Mathematical Logic

Part Two

Announcements

- Problem Set 2 and Checkpoint 3 graded.
  - Will be returned at end of lecture.
- Problem Set 3 due this Friday at 2:15PM.
  - Stop by office hours questions!
  - Email cs103-aut1213-staff@lists.stanford.edu with questions!

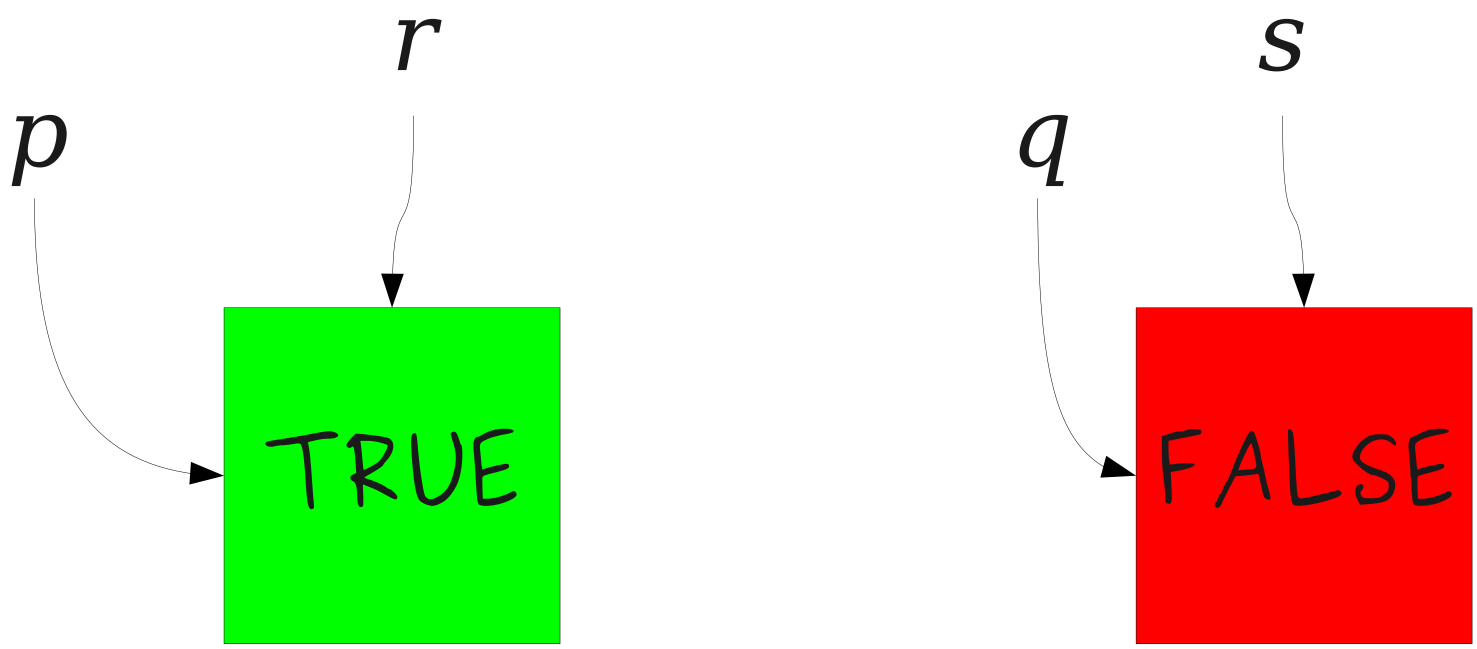
First-Order Logic

What is First-Order Logic?

- **First-order logic** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - **predicates** that describe properties of objects, and
  - **functions** that map objects to one another,
  - **quantifiers** that allow us to reason about multiple objects simultaneously.

The Universe of Propositional Logic

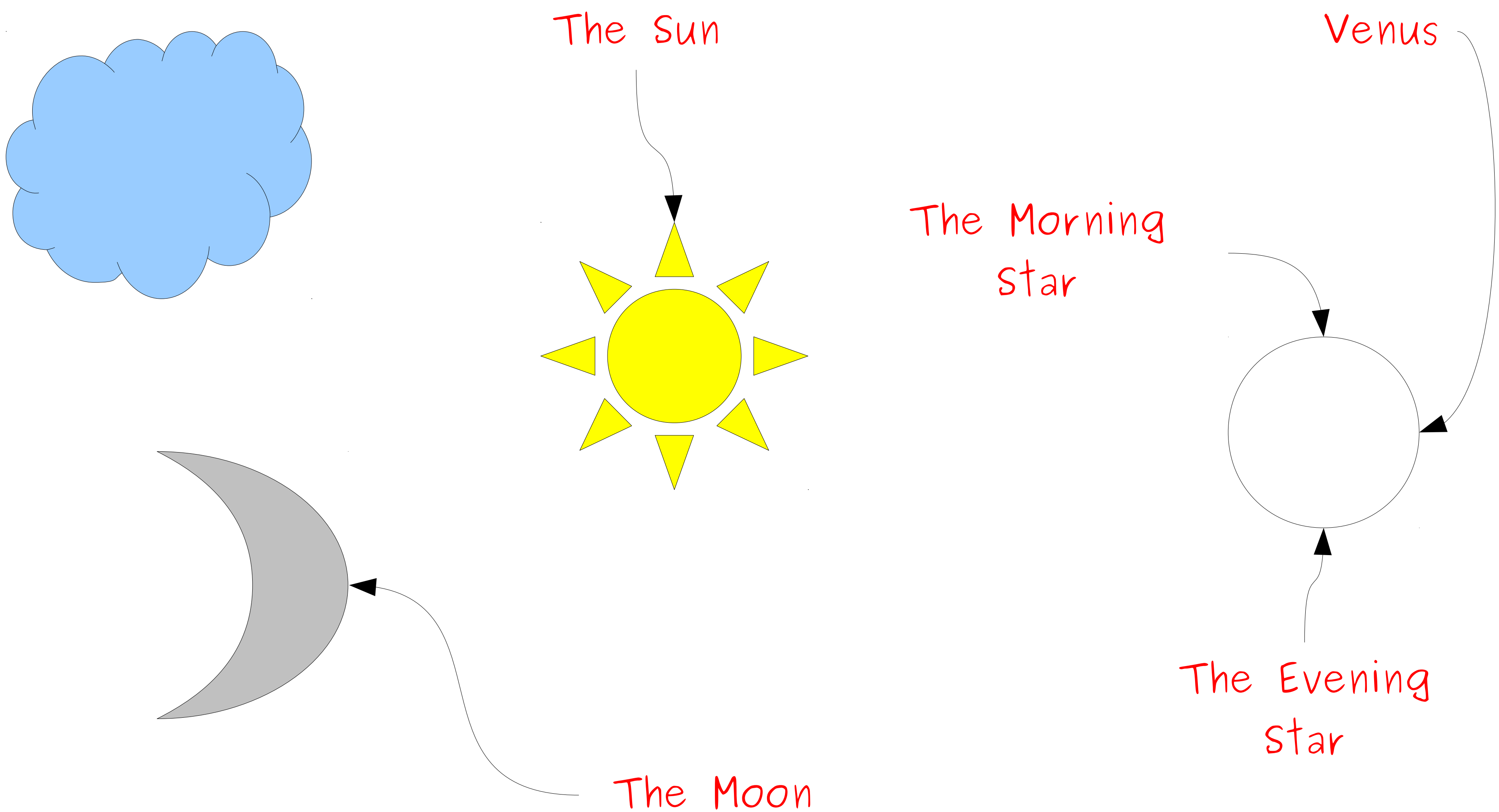
$$p \wedge q \rightarrow \neg r \vee \neg s$$



Propositional Logic

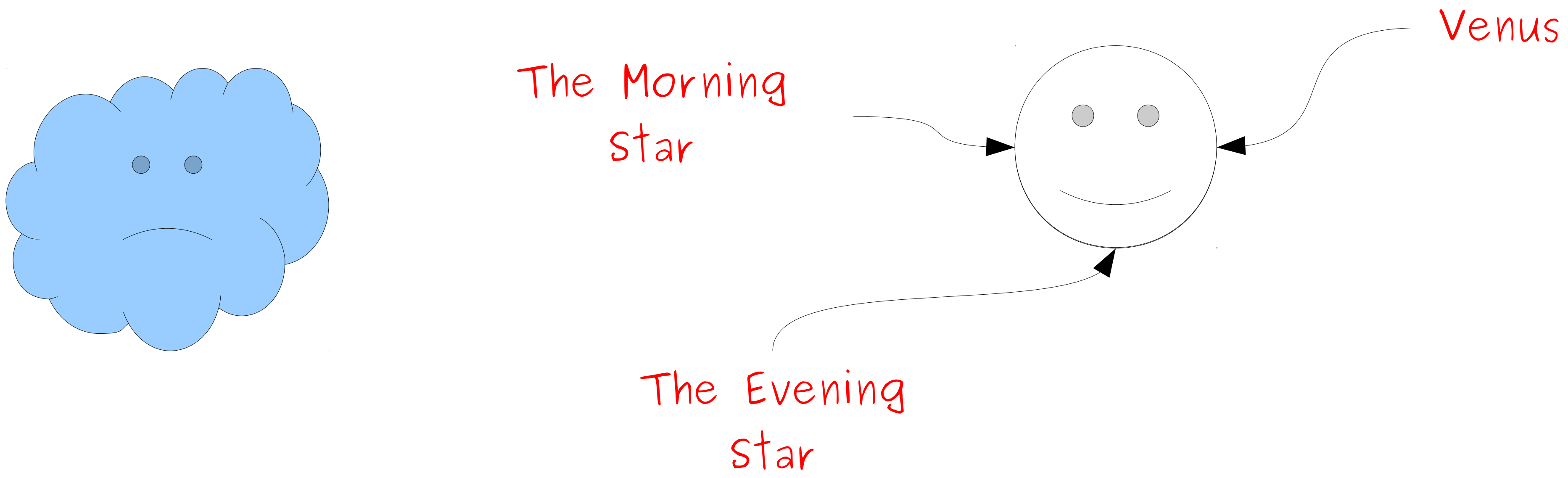
- In propositional logic, each variable represents a **proposition**, which is either true or false.
- Consequently, we can directly apply connectives to propositions:
  - $p \to q$
  - $\neg p \land q$
- The truth or falsity of a statement can be determined by plugging in the truth values for the input propositions and computing the result.
- We can see all possible truth values for a statement by checking all possible truth assignments to its variables.

The Universe of First-Order Logic



First-Order Logic

- In first-order logic, each variable refers to some object in a set called the **domain of discourse**.
- Some objects may have multiple names.
- Some objects may have no name at all.



Propositional vs. First-Order Logic

- Because propositional variables are either true or false, we can directly apply connectives to them.
  - $p \to q$
  - $\neg p \leftrightarrow q \land r$
- Because first-order variables refer to arbitrary objects, it does not make sense to apply connectives to them.
  - $Venus \to Sun$
  - $137 \leftrightarrow \neg 42$
- ***This is not C!***

Reasoning about Objects

- To reason about objects, first-order logic uses **predicates**.
- Examples:
  - *GottaGetDownOn*(*Friday*)
  - *LookingForwardTo*(*Weekend*)
  - *ComesAfterwards*(*Sunday*, *Saturday*)
- Predicates can take any number of arguments, but each predicate has a fixed number of arguments (called its **arity**)
- Applying a predicate to arguments produces a proposition, which is either true or false.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$LikesToEat(V, M) \land Near(V, M) \to WillEat(V, M)$$

$$Cute(t) \to Dikdik(t) \lor Kitty(t) \lor Puppy(t)$$

$$x < 8 \to x < 137$$

The notation **$x < 8$** is just a shorthand for something like **LessThan(x, 8)**. Binary predicates in math are often written like this, but symbols like < are not a part of first-order logic.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as → and ¬ are.
- Examples:

$$\mathit{MorningStar} = \mathit{EveningStar}$$

$$\mathit{Glenda} = \mathit{GoodWitchOfTheNorth}$$

- Equality can only be applied to **objects**; to see if **propositions** are equal, use ↔.

For notational simplicity, define $\neq$ as

$$x \neq y \quad \equiv \quad \neg(x = y)$$

Expanding First-Order Logic

$$x < 8 \land y < 8 \to x + y < 16$$

Why is this allowed?

Functions

- First-order logic allows **functions** that return objects associated with other objects.
- Examples:

$$x + y$$

$$LengthOf(path)$$

$$MedianOf(x, y, z)$$

- As with predicates, functions can take in any number of arguments, but each function has a fixed arity.
- Functions evaluate to **objects**, not **propositions**.
- There is no syntactic way to distinguish functions and predicates; you'll have to look at how they're used.

How would we translate the statement

"For any natural number n,
n is even iff n^2 is even"

into first-order logic?

Quantifiers

- The biggest change from propositional logic to first-order logic is the use of **quantifiers**.
- A **quantifier** is a statement that expresses that some property is true for some or all choices that could be made.
- Useful for statements like “for every action, there is an equal and opposite reaction.”

“For any natural number n,
n is even iff n^2 is even”

$$\forall n.\ (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the **universal quantifier** and says “for any choice of n, the following is true.”

The Universal Quantifier

- A statement of the form **$\forall x.\ \psi$** asserts that for **every** choice of x in our domain, ψ is true.
- Examples:

$$\forall v.\ (Velociraptor(v) \to WillEat(v, me))$$

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow \neg Odd(n)))$$

$$Tallest(x) \to \forall y.\ (x \neq y \to IsShorterThan(y, x))$$

Some velociraptor can open windows.

$$\exists v.\ (\mathit{Velociraptor}(v) \land \mathit{OpensWindows}(v))$$

$\exists$ is the **existential quantifier** and says "for some choice of v, the following is true."

The Existential Quantifier

- A statement of the form **$\exists x.\ \psi$** asserts that for **some** choice of x in our domain, ψ is true.
- Examples:

$$\exists x.\ (Even(x) \land Prime(x))$$

$$\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$$

$$(\exists x.\ Appreciates(x, me)) \to Happy(me)$$

Operator Precedence (Again)

- When writing out a formula in first-order logic, the quantifiers $\forall$ and $\exists$ have precedence just below $\neg$.
- Thus

$$\forall x.\ P(x) \lor R(x) \to Q(x)$$

is interpreted as

$$((\forall x.\ P(x)) \lor R(x)) \to Q(x)$$

rather than

$$\forall x.\ ((P(x) \lor R(x)) \to Q(x))$$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Everyone loves someone else."

$$\forall x.\ \exists y.\ (x \neq y \land Loves(x, y))$$

For any person

There is some person

Who isn't them

That they love

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “There is someone everyone else loves.”

$$\exists y.\ \forall x.\ (x \neq y \to Loves(x, y))$$

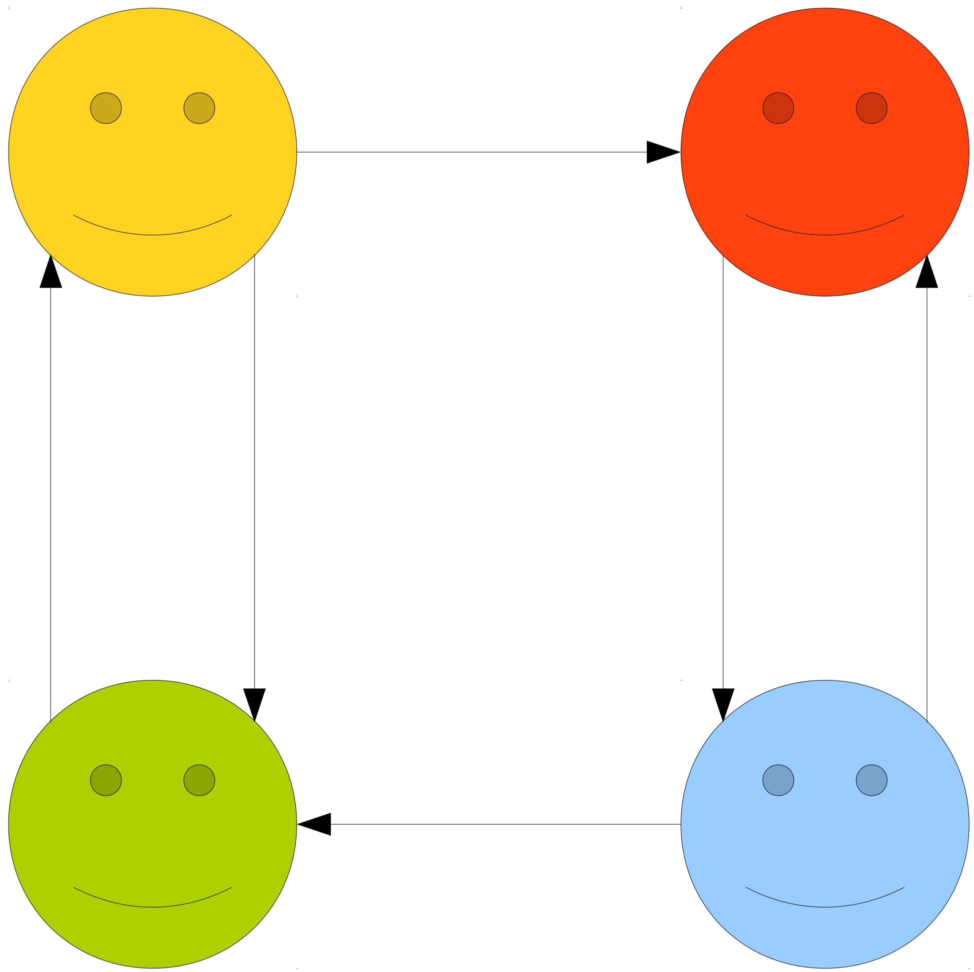
There is some person

that everyone

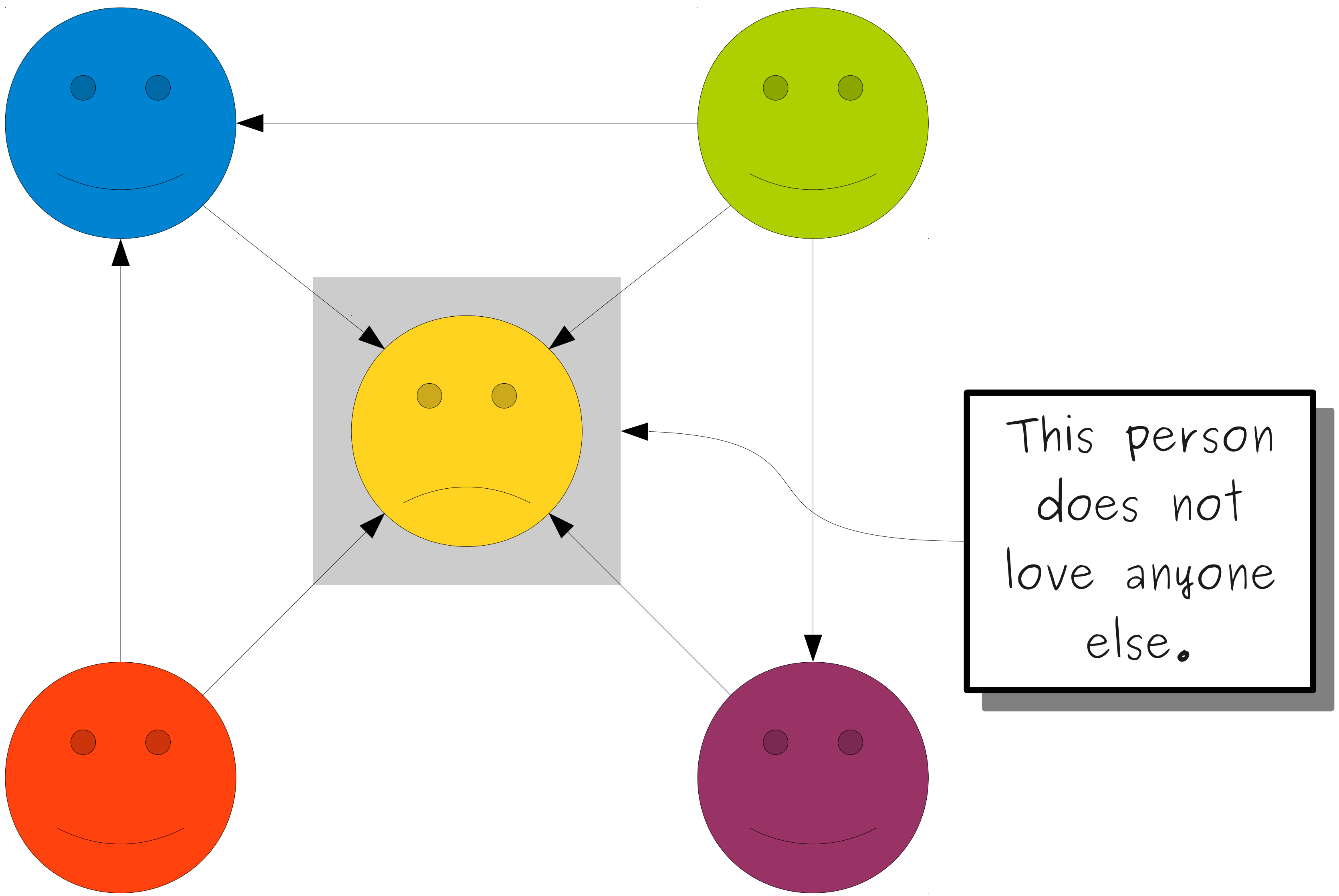
who isn't them

loves

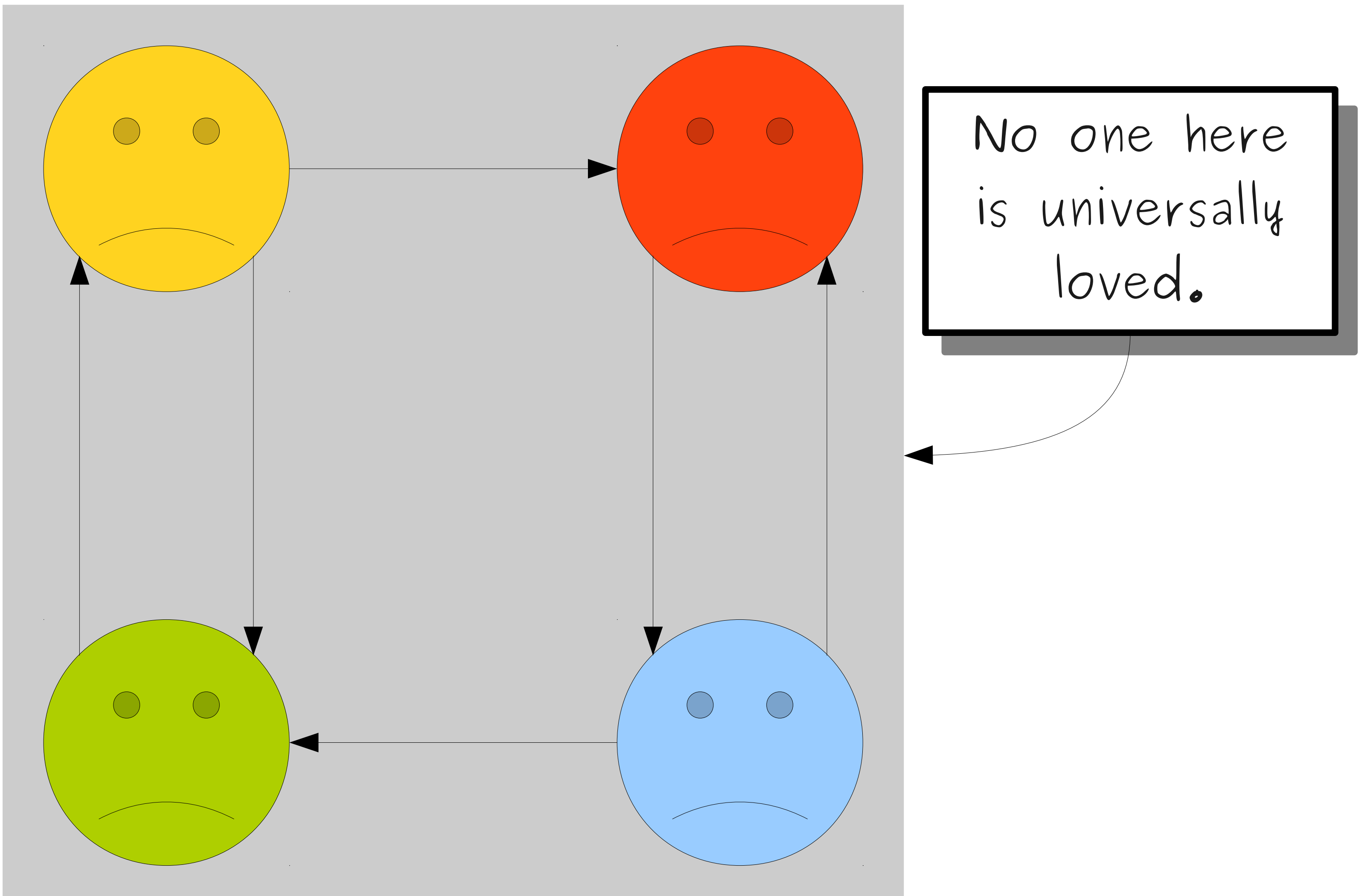
$$\forall x.\ \exists y.\ (x \neq y \land Loves(x, y))$$



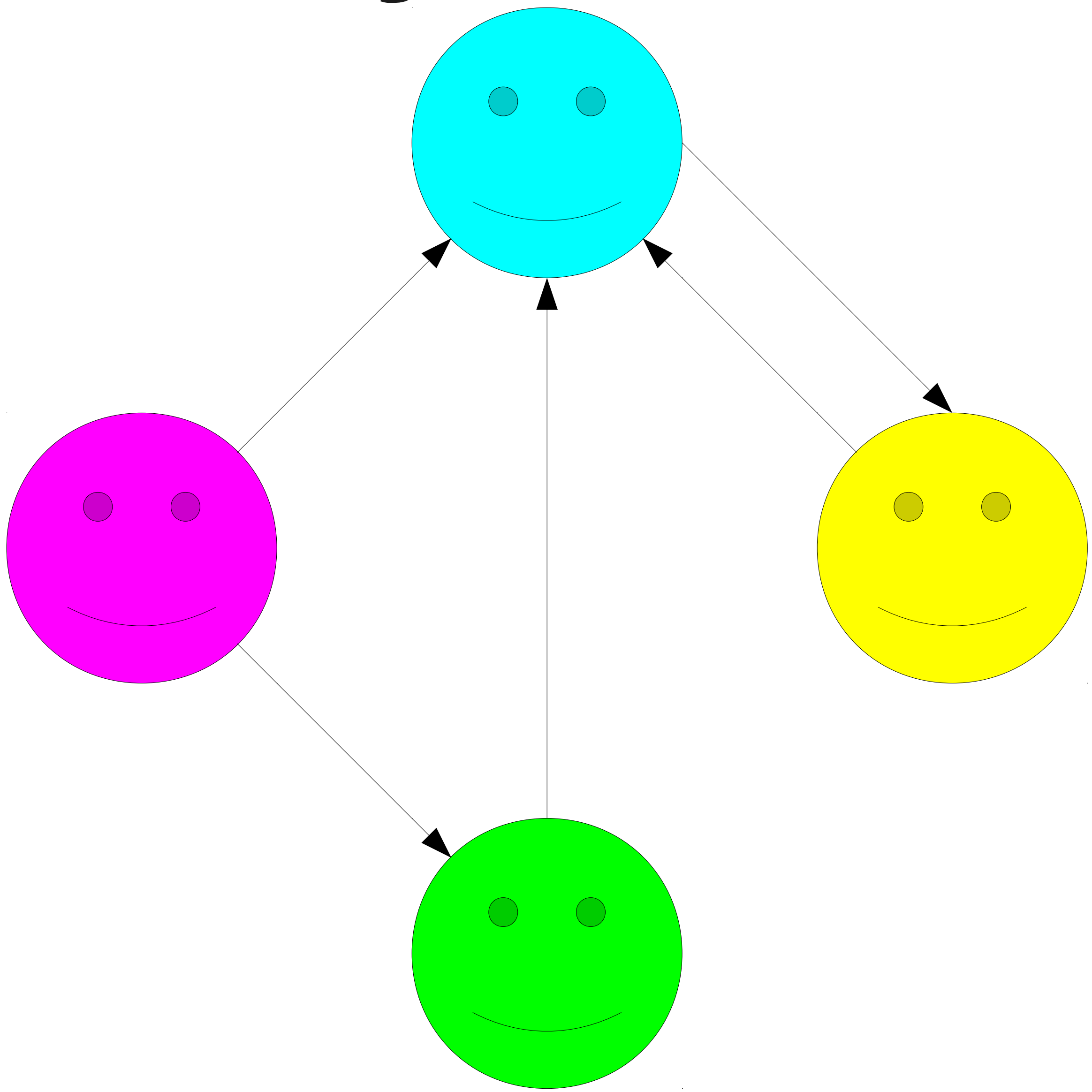
$$\exists y.\ \forall x.\ (x \neq y \rightarrow Loves(x, y))$$



$$\forall x.\ \exists y.\ (x \neq y \land Loves(x, y))$$



$$(\forall x.\ \exists y.\ (x \neq y \land Loves(x, y))) \land (\exists y.\ \forall x.\ (x \neq y \to Loves(x, y)))$$



The statement

$$\forall x.\ \exists y.\ P(x, y)$$

means “For any choice of *x*, there is **some** choice of *y* where *P*(*x*, *y*).”

The statement

$$\boldsymbol{\exists y.\ \forall x.\ P(x,\ y)}$$

means "There is some choice of y where for **any** choice of x, $P(x, y)$."

Order matters when mixing existential and universal quantifiers!

A Note on the Checkpoints…

This Doesn't Work!

Theorem: If R is transitive, then R^{-1} is transitive.
Proof: Consider any a, b, and c such that aRb and bRc. Since R is transitive, we have aRc. Since aRb and bRc, we have $bR^{-1}a$ and $cR^{-1}b$. Since we have aRc, we have $cR^{-1}a$. Thus $cR^{-1}b$, $bR^{-1}a$, and $cR^{-1}a$. ■

This proves

$$\forall a.\ \forall b.\ \forall c.\ (aRb \land bRc \to cR^{-1}b \land bR^{-1}a \land cR^{-1}a)$$

You need to show

$$\forall a.\ \forall b.\ \forall c.\ \ (aR^{-1}b \land bR^{-1}c \to aR^{-1}c)$$

Don't get tripped up by definitions!

To directly prove that $p \to q$,
assume p and prove q.

A Correct Proof

$$\forall a.\ \forall b.\ \forall c.\ \ (aR^{-1}b \land bR^{-1}c \to aR^{-1}c)$$

Theorem: If R is transitive, then R^{-1} is transitive.
Proof: Consider any a, b, and c such that $aR^{-1}b$ and $bR^{-1}c$. We will prove $aR^{-1}c$. Since $aR^{-1}b$ and $bR^{-1}c$, we have that bRa and cRb. Since cRb and bRa, by transitivity we know cRa. Since cRa, we have $aR^{-1}c$, as required. ■

Back to First-Order Logic…

A Bad Translation

Everyone who can outrun velociraptors won't get eaten.

$$\forall x.\ (FasterThanVelociraptors(x) \land \neg WillBeEaten(x))$$

What happens if x refers to someone slower than velociraptors who does get eaten?

A Better Translation

Everyone who can outrun velociraptors won't get eaten.

$$\forall x.\ (FasterThanVelociraptors(x) \to \neg WillBeEaten(x))$$

“Whenever $P(x)$, then $Q(x)$”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Another Bad Translation

There is some velociraptor that can open windows and eat me.

$$\exists x. \ (Velociraptor(x) \land OpensWindows(x) \to EatsMe(x))$$

What happens if

1. The above statement is false, but
2. x refers to me (I'm not a velociraptor!)

A Better Translation

There is some velociraptor that can open windows and eat me.

$$\exists x. \ (\mathit{Velociraptor}(x) \land \mathit{OpensWindows}(x) \land \mathit{EatsMe}(x))$$

"There is some $P(x)$ where $Q(x)$"

translates as

$$\exists x.\ (P(x) \land Q(x))$$

The Takeaway Point

- Be careful when translating statements into first-order logic!
- $\forall$ is usually paired with $\rightarrow$.
- $\exists$ is usually paired with $\land$.

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

means "for any element x of set S, $P(x)$ holds."

- This is not technically a part of first-order logic; it is a shorthand for

$$\forall x.\ (x \in S \to P(x))$$

- How might we encode this concept?

$$\exists x \in S.\ P(x)$$

Answer: $\exists x.\ (x \in S \land P(x))$.

Note the use of ∧ instead of → here.

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ \varphi$$

$$\exists x \in S.\ \varphi$$

  is allowed for quantifying over sets.
- In CS103, please do not use variants of this syntax.
- Please don't do things like this:

$$\forall x \text{ with } P(x).\ Q(x)$$

$$\forall y \text{ such that } P(y) \land Q(y).\ R(y).$$

Translating into First-Order Logic

- First-order logic has great expressive power and is often used to formally encode mathematical definitions.
- Let's go provide rigorous definitions for the terms we've been using so far.

Set Theory

"Two sets are equal iff they contain the same elements."

$$\forall S.\ \forall T.\ (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T))$$

These quantifiers are critical here, but they don't appear anywhere in the English. Many statements asserting a general claim is true are implicitly universally quantified.