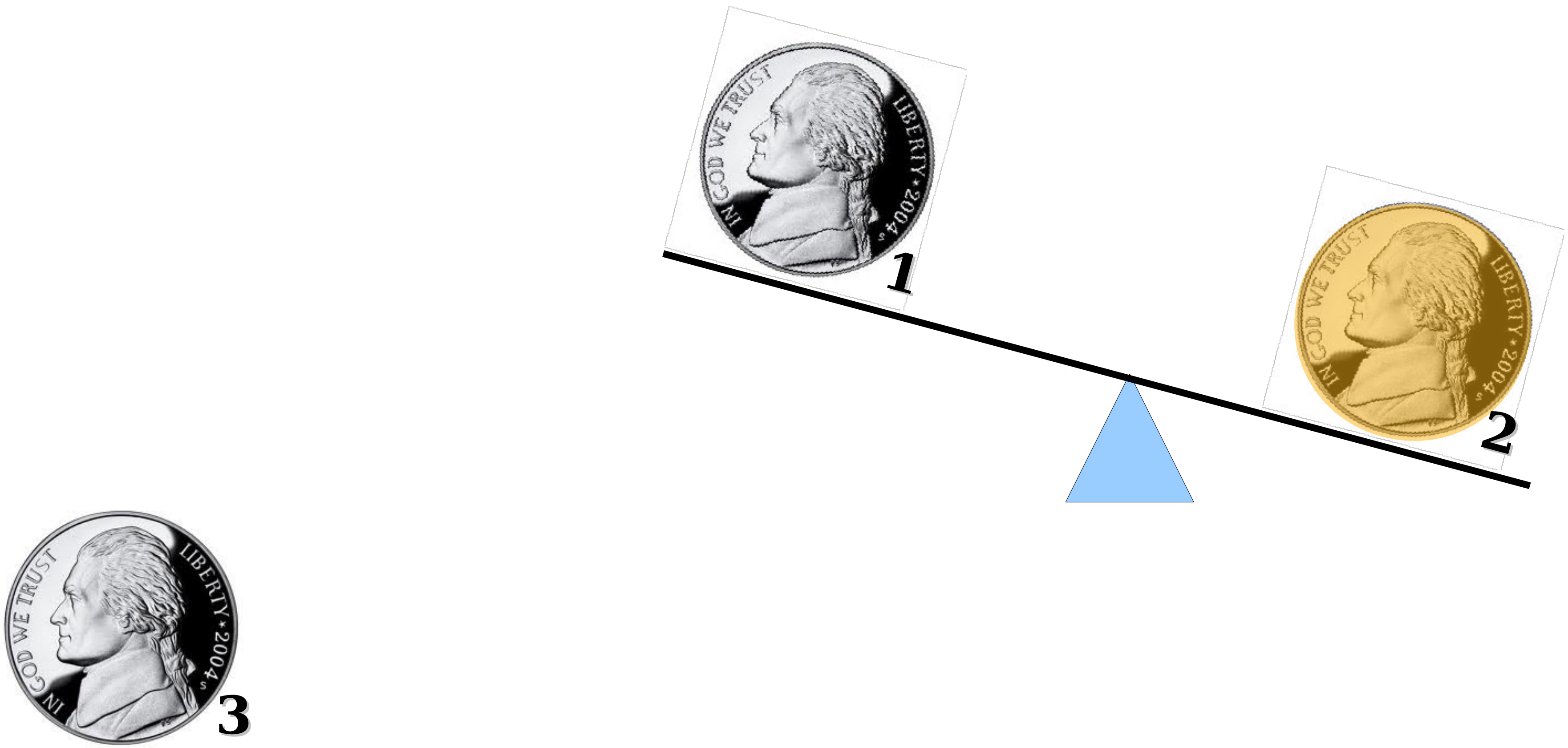

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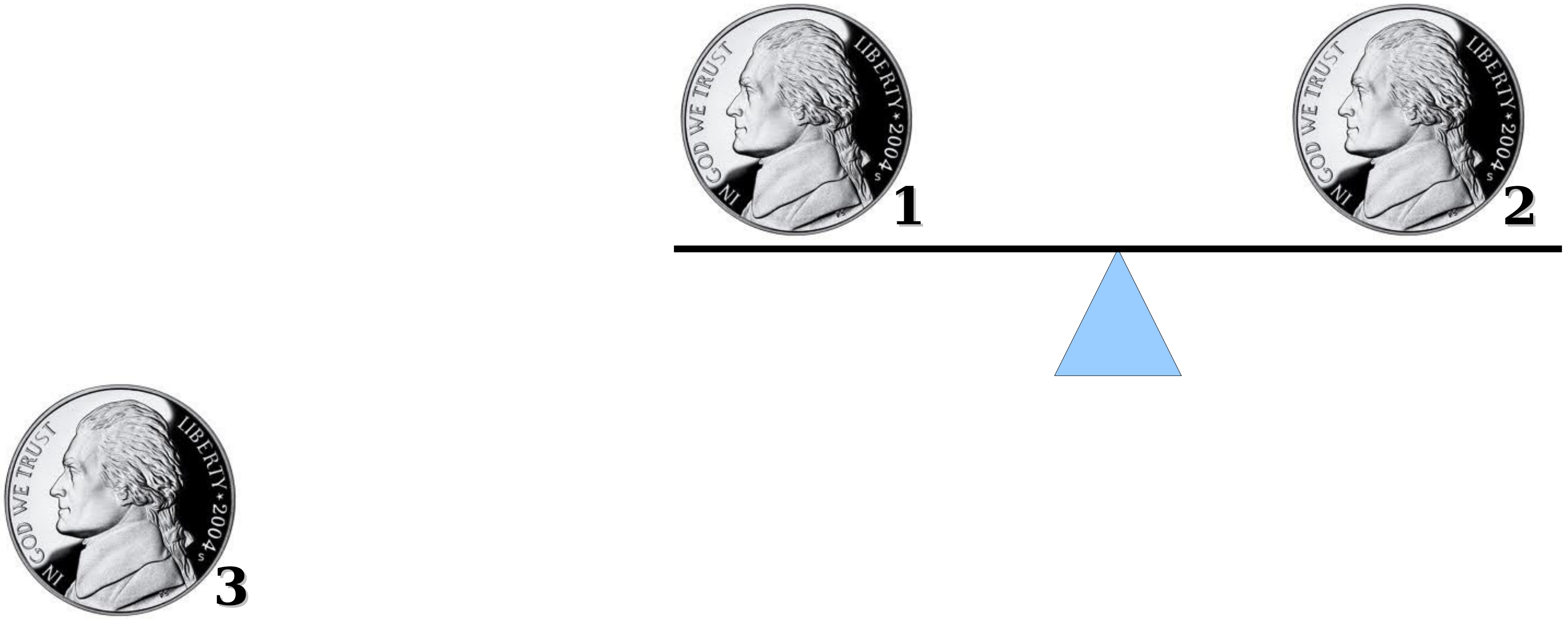

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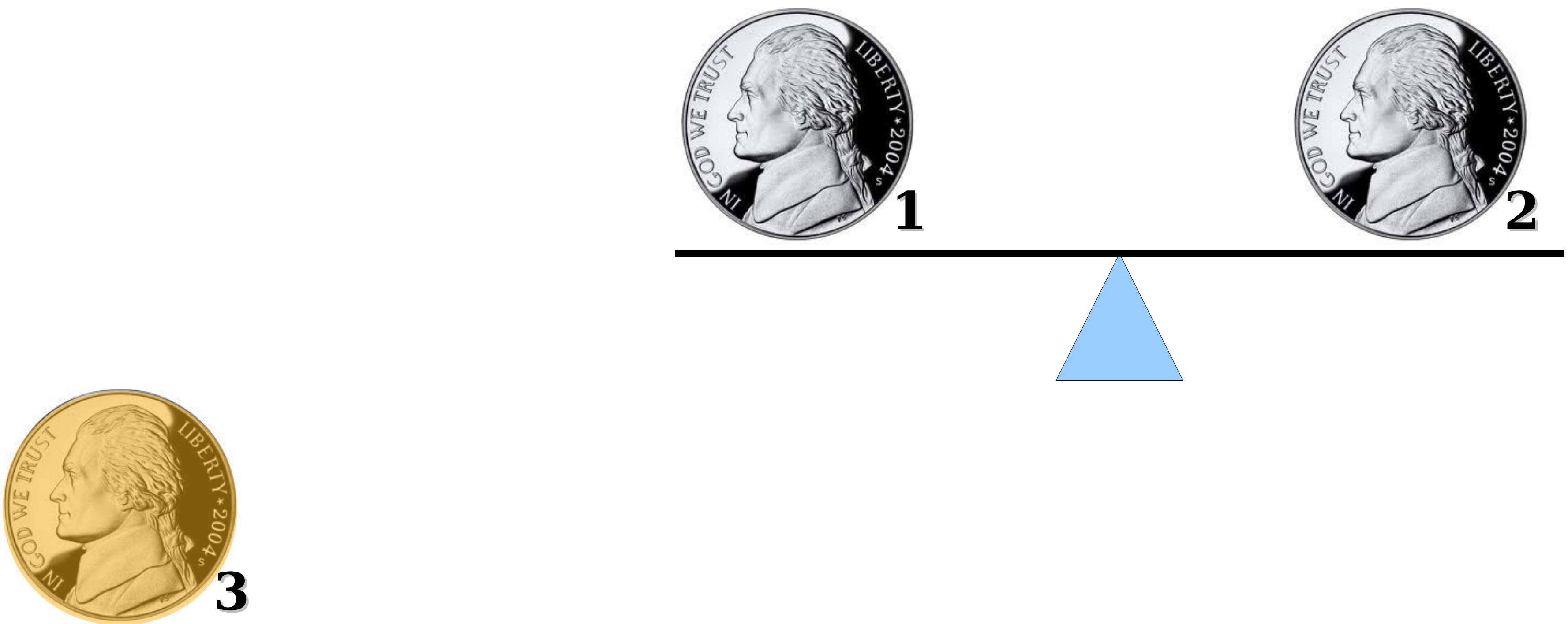
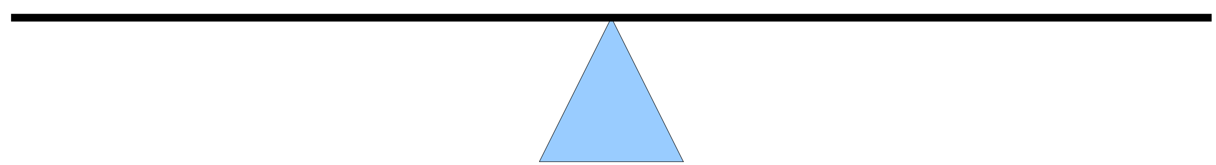


A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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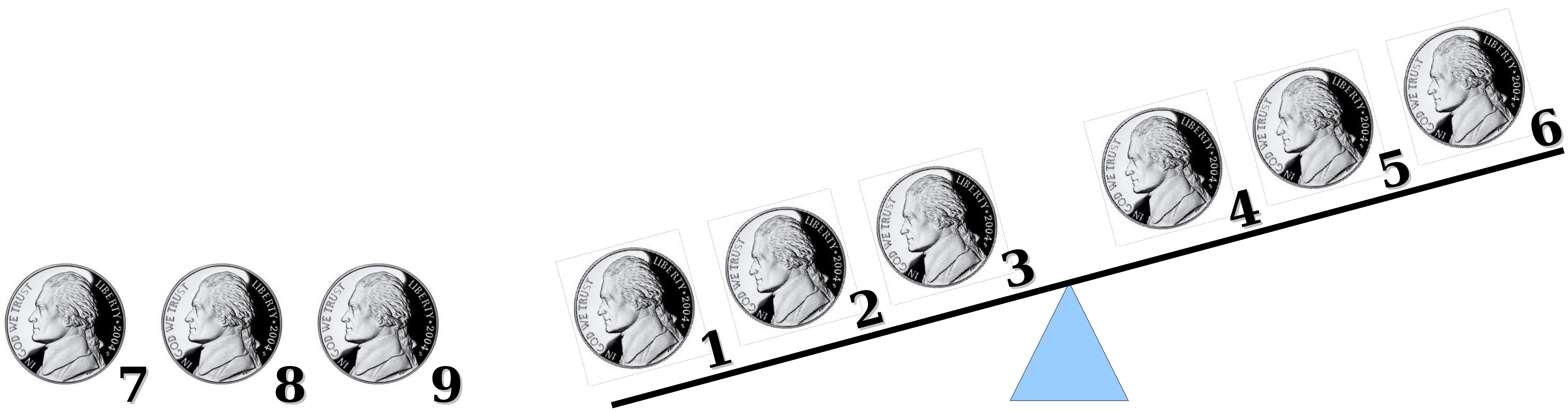

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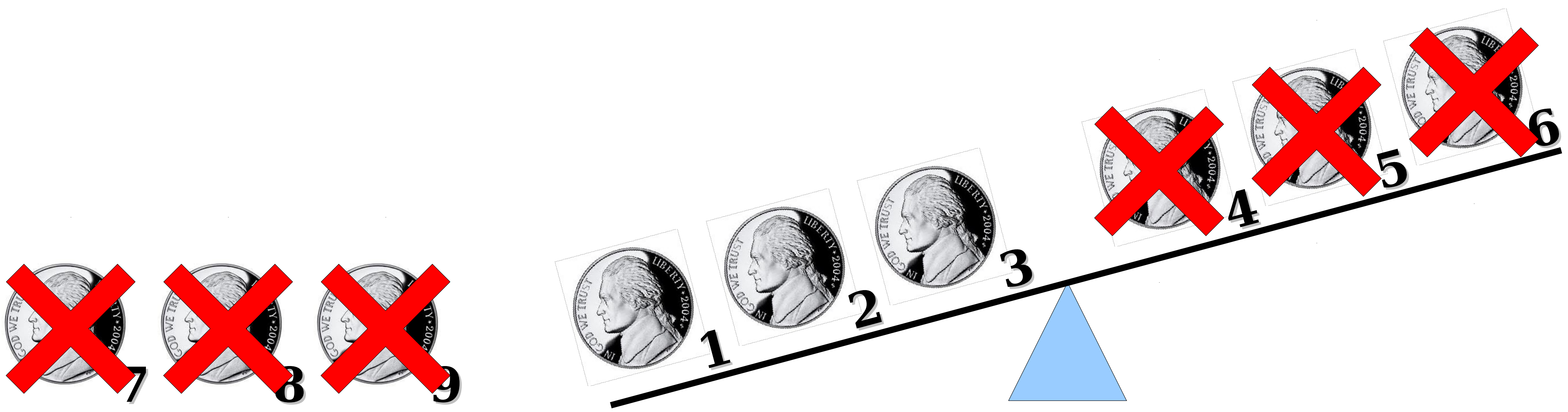

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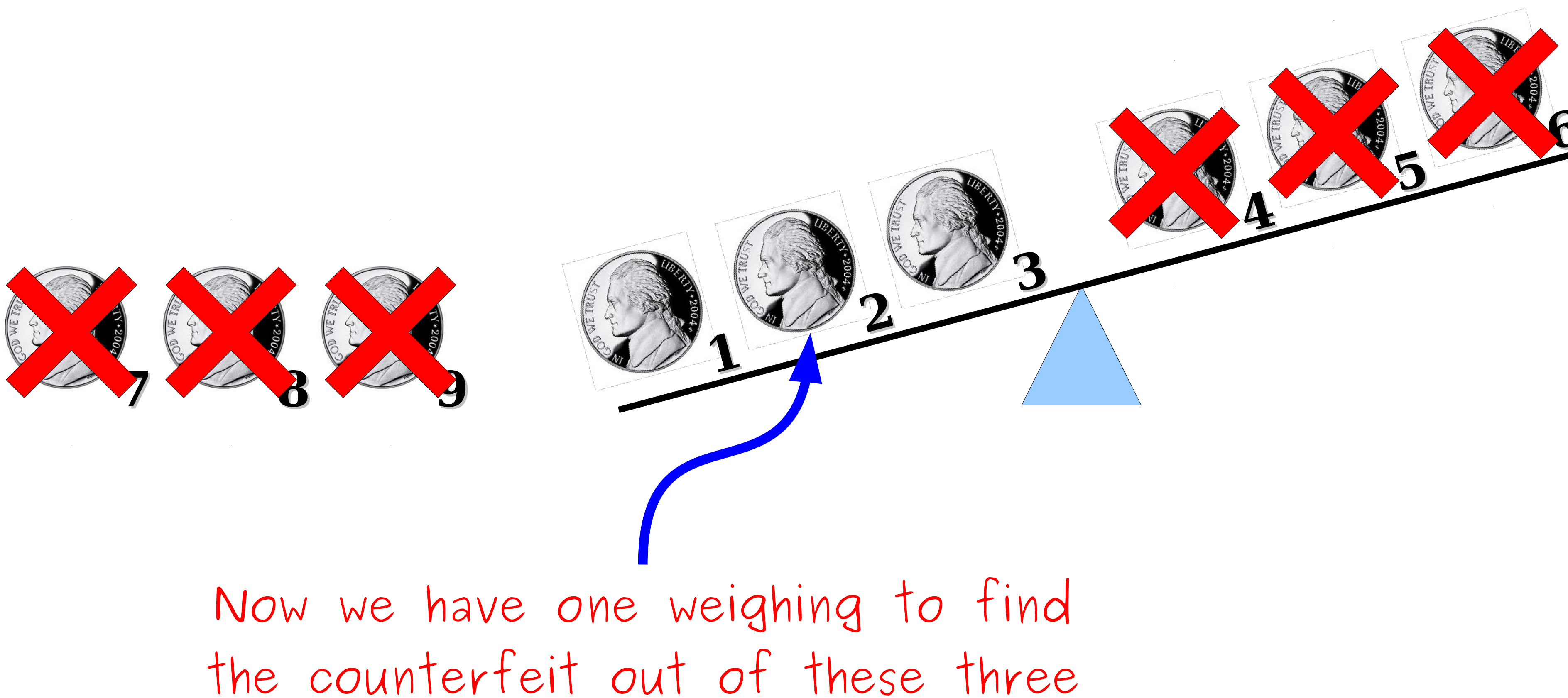

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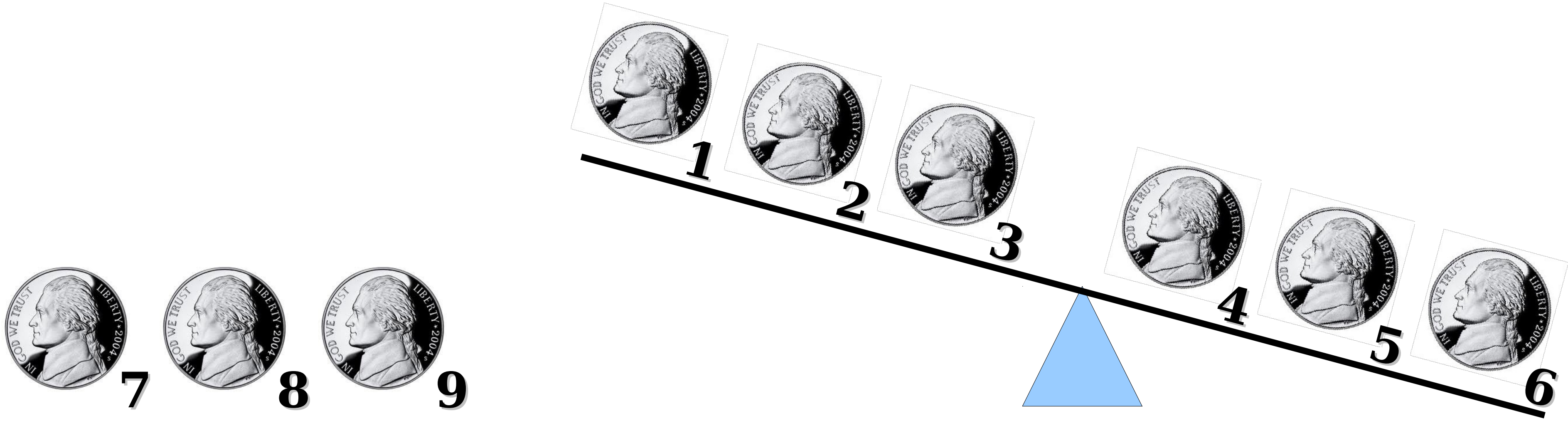

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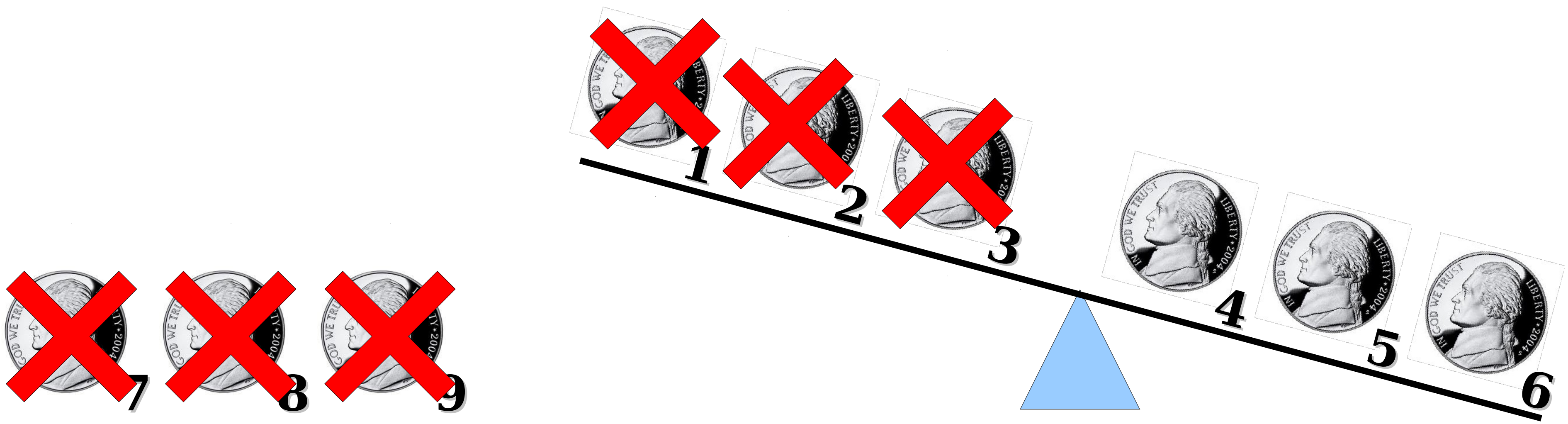

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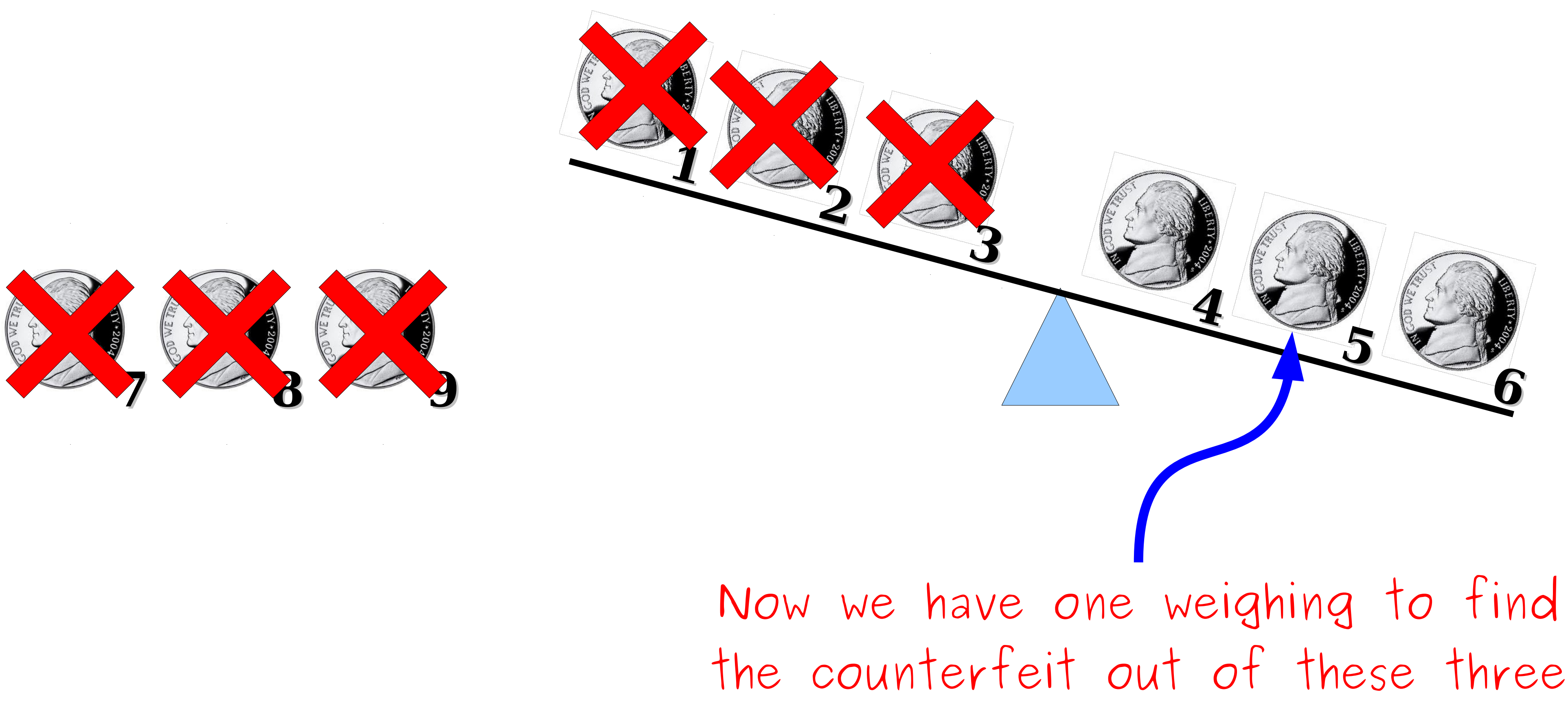

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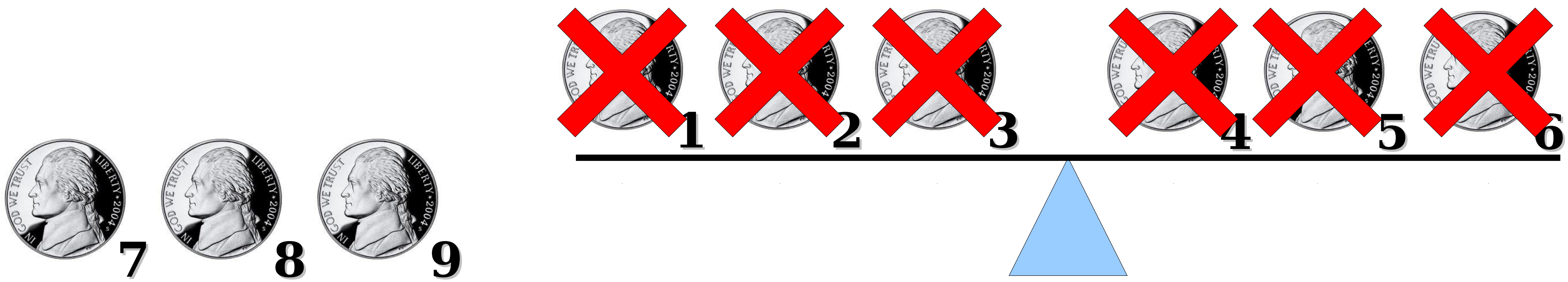

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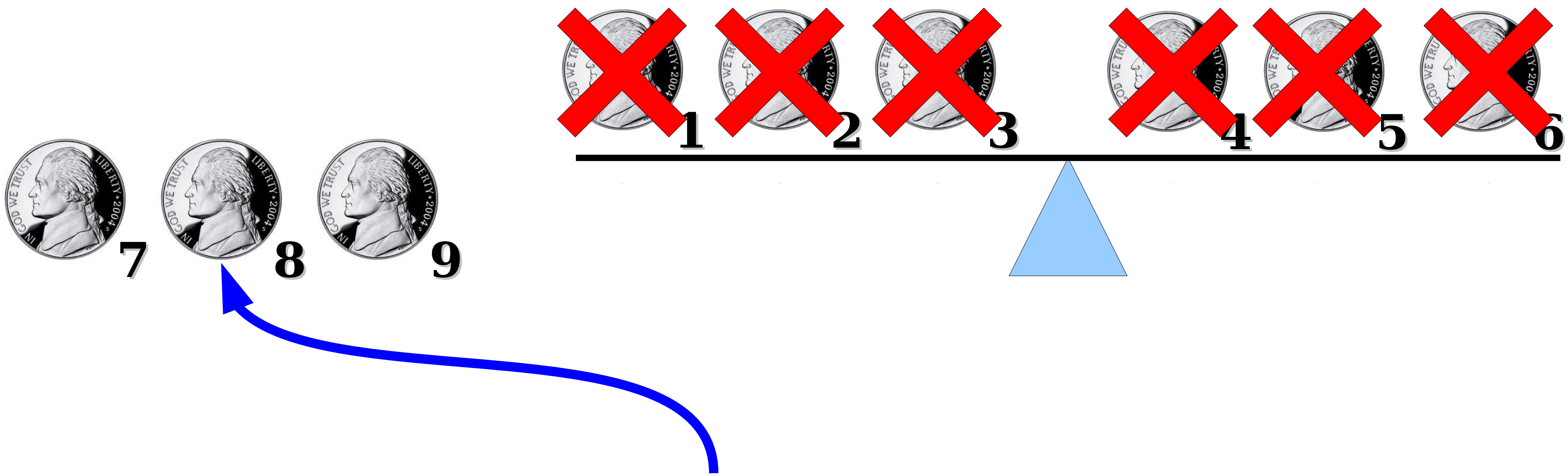

Now we have one weighing to find the counterfeit out of these three

If we have n weighings on the scale, what is the largest number of coins out of which we can find the counterfeit?

A Pattern

- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One coin**, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

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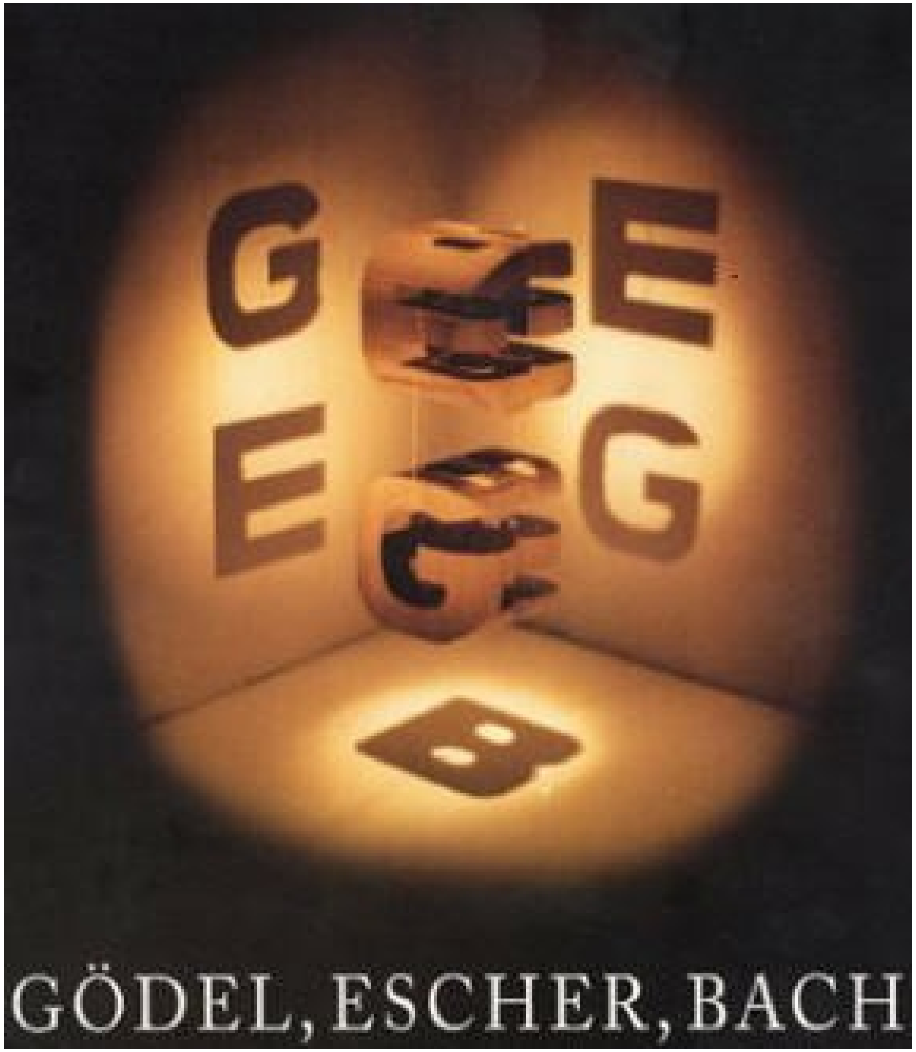
The **MU** Puzzle

Gödel, Escher Bach: An Eternal Golden Braid



- Pulitzer-Prize winning book exploring recursion, computability, and consciousness.
- Written by Douglas Hofstadter, computer scientist at Indiana University.
- A great (but dense!) read.

The **MU** Puzzle

- Begin with the string **MI**.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the **M**: for example, **MIIU** becomes **MIIUIIU** or **MI** becomes **MII**.
  - Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**
  - Append **U** to the string if it ends in **I**: **MI** becomes **MIU**
  - Remove any **UU**: **MUUU** becomes **MU**
- **Question**: How do you transform **MI** to **MU**?

A) Double the contents of the string after **`M`**.

B) Replace **`III`** with **`U`**.

C) Remove **`UU`**

D) Append **`U`** if the string ends in **`I`**.

MI

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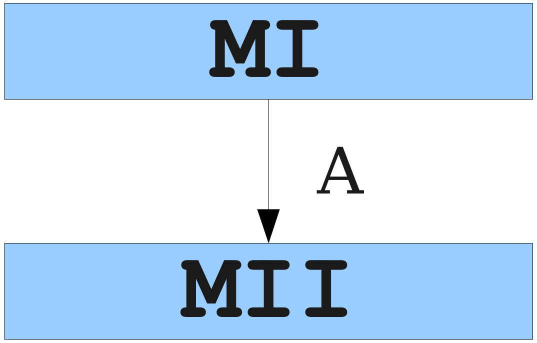

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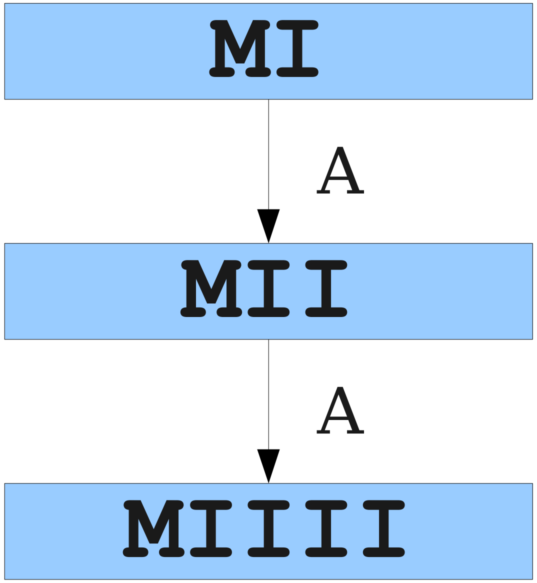

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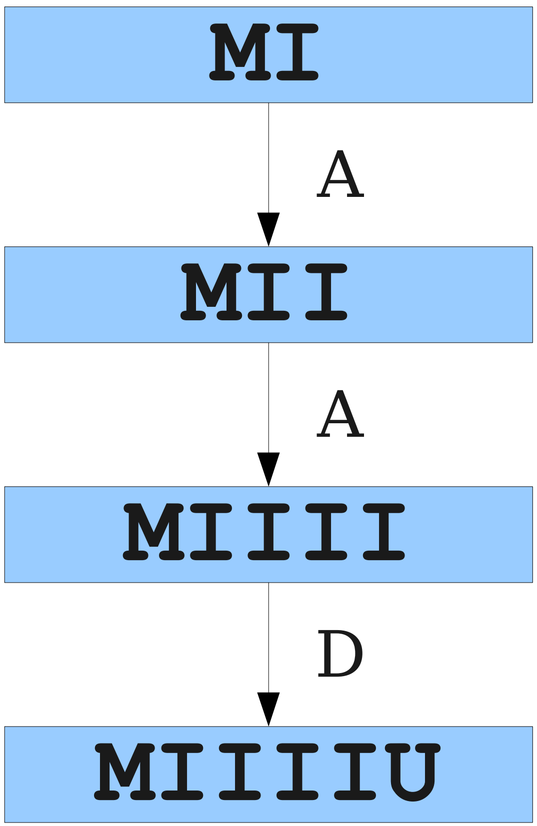

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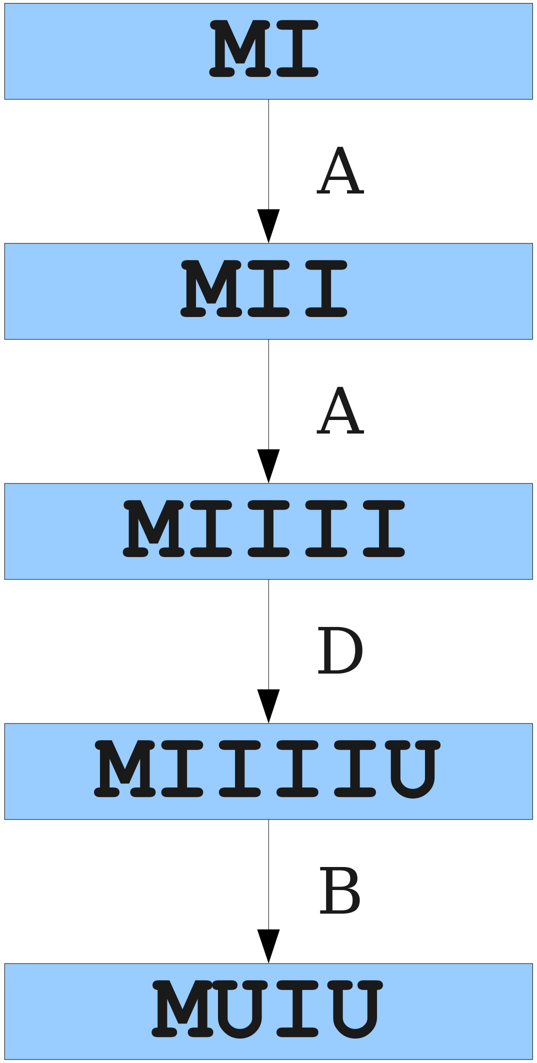

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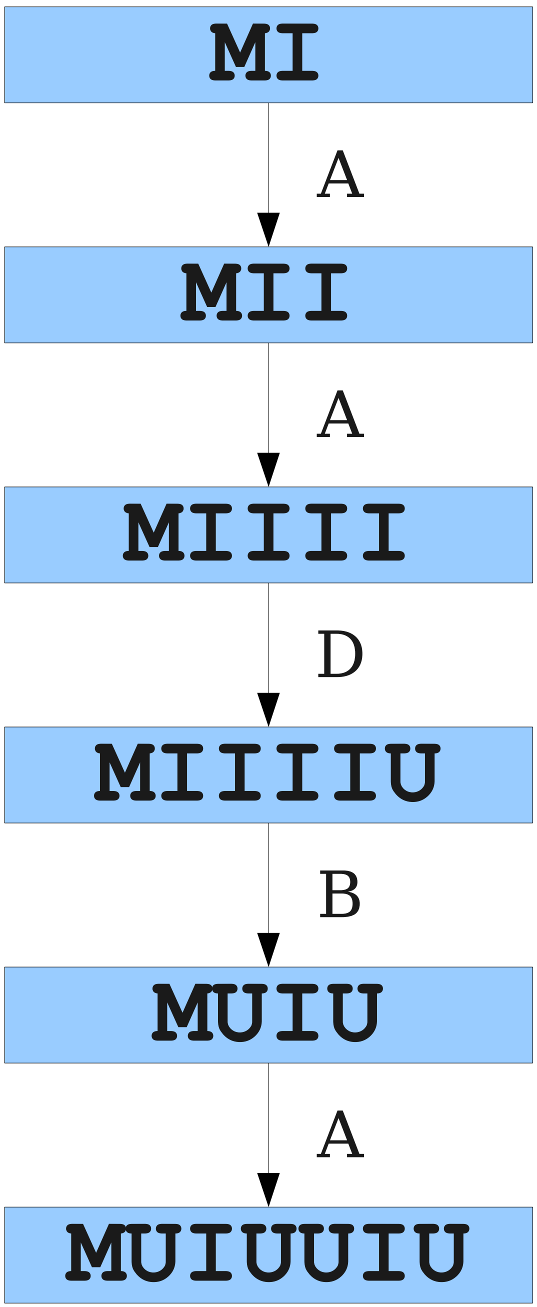

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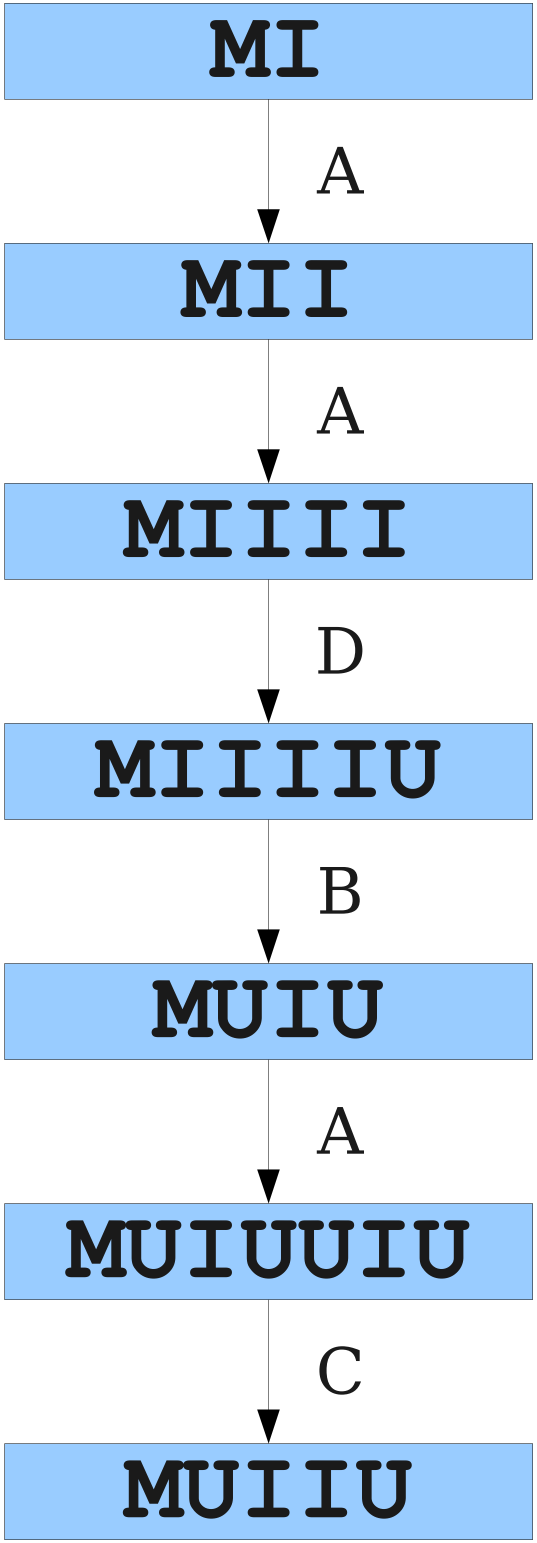

Try It!

Starting with **`MI`**, apply these operations to make **`MU`**:

A) Double the contents of the string after **`M`**.

B) Replace **`III`** with **`U`**.

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D) Append **`U`** if the string ends in **`I`**.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting `I`'s

Counting I's

MI
MII
MIIII
MIIIIU
MIIIIUIIIIU
MIIIIUUIU
MIIIIUUIUIIIIUUIU
MUIUUIUIIIIUUIU

Counting **I**'s

MI	1
MII	2
MIIII	4
MIIIIU	4
MIIIIUIIIIU	8
MIIIIUUIU	5
MIIIIUUIUIIIIUUIU	10
MUIUUIUIIIIUUIU	7

Counting **I**'s

String	Number of I's
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MIIII	4
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None of these are multiples of three...

The Key Insight

- Initially, the number of `I`'s is **not** a multiple of three.
- To make `MU`, the number of `I`'s must end up as a multiple of three.
- Can we *ever* make the number of `I`'s a multiple of three?

Lemma: Beginning with **`MI`** and applying any legal sequence of moves, the number of **`I`**'s is never a multiple of 3.

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Thus any sequence of $n + 1$ moves starting with **MI** ends with the number of **I**'s not a multiple of three.

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Thus any sequence of $n + 1$ moves starting with **MI** ends with the number of **I**'s not a multiple of three. Thus $P(n + 1)$ holds, completing the induction. ■

Theorem: The **`MU`** puzzle has no solution.

Proof: By contradiction; assume it has a solution. By our lemma, the number of **`I`**'s in the final string must not be a multiple of 3. However, for the solution to be valid, the number of **`I`**'s must be 0, which is a multiple of 3. We have reached a contradiction, so our assumption was wrong and the MU puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a **loop invariant**.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!