Mathematical Induction

Part Two

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

$P(0)$ is true

If it starts …

… and it keeps going …

and

For any $n \in \mathbb{N}$, we have $P(n) \rightarrow P(n+1)$

Then

… then it's always true.

For any $n \in \mathbb{N}$, $P(n)$ is true.

Theorem: For any natural number n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: By induction. Let $P(n)$ be

$$P(n) \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

For our base case, we need to show $P(0)$ is true, meaning that

$$\sum_{i=1}^{0} i = \frac{0(0+1)}{2}$$

Since the empty sum is defined to be 0, this claim is true.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, so

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We need to show that $P(n + 1)$ holds, meaning that

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

To see this, note that

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Thus $P(n + 1)$ is true, completing the induction. ■

Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is,
  - that $P(0)$ is true, and that
  - whenever $P(n)$ is true, $P(n + 1)$ is true,

  the proof is usually valid.

Theorem: For any natural number n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: By induction on n. For our base case, if n = 0, note that

$$\sum_{i=1}^{0} i = \frac{0(0+1)}{2} = 0$$

and the theorem is true for 0.

For the inductive step, assume that for some n the theorem is true. Then we have that

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

so the theorem is true for n + 1, completing the induction. ■

A Variant of Induction

n^2 versus 2^n

$0^2 = 0$	$2^0 = 1$
$1^2 = 1$	$2^1 = 2$
$2^2 = 4$	$2^2 = 4$
$3^2 = 9$	$2^3 = 8$
$4^2 = 16$	$2^4 = 16$
$5^2 = 25$	$2^5 = 32$
$6^2 = 36$	$2^6 = 64$
$7^2 = 49$	$2^7 = 128$
$8^2 = 64$	$2^8 = 256$
$9^2 = 81$	$2^9 = 512$
$10^2 = 100$	$2^{10} = 1024$

n^2 versus 2^n

$0^2 = 0$	$<$	$2^0 = 1$
$1^2 = 1$	$<$	$2^1 = 2$
$2^2 = 4$	$=$	$2^2 = 4$
$3^2 = 9$	$>$	$2^3 = 8$
$4^2 = 16$	$=$	$2^4 = 16$
$5^2 = 25$	$<$	$2^5 = 32$
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2^n is <u>much</u> bigger here. Does the trend continue?

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$$(n + 1)^2 = n^2 + 2n + 1$$

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Since $n \geq 5$, we

$(n + 1)^2 = n$ …

$< n$ … $5 \leq n)$

$= n$ …

$< n^2 + n^2$ *(since $3n < 5n \leq n^2$)*

$= 2n^2$

Why is this allowed?

So $(n + 1)^2 < 2n^2$. Now, by our inductive hypothesis, we know that $n^2 < 2^n$. This means that

$(n + 1)^2 < 2n^2$ *(from above)*

$< 2(2^n)$ *(by the inductive hypothesis)*

$= 2^{n+1}$

Completing the induction. ■

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Remember: $A \to B$ means "whenever A is true, B is true." If B is always true, $A \to B$ is true for any A.

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- We explicitly proved $P(5)$, so $P(4) \to P(5)$

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Again, $A \to B$ is automatically true if B is always true.

Why is this Legal?

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- $P(4)$ is trivially true, so $P(3) \to P(4)$
- We explicitly proved $P(5)$, so $P(4) \to P(5)$
- For any $n \geq 5$, we explicitly proved that $P(n) \to P(n + 1)$.

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- Let $P(n)$ be "Either $n < 5$ or $n^2 < 2^n$."
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- $P(4)$ is trivially true, so $P(3) \to P(4)$
- We explicitly proved $P(5)$, so $P(4) \to P(5)$
- For any $n \geq 5$, we explicitly proved that $P(n) \to P(n + 1)$.
- Thus $P(0)$ and for any $n \in \mathbb{N}$, $P(n) \to P(n + 1)$, so by induction $P(n)$ is true for all natural numbers n.

Induction Starting at k

- To prove that $P(n)$ is true for all natural numbers greater than or equal to k:
  - Show that $P(k)$ is true.
  - Show that for any $n \geq k$, that $P(n) \to P(n + 1)$.
  - Conclude $P(k)$ holds for all natural numbers greater than or equal to k.
- You don't need to justify why it's okay to start from k.

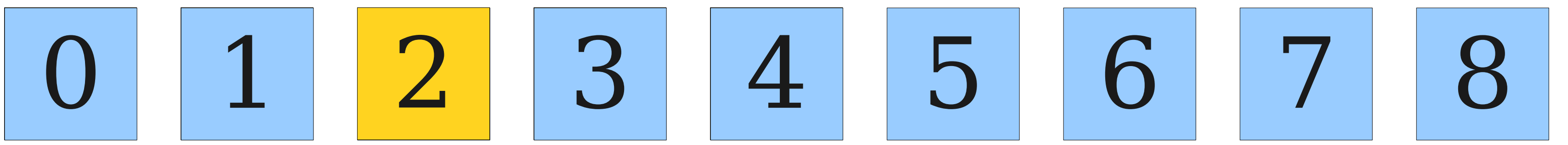
An Important Observation

One Major Catch

One Major Catch

One Major Catch

One Major Catch



One Major Catch

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0 1 2 3 4 5 6 7 8

In an inductive proof, to prove P(5), we can only assume P(4). We cannot rely on any of our earlier results!

Strong Induction

The **principle of strong induction** states that if for some property $P(n)$, we have that

$\boldsymbol{P(0)}$ **is true**

and

For any $\boldsymbol{n \in \mathbb{N}}$ **with** $\boldsymbol{n \neq 0}$**,**
if $\boldsymbol{P(n')}$ **is true for all** $\boldsymbol{n' < n}$**, then**
$\boldsymbol{P(n)}$ **is true**

then

For any $\boldsymbol{n \in \mathbb{N}}$**,** $\boldsymbol{P(n)}$ **is true.**

The **principle of strong induction** states that if for some property $P(n)$, we have that

$P(0)$ is true

Assume that P(n) holds for all natural numbers smaller than n.

and

For any $n \in \mathbb{N}$ with $n \neq 0$, if $P(n')$ is true for all $n' < n$, then $P(n)$ is true

then

For any $n \in \mathbb{N}$, $P(n)$ is true.

Using Strong Induction

0	1	2	3	4	5	6	7	8

Using Strong Induction

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0	1	2	3	4	5	6	7	8

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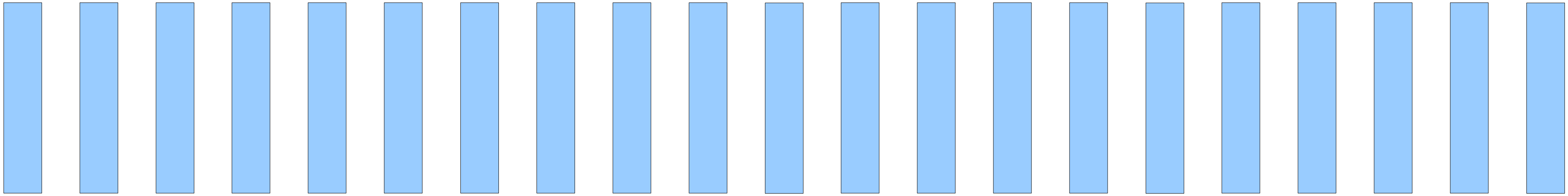
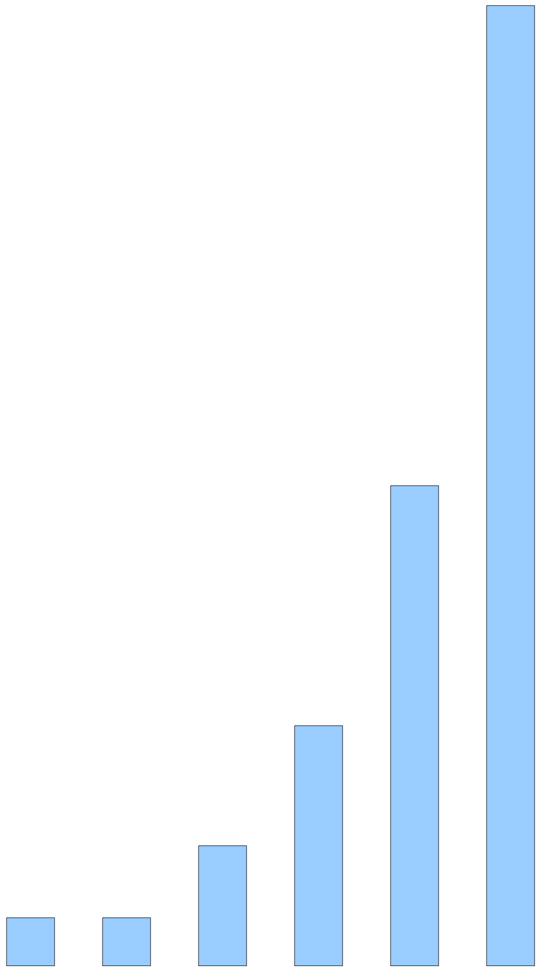
0	1	2	3	4	5	6	7	8

Using Strong Induction

Induction and Dominoes



Strong Induction and Dominoes



Weak and Strong Induction

- **Weak induction** (regular induction) is good for showing that some property holds by incrementally adding in one new piece.
- **Strong induction** is good for showing that some property holds by breaking a large structure down into multiple small pieces.

Proof by Strong Induction

- State that you are attempting to prove something by strong induction.
- State what your choice of $P(n)$ is.
- Prove the base case:
  - State what $P(0)$ is, then prove it.
- Prove the inductive step:
  - State that you assume for all $0 \leq n' < n$, that $P(n')$ is true.
  - State what $P(n)$ is. *(this is what you're trying to prove)*
  - Go prove $P(n)$.

Application: **Binary Numbers**

Binary Numbers

- The **binary number system** is base 2.
- Every number is represented as 1s and 0s encoding various powers of two.
- Examples:
  - $100_2 = 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 4$
  - $11011_2 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 27$
- Enormously useful in computing; almost all computers do computation on binary numbers.
- Question: How do we know that every natural number can be written in binary?

Justifying Binary Numbers

- To justify the binary representation, we will prove the following result:

Every natural number *n*
can be expressed as the sum
of distinct powers of two.

- This says that there's *at least* one way to write a number in binary; we'd need a separate proof to show that there's *exactly* one way to do it.
- So how do we prove this?

One Proof Idea

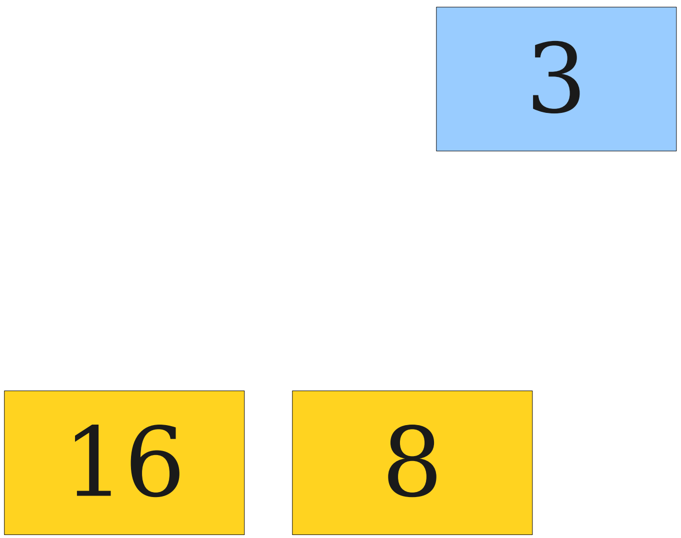
27

One Proof Idea

11

16

One Proof Idea



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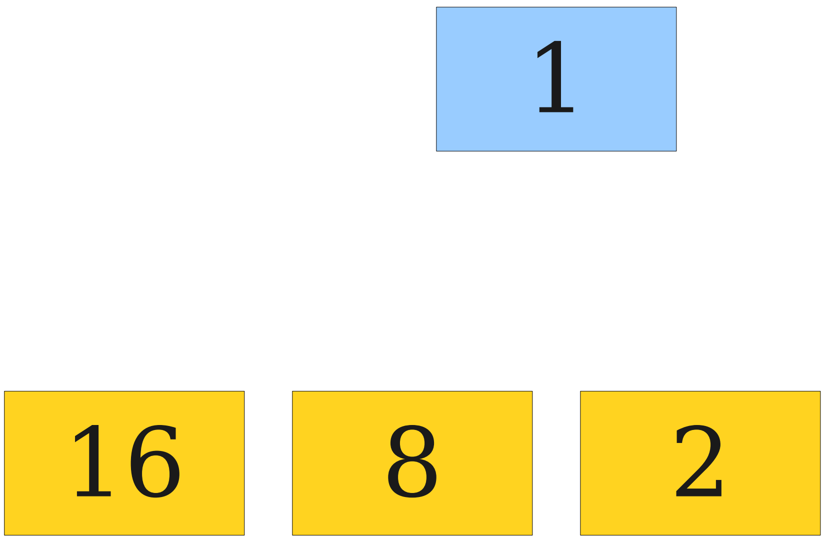

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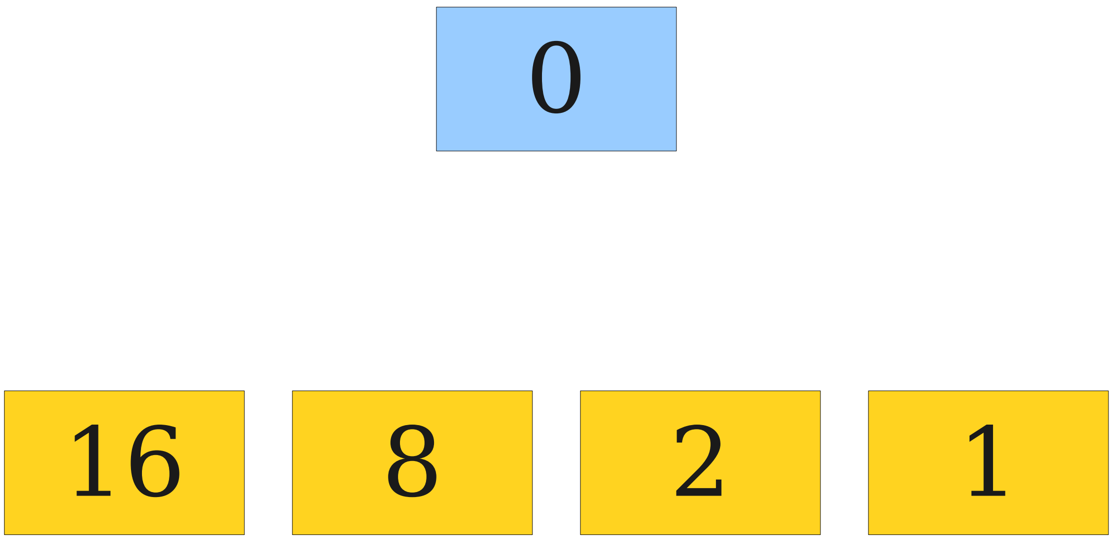

General Idea

- Repeatedly subtract out the largest power of two less than the number.
- Can't subtract 2^n twice for any n; otherwise, you could have subtracted 2^{n+1}.
- Eventually, we reach 0; the number is then the sum of the powers of two that we subtracted.
- How do we formalize this as a proof?

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Notice the stronger version of the induction hypothesis. We're now showing that **$P(n')$** is true for all natural numbers in the range **$0 \leq n' < n$**. We'll use this fact later on.

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Let 2^k be the greatest power of two such that $2^k \leq n$. Consider $n - 2^k$.

Here's the key step of the proof. If we can show that

$$\mathbf{0 \leq n - 2^k < n}$$

then we can use the inductive hypothesis to claim that $\mathbf{n - 2^k}$ is a sum of distinct powers of two.

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Here is where strong induction kicks in. We use the fact that any smaller number can be written as the sum of distinct powers of two to show that $\boldsymbol{n - 2^k}$ can be written as the sum of distinct powers of two.

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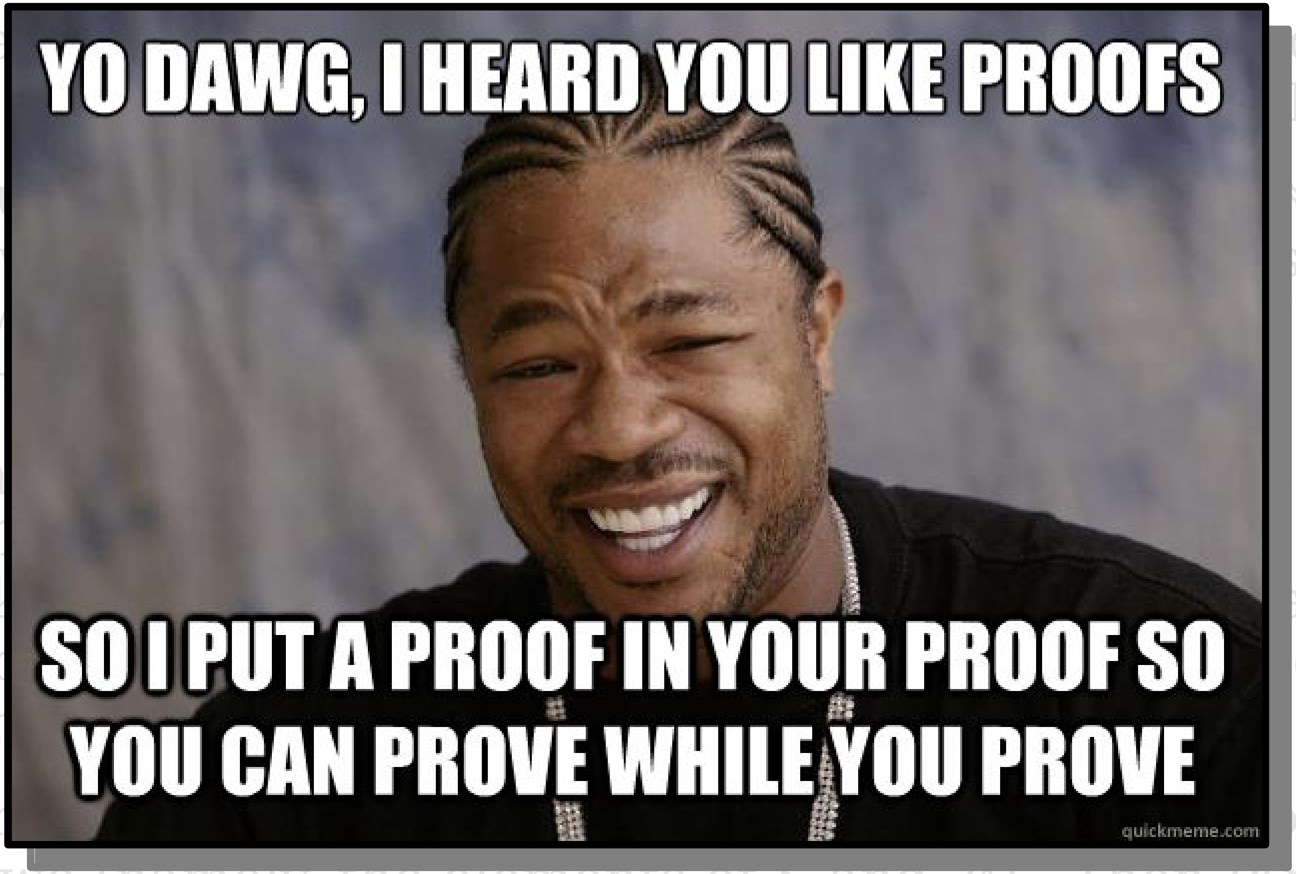

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Application: **Continued Fractions**

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Continued Fractions

$$\frac{1}{4 + \frac{1}{1 + \frac{1}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\cfrac{3}{2}}}$$

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Continued Fractions

$$\frac{1}{\frac{14}{3}}$$

Continued Fractions

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Continued Fractions

$$\frac{3}{14}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{2}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{2}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$\frac{137}{42}$$

Continued Fractions

- A **continued fraction** is an expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}}}$$

- Formally, a continued fraction is either
  - An integer *n*, or
  - *n* + 1 / *F*, where *n* is an integer and *F* is a continued fraction.
- Continued fractions have numerous applications in number theory and computer science.
- (They're also really fun to write!)

Fun with Continued Fractions

- Every rational number, including negative rational numbers, has a continued fraction representation.
- Harder result: every *irrational* number has an (infinite) continued fraction representation.
- Even harder result: If we truncate an infinite continued fraction for an irrational number, we can get progressively better approximations of that number.

π as a Continued Fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}}}}}}$$

Approximating π

Approximating π

$\pi = 3$

3 = **3**.0000...

Approximating π

$\pi = 3$

$3 = \mathbf{3}.0000\ldots$

> And he made the Sea of cast bronze, ten cubits from one brim to the other; it was completely round. [… A] line of thirty cubits measured its circumference.
>
> 1 Kings 7:23, New King James Translation

Approximating π

$$\pi = 3 + \cfrac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Approximating π

$$\pi = 3 + \cfrac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Greek mathematician **Archimedes** knew of this approximation, circa 250 BCE

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

Chinese mathematician 祖沖之 (Zu Chongzhi) discovered this approximation in the early fifth century; this was the best approximation of pi for over a thousand years.

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

3 = **3**.0000...

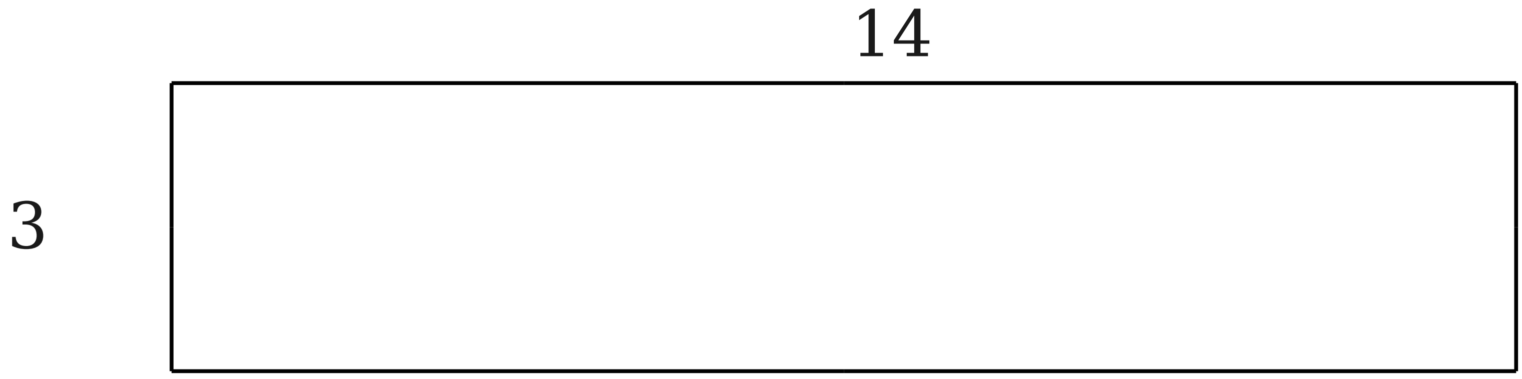
22/7 = **3.14**2857...

336/106 = **3.1415**094...

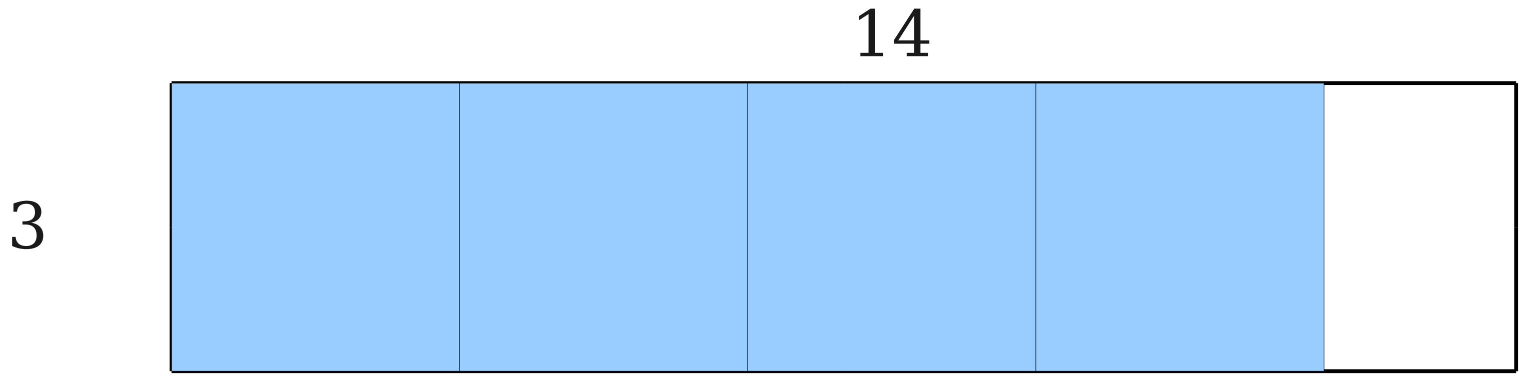
355/113 = **3.141592**92...

103993/33102 = **3.141592653**0...

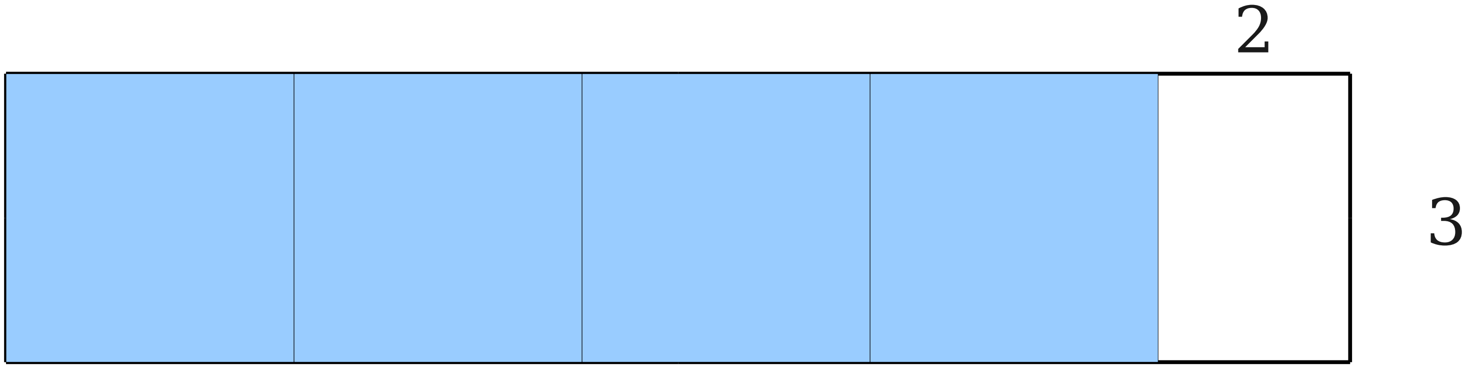
More Continued Fractions



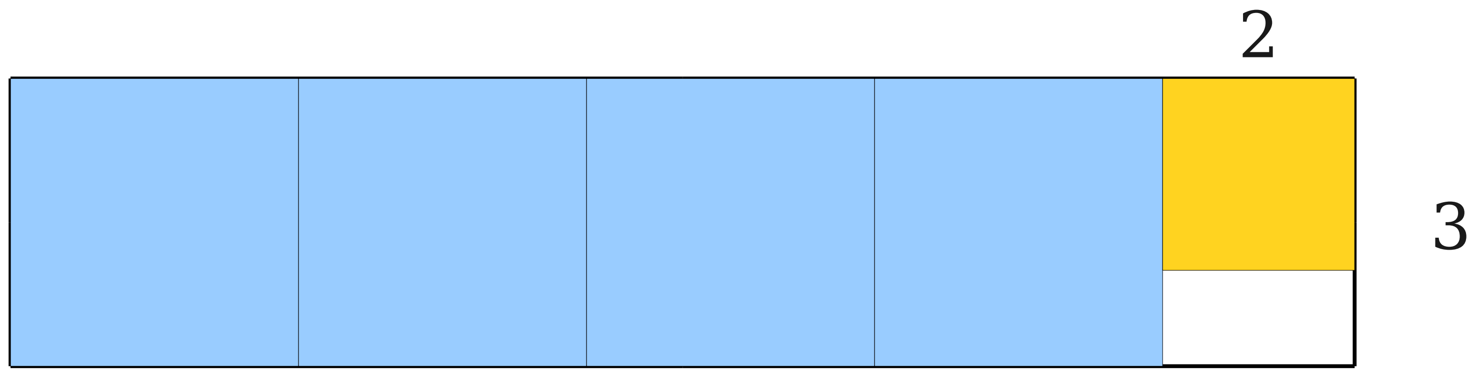
More Continued Fractions



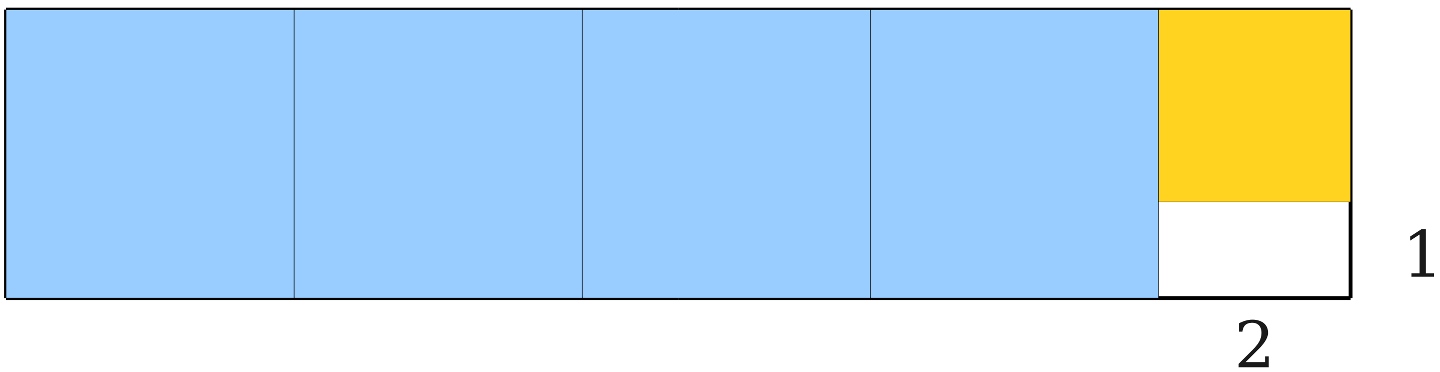
More Continued Fractions



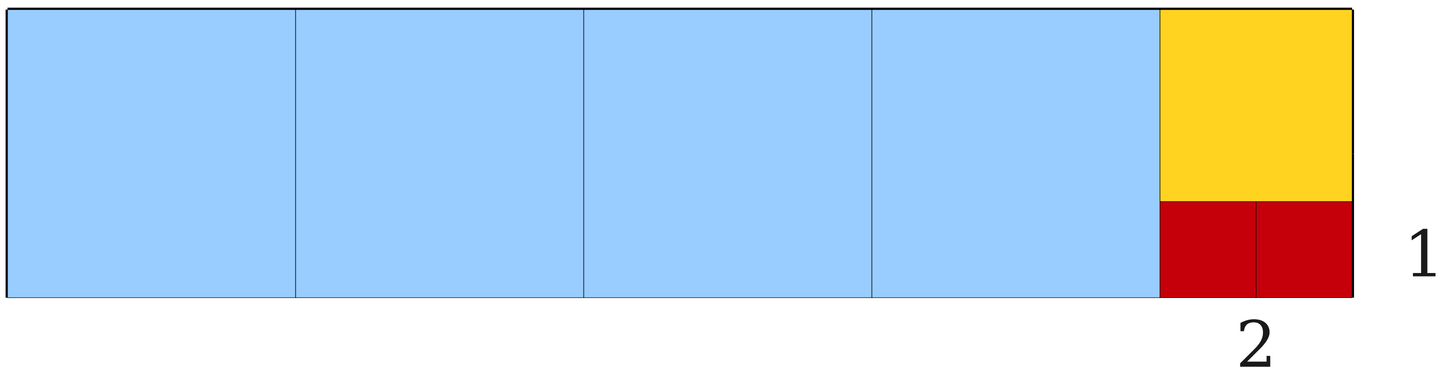
More Continued Fractions



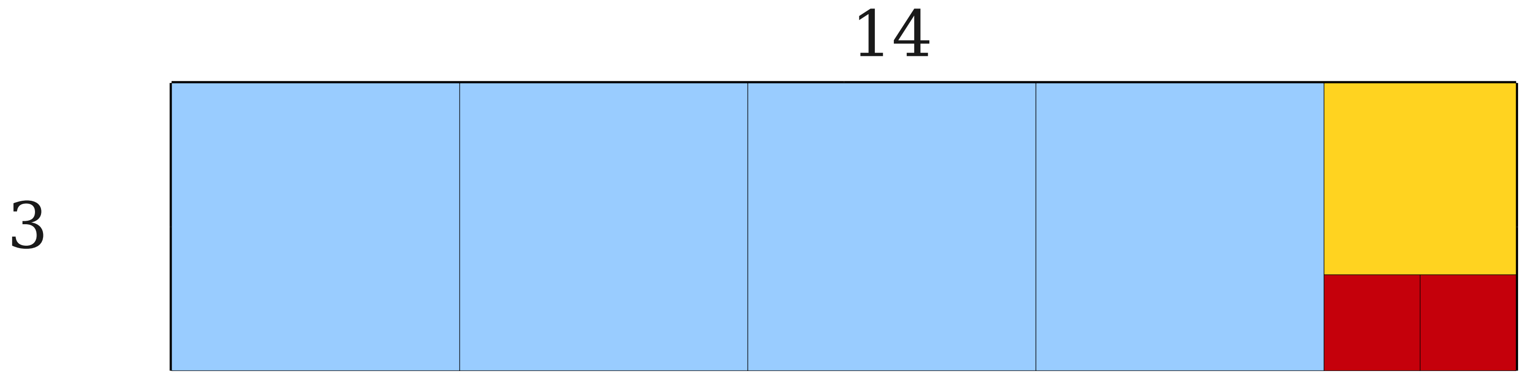
More Continued Fractions



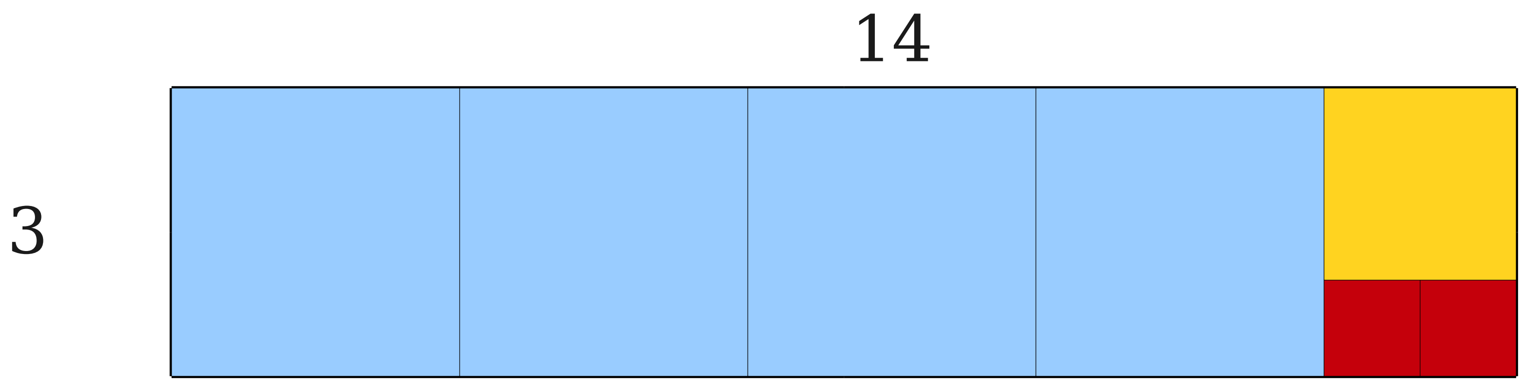
More Continued Fractions



More Continued Fractions

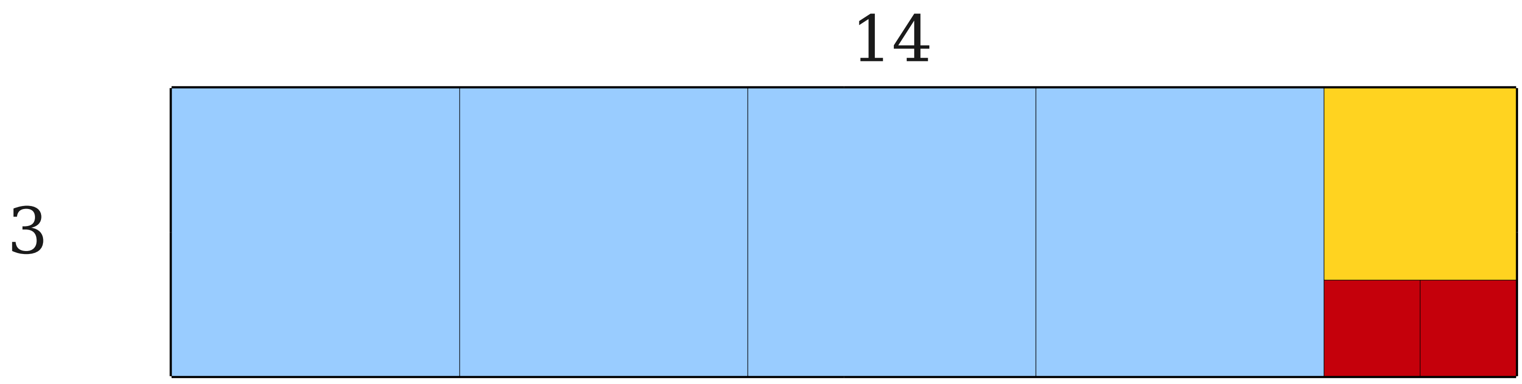


More Continued Fractions



$$\frac{3}{14} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

More Continued Fractions

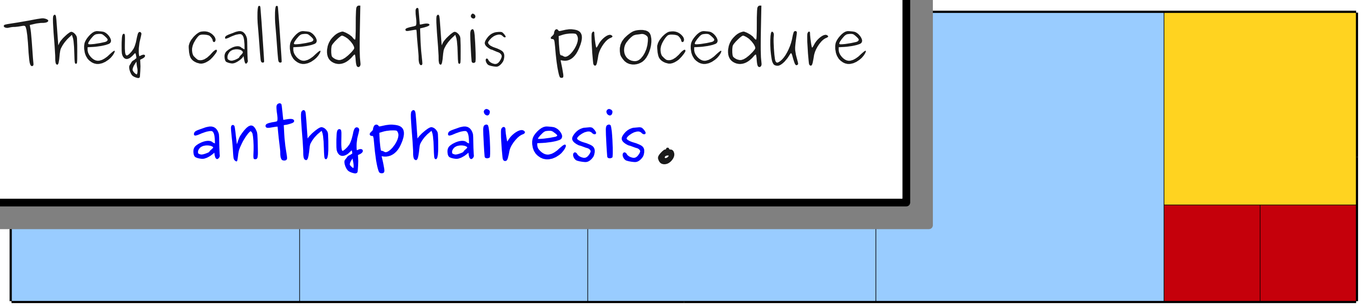


$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

More Continued Fractions

The Ancient Greeks knew about this connection. They called this procedure anthyphairesis.

3



$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}}}}}$$

An Interesting Continued Fraction

$x = 1$

An Interesting Continued Fraction

$$x = 1 + \frac{1}{1\cdot}$$

1 / 1

2 / 1

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1}}$$

1 / 1

2 / 1

3 / 2

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1.}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

21 / 13

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}$$

1 / 1
2 / 1
3 / 2
5 / 3
8 / 5
13 / 8
21 / 13
34 / 21

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}$$

1 / 1
2 / 1
3 / 2
5 / 3
8 / 5
13 / 8
21 / 13
34 / 21

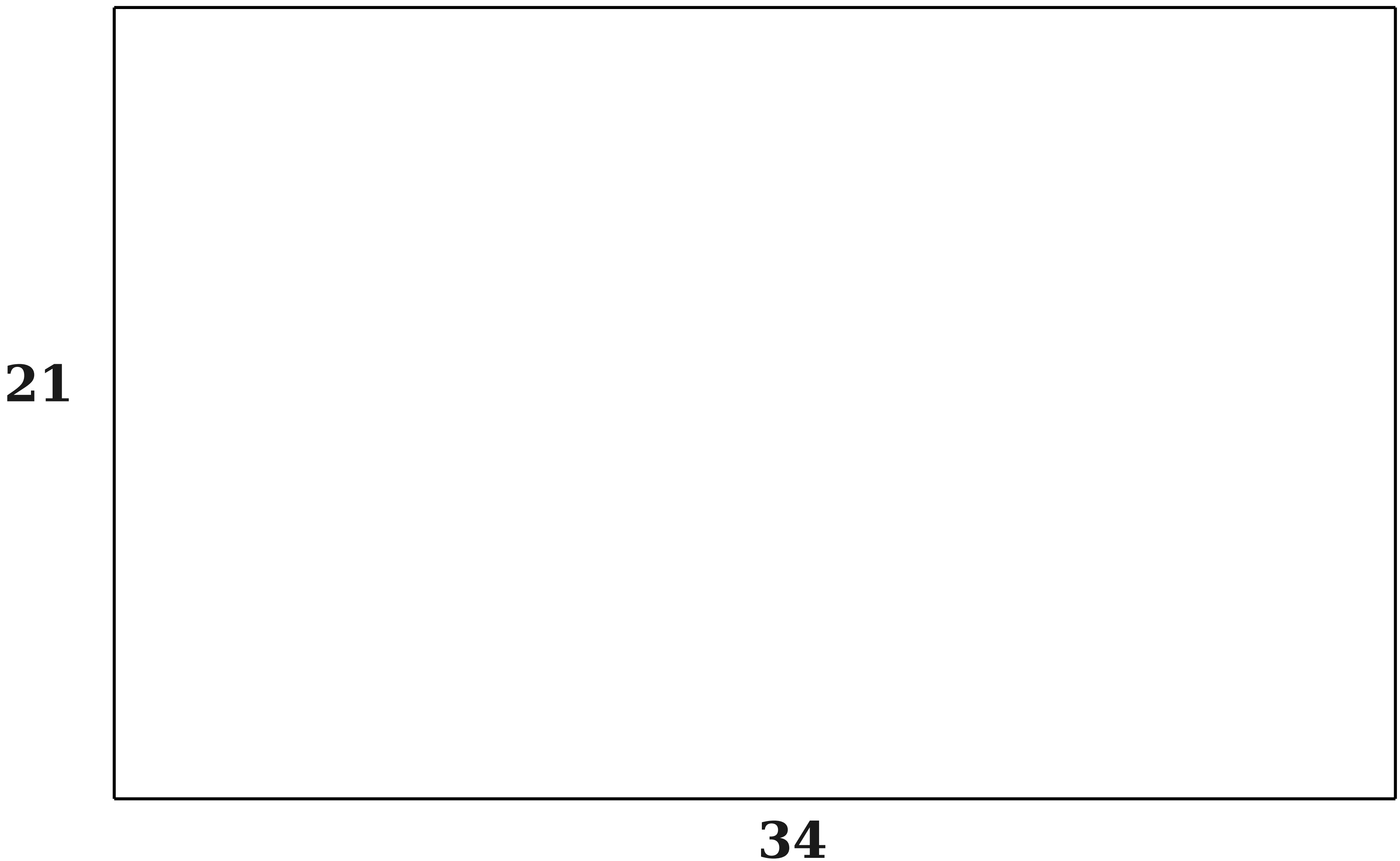
Each fraction is the ratio of consecutive Fibonacci numbers!

The Golden Ratio

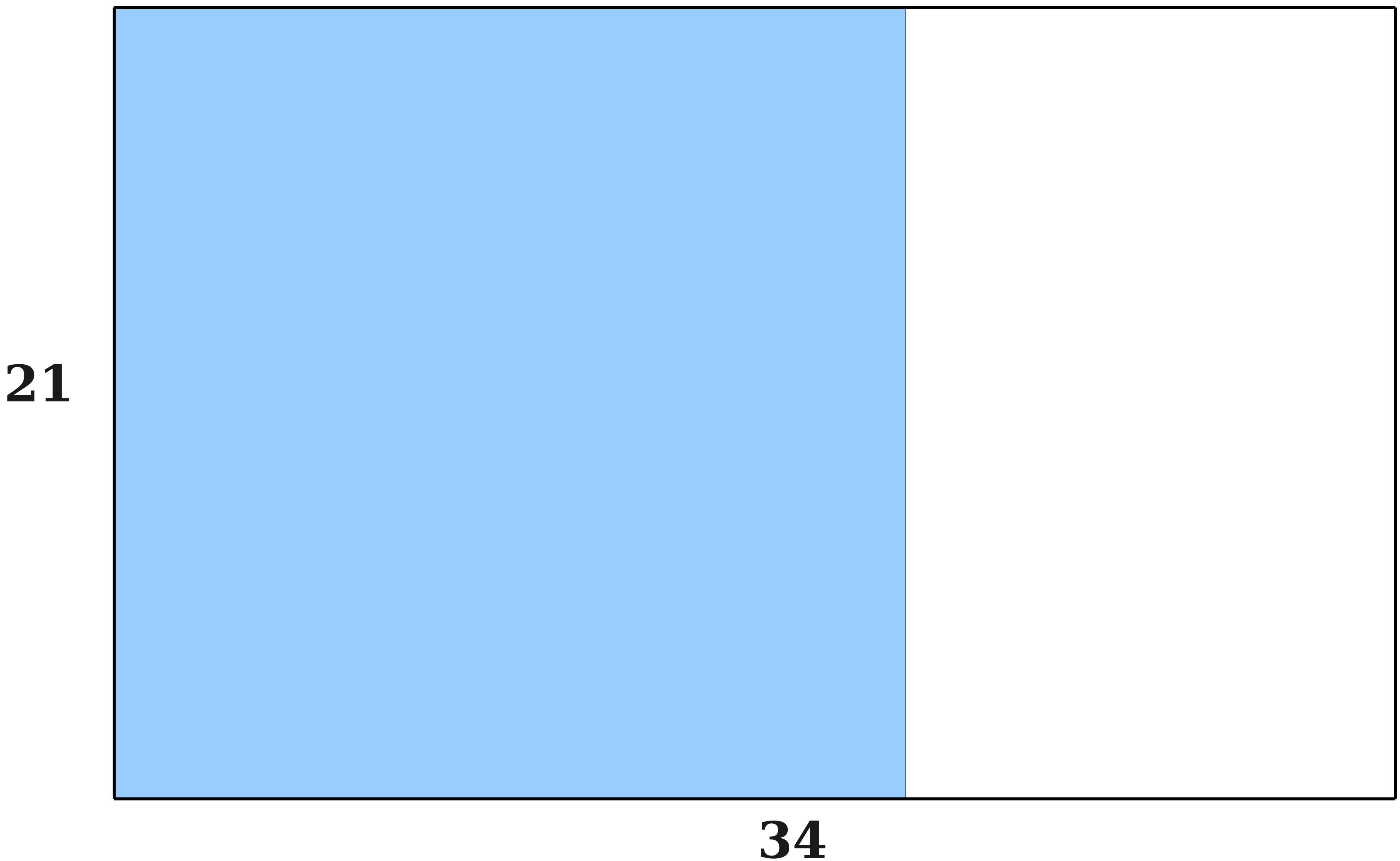
$$\varphi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

$$\varphi \approx 1.61803399$$

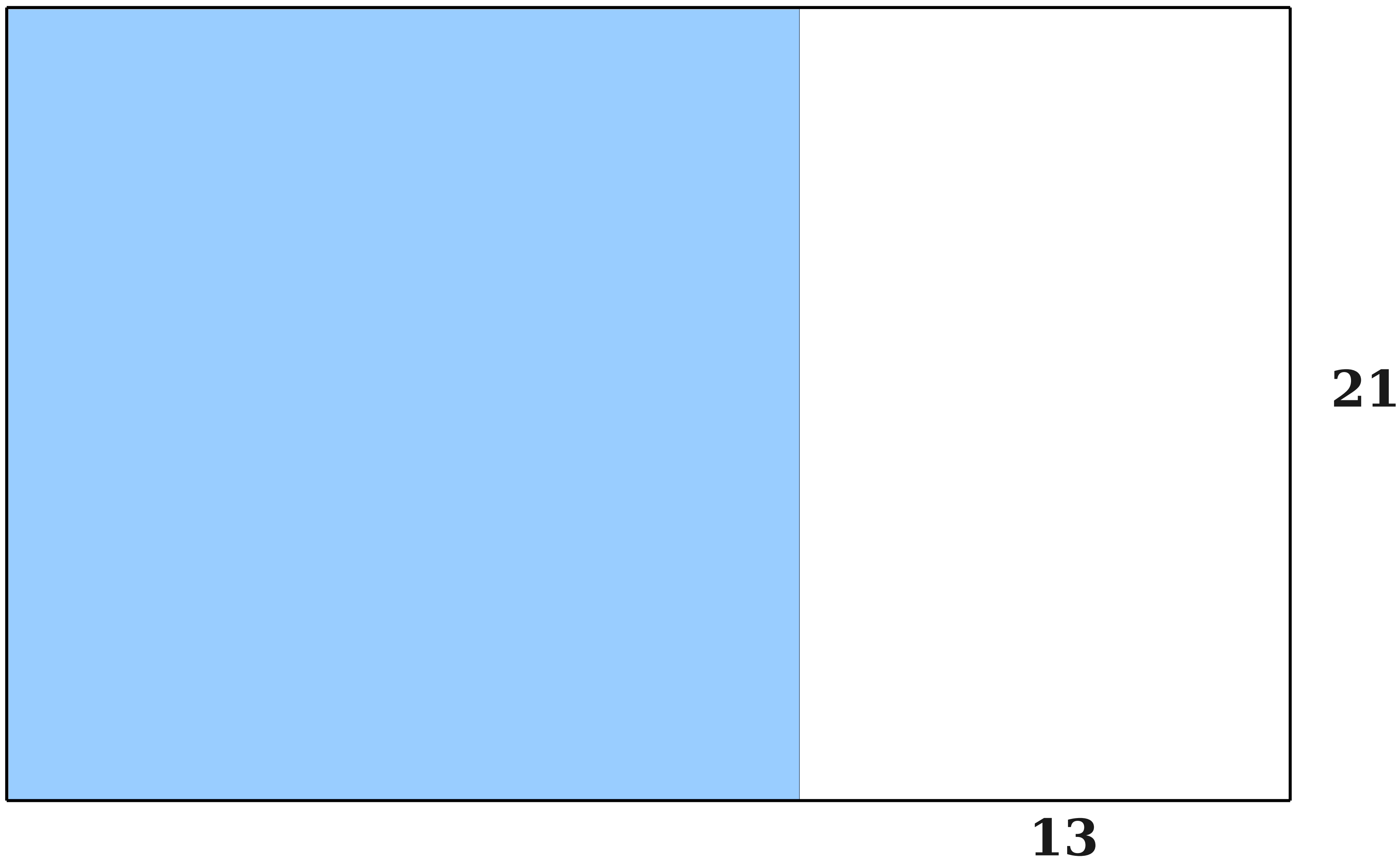
The Golden Ratio



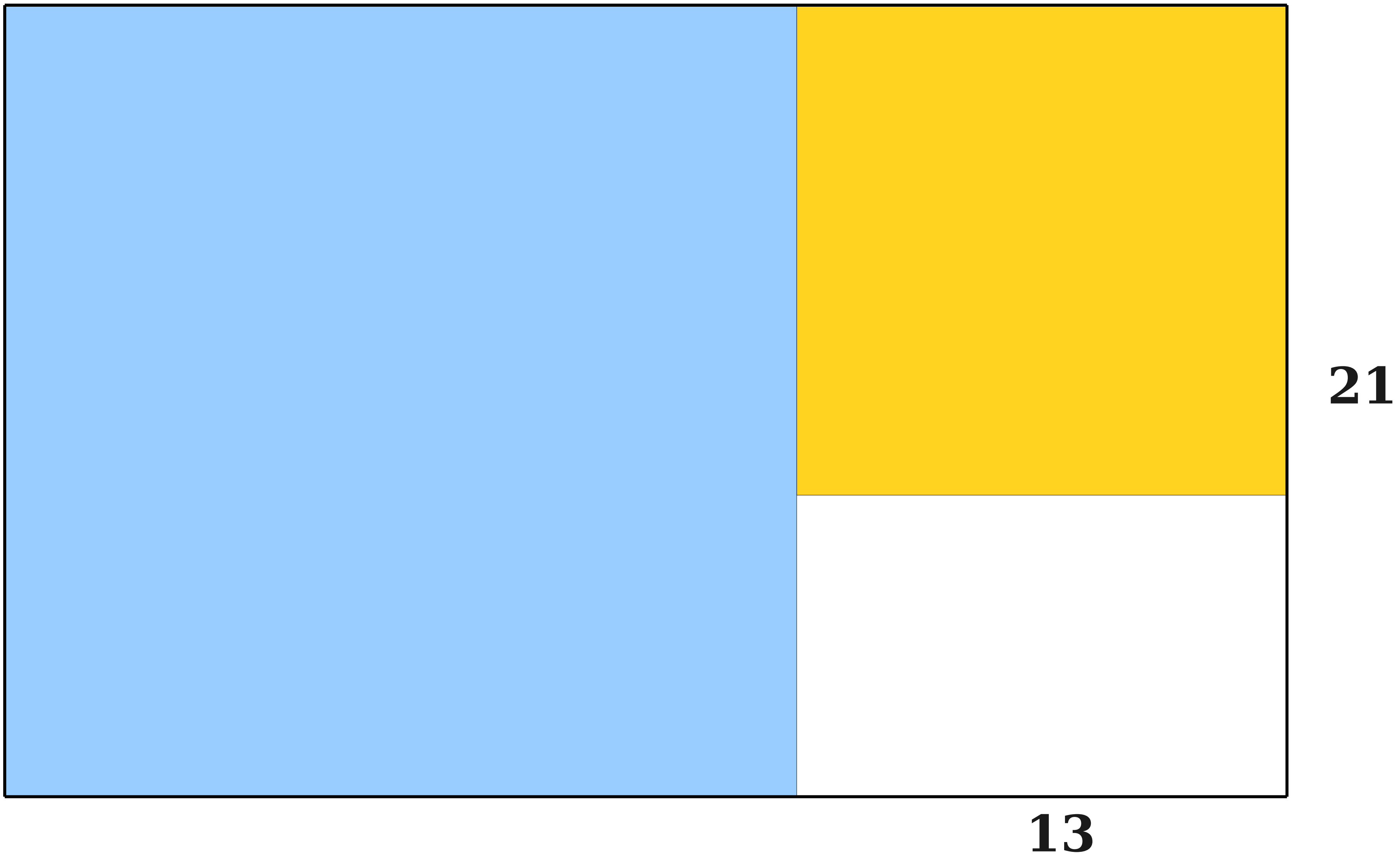
The Golden Ratio



The Golden Ratio



The Golden Ratio



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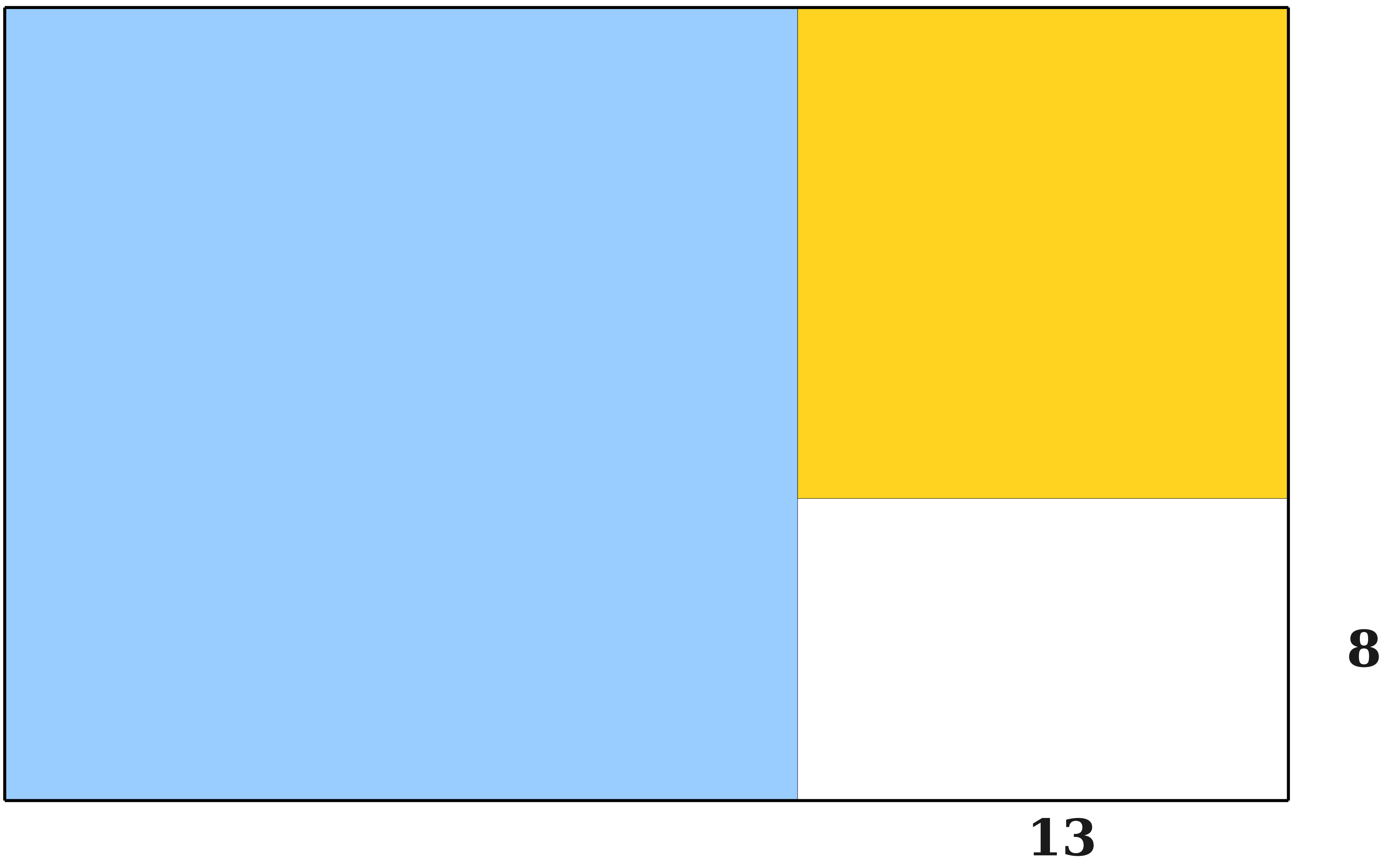

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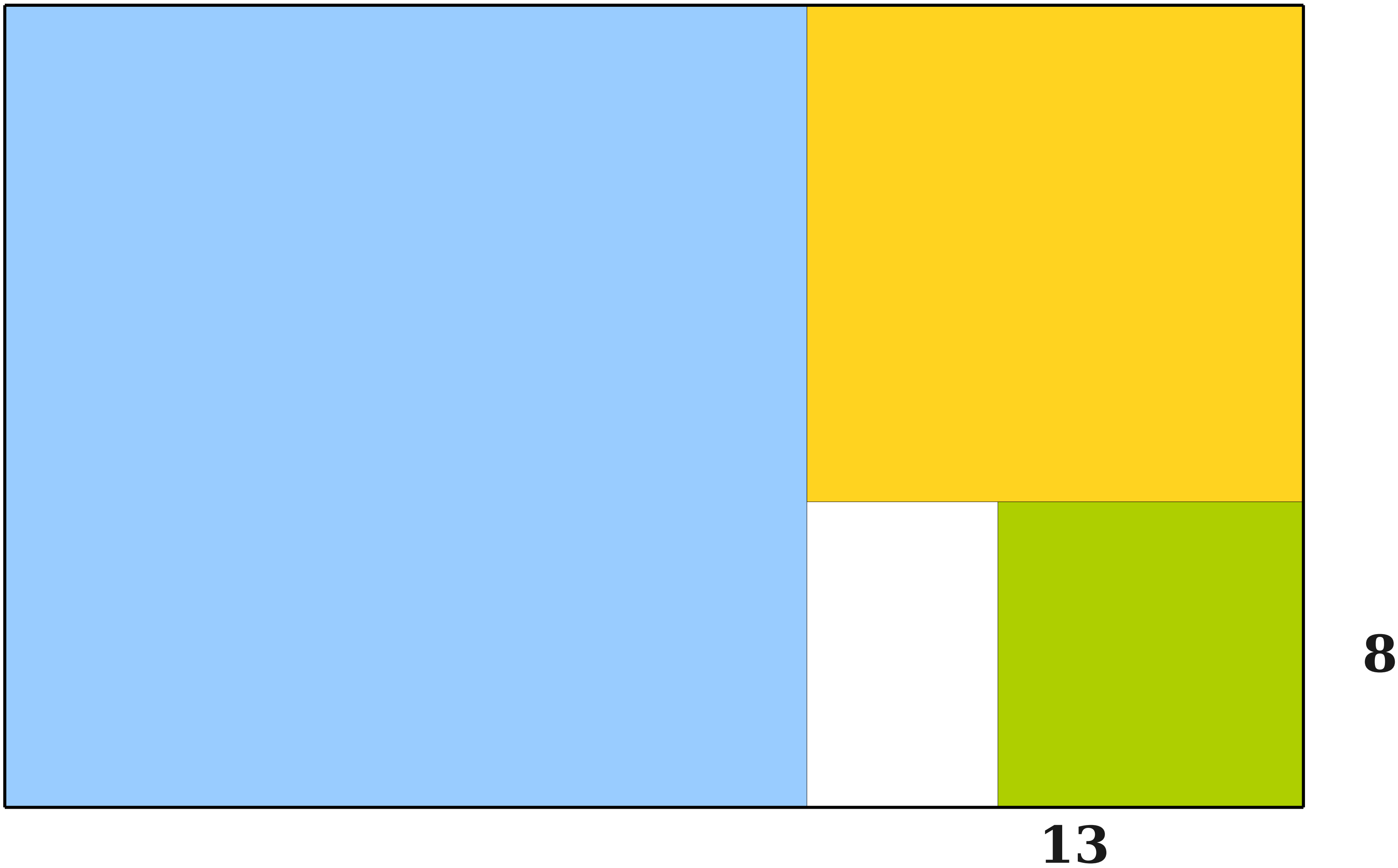

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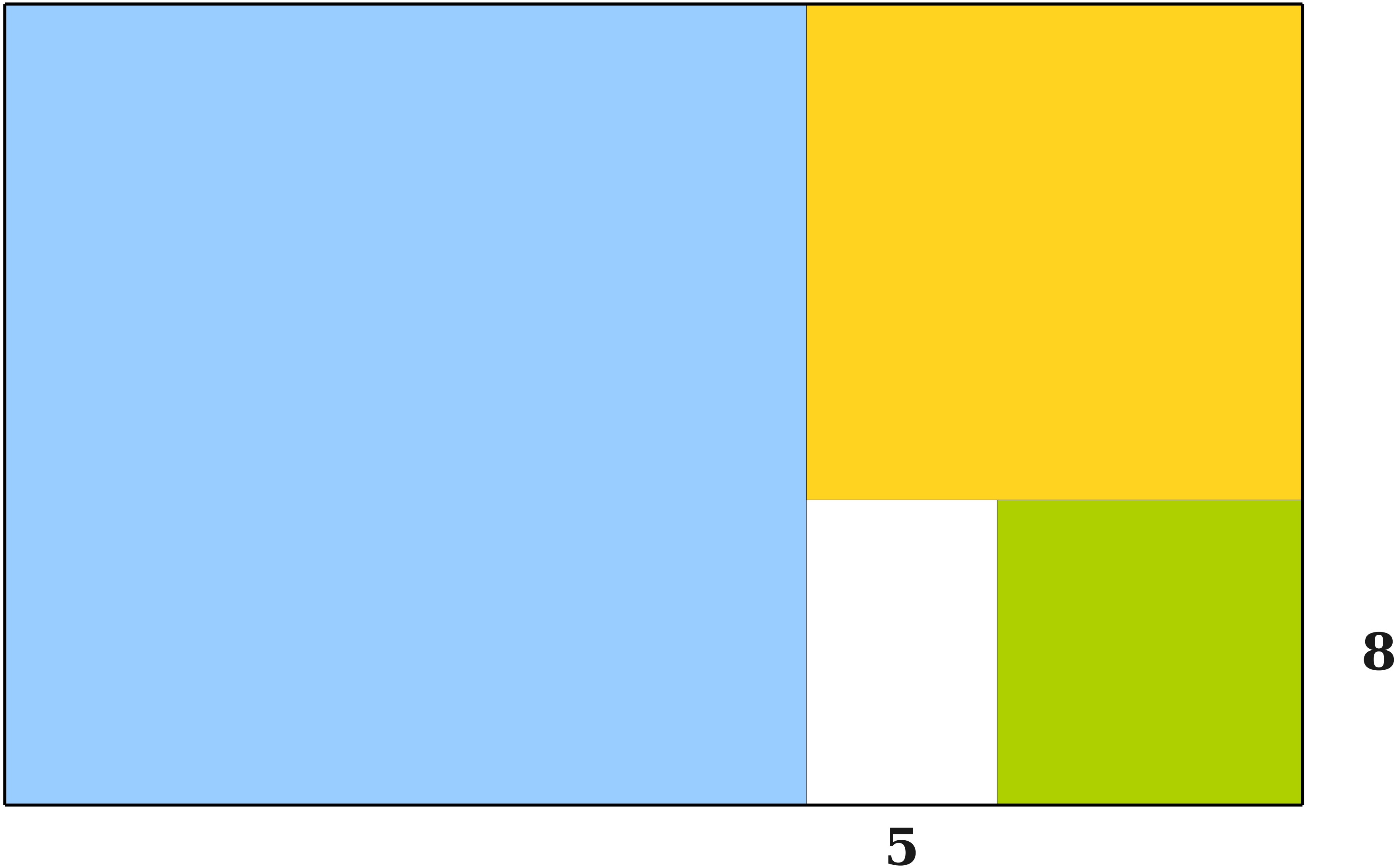
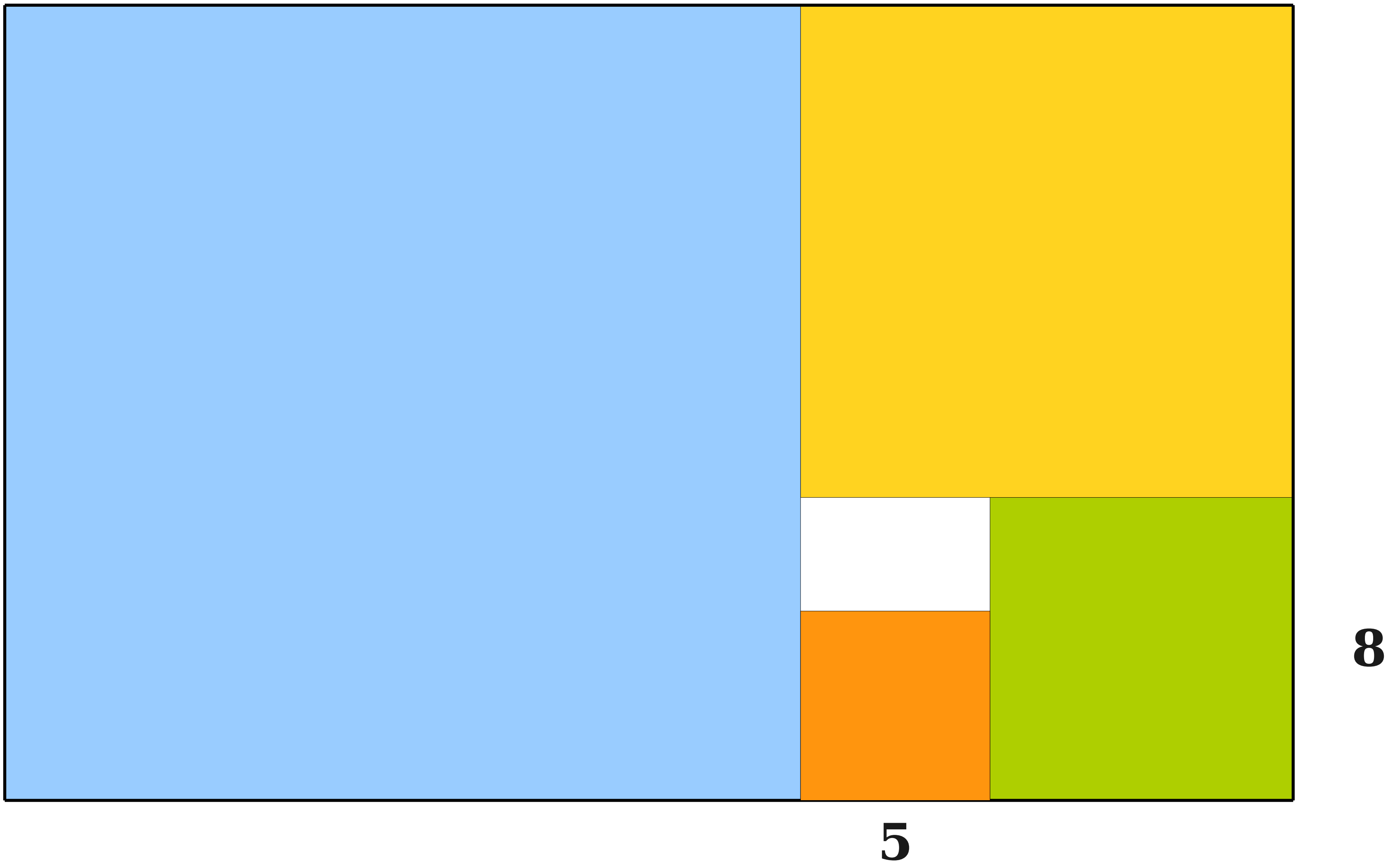


The Golden Ratio



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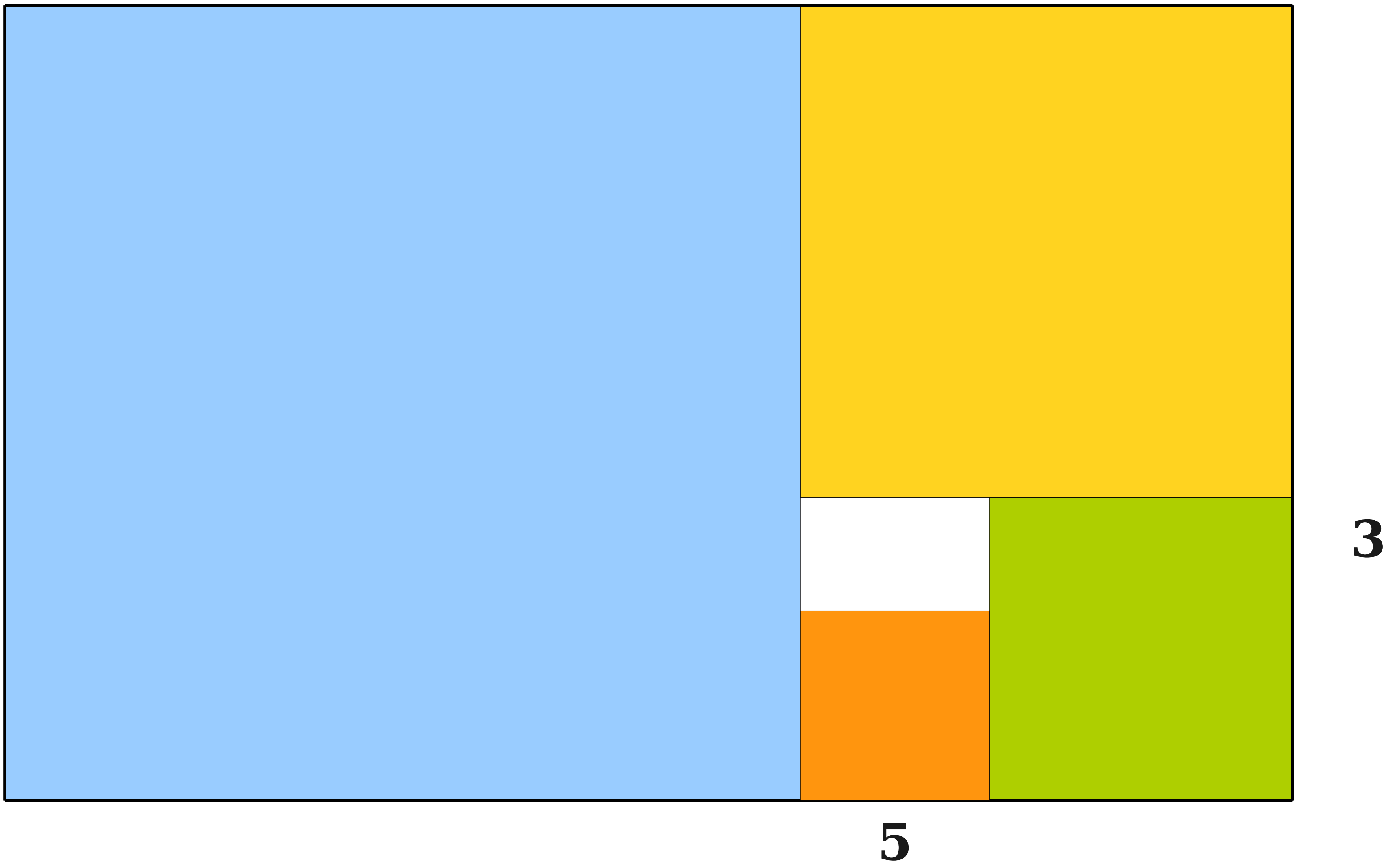
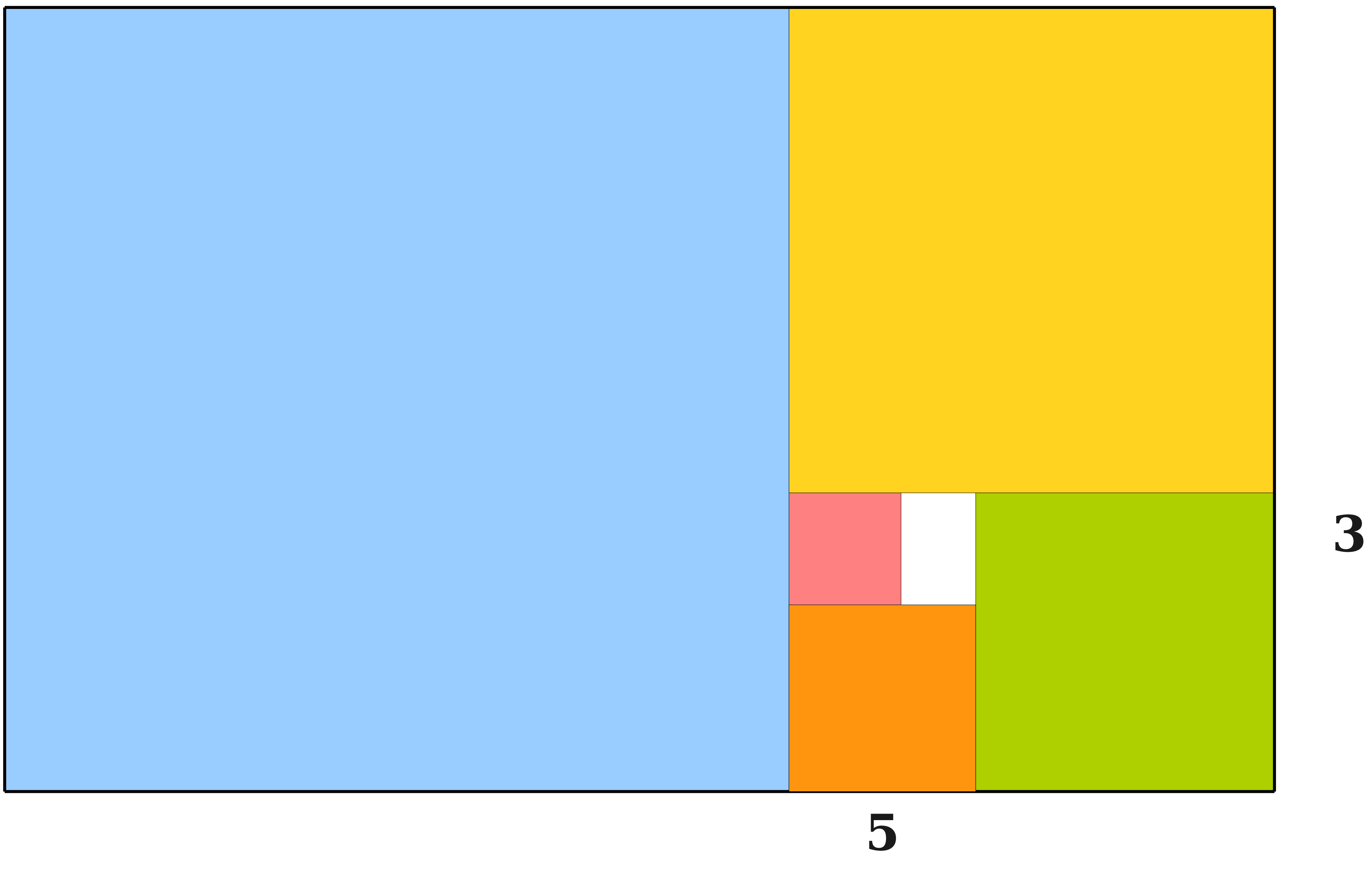
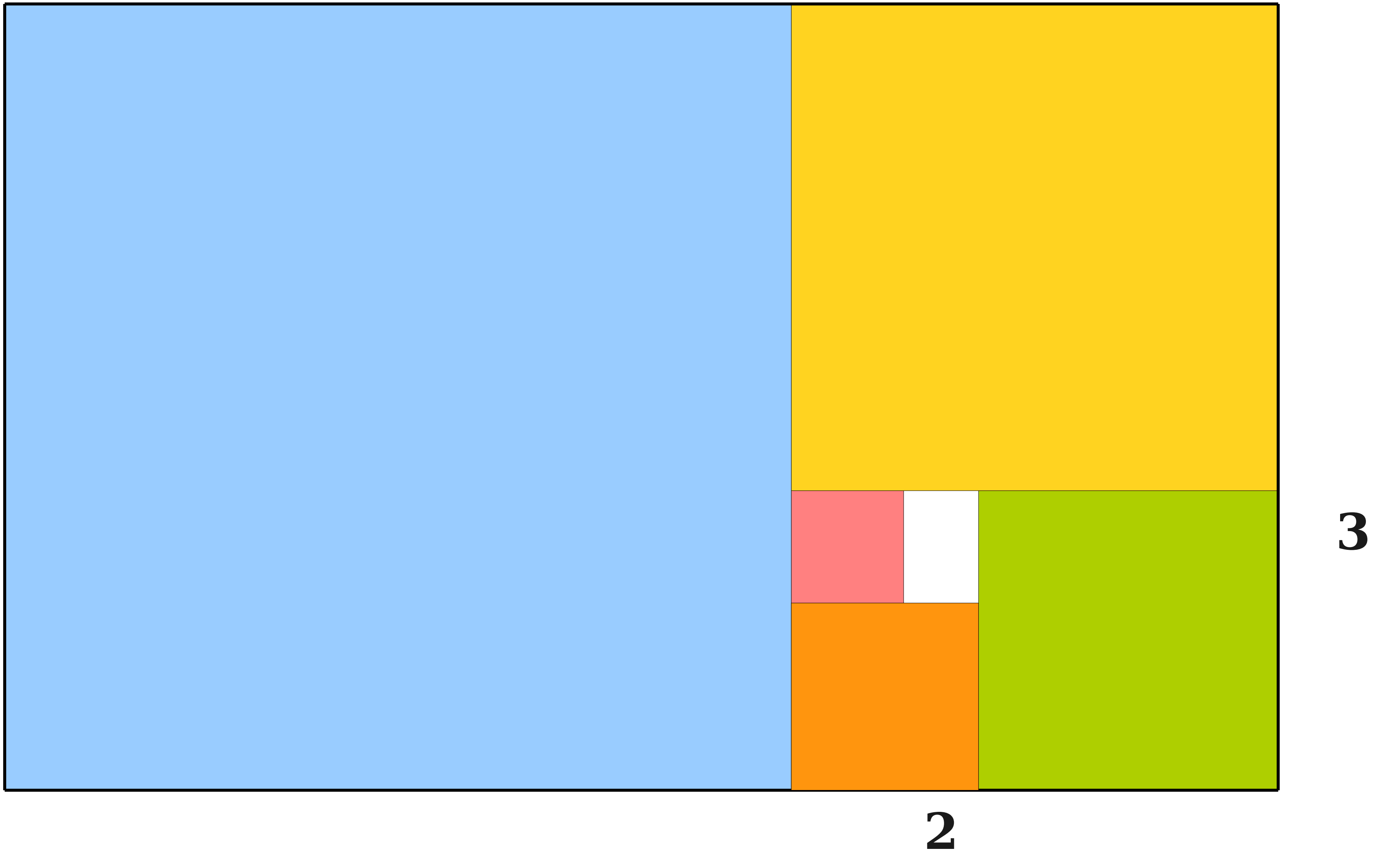


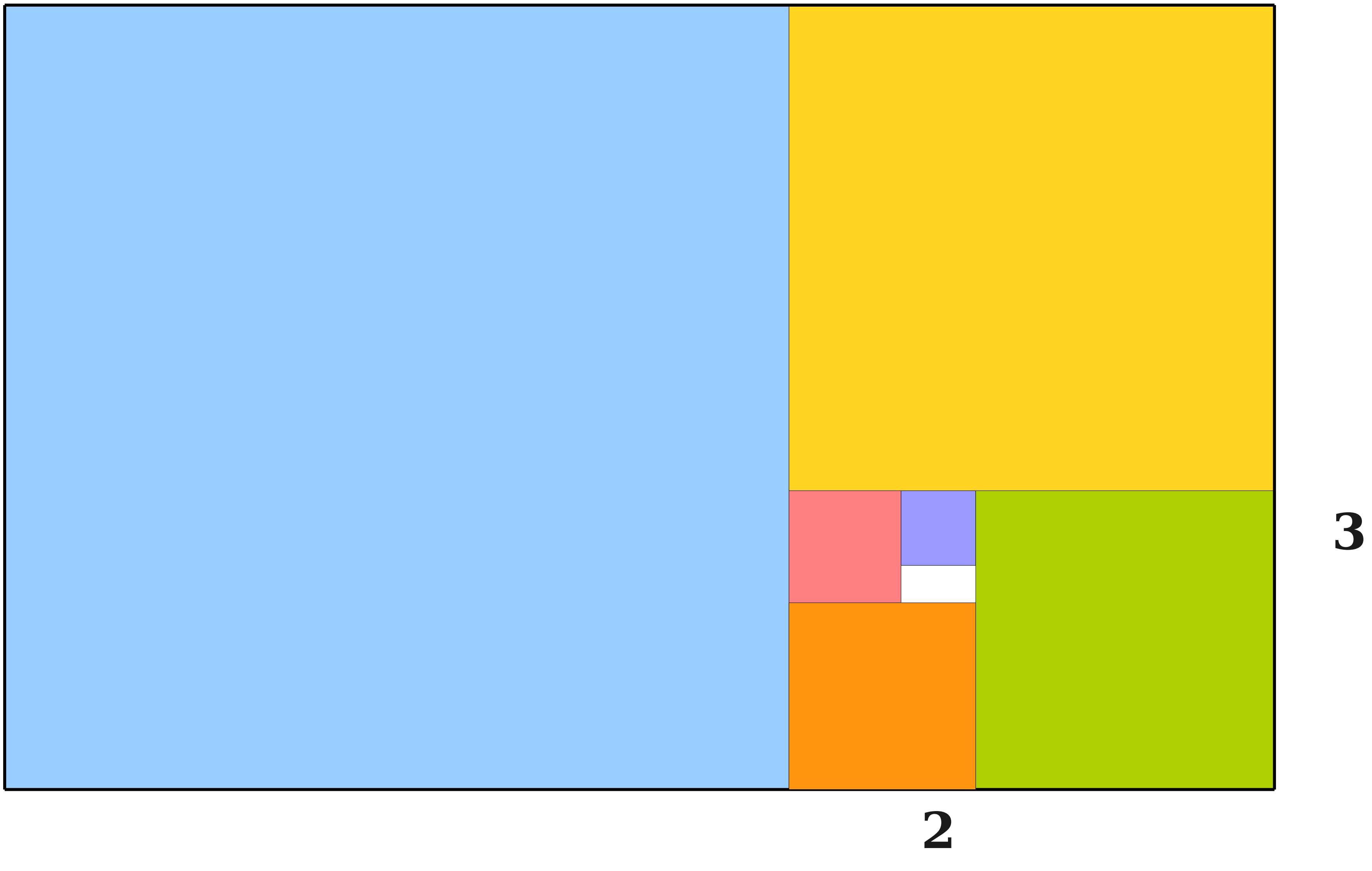
The Golden Ratio



The Golden Ratio



The Golden Ratio



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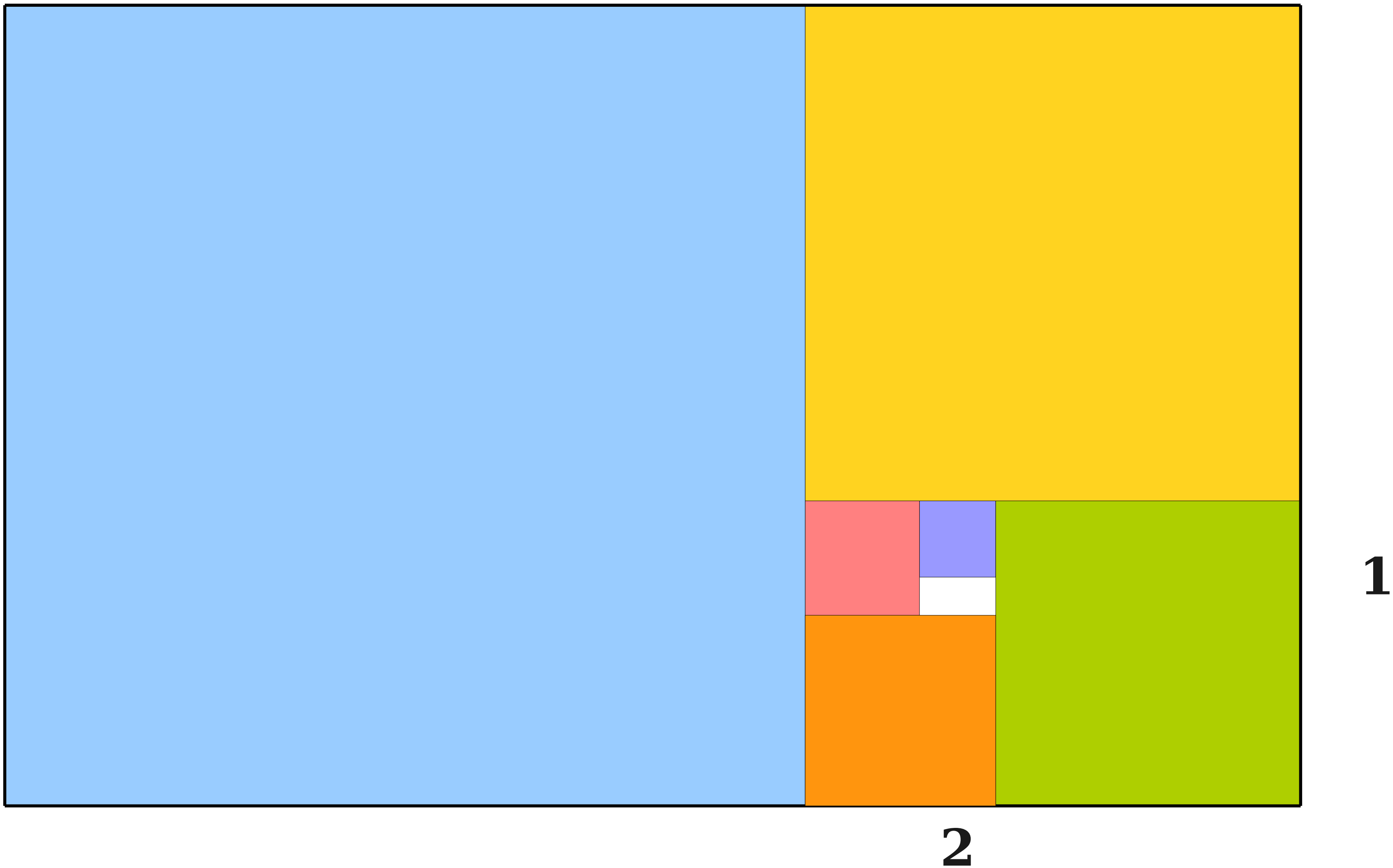

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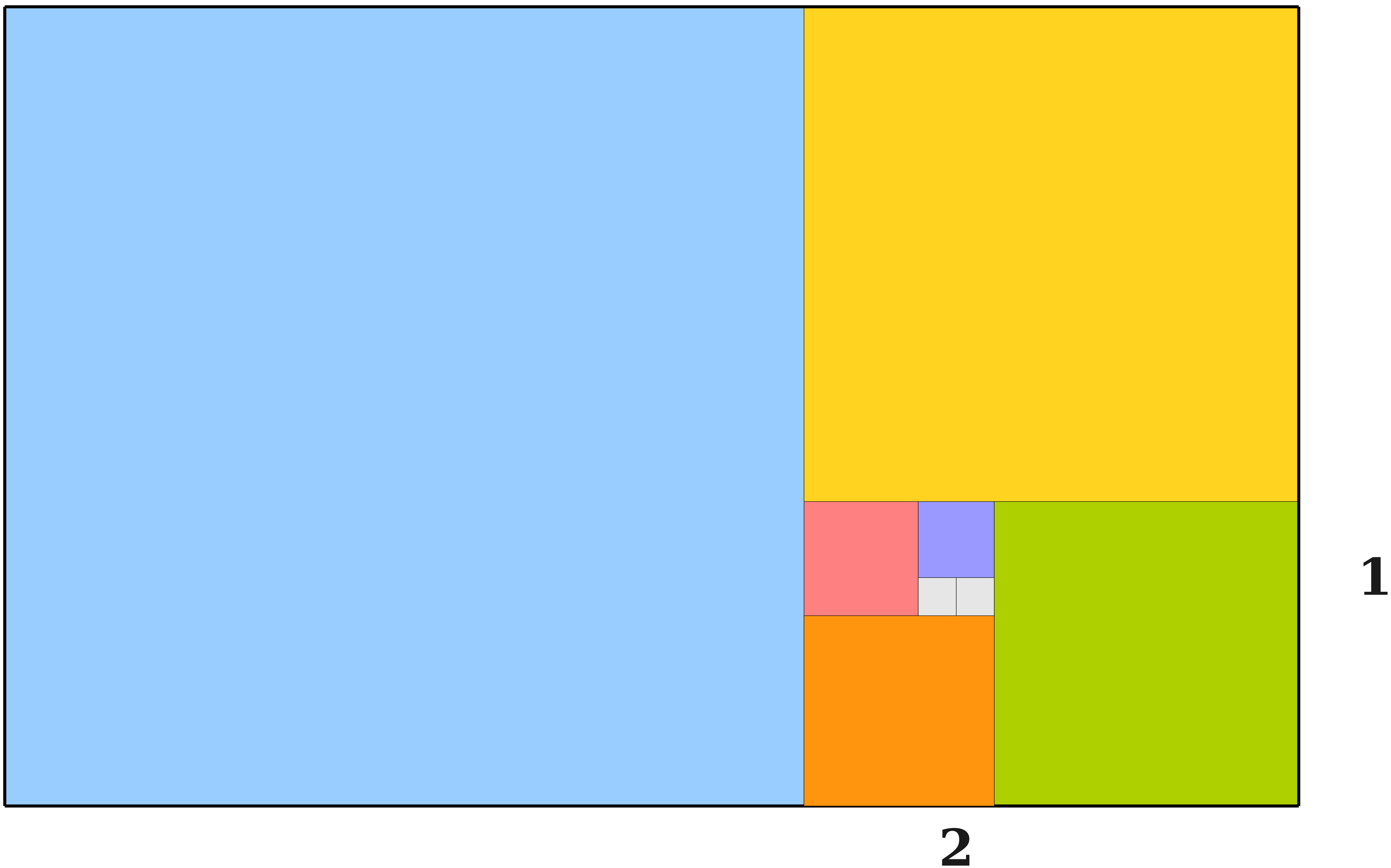
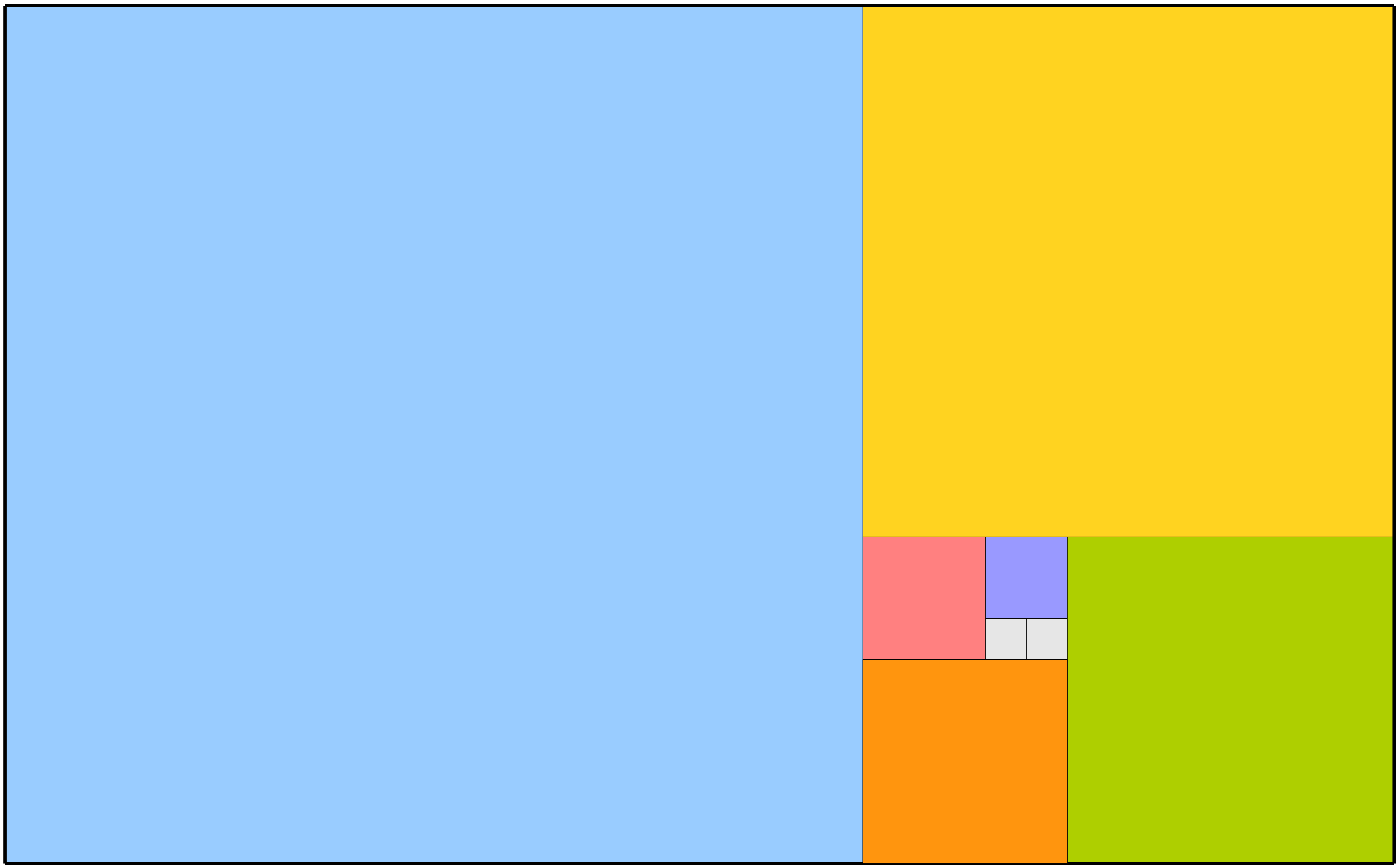
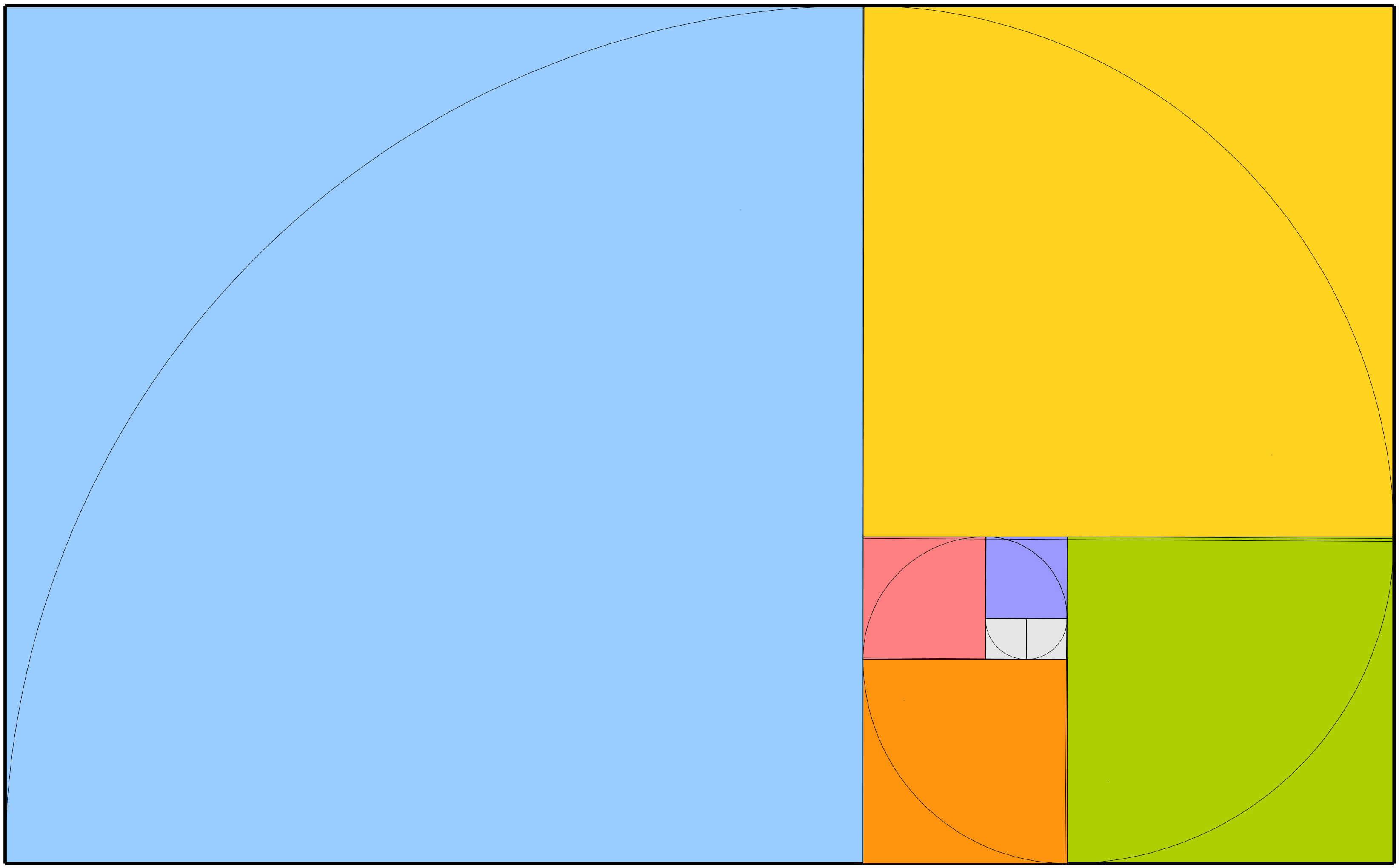


The Golden Ratio



The Golden Spiral



How do we prove all rational numbers have continued fractions?

Constructing a Continued Fraction

$$\frac{7}{9}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \frac{7}{9}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\frac{9}{7}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\frac{9}{7}}$$

$$\frac{9}{7}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \frac{1}{\frac{9}{7}}$$

$$\frac{9}{7} = 1 + \frac{2}{7}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \frac{1}{\frac{9}{7}}$$

$$\frac{9}{7} = 1 + \frac{1}{\frac{7}{2}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\frac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\frac{7}{2}}$$

$$\frac{7}{2}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\cfrac{7}{2}}$$

$$\frac{7}{2} = 3 + \frac{1}{2}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\frac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\frac{7}{2}}$$

$$\frac{7}{2} = 3 + \frac{1}{2}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \frac{1}{\frac{9}{7}}$$

$$\frac{9}{7} = 1 + \frac{1}{3 + \frac{1}{2}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \frac{1}{\frac{9}{7}}$$

$$\frac{9}{7} = 1 + \frac{1}{3 + \frac{1}{2}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

$$\frac{9}{7}$$

$$\frac{7}{2}$$

$$\frac{2}{1}$$

Constructing a Continued Fraction

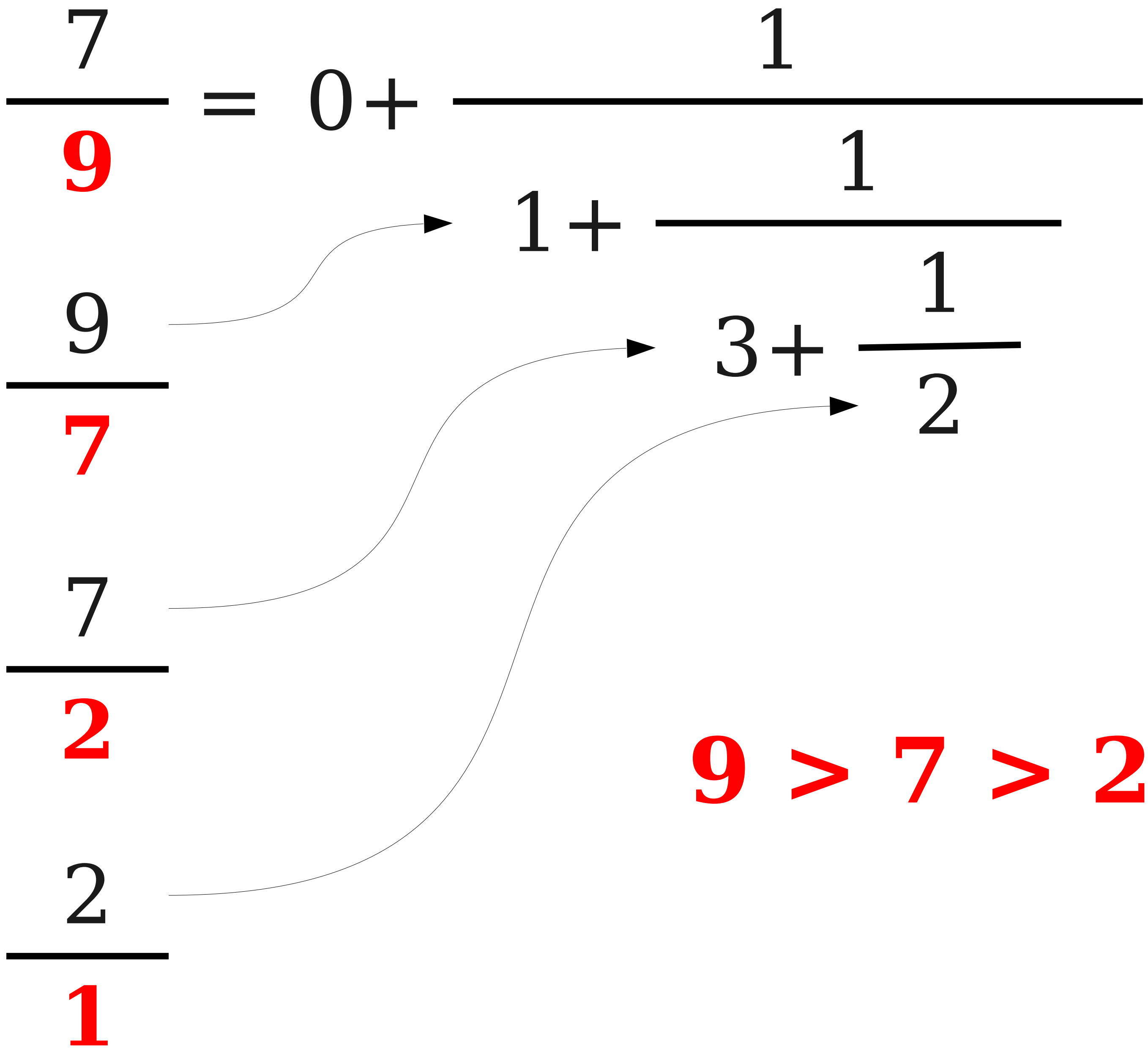
$$\frac{7}{\mathbf{9}} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

$$\frac{9}{\mathbf{7}}$$

$$\frac{7}{\mathbf{2}}$$

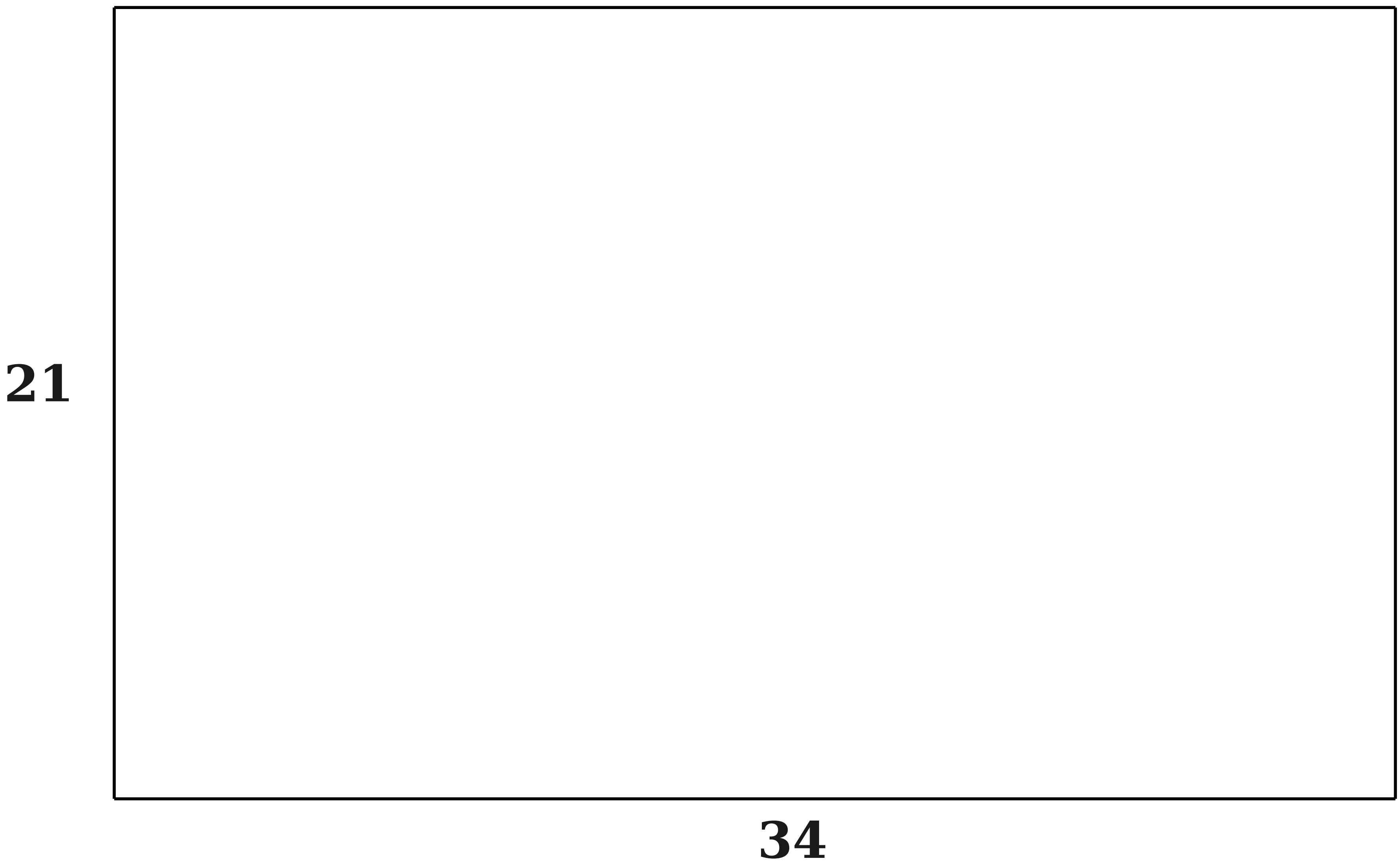
$$\frac{2}{\mathbf{1}}$$

Constructing a Continued Fraction

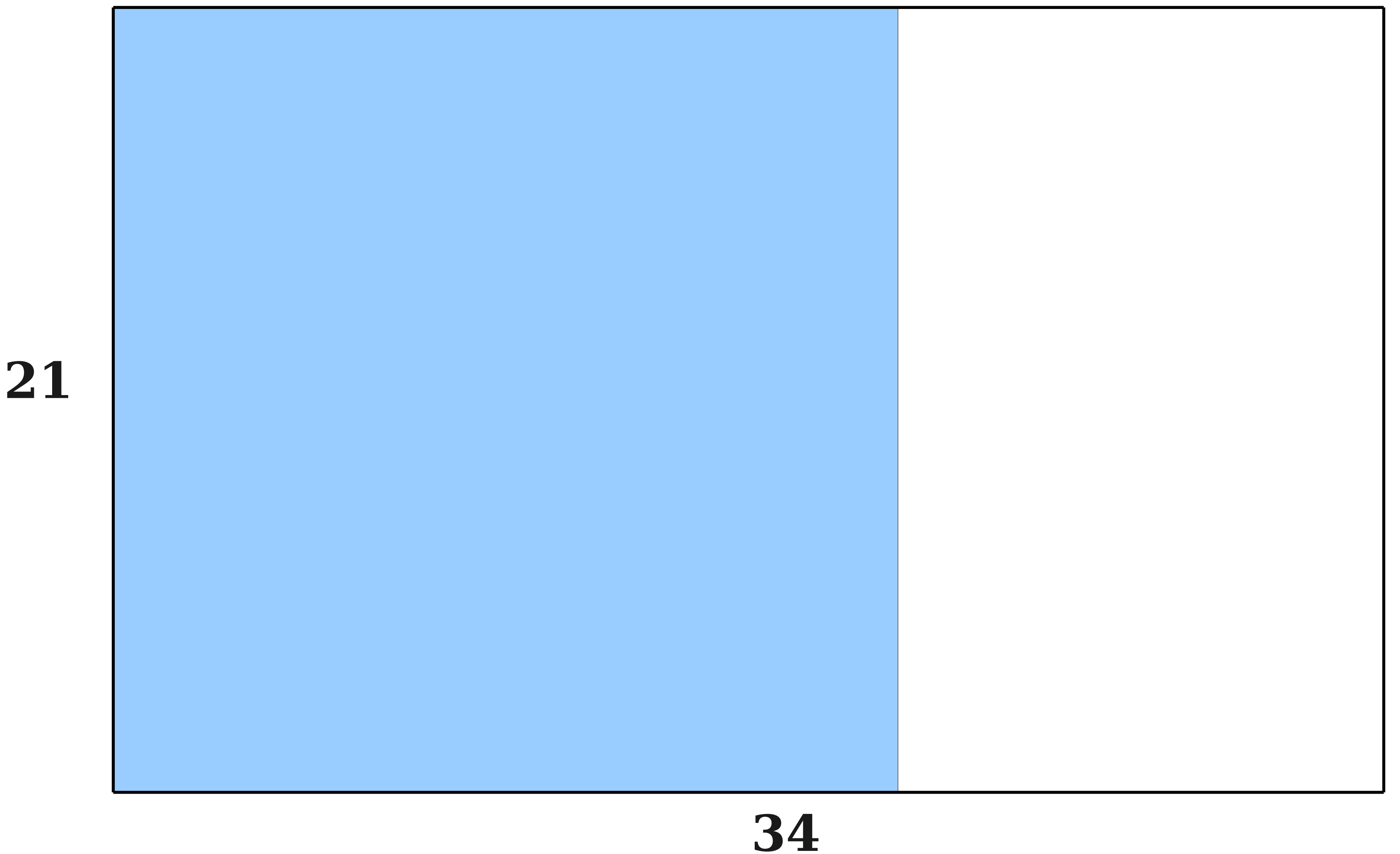


9 > 7 > 2 > 1

The Golden Ratio



The Golden Ratio



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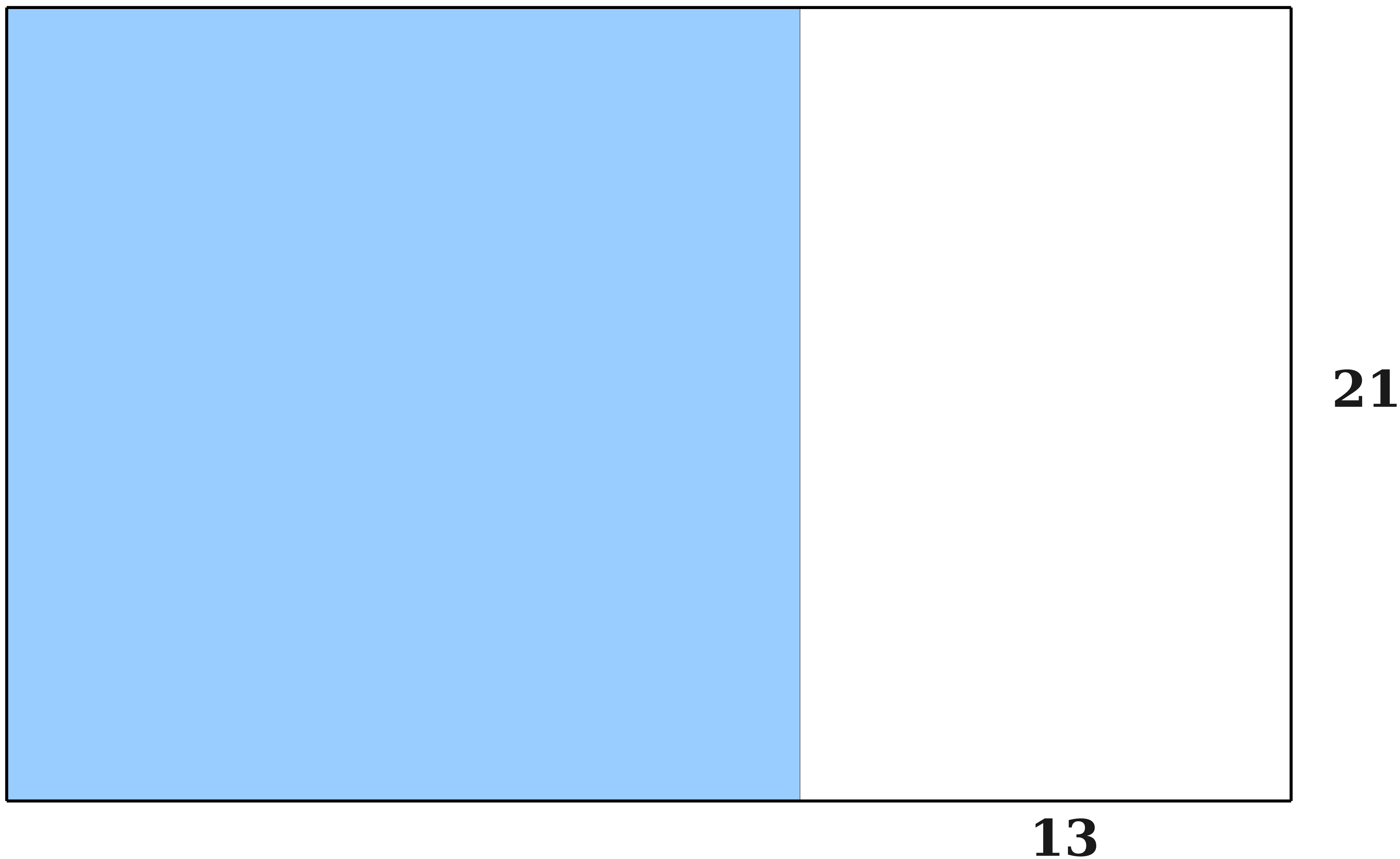

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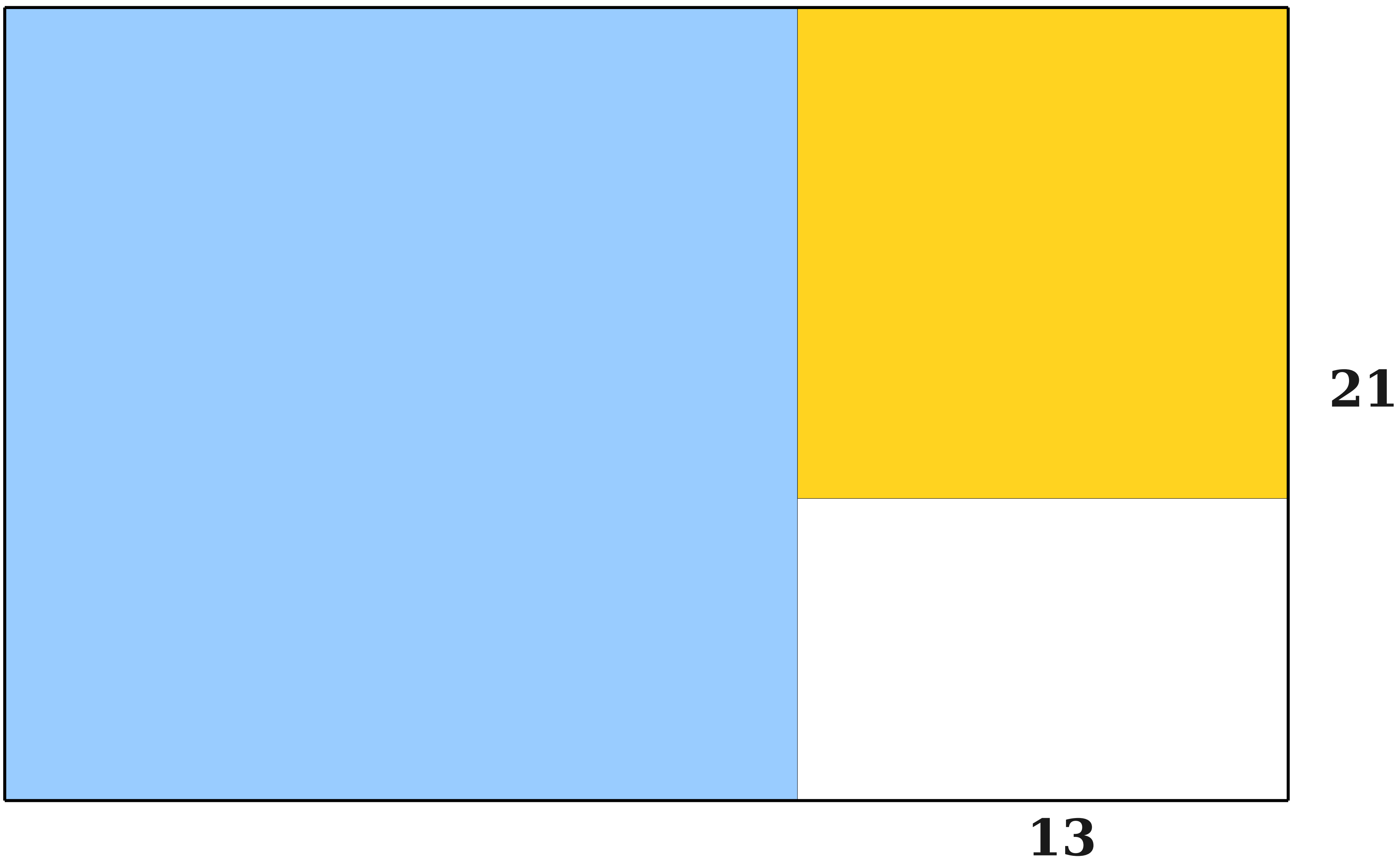
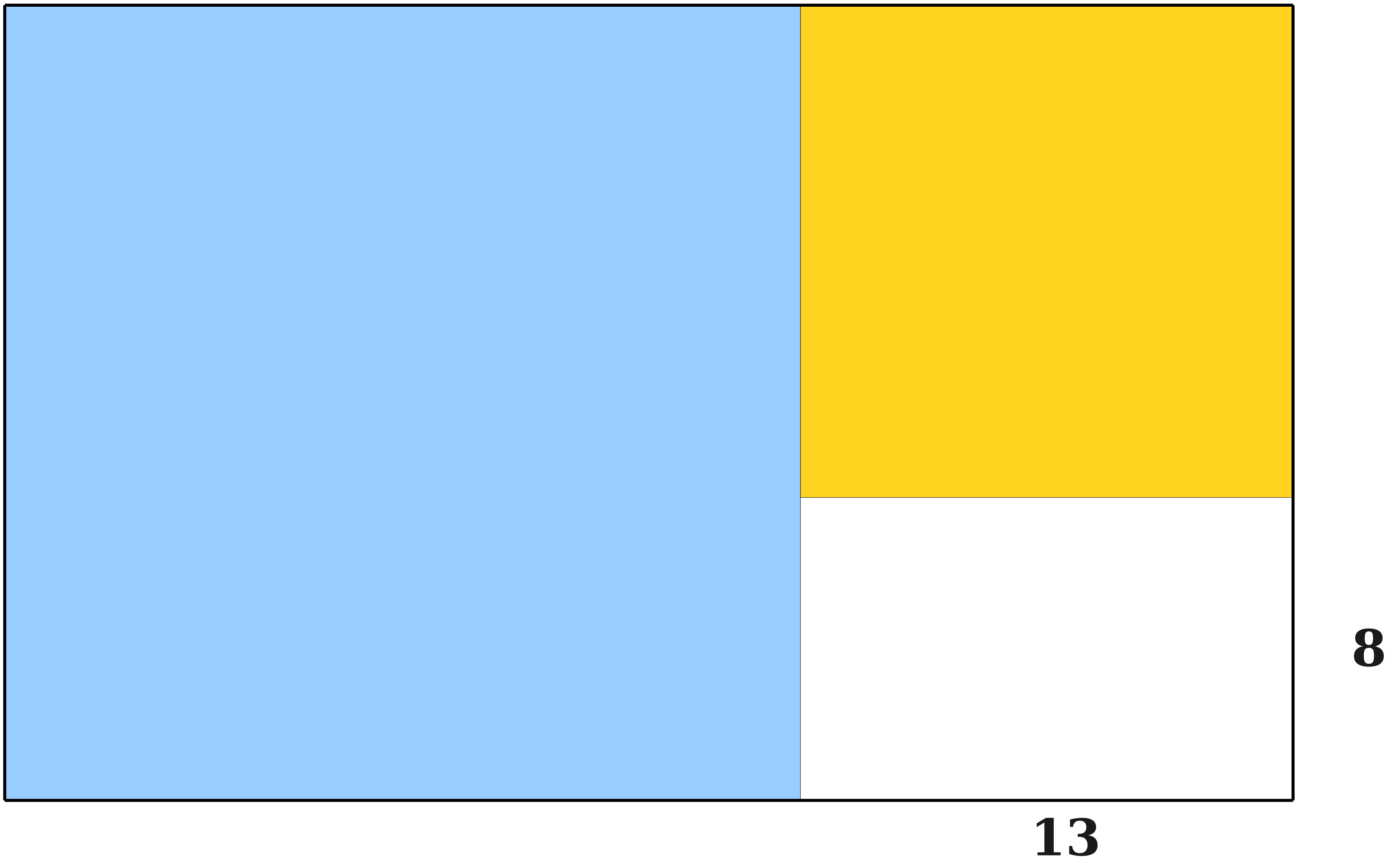


The Golden Ratio



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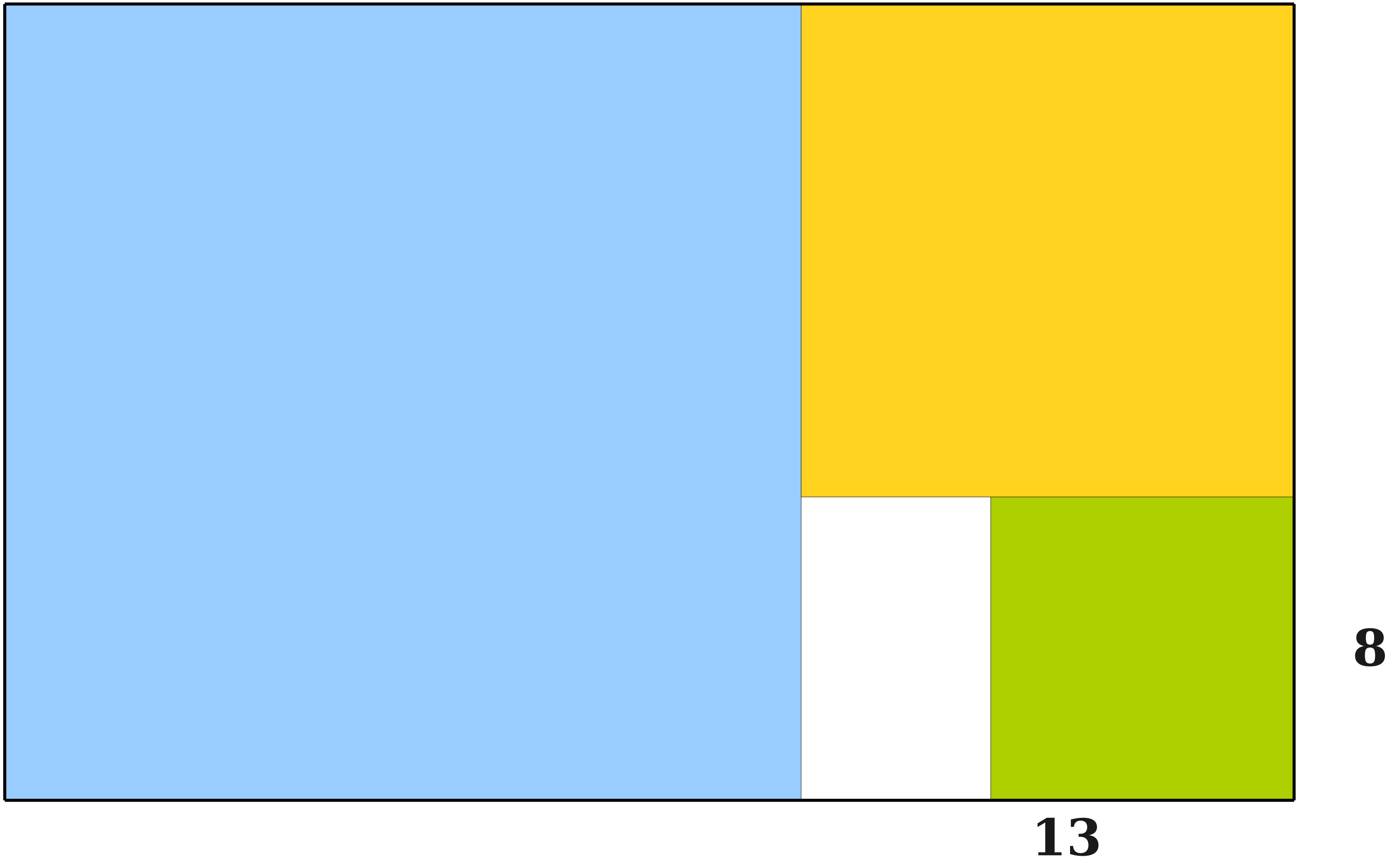

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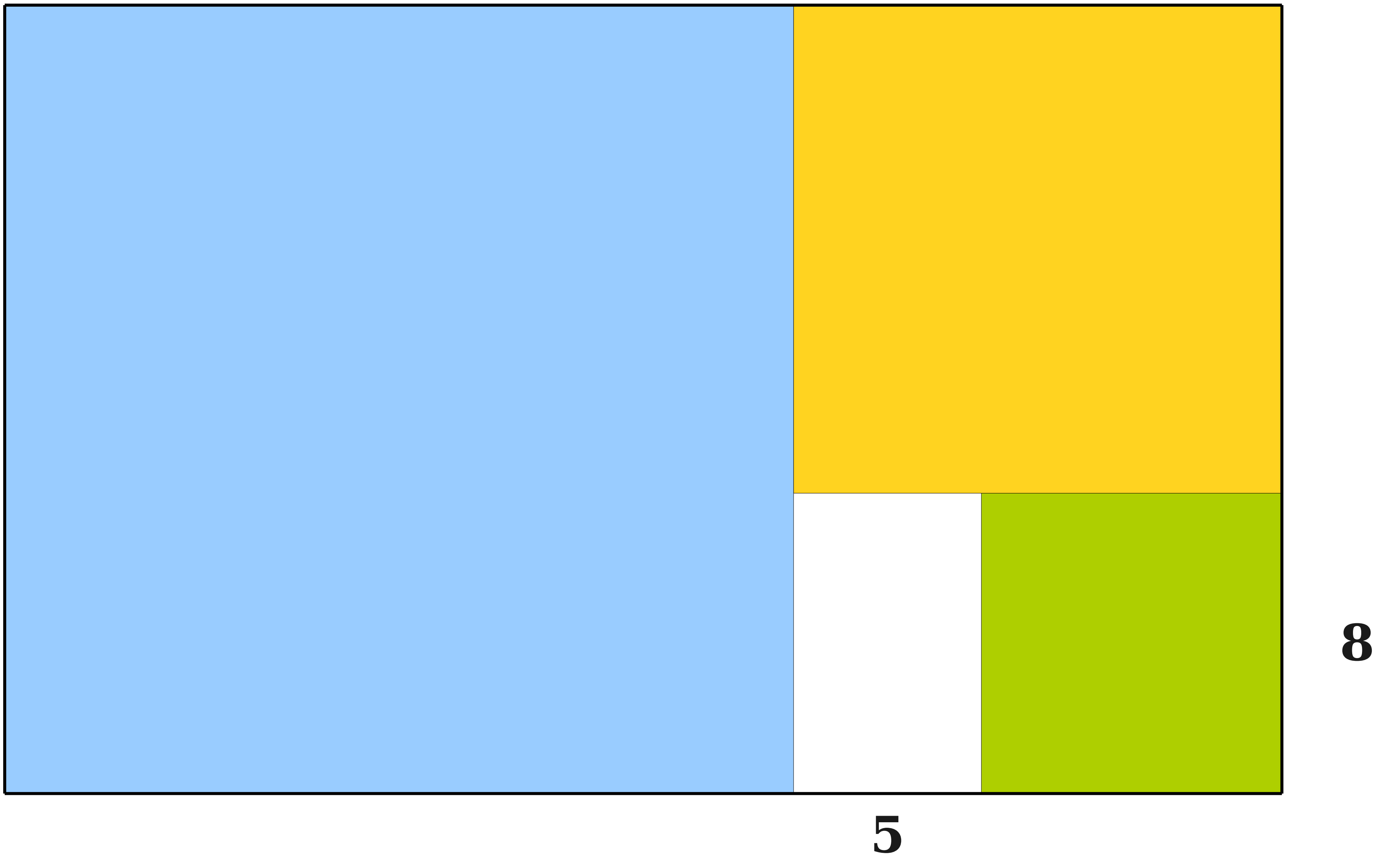
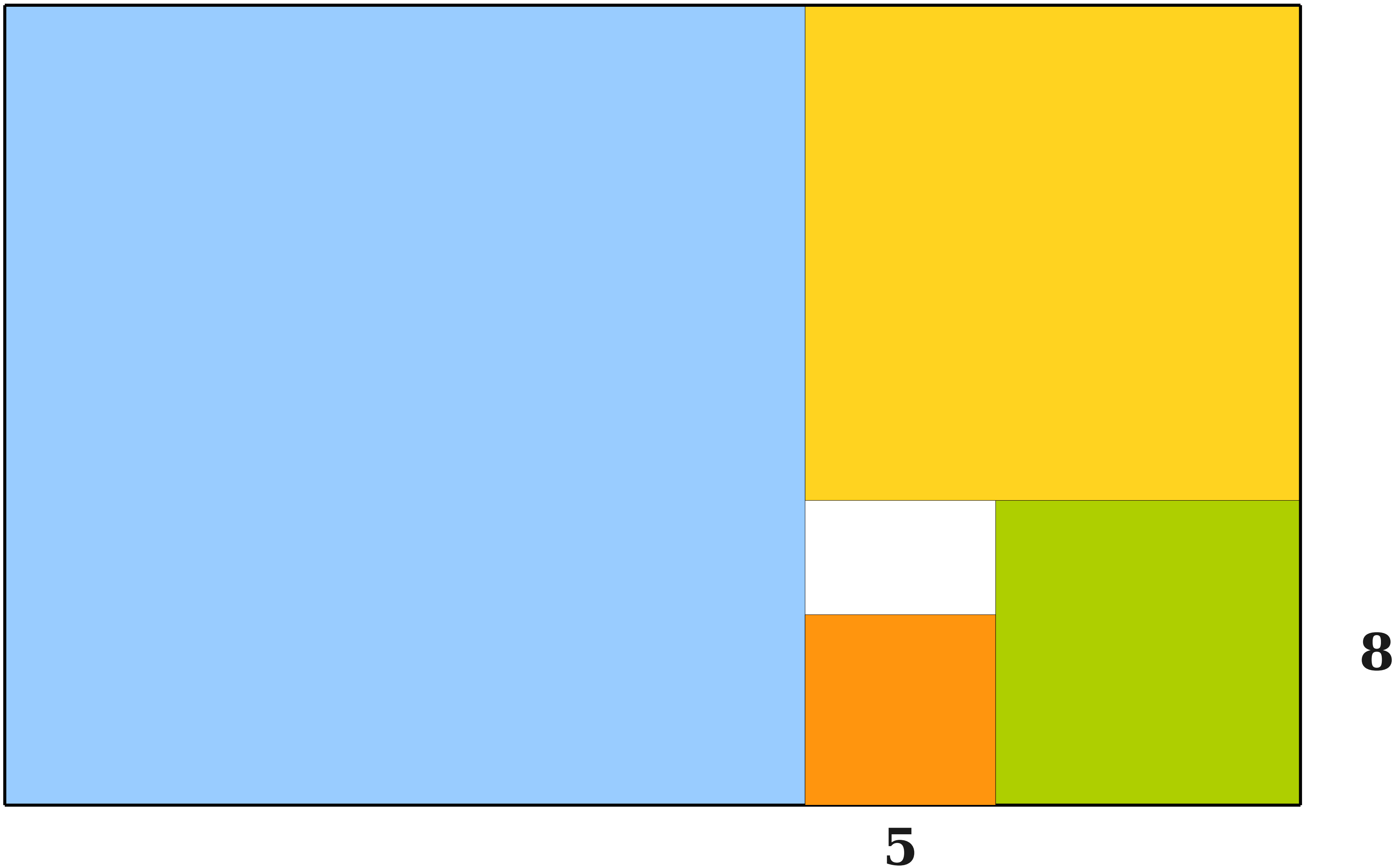


The Golden Ratio



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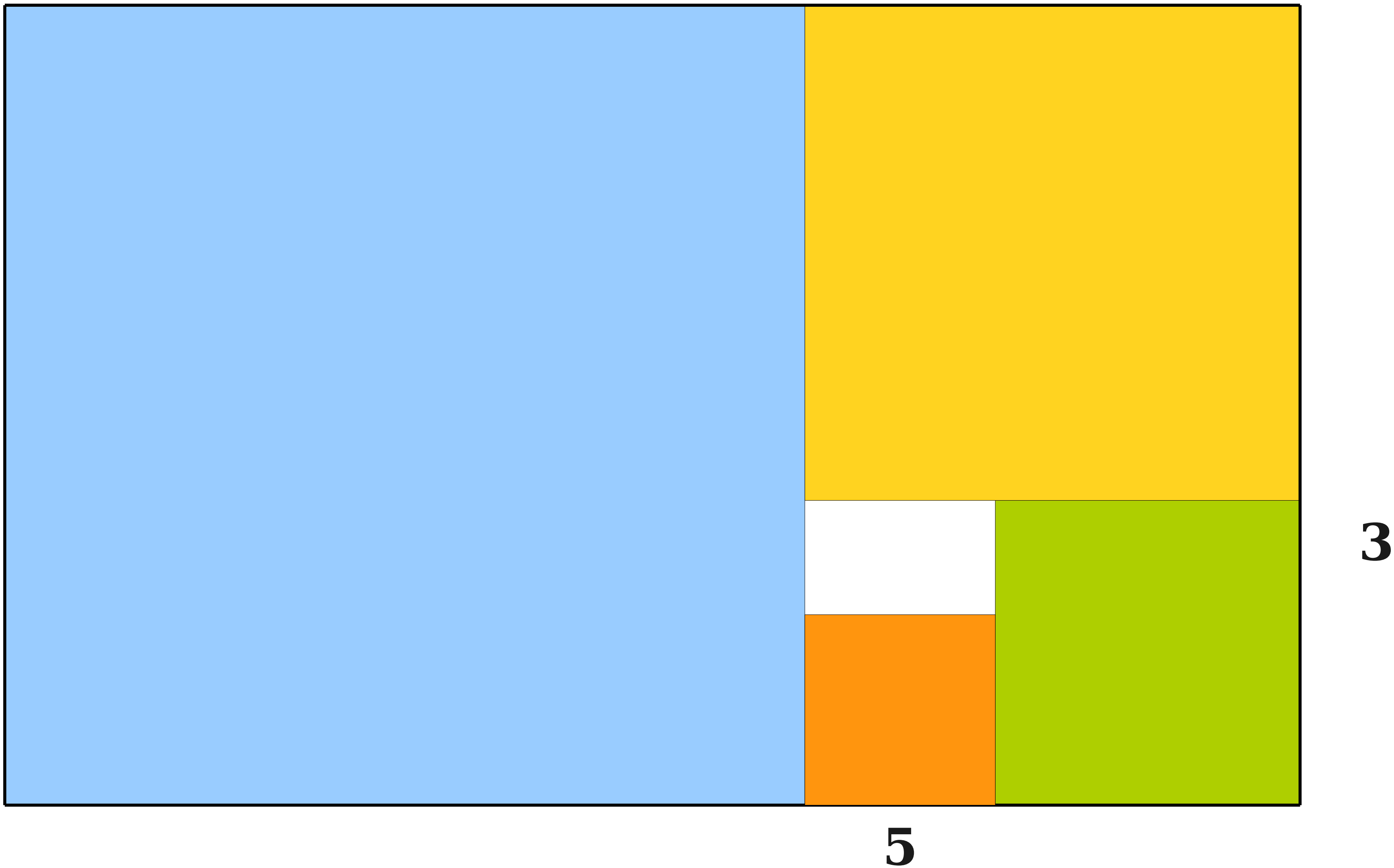
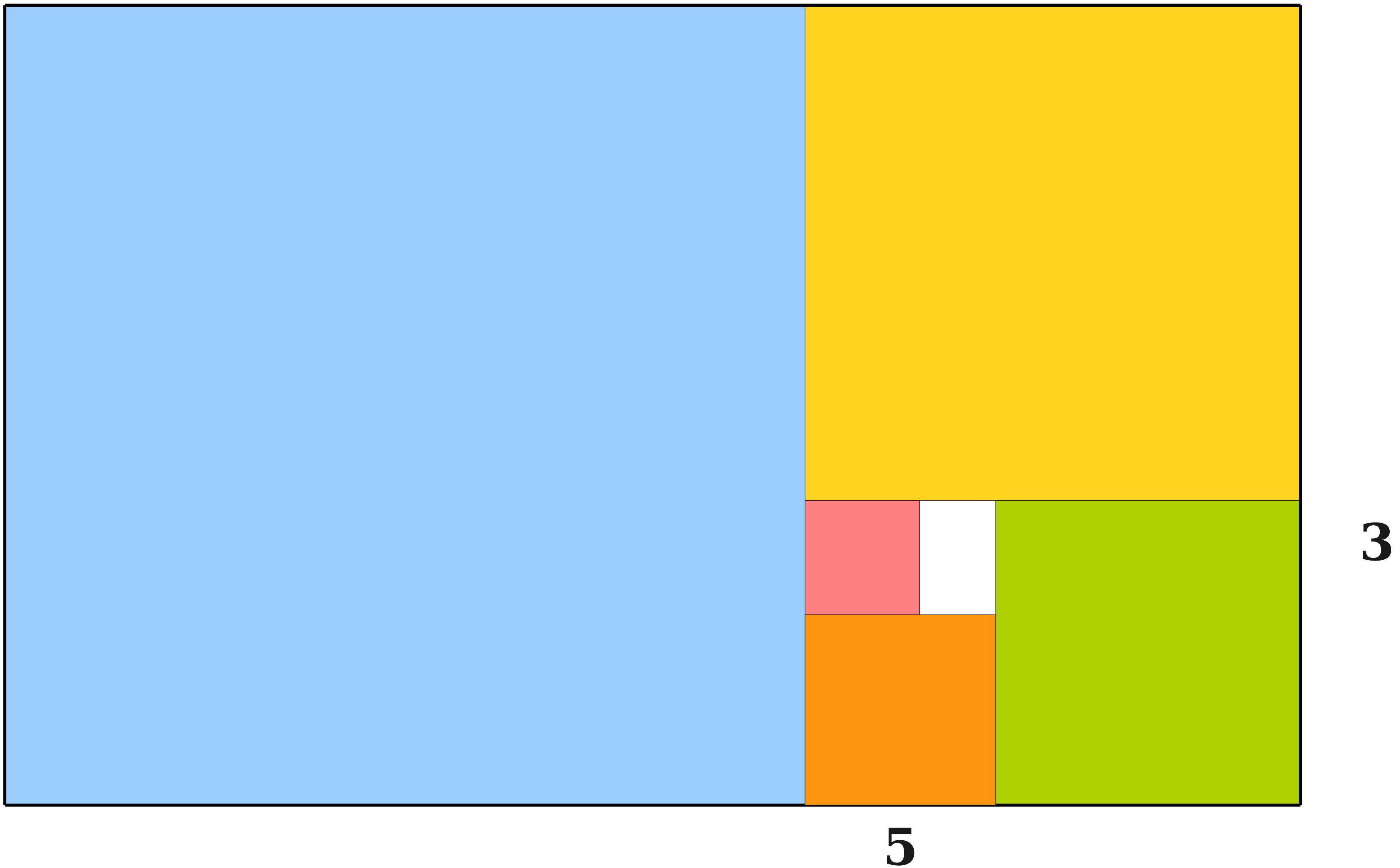
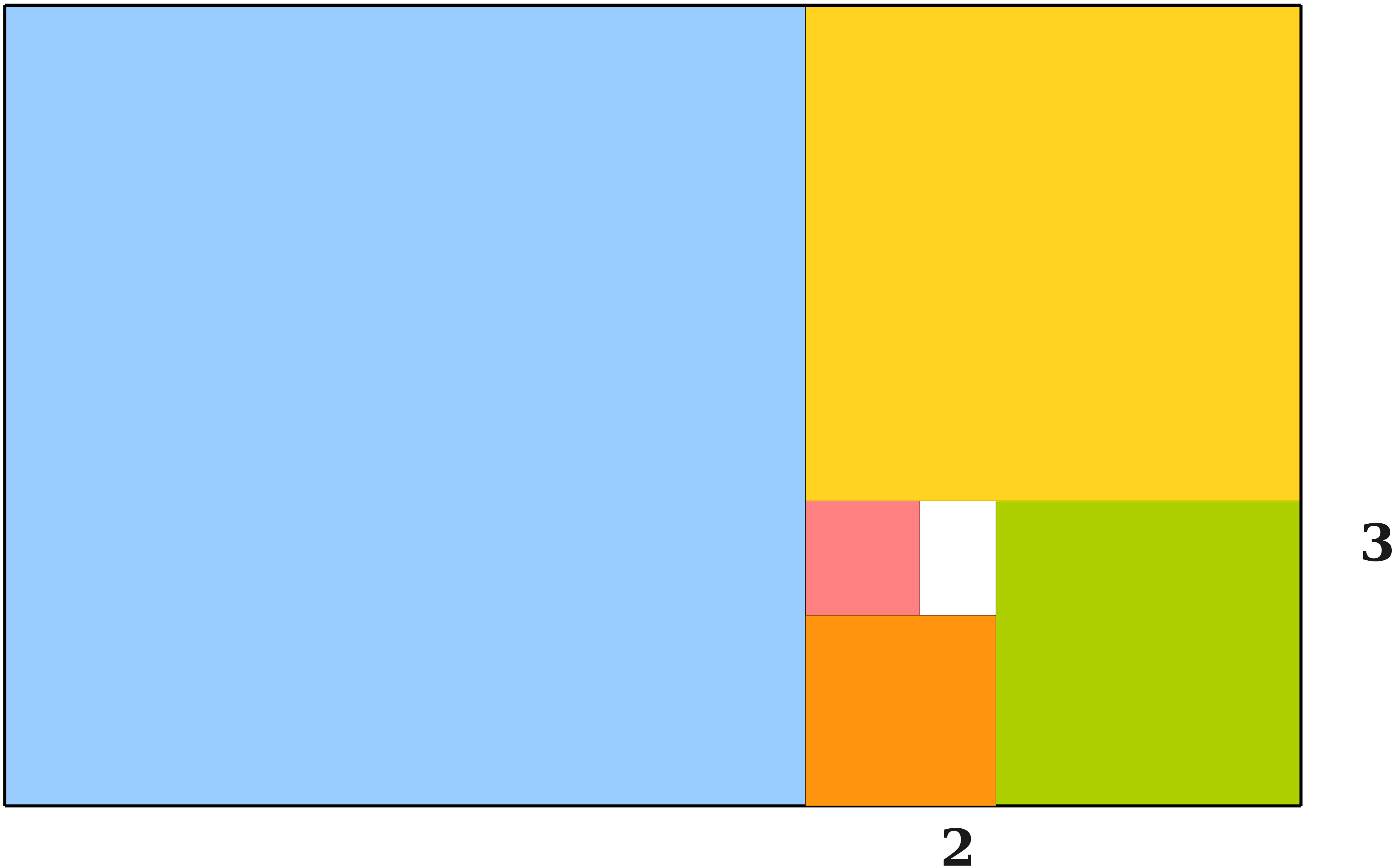


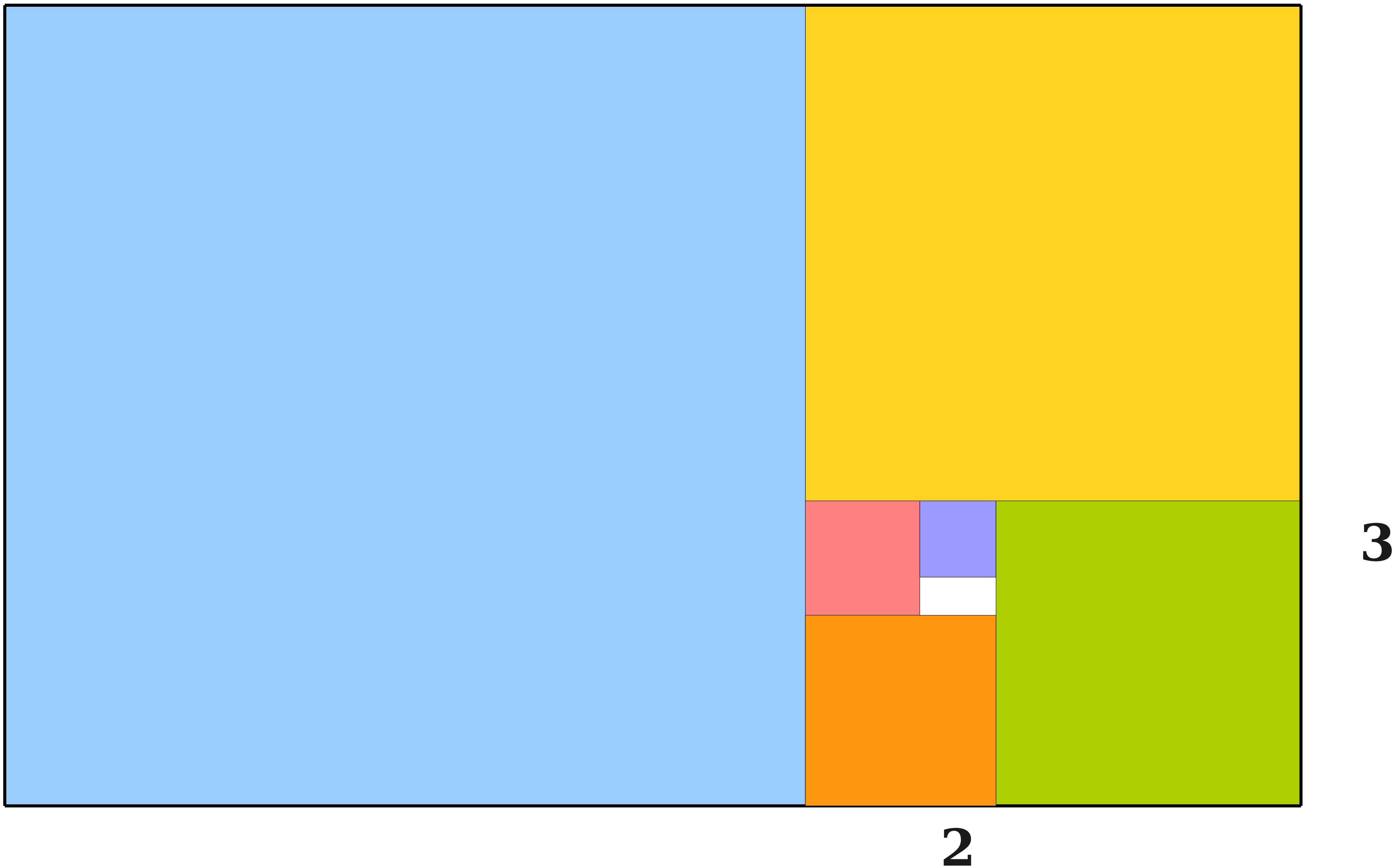
The Golden Ratio



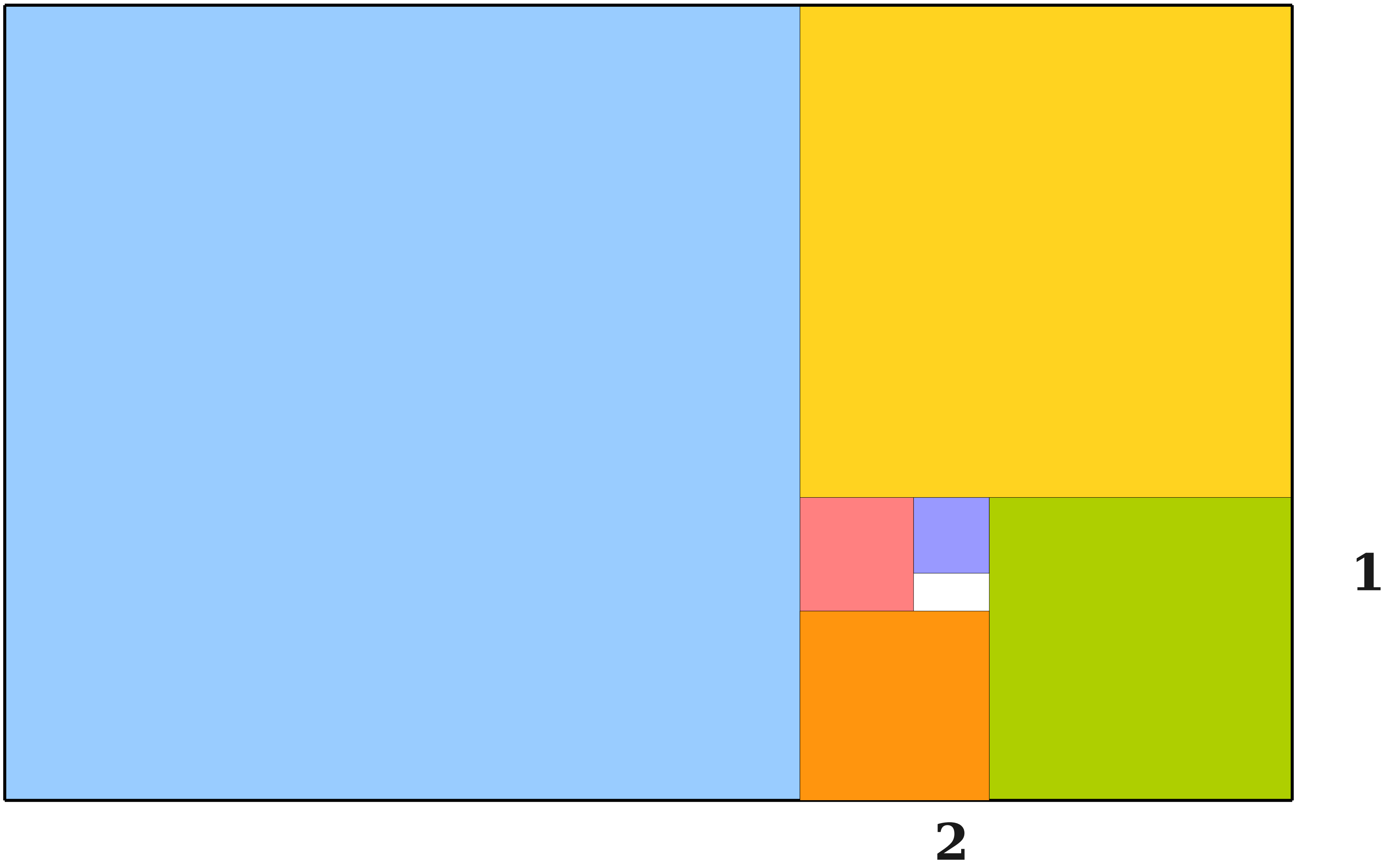
The Golden Ratio



The Golden Ratio



The Golden Ratio



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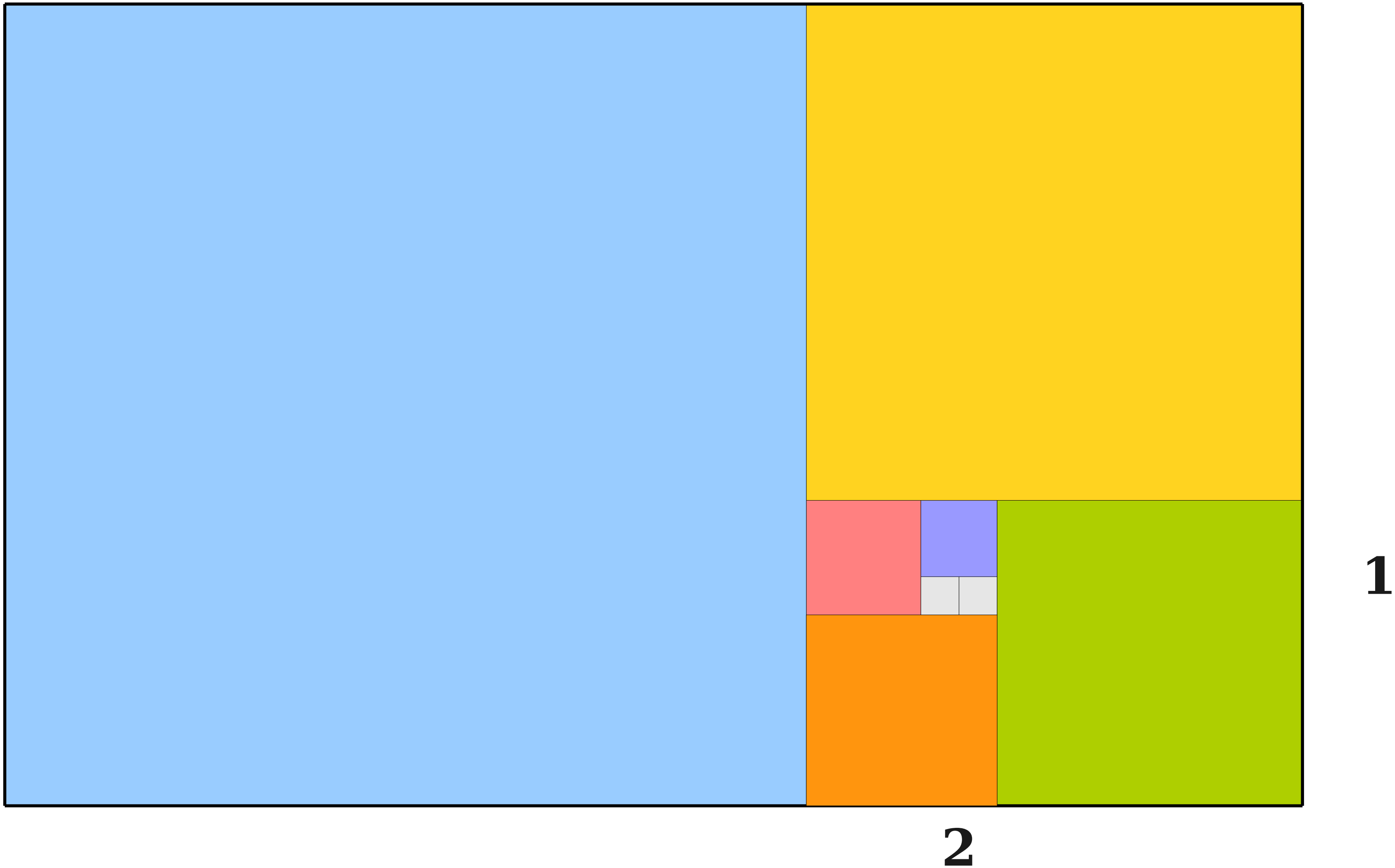

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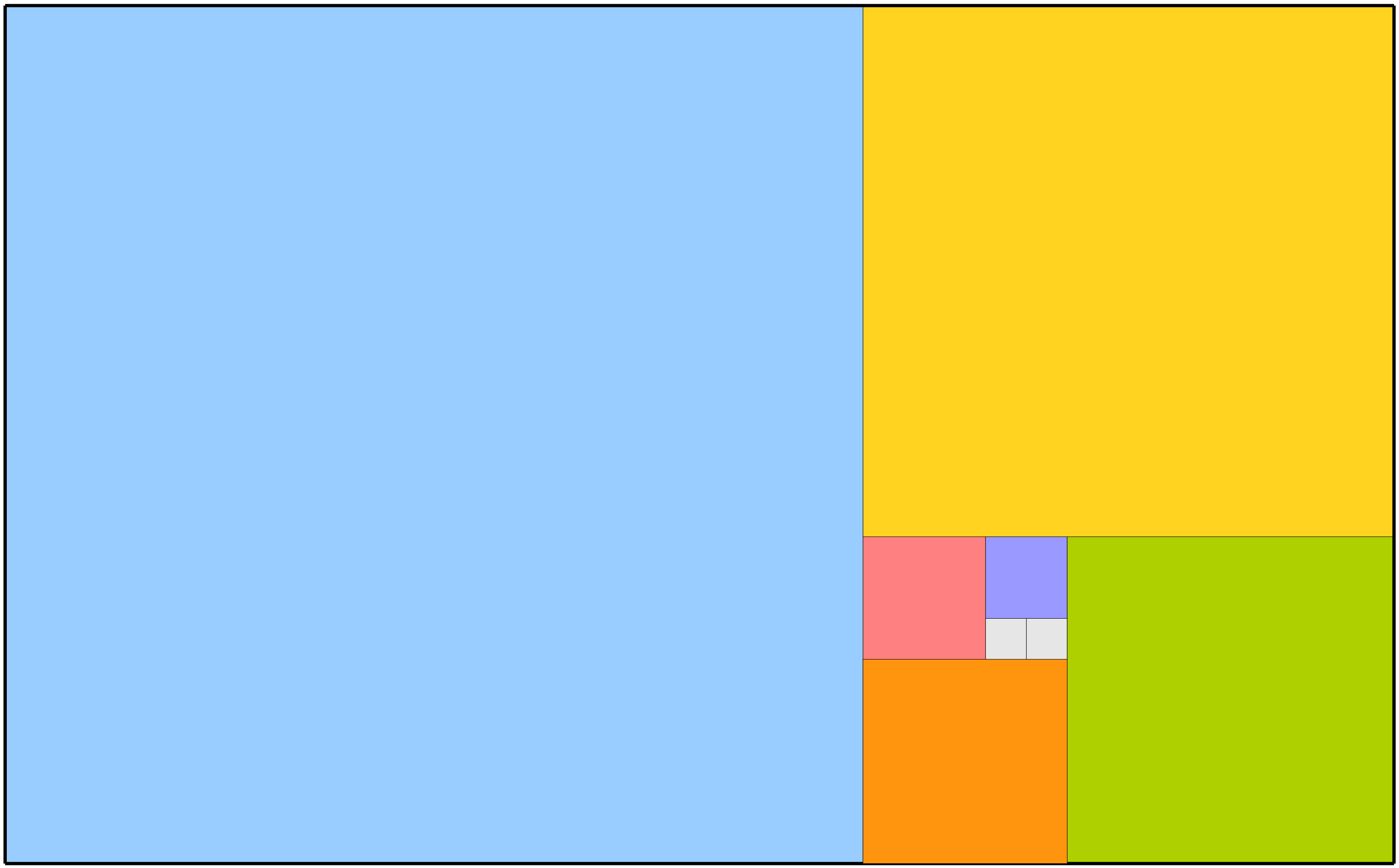

The Division Algorithm

- For any integers a and b, with $b > 0$, there exists **unique** integers q and r such that

$$a = qb + r$$

and

$$0 \leq r < b$$

- q is the **quotient** and r is the **remainder**.
- Given $a = 11$ and $b = 4$: $11 = \mathbf{2} \cdot 4 + \mathbf{3}$
- Given $a = -137$ and $b = 42$: $-137 = \mathbf{-4} \cdot 42 + \mathbf{31}$

Theorem: Every rational has a continued fraction.

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Proof: By strong induction.

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> We use that $r < d$ to justify using the inductive hypothesis.
>
> Since our induction starts at 1, we also have to show that $r \geq 1$. Otherwise we might be out of the range of where the inductive hypothesis holds.

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For more on continued fractions:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html

Next Time

- **Graphs and Relations**
  - Representing structured data.
  - Categorizing how objects are connected.