Order Relations
and
Functions

Problem Session Tonight

7:00PM – 7:50PM
380-380X

Optional, but highly recommended!

“x is larger than y”

“x is tastier than y”

“x is faster than y”

“x is a subset of y”

“x divides y”

“x is a part of y”

Informally

An **order relation** is a relation that ranks elements against one another.

Do <u>not</u> use this definition in proofs!
It's just an intuition!

Properties of Order Relations

$$x \leq y$$

$$x \leq y \quad \textbf{and} \quad y \leq z$$

$$x \leq z$$

Transitivity

Properties of Order Relations

$$x \leq y$$

$$x \leq x$$

Reflexivity

Properties of Order Relations

$x \leq y$

$19 \leq 21$

~~**$21 \leq 19$**~~?

Properties of Order Relations

$$x \leq y$$

$42 \leq 137$

~~**137 ≤ 42**~~?

Properties of Order Relations

$x \leq y$

137 ≤ 137

137 ≤ 137

Antisymmetry

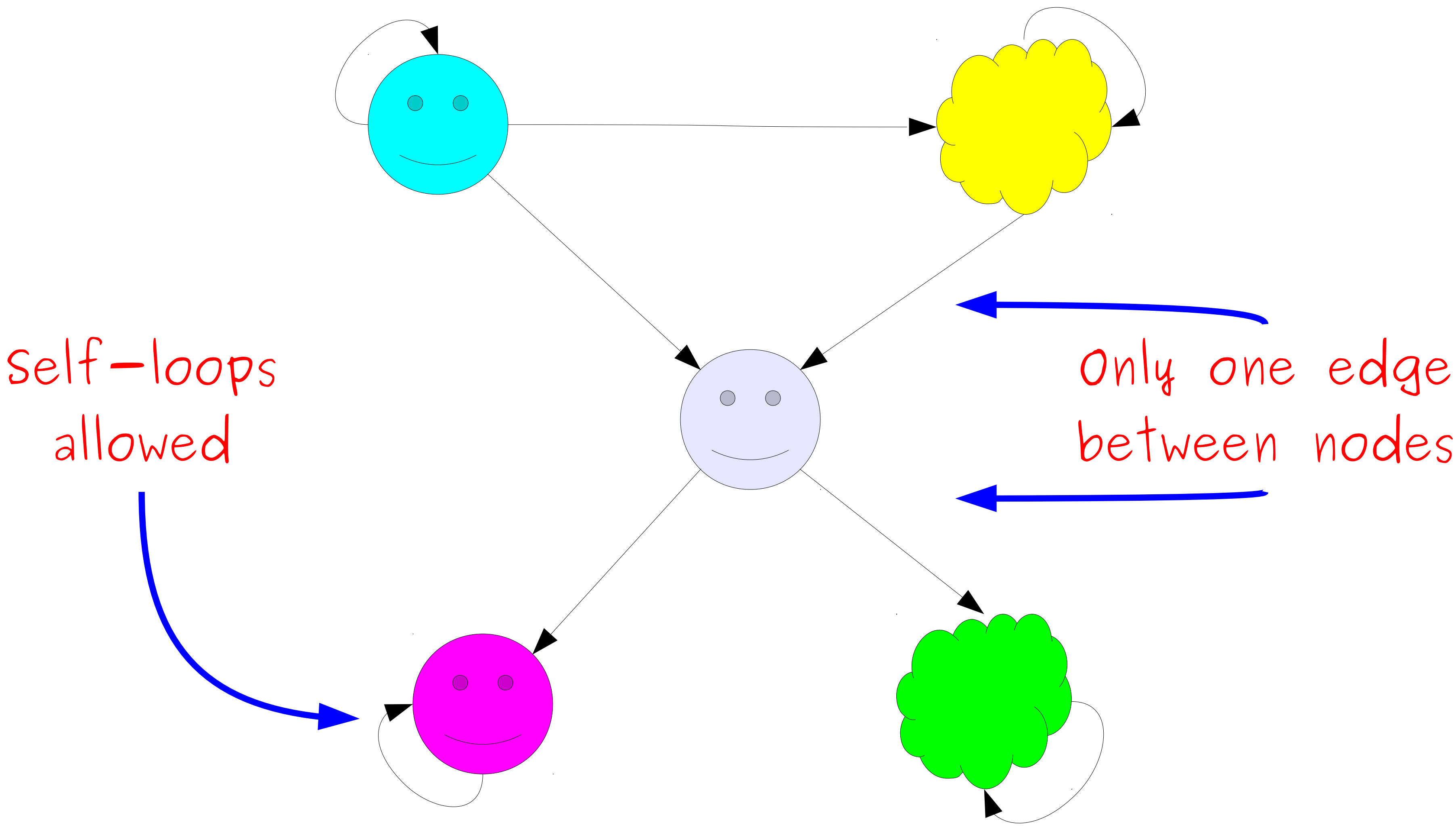
A binary relation R over a set A is called **antisymmetric** iff

For any $x \in A$ and $y \in A$,
If xRy and $y \neq x$, then $y \not R x$.

Equivalently:

For any $x \in A$ and $y \in A$,
if xRy and yRx, then $x = y$.

An Intuition for Antisymmetry



For any $x \in A$ and $y \in A$,
If xRy and $y \neq x$, then $y \not{R} x$.

Partial Orders

- A binary relation R is a **partial order** over a set A iff it is
  - **reflexive**,
  - **antisymmetric**, and
  - **transitive**.
- A pair (A, R), where R is a partial order over A, is called a **partially ordered set** or **poset**.

2012 Summer Olympics

Gold	Silver	Bronze	Total
46	29	29	104
38	27	23	88
29	17	19	65
24	26	32	82
13	8	7	28
11	19	14	44
11	11	12	34

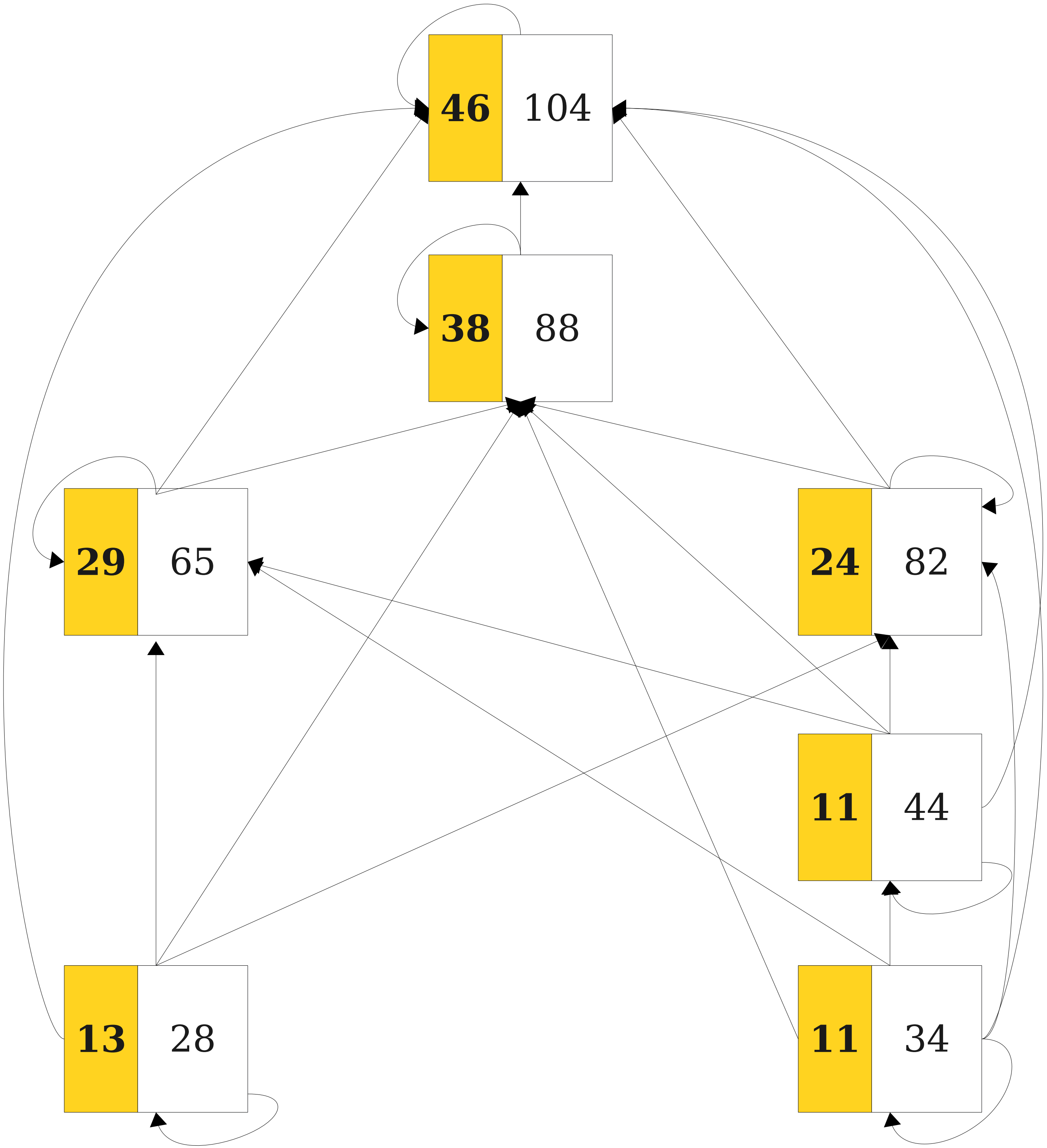
Inspired by http://tartarus.org/simon/2008-olympics-hasse/
Data from http://www.london2012.com/medals/medal-count/

Define the relationship

$$(\mathbf{gold}_0, \mathbf{total}_0)\,\boldsymbol{R}\,(\mathbf{gold}_1, \mathbf{total}_1)$$

to be true when

$$\mathbf{gold}_0 \leq \mathbf{gold}_1 \text{ and } \mathbf{total}_0 \leq \mathbf{total}_1$$


46
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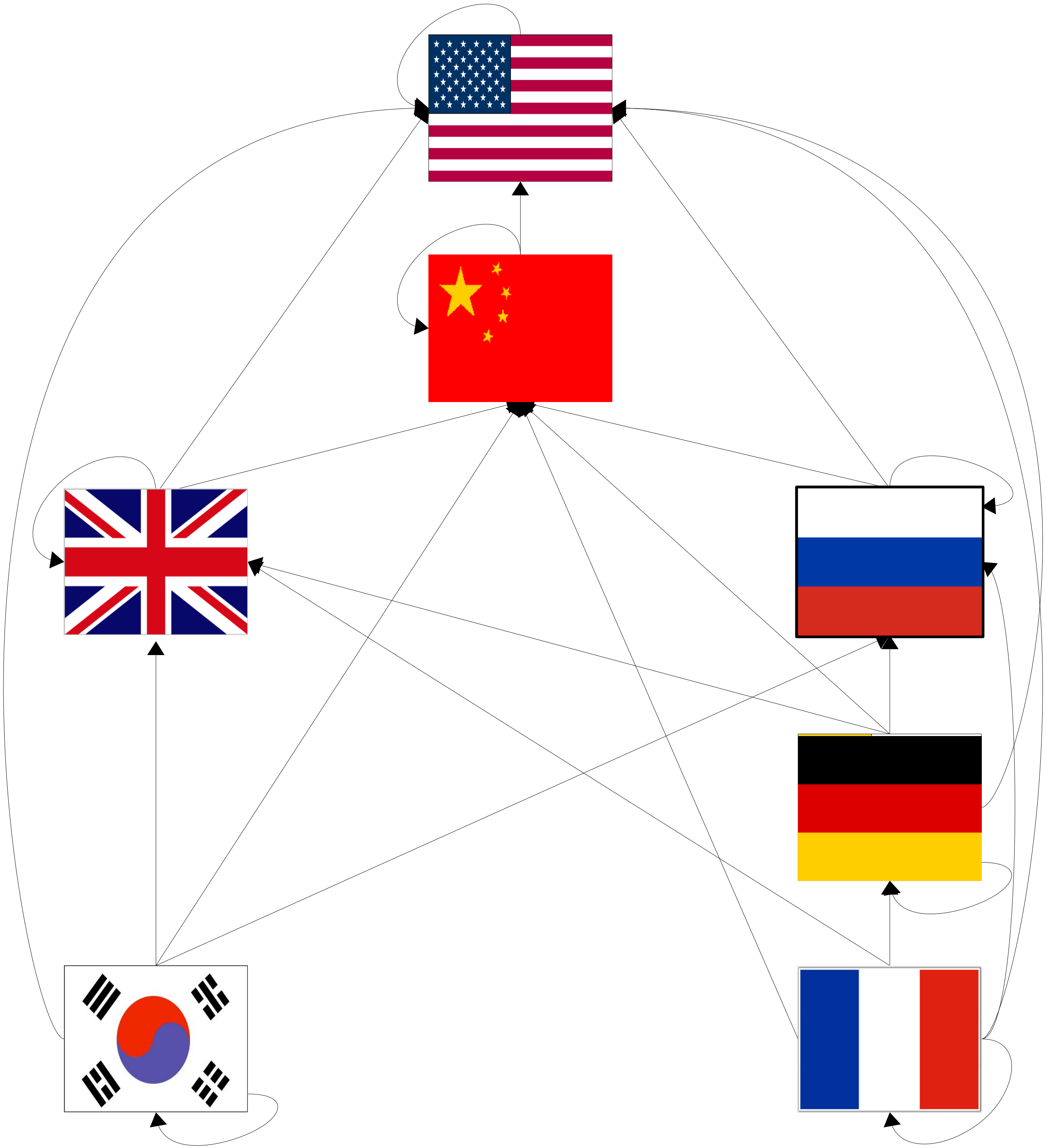
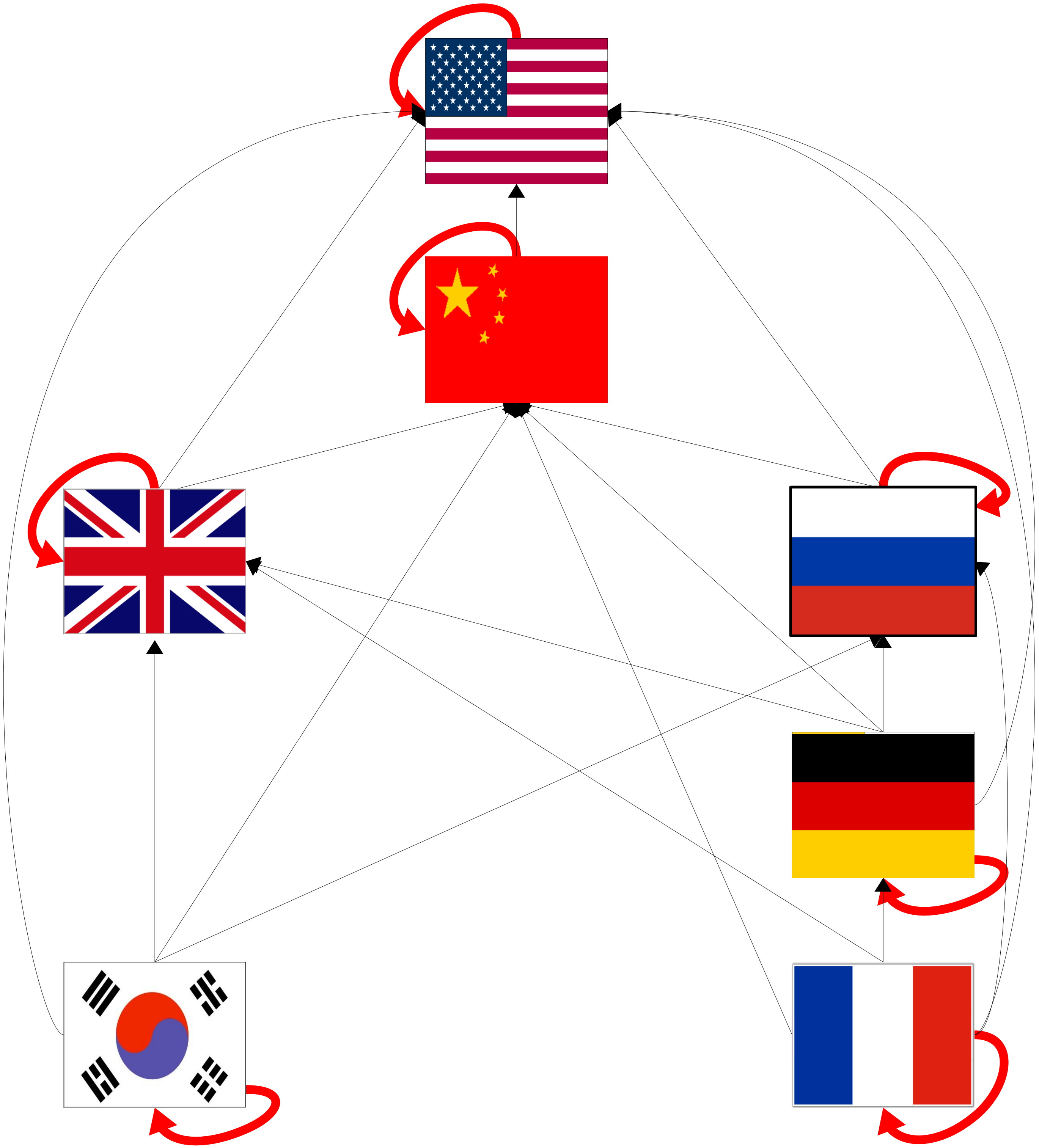
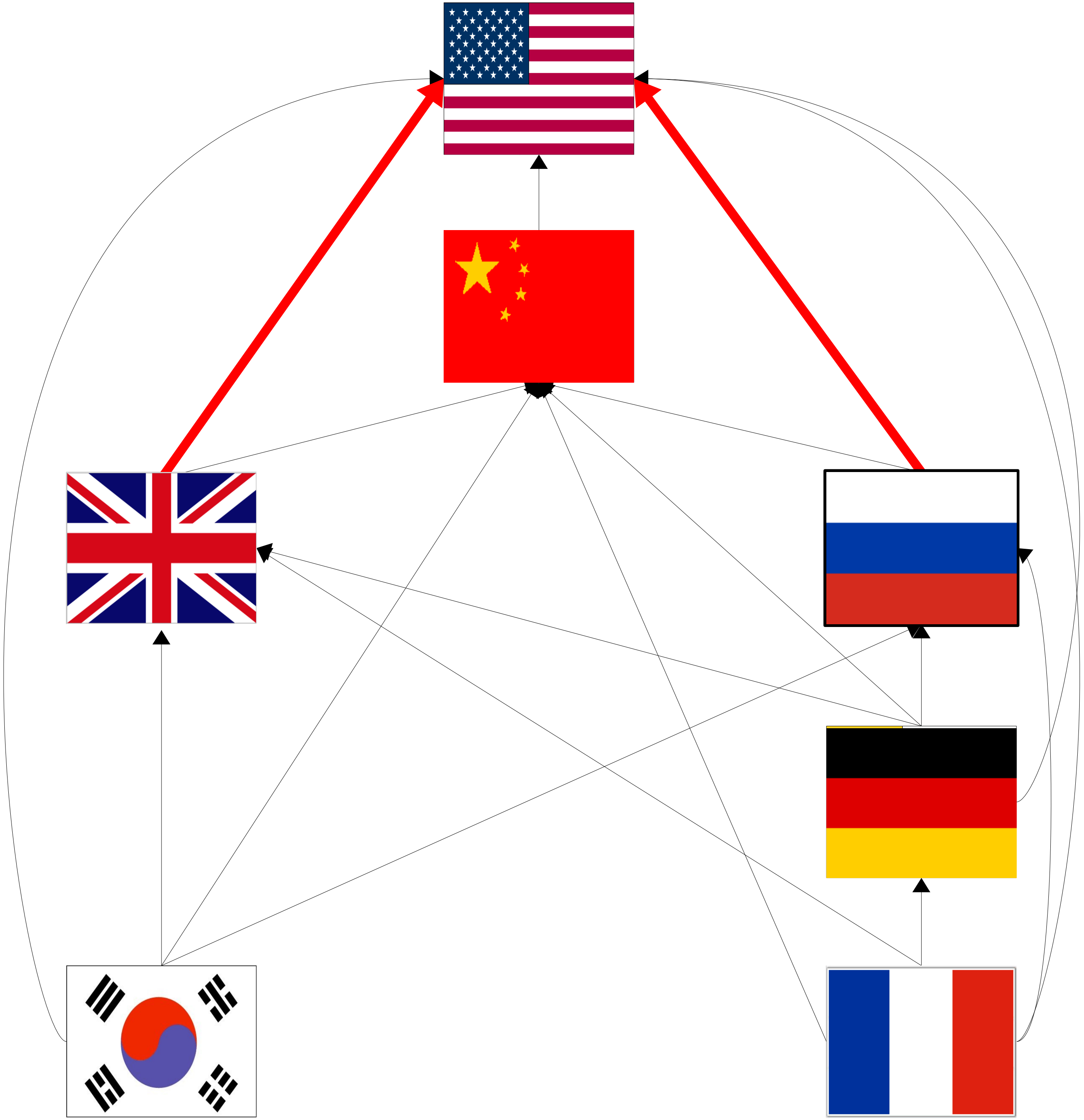
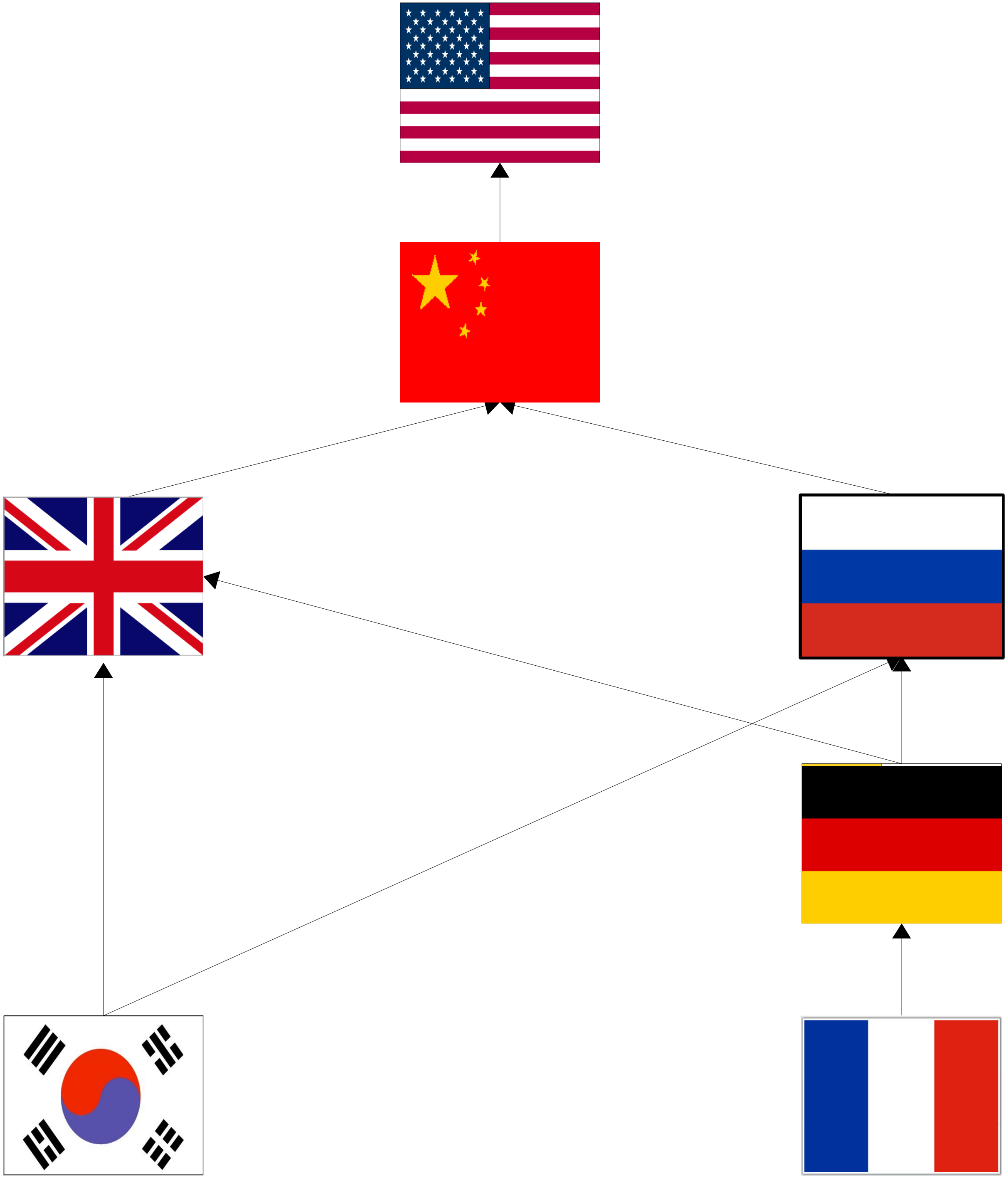


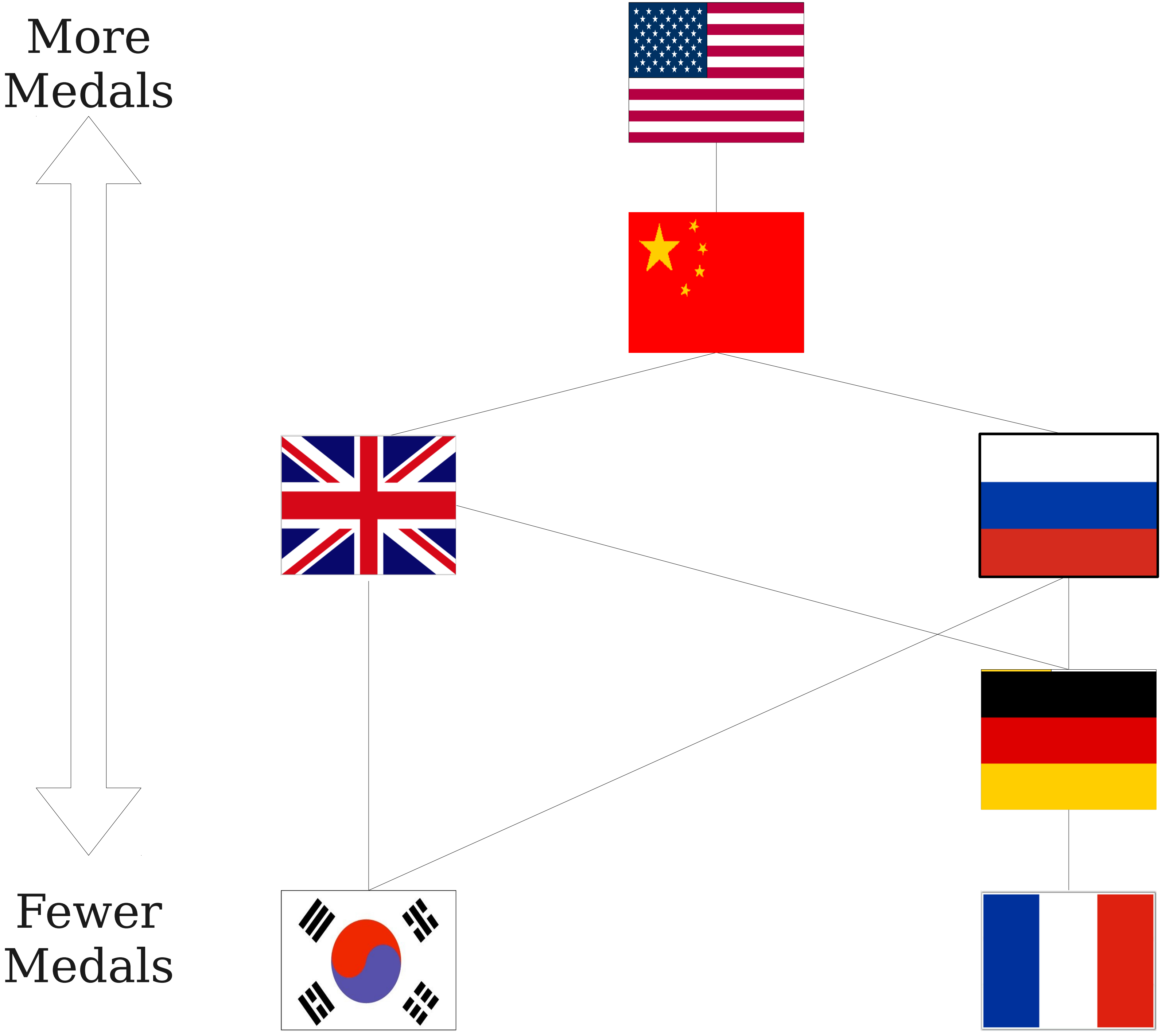
Partial and Total Orders

- A binary relation R over a set A is called **total** iff for any $x \in A$ and $y \in A$, that xRy or yRx.
  - It's possible for both to be true.
- A binary relation R over a set A is called a **total order** iff it is a partial order and it is total.
- Examples:
  - Integers ordered by $\leq$.
  - Strings ordered alphabetically.



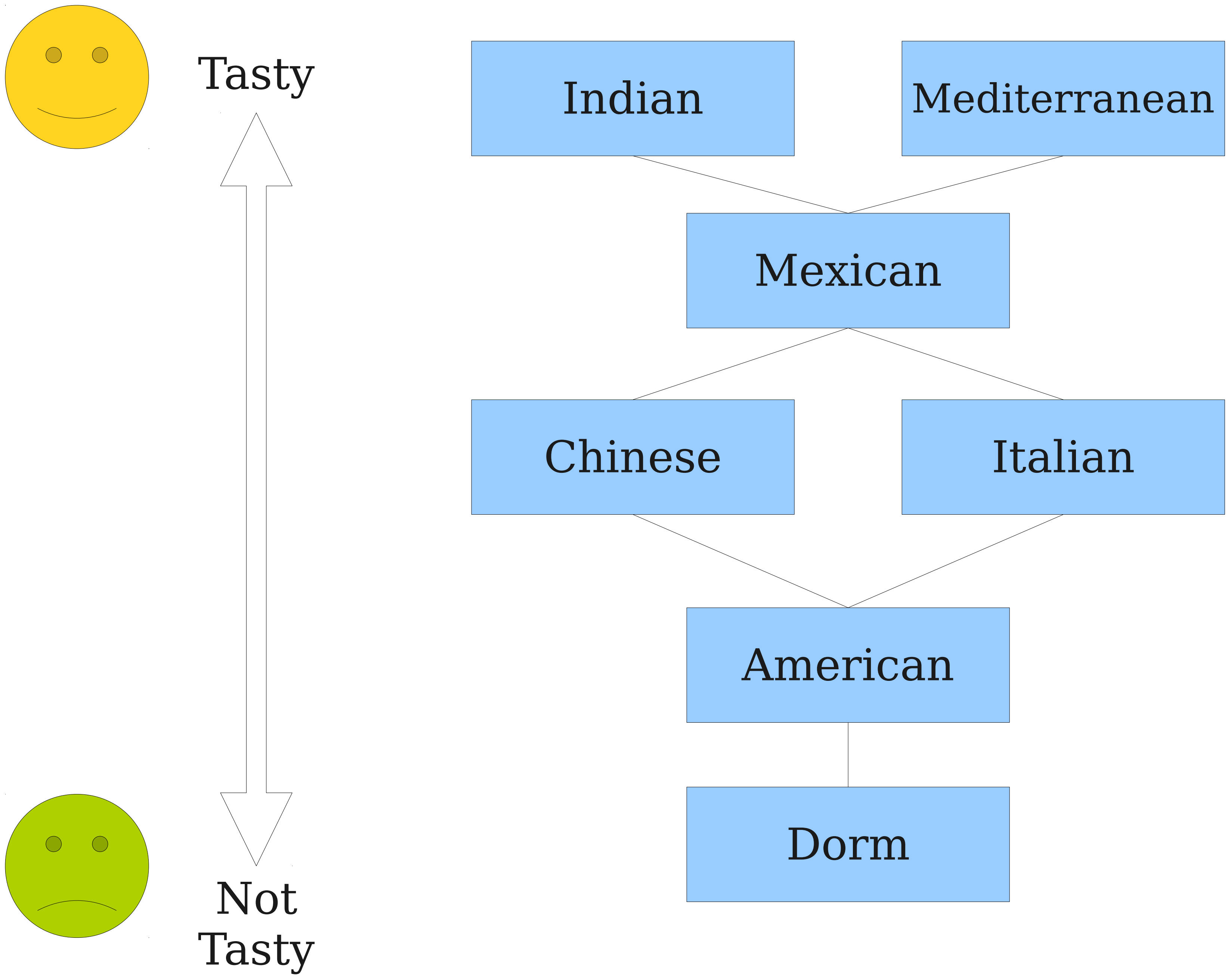




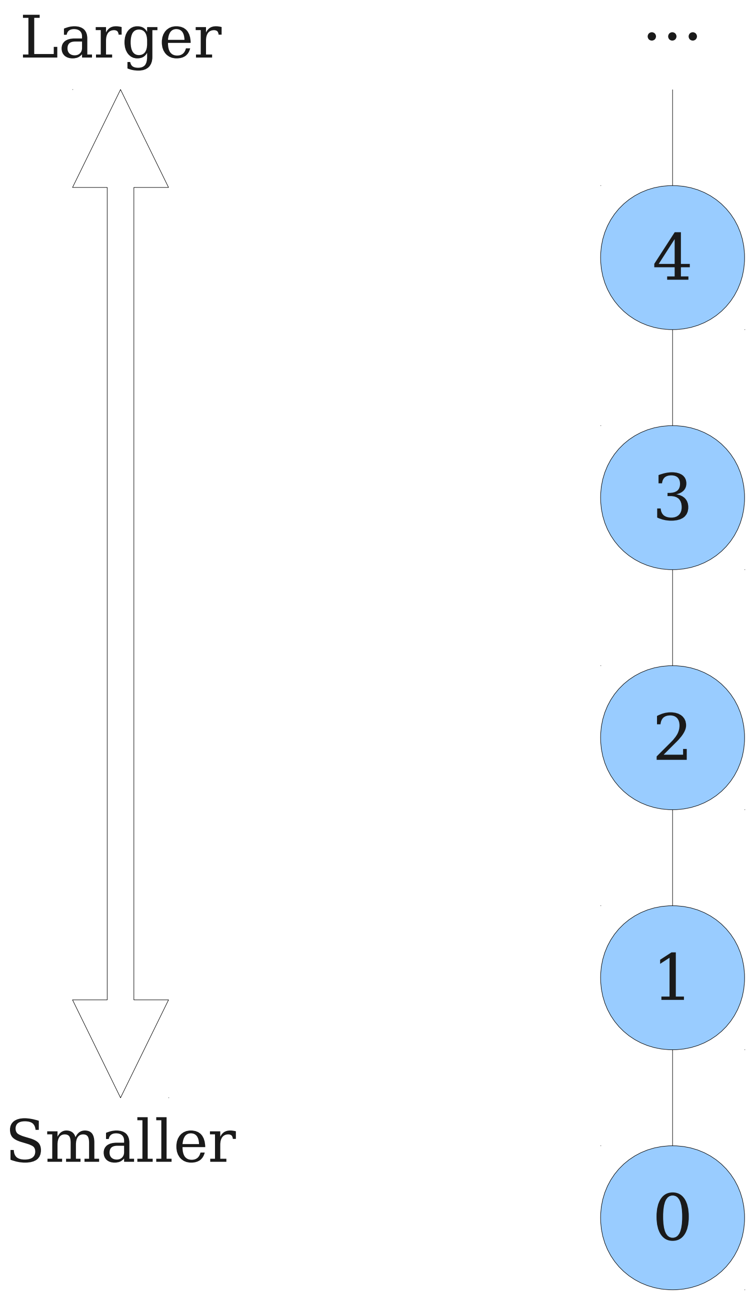

More Medals
Fewer Medals

Hasse Diagrams

- A **Hasse diagram** is a graphical representation of a partial order.
- No self-loops: by **reflexivity**, we can always add them back in.
- Higher elements are bigger than lower elements: by **antisymmetry**, the edges can only go in one direction.
- No redundant edges: by **transitivity**, we can infer the missing edges.

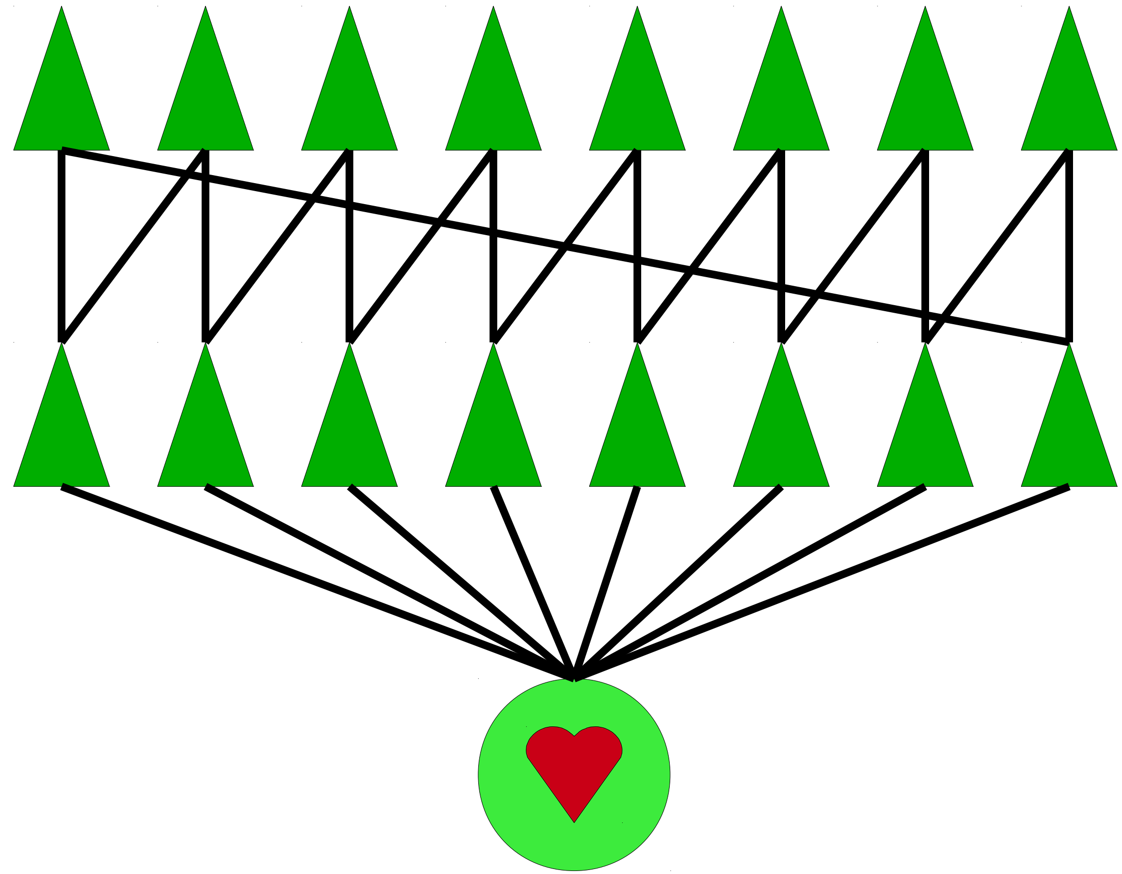


Tasty
Not
Tasty
Indian
Mediterranean
Mexican
Chinese
Italian
American
Dorm



Larger
Smaller
...
4
3
2
1
0

Hasse Artichokes



Summary of Order Relations

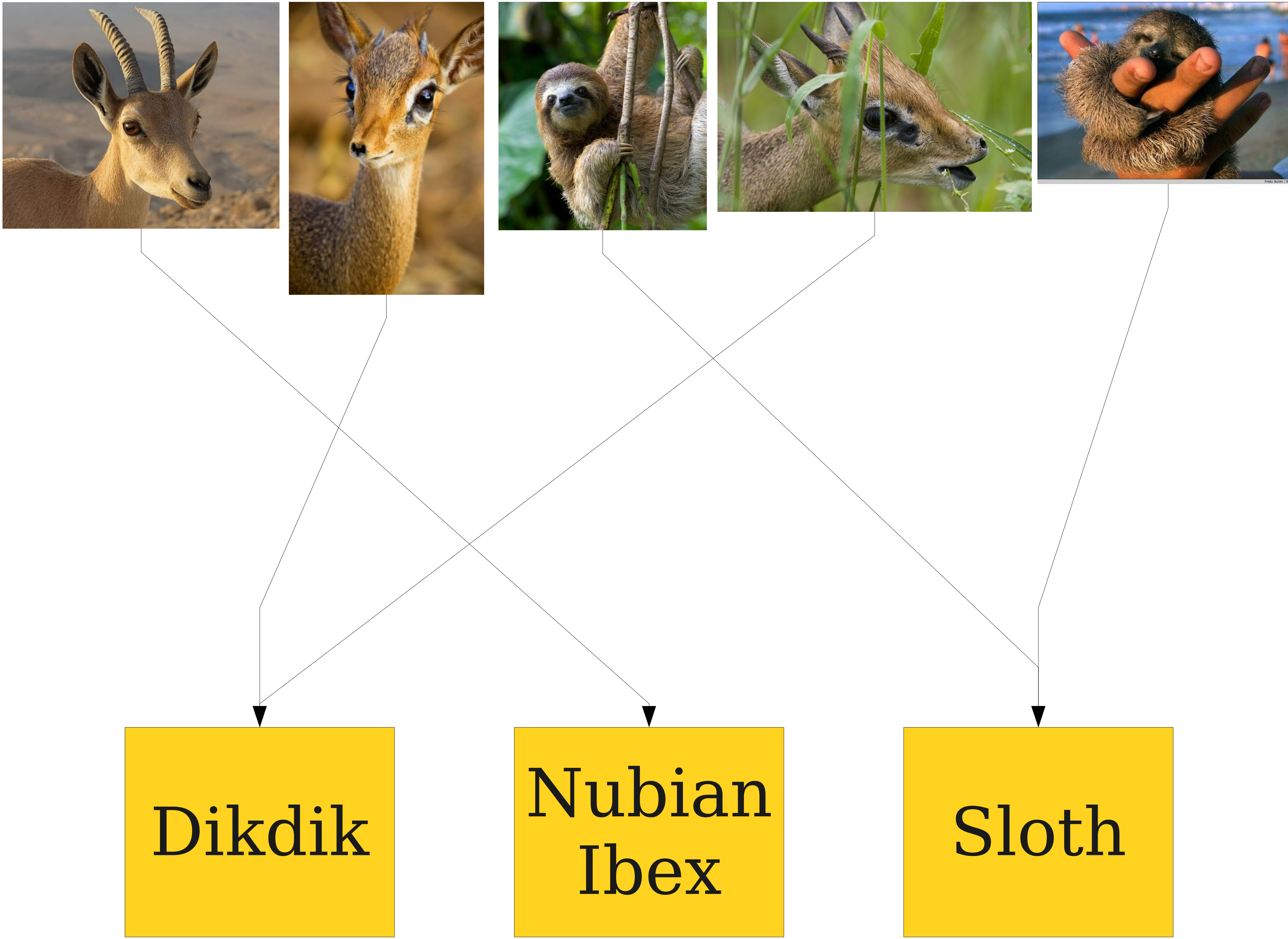
- A **partial order** is a relation that is reflexive, antisymmetric, and transitive.
- A **Hasse diagram** is a drawing of a partial order that has no self-loops, arrowheads, or redundant edges.
- A **total order** is a partial order in which any pair of elements are comparable.

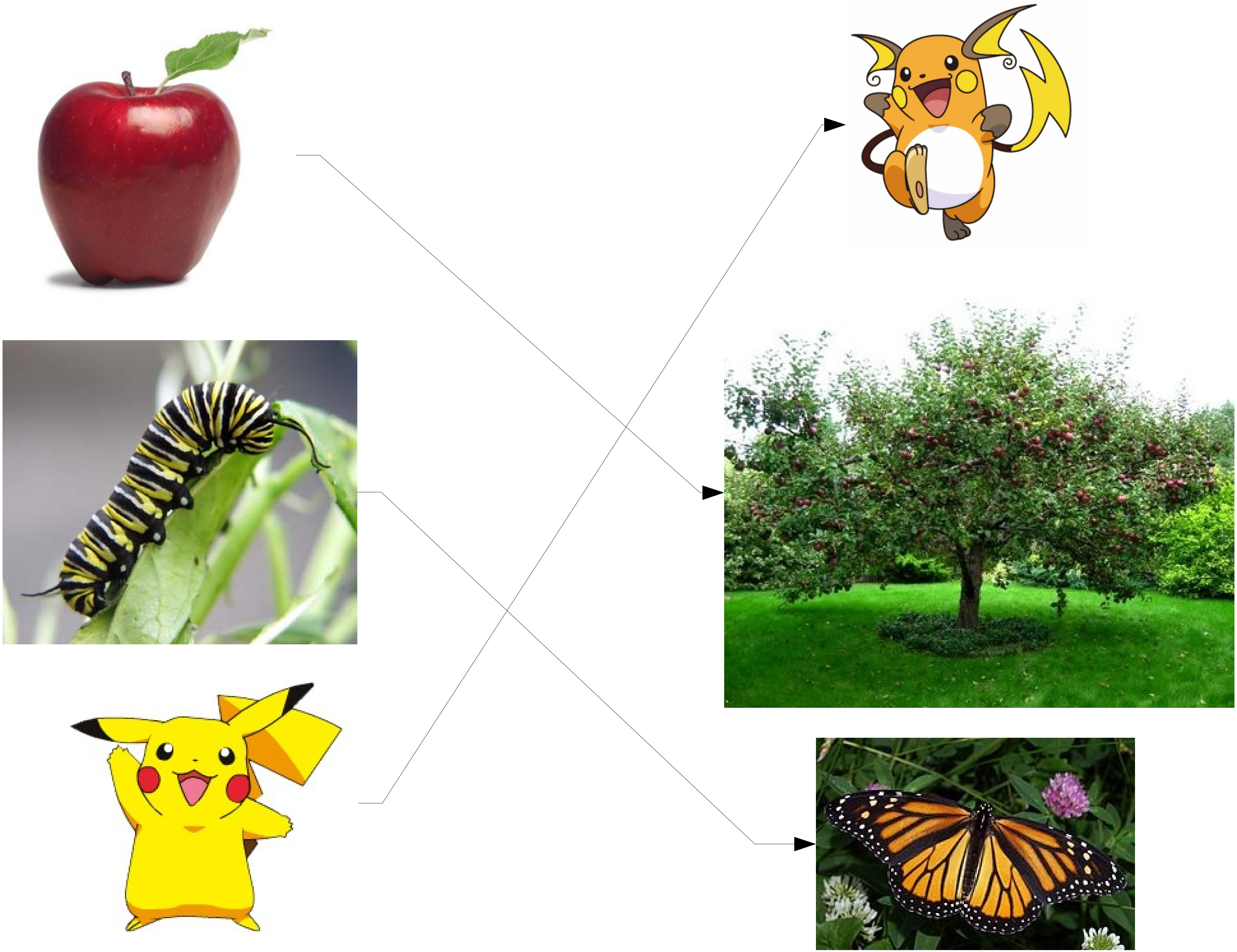
For More on the Olympics:

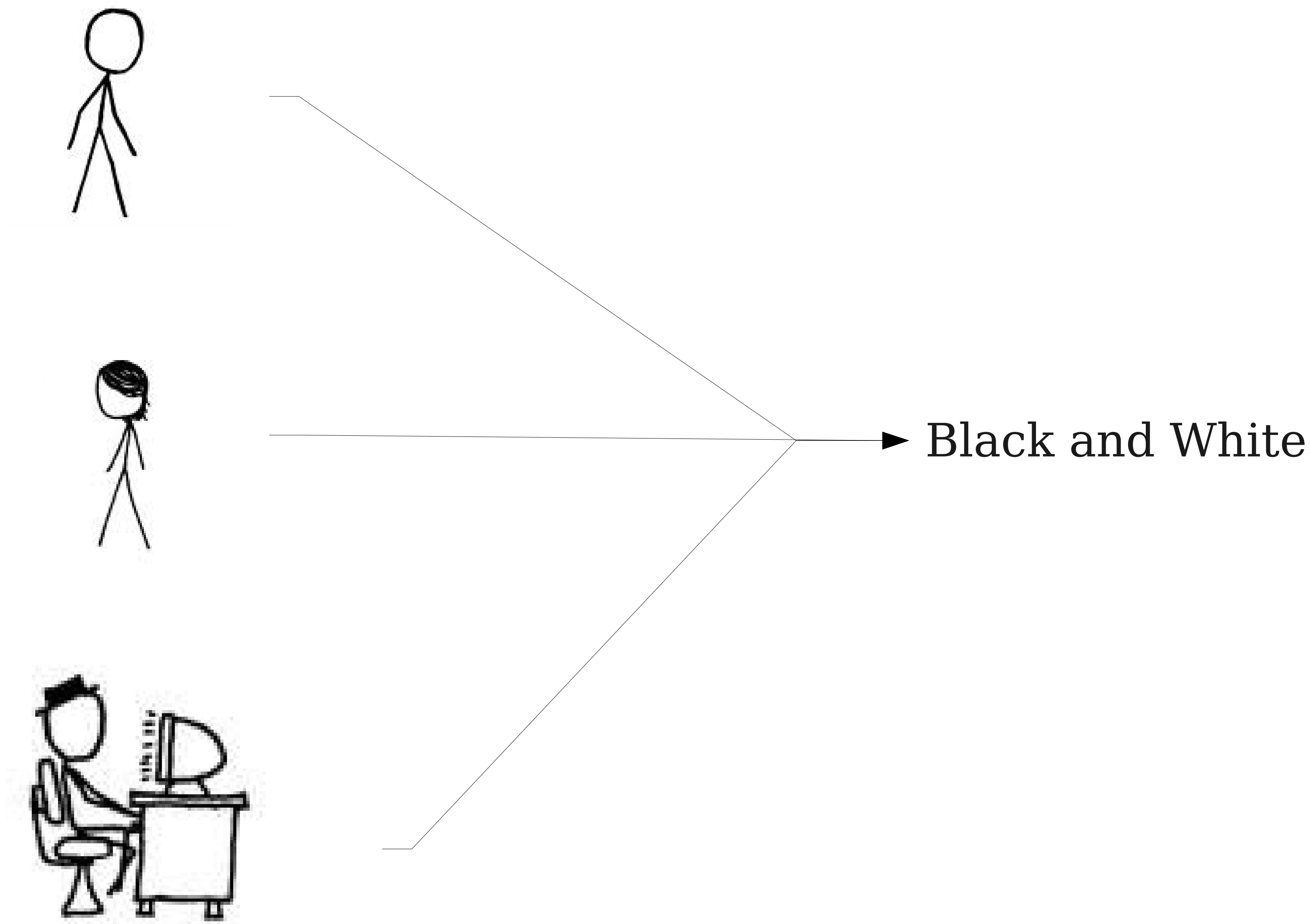
http://www.nytimes.com/interactive/2012/08/07/sports/olympics/the-best-and-worst-countries-in-the-medal-count.html

Functions

A **function** is a means of associating each object in one set with an object in some other set.


Dikdik
Nubian Ibex
Sloth



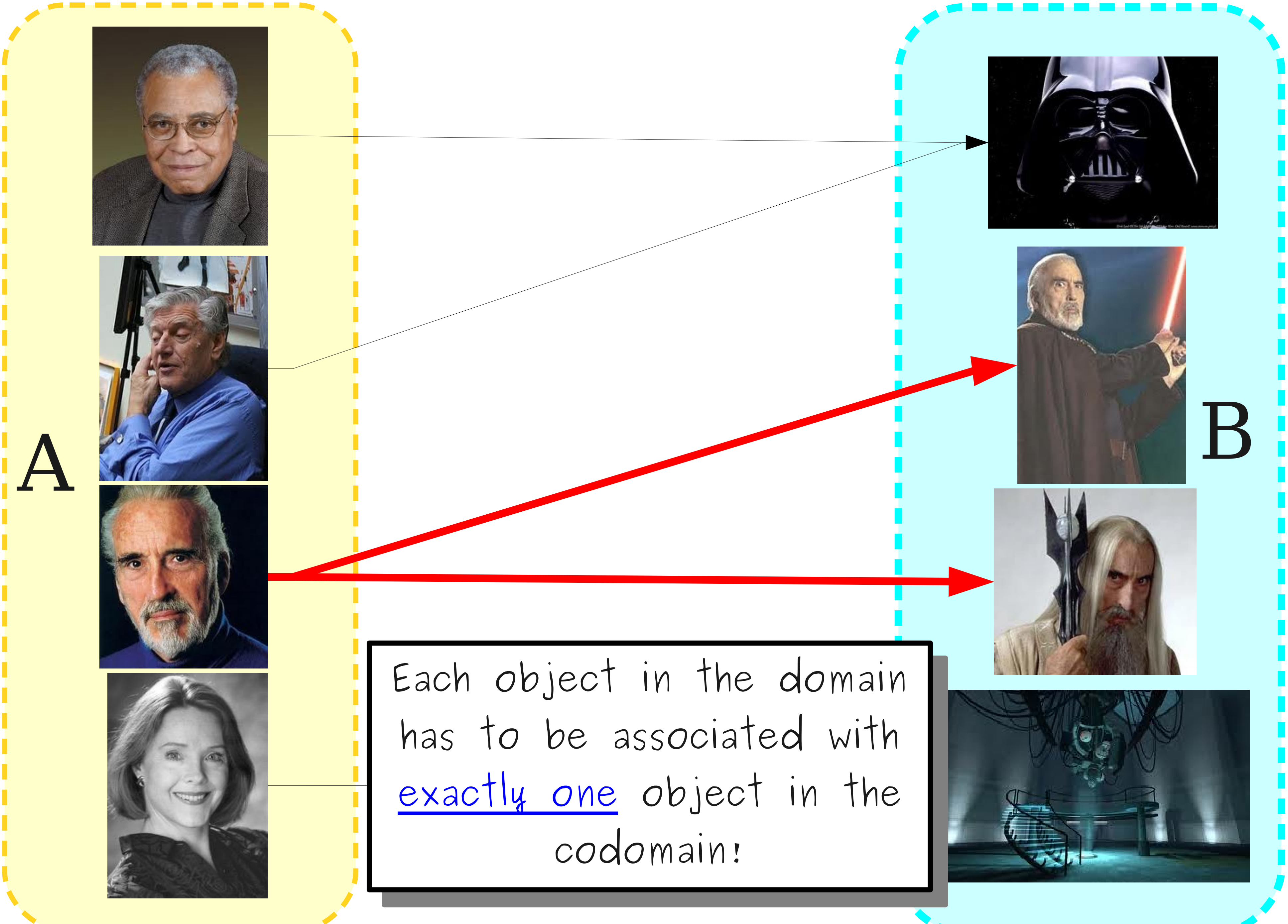

Black and White

Terminology

- A **function** f is a mapping such that every value in A is associated with a unique value in B.
  - For every $a \in A$, there exists some $b \in B$ with $f(a) = b$.
  - If $f(a) = b_0$ and $f(a) = b_1$, then $b_0 = b_1$.
- If f is a function from A to B, we sometimes say that f is a **mapping** from A to B.
  - We call A the **domain** of f.
  - We call B the **codomain** of f.
    - We'll discuss "range" in a few minutes.
- We denote that f is a function from A to B by writing

$$f : A \to B$$

Is This a Function from A to B?



Each object in the domain has to be associated with exactly one object in the codomain!

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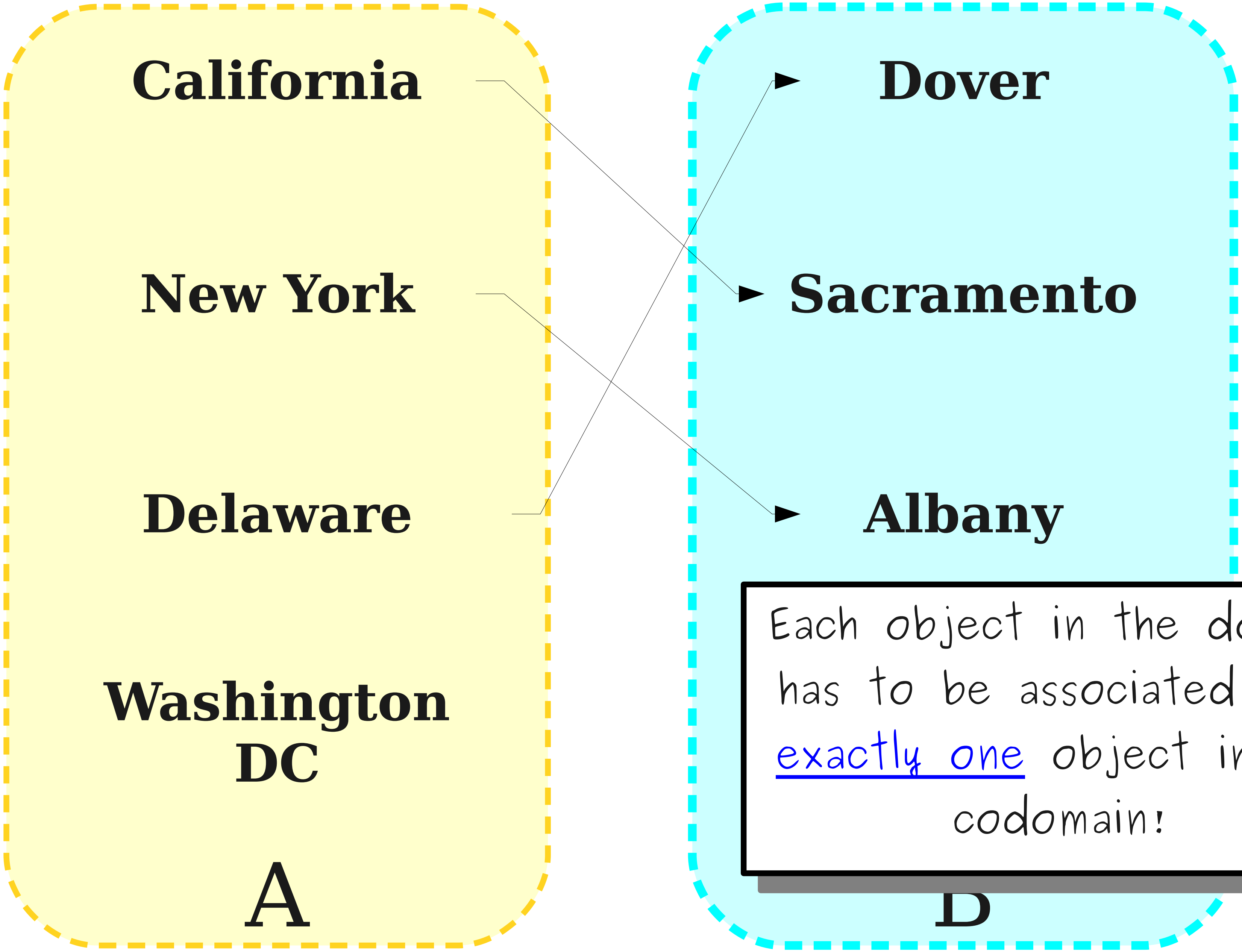

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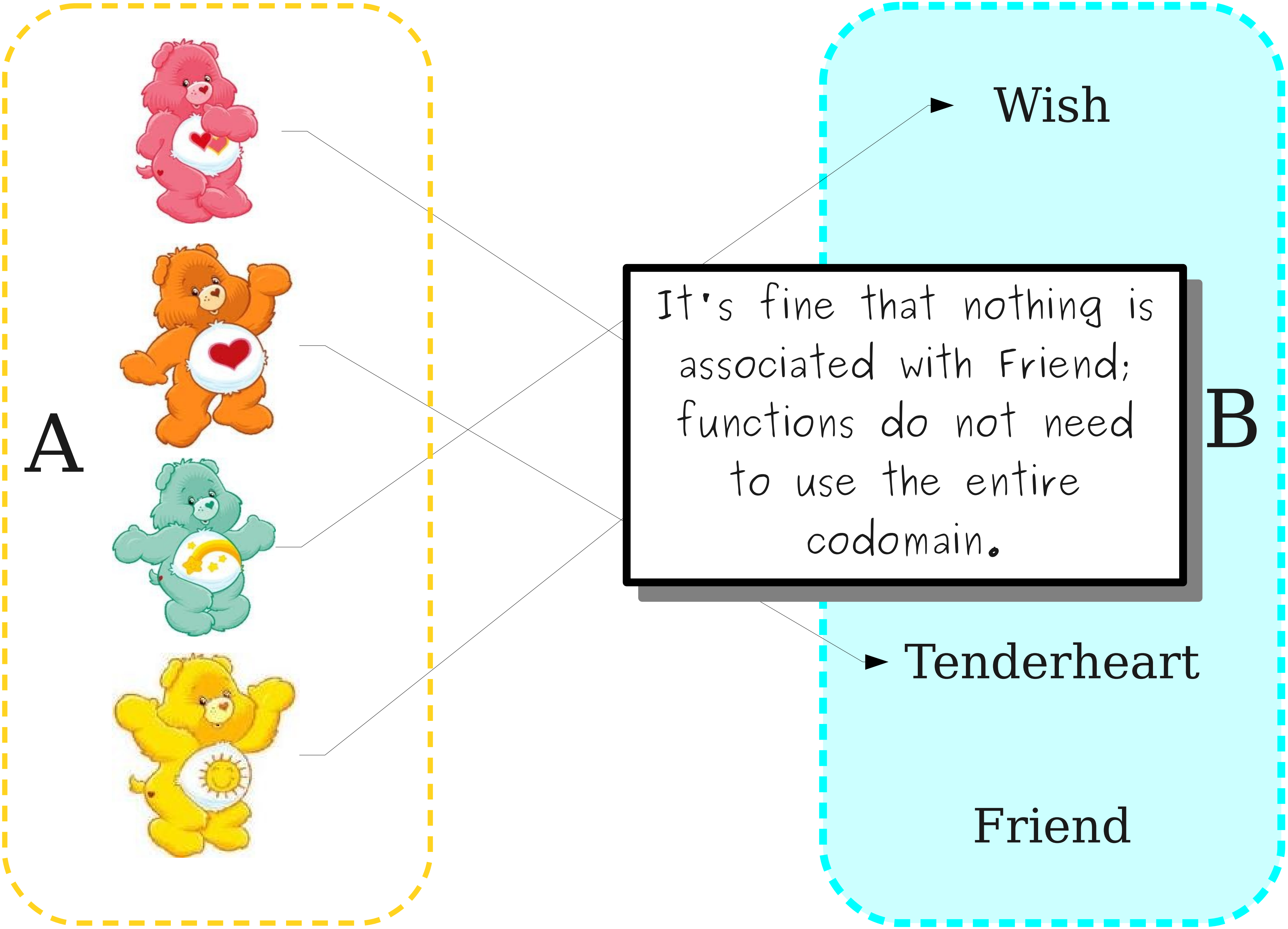

Defining Functions

- Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.
- Examples:
  - $f(n) = n + 1$, where $f : \mathbb{Z} \to \mathbb{Z}$
  - $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
  - $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$
- When defining a function it is always a good idea to verify that
  - The function is uniquely defined for all elements in the domain, and
  - The function's output is always in the codomain.

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This is the ceiling function – the smallest integer greater than or equal to x. For example, $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil \pi \rceil = 4$.

When defining a function it is always a good idea to verify that

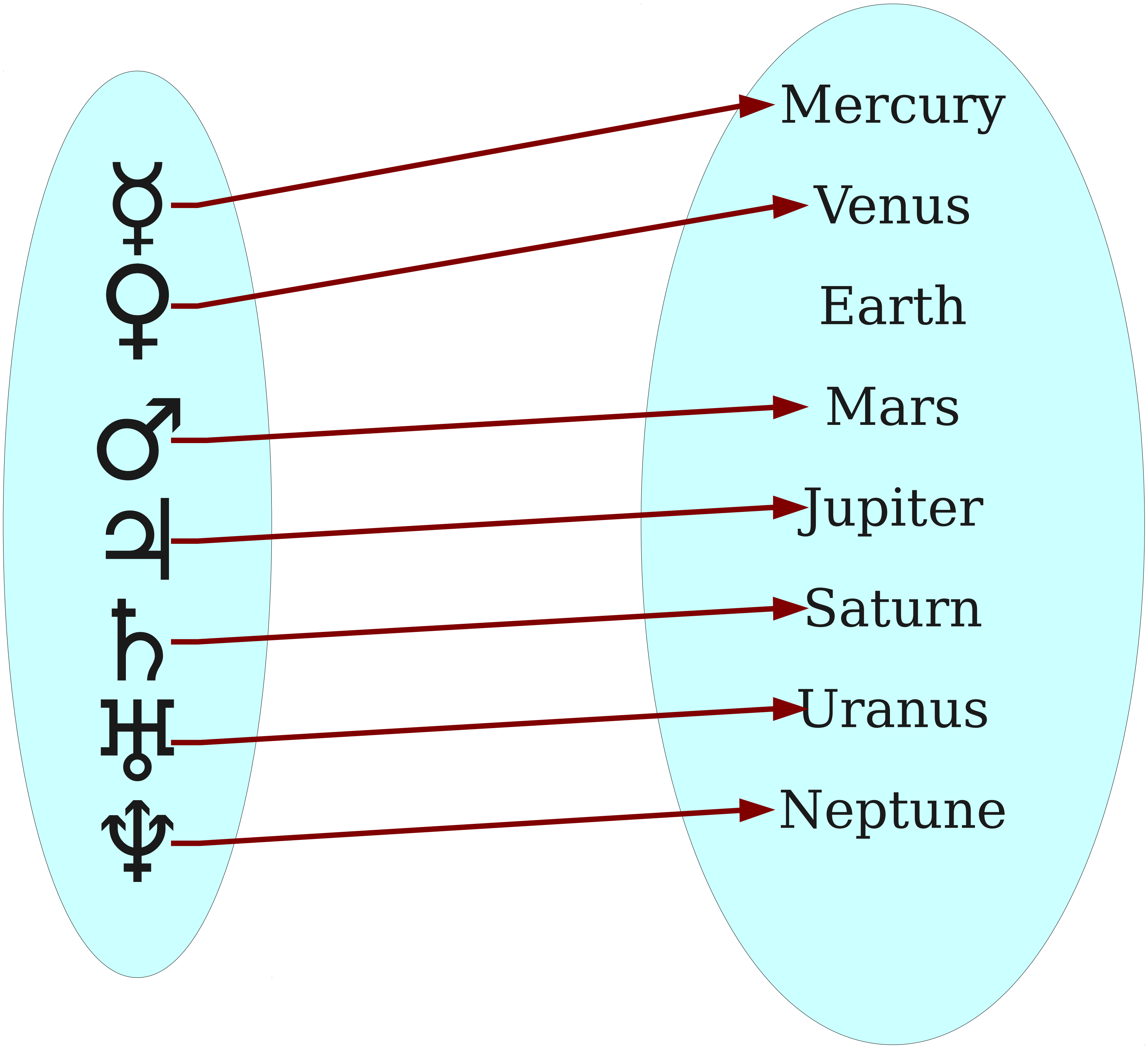
- The function is uniquely defined for all elements in the domain, and
- The function's output is always in the codomain.

Piecewise Functions

- Functions may be specified **piecewise**, with different rules applying to different elements.
- Example:

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{otherwise} \end{cases}$$

- When defining a function piecewise, it's up to you to confirm that it defines a legal function!

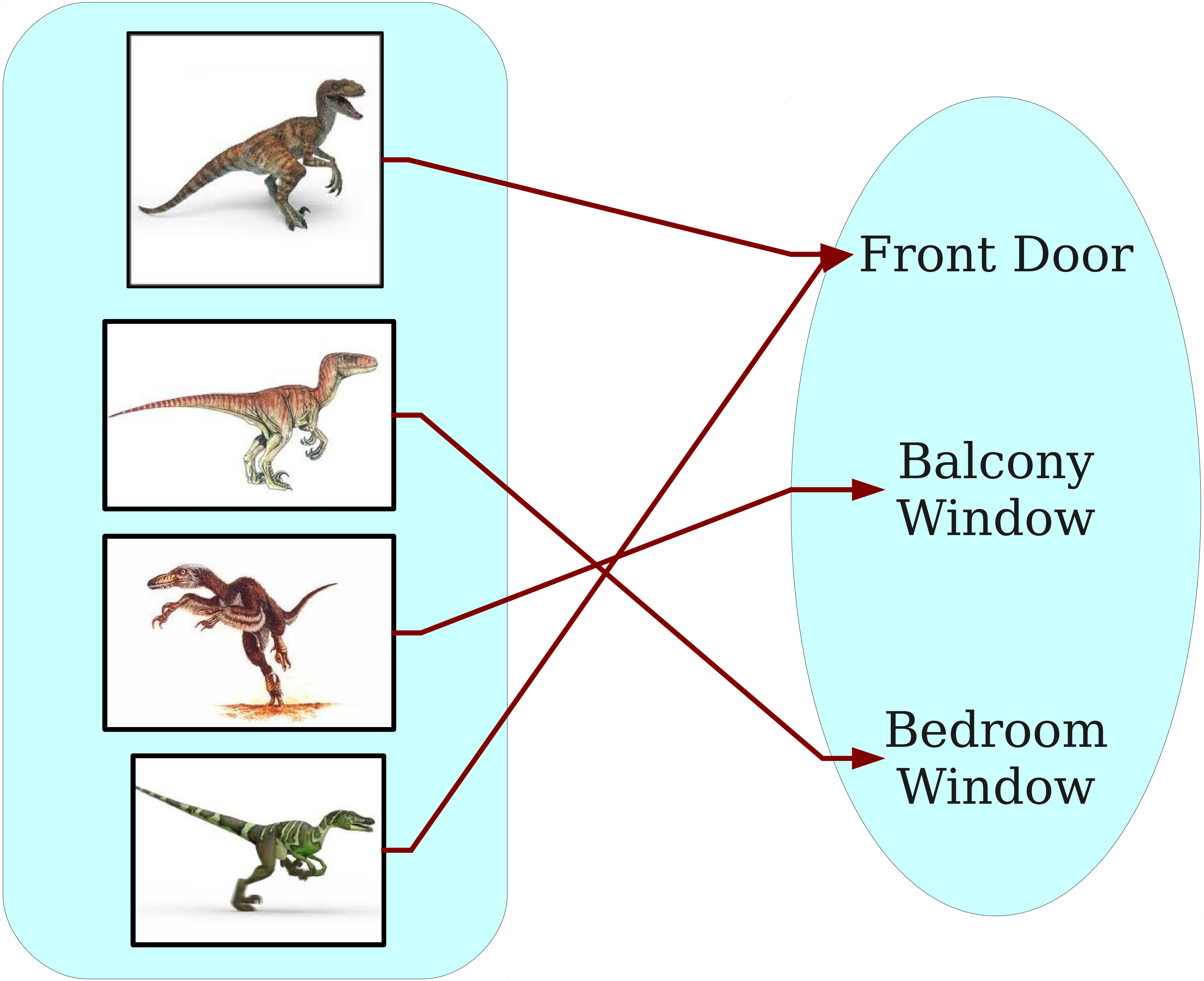

Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune

Injective Functions

- A function $f : A \to B$ is called **injective** (or **one-to-one**) if each element of the codomain has at most one element of the domain associated with it.
  - A function with this property is called an **injection**.
- Formally:

$$\textbf{If } f(x_0) = f(x_1)\textbf{, then } x_0 = x_1$$

- An intuition: injective functions label the objects from A using names from B.


Front Door
Balcony Window
Bedroom Window

Surjective Functions

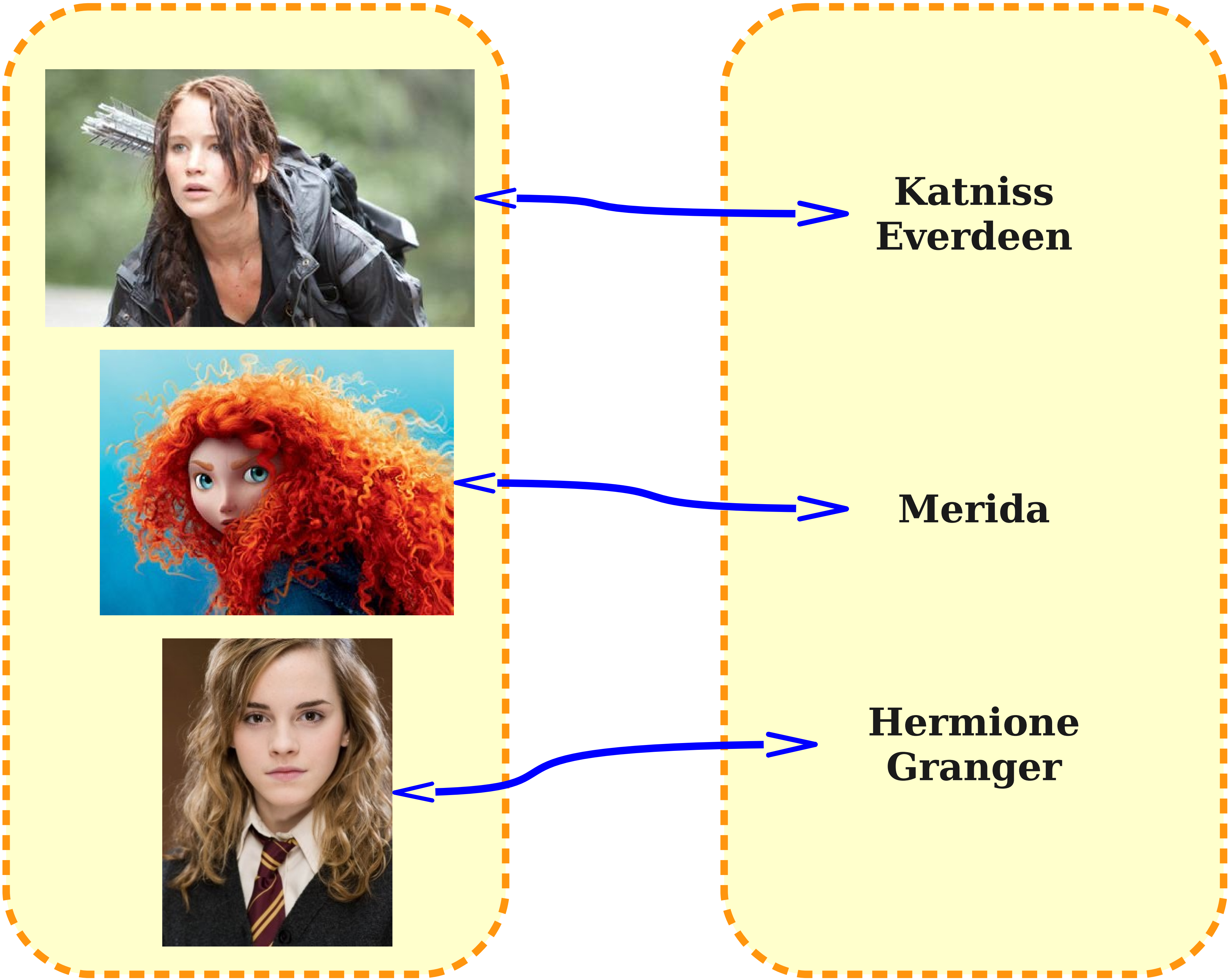
- A function $f : A \to B$ is called **surjective** (or **onto**) if each element of the codomain has at least one element of the domain associated with it.
  - A function with this property is called a **surjection**.
- Formally:

For any $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

- An intuition: surjective functions cover every element of B with at least one element of A.

Injections and Surjections

- An injective function associates **at most** one element of the domain with each element of the codomain.
- A surjective function associates **at least** one element of the domain with each element of the codomain.
- What about functions that associate **exactly one** element of the domain with each element of the codomain?


Katniss Everdeen
Merida
Hermione Granger

Bijections

- A function that associates each element of the codomain with a unique element of the domain is called **bijective**.
  - Such a function is a **bijection**.
- Formally, a bijection is a function that is both **injective** and **surjective**.
- A bijection is a one-to-one correspondence between two sets.

Compositions


www.microsoft.com
www.apple.com
www.google.com
Microsoft®
Google™
zune


www.microsoft.com
www.apple.com
www.google.com
zune
Google
Gmail
Maps
YouTube
Market

Function Composition

- Let $f : A \to B$ and $g : B \to C$.
- The **composition of f and g** (denoted **$g \circ f$**) is the function $g \circ f : A \to C$ defined as

$$(g \circ f)(x) = g(f(x))$$

- Note that f is applied first, but f is on the right side!
- Function composition is **associative**:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Function Composition

- Suppose $f : A \to A$ and $g : A \to A$.
- Then both $g \circ f$ and $f \circ g$ are defined.
- Does $g \circ f = f \circ g$?
- **In general, no:**
  - Let $f(x) = 2x$
  - Let $g(x) = x + 1$
  - $(g \circ f)(x) = g(f(x)) = g(2x) = 2x + 1$
  - $(f \circ g)(x) = f(g(x)) = f(x + 1) = 2x + 2$

Cardinality Revisited

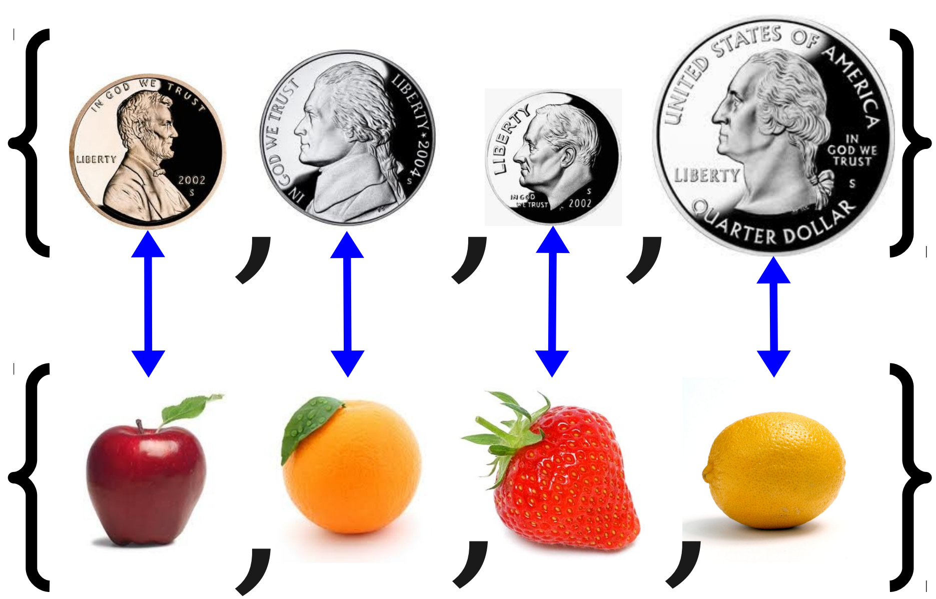
Cardinality

- Recall (from *lecture one!*) that the **cardinality** of a set is the number of elements it contains.
  - Denoted |S|.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200, 300\}| = 3$
- For infinite sets, we introduce **infinite cardinals** to denote the size of sets:
  - $|\mathbb{N}| = \aleph_0$

Comparing Cardinalities

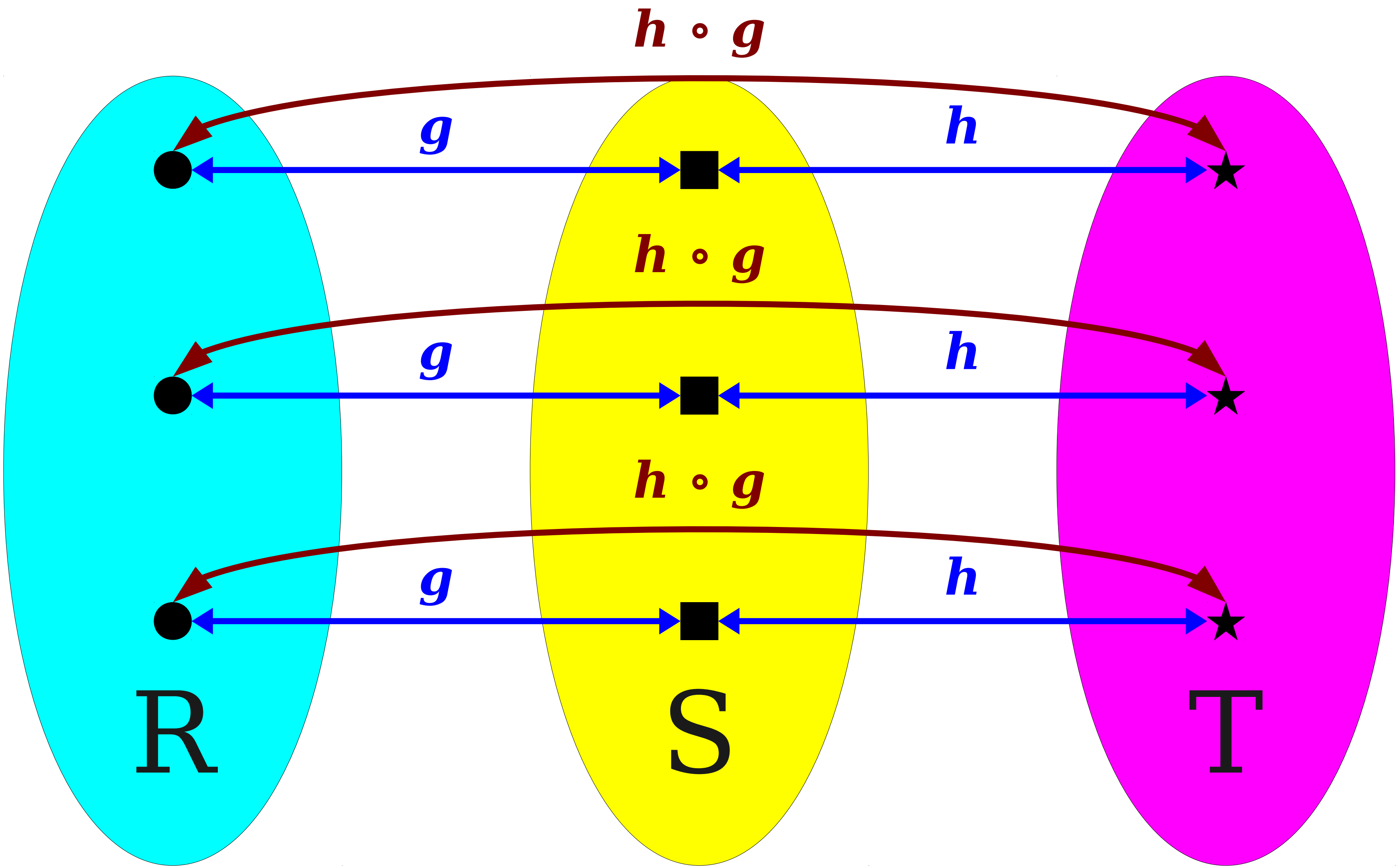
- The relationships between set cardinalities are defined in terms of functions between those sets.
- $|S| = |T|$ is defined using bijections.

$|S| = |T|$ iff there is a bijection $f : S \to T$



Theorem: If $|R| = |S|$ and $|S| = |T|$, then $|R| = |T|$.

Proof: We will exhibit a bijection $f : R \to T$. Since $|R| = |S|$, there is a bijection $g : R \to S$. Since $|S| = |T|$, there is a bijection $h : S \to T$.



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Let $f = h \circ g$; this means that $f : R \to T$. We prove that f is a bijection by showing that it is injective and surjective.

To see that f is injective, suppose that $f(r_0) = f(r_1)$. We will show that $r_0 = r_1$. Since $f(r_0) = f(r_1)$, we know $(h \circ g)(r_0) = (h \circ g)(r_1)$. By definition of composition, we have $h(g(r_0)) = h(g(r_1))$. Since h is a bijection, h is injective. Thus since $h(g(r_0)) = h(g(r_1))$, we have that $g(r_0) = g(r_1)$. Since g is a bijection, g is injective, so because $g(r_0) = g(r_1)$ we have that $r_0 = r_1$. Therefore, f is injective.

To see that f is surjective, consider any $t \in T$. We will show that there is some $r \in R$ such that $f(r) = t$. Since h is a bijection from S to T, h is surjective, so there is some $s \in S$ such that $h(s) = t$. Since g is a bijection from R to S, g is surjective, so there is some $r \in R$ such that $g(r) = s$. Thus $f(r) = (h \circ g)(r) = h(g(r)) = h(s) = t$ as required, so f is surjective.

Since f is injective and surjective, it is bijective. Thus there is a bijection from R to T, so $|R| = |T|$. ■

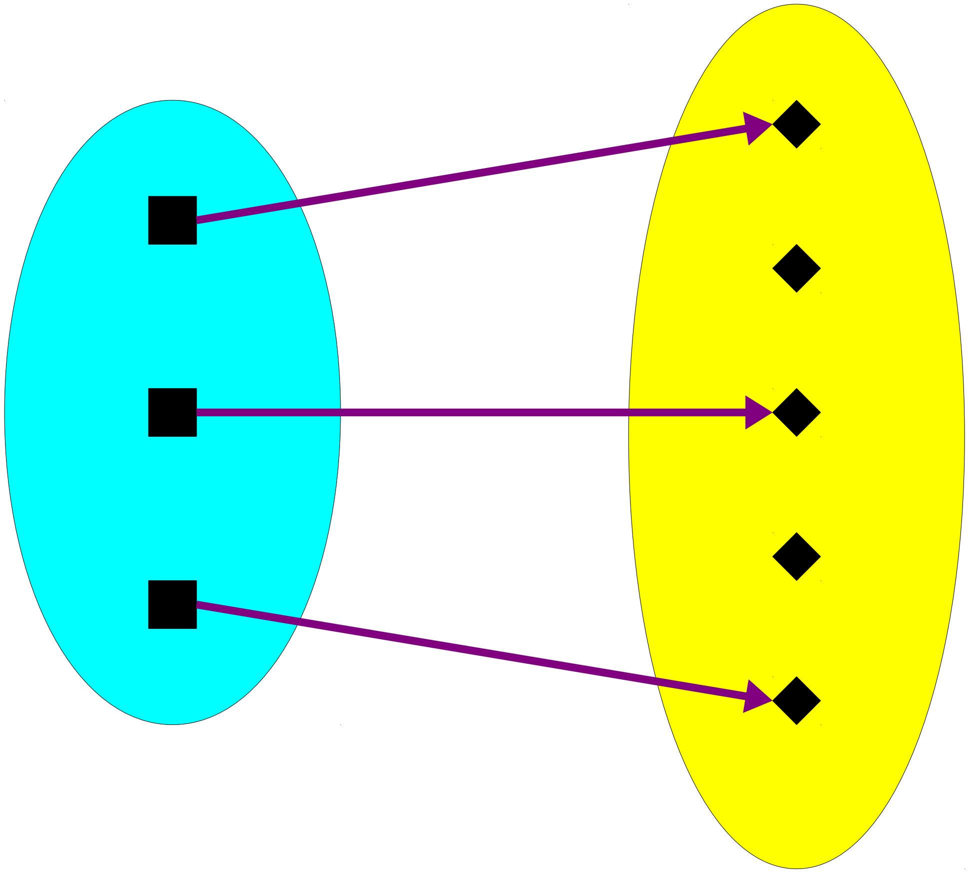
Properties of Cardinality

- Equality of cardinality is an equivalence relation. For any sets R, S, and T:
  - $|S| = |S|$. ***(reflexivity)***
  - If $|S| = |T|$, then $|T| = |S|$. ***(symmetry)***
  - If $|R| = |S|$ and $|S| = |T|$, then $|R| = |T|$. ***(transitivity)***

Comparing Cardinalities

- We define $|S| \leq |T|$ as follows:

$|S| \leq |T|$ iff there is an injection $f : S \to T$



Comparing Cardinalities

- We define $|S| \leq |T|$ as follows:

$|S| \leq |T|$ iff there is an injection $f : S \to T$

- The ≤ relation over set cardinalities is a total order. For any sets R, S, and T:
  - $|S| \leq |S|$. ***(reflexivity)***
  - If $|R| \leq |S|$ and $|S| \leq |T|$, then $|R| \leq |T|$. ***(transitivity)***
  - If $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$. ***(antisymmetry)***
  - Either $|S| \leq |T|$ or $|T| \leq |S|$. ***(totality)***
- These last two proofs are **extremely hard**.
  - The antisymmetry result is the **Cantor-Bernstein-Schroeder Theorem**; a fascinating read, but beyond the scope of this course.
  - Totality requires the **axiom of choice**. Take Math 161 for more details.