Finite Automata


BRACE YOURSELF
MIDTERM IS COMING
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Midterm Logistics

- Midterm is next **Monday, October 29** from **7PM – 10PM** in **Cubberly Auditorium**.
  - Open-book, open-note, open-computer, closed-network.
  - Covers material up through and including today's lecture.
- Practice exam available now; solutions will be released on Wednesday.
- If you need to take the exam at an alternate time, email the course staff no later than tomorrow at 2:15PM.

CS103

Mathematical Foundations of Computing

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...r Out

...ent out today. The checkpoint ...is Monday, October 22 at the ...is graded on a received / not ...he remaining problems are due ...er 26 at the start of class.

...plays around with logic. You'll

Handouts

00: Course Information
01: Syllabus
02: Prior Experience Survey
07: Diagonalization

Assignments

Problem Set 1
(checkpoint solutions)

Resources

Course Notes
Definitions and Theorems
Office Hours Schedule
Lecture Videos
Grades

Lectures

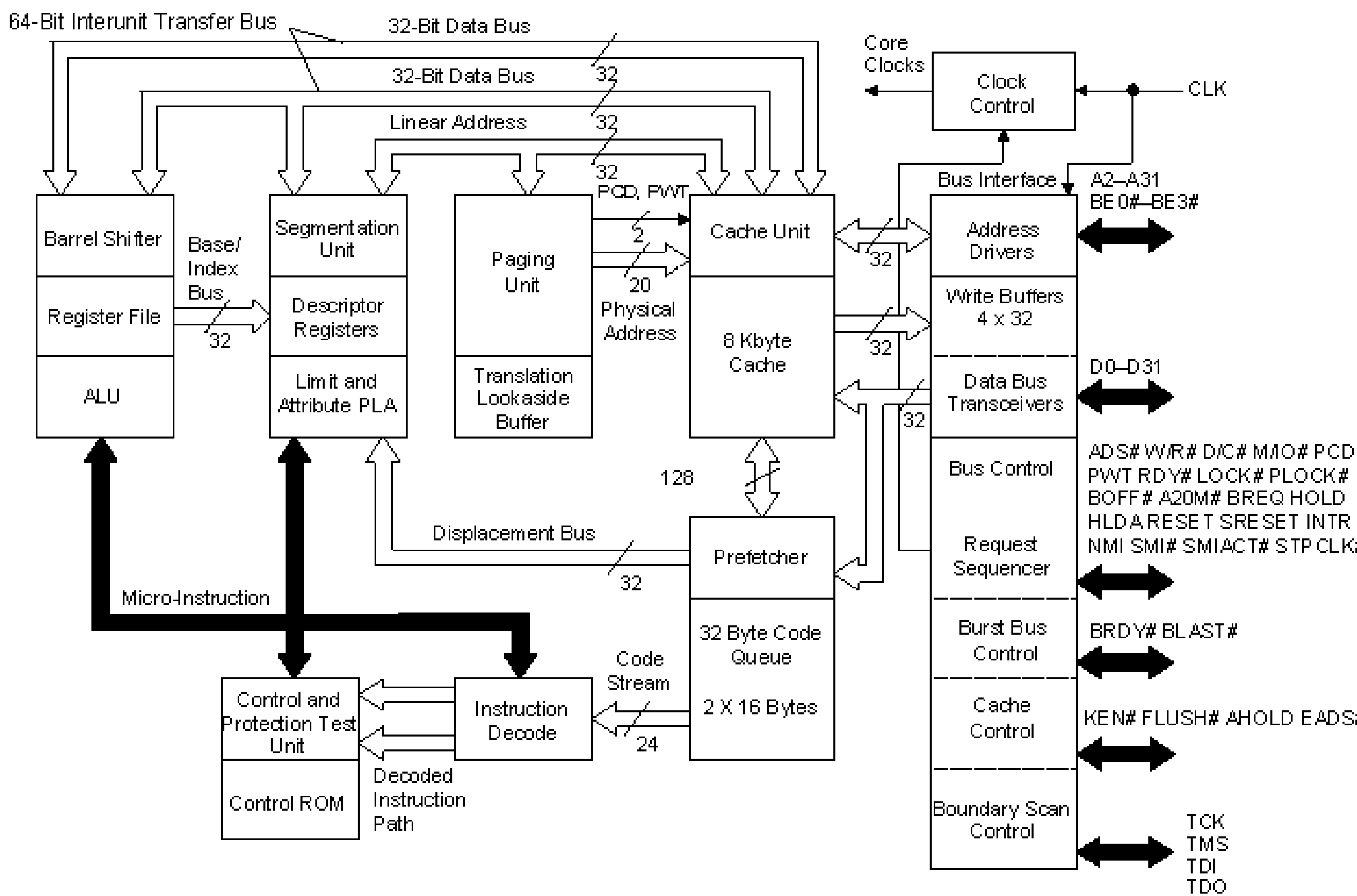
00: Set Theory

Computability Theory

What problems can we solve with a computer?

What kind of computer?

Computers are Messy



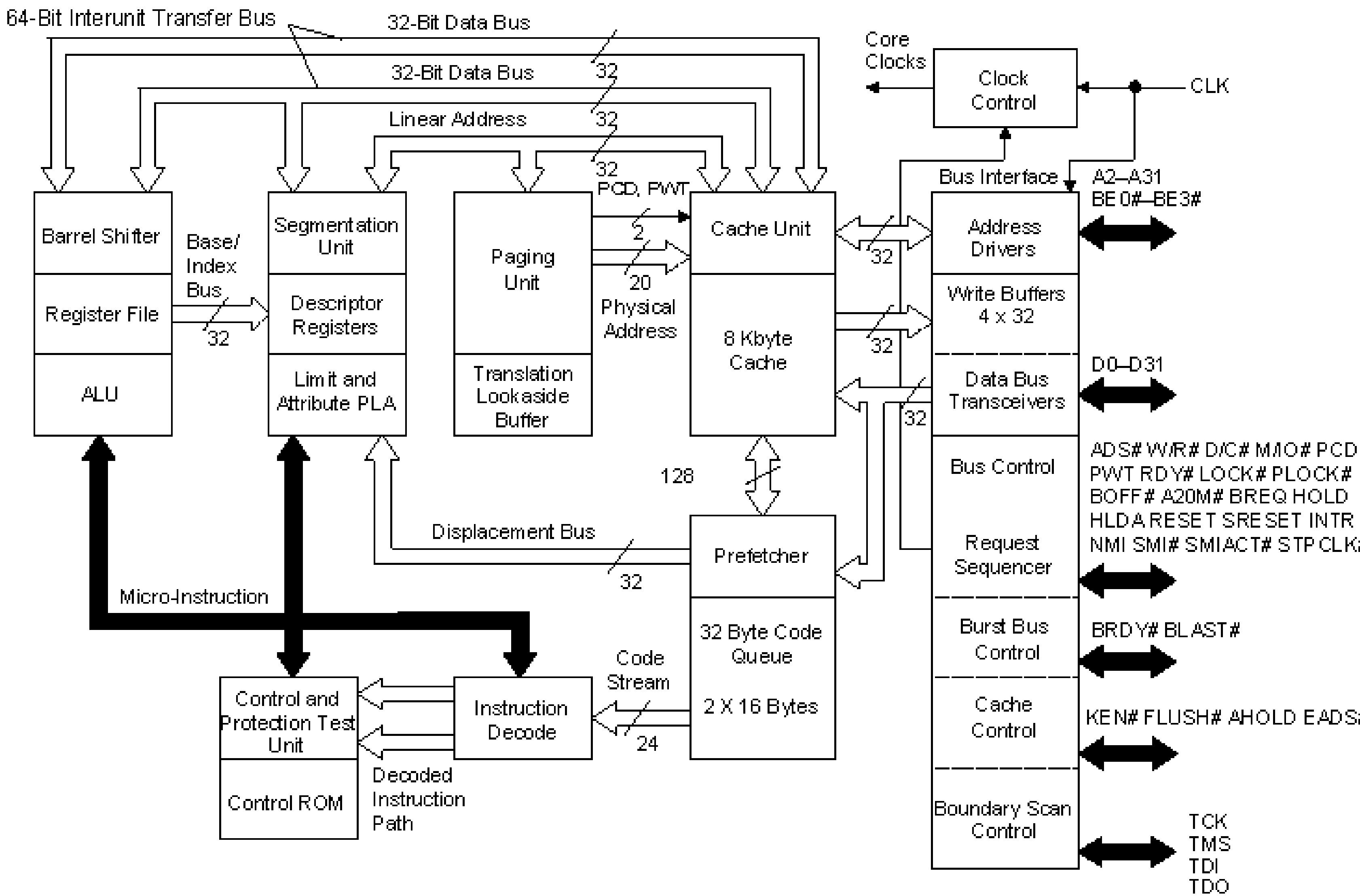
http://www.intel.com/design/intarch/prodbref/272713.htm

We need a simpler way of
discussing computing machines.

An **automaton** (plural: **automata**) is a mathematical model of a computing device.

Automata make it possible to reason about computability by **abstracting away** the implementation complexity of real computing systems.

Computers are Messy



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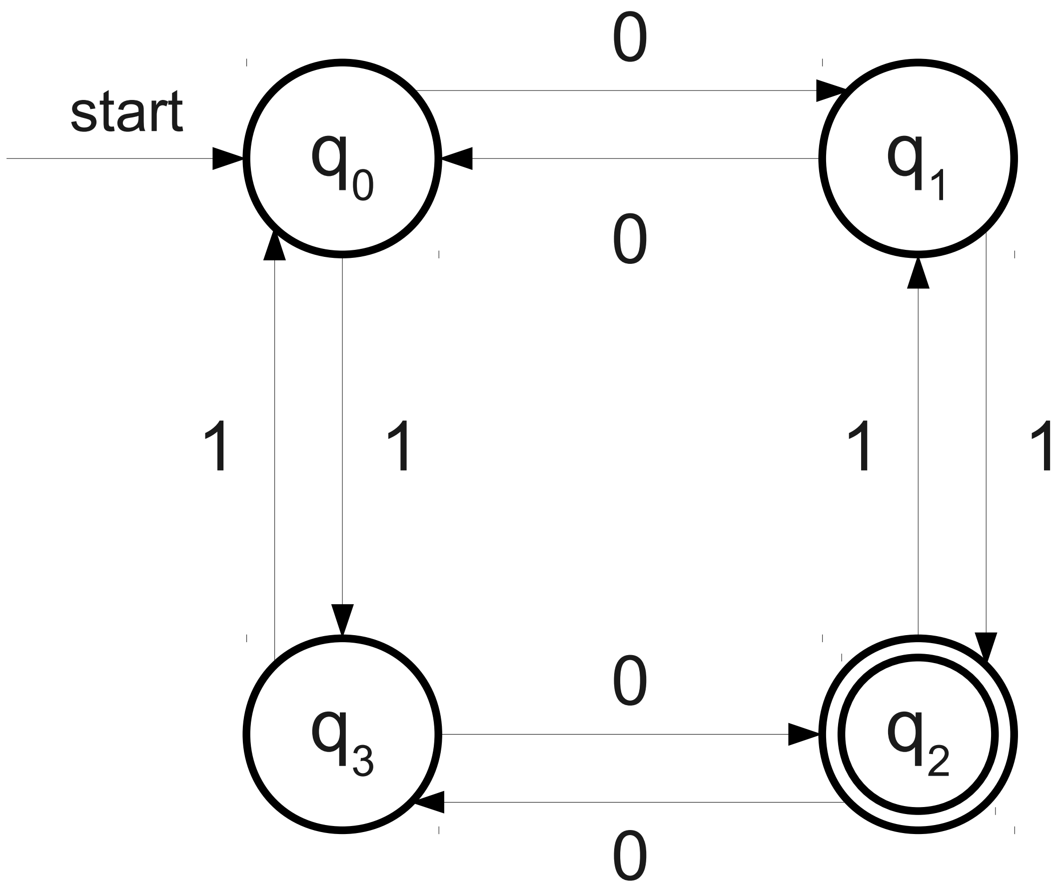
Automata are Clean



Why Build Models?

- The models of computation we will explore in this class correspond to different conceptions of what a computer could do.
- **Finite automata** (this week) are an abstraction of computers with finite resource constraints.
  - Provide upper bounds for the computing machines that we can actually build.
- **Pushdown automata** and **Turing machines** (next two weeks) are an abstraction of computers with unbounded resources.
  - Provide upper bounds for what we could **ever** hope to accomplish.

What problems can we solve with a computer?

What is a "problem?"

Problems with Problems

- Before we can talk about what problems we can solve, we need a formal definition of a "problem."
- We want a definition that
  - corresponds to the problems we want to solve,
  - captures a large class of problems, and
  - is mathematically simple to reason about.
- No one definition has all three properties.

Decision Problems

- In this class, we will consider **decision problems**, problems with yes/no answers.
- Examples:
  - Does 137 + 42 have 3 as a divisor?
  - Is P the shortest path from u to v?
  - Is District 12 better than District 1?
  - Do we have to get down on Friday?
- More realistic example: SAT.

Decision and Function Problems

- Decision problems do not encompass all possible problems.
  - Example: “What is 2 + 2?” is not a decision problem.
- These more general problems are called **function problems**.
- For now, we'll ignore function problems. We'll revisit them toward the end of the quarter.

Why Decision Problems

- Why restrict ourselves to decision problems?
- Many nice mathematical properties:
  - All answers are just one bit, so machines can produce answers more easily.
  - No need to worry about what formats the answers will be provided in.
  - Easy to use as subroutines.
- If we can't solve a decision problem, the question must be so hard that we can't even get a one-bit answer back!

How do we encode problems?

Strings

- An **alphabet** is a finite set of **characters**.
  - Typically, we use the symbol **Σ** to refer to an alphabet.
- A **string** is a finite sequence of characters drawn from some alphabet.
- Example: If $\Sigma = \{0, 1\}$, some valid strings include
  - 0
  - 1110100100001000000001
  - 1101110010110111
- The **empty string** contains no characters and is denoted **ε**.

Languages

- A **formal language** is a set of strings.
- We say that L is a **language over Σ** if it is a set of strings formed from characters in Σ.
- Example: The language of palindromes over Σ = {0, 1, 2} is the set

  {ε, 0, 1, 2, 00, 11, 22, 000, 010, 020, 101, ... }
- The set of all strings composed from letters in Σ is denoted **Σ***.
- L is a language over Σ iff $L \subseteq \Sigma^*$.

Decision Problems and Languages

- Languages give a compact and flexible way to encode decision problems.
- Consider these examples:
  - $\{ p \mid p$ is a binary representation of a prime number $\}$
  - $\{ w \mid w$ is a textual encoding of (x, y), where $x < y \}$
  - $\{ x \mid x$ is a textual encoding of a graph G and a path P, where P is the longest path in $G \}$
- Any decision problem can be represented by a language of strings encoding inputs to which the answer is "yes."
- All the automata we will discuss in this class will be machines for answering the question "is string x in language L?"

To Summarize

- An **automaton** is an idealized mathematical computing machine.
- A **language** is a set of strings.
- A **decision problem** is a yes/no question (though it can be quite complex).
- The automata we will study will accept as input a string and (attempt to) output whether that string is contained in a particular language.

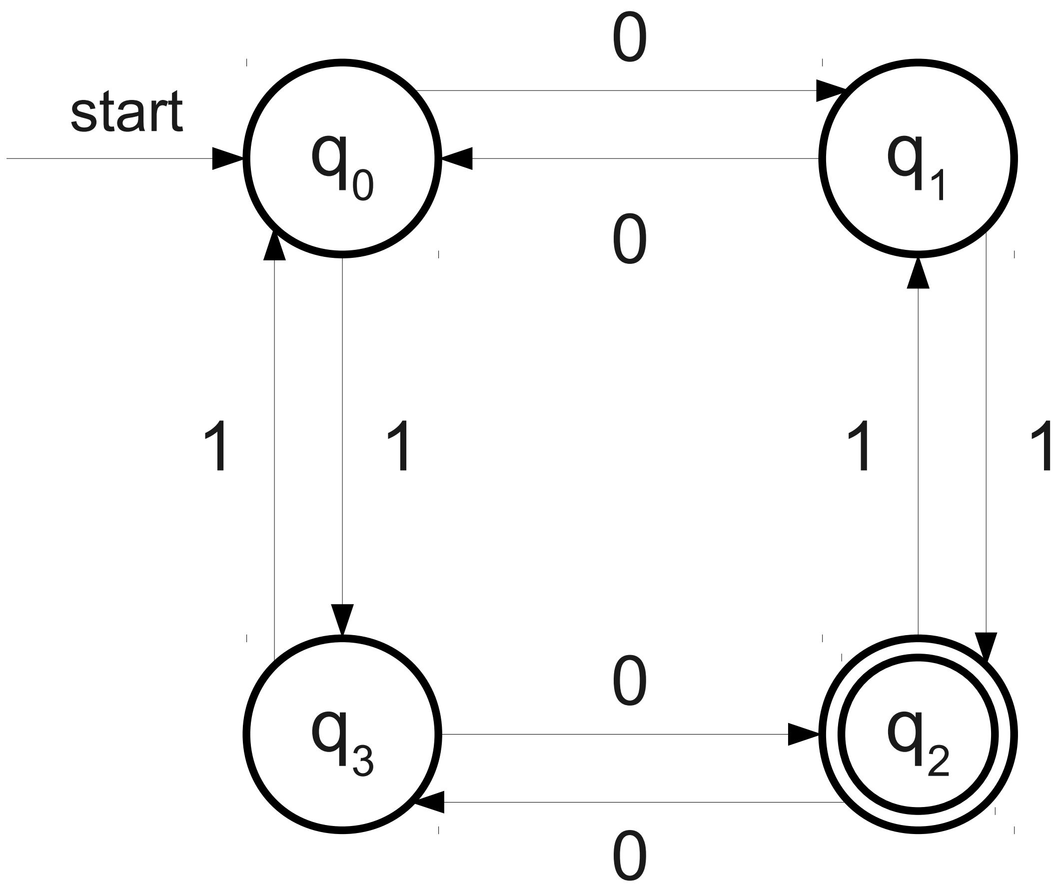
What problems can we solve with a computer?

Finite Automata

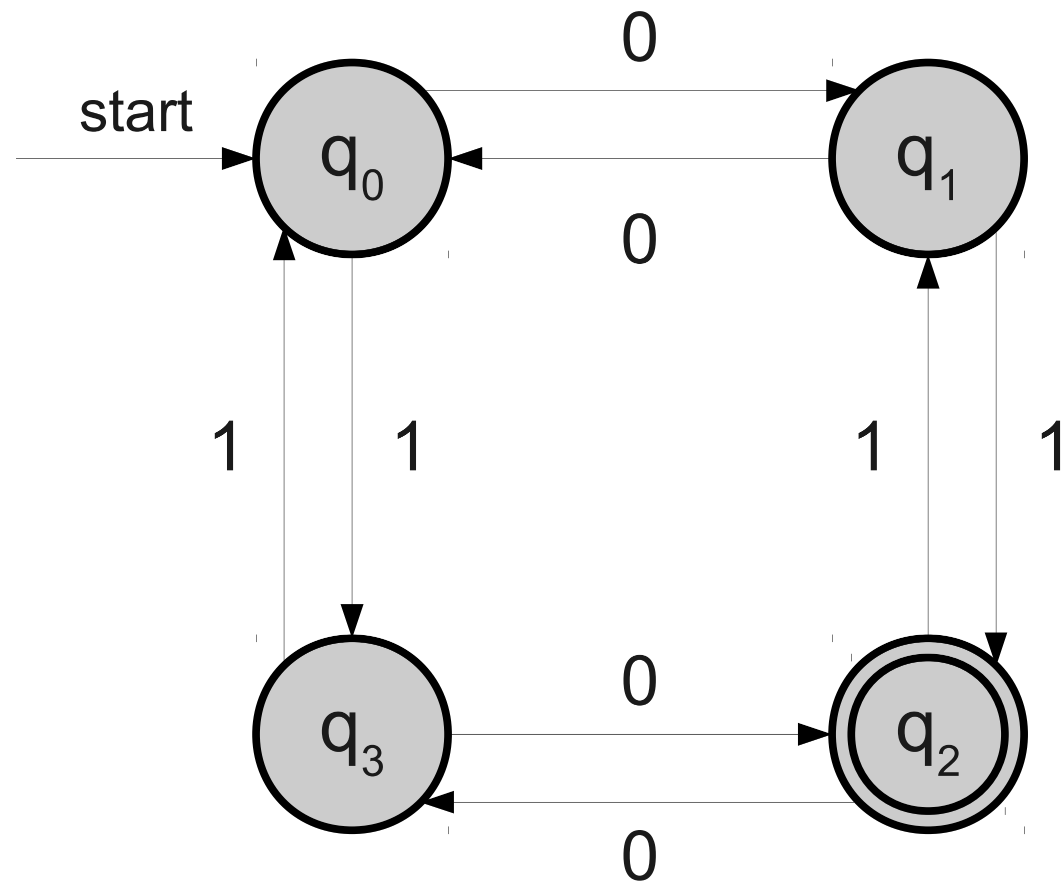
A **finite automaton** is a mathematical machine for determining whether a string is contained within some language.

Each finite automaton consists of a set of **states** connected by **transitions**.

A Simple Finite Automaton

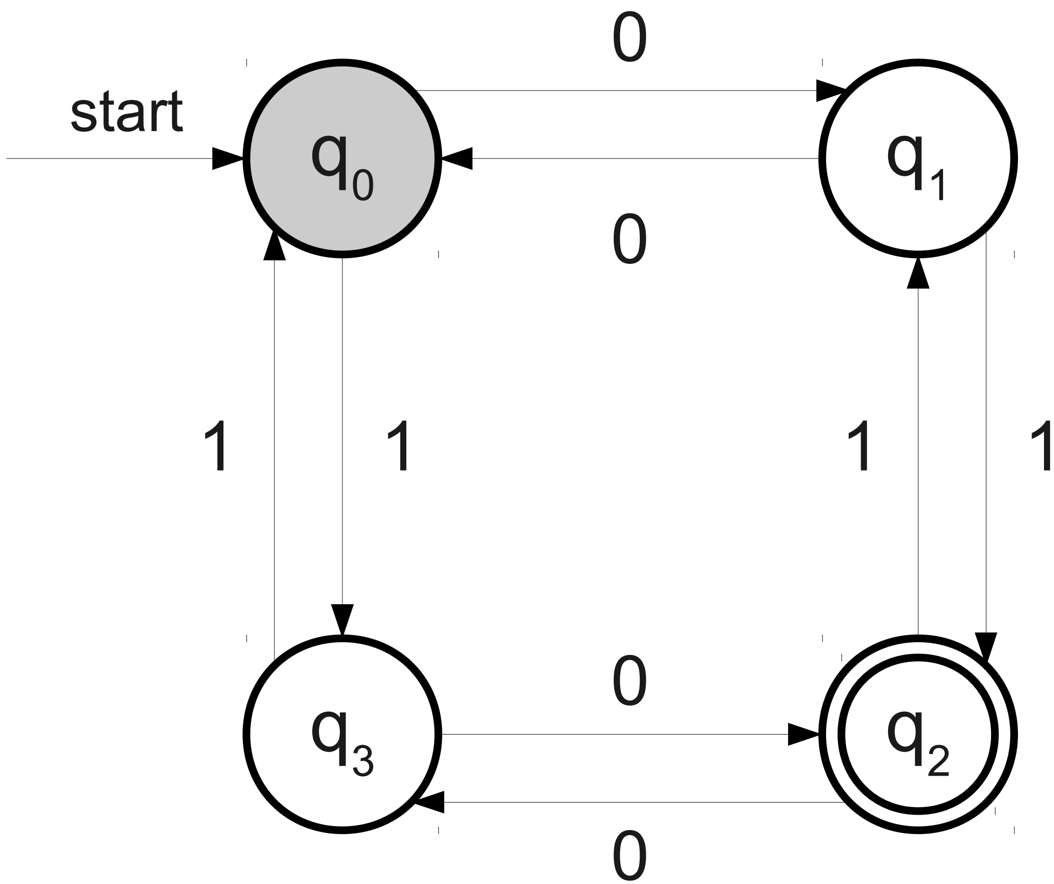


A Simple Finite Automaton



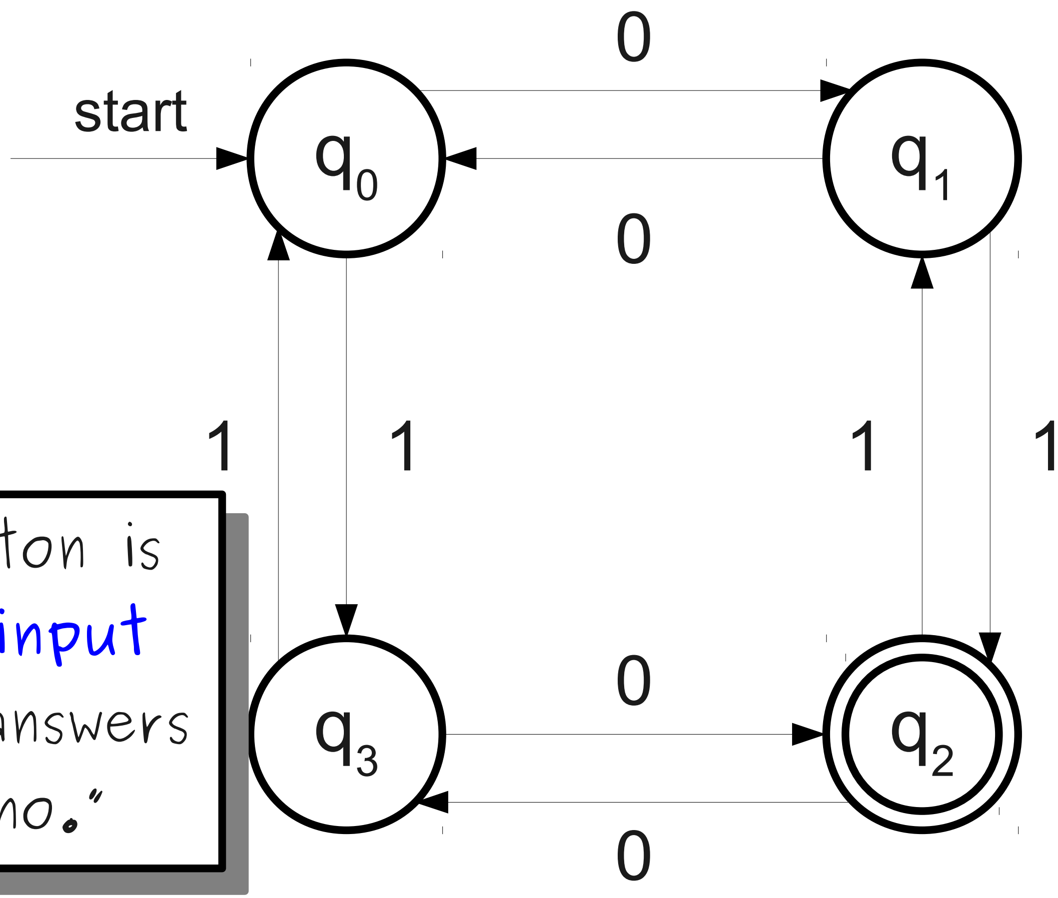
Each circle represents a **state** of the automaton.

A Simple Finite Automaton



One special state is designated as the **start state**.

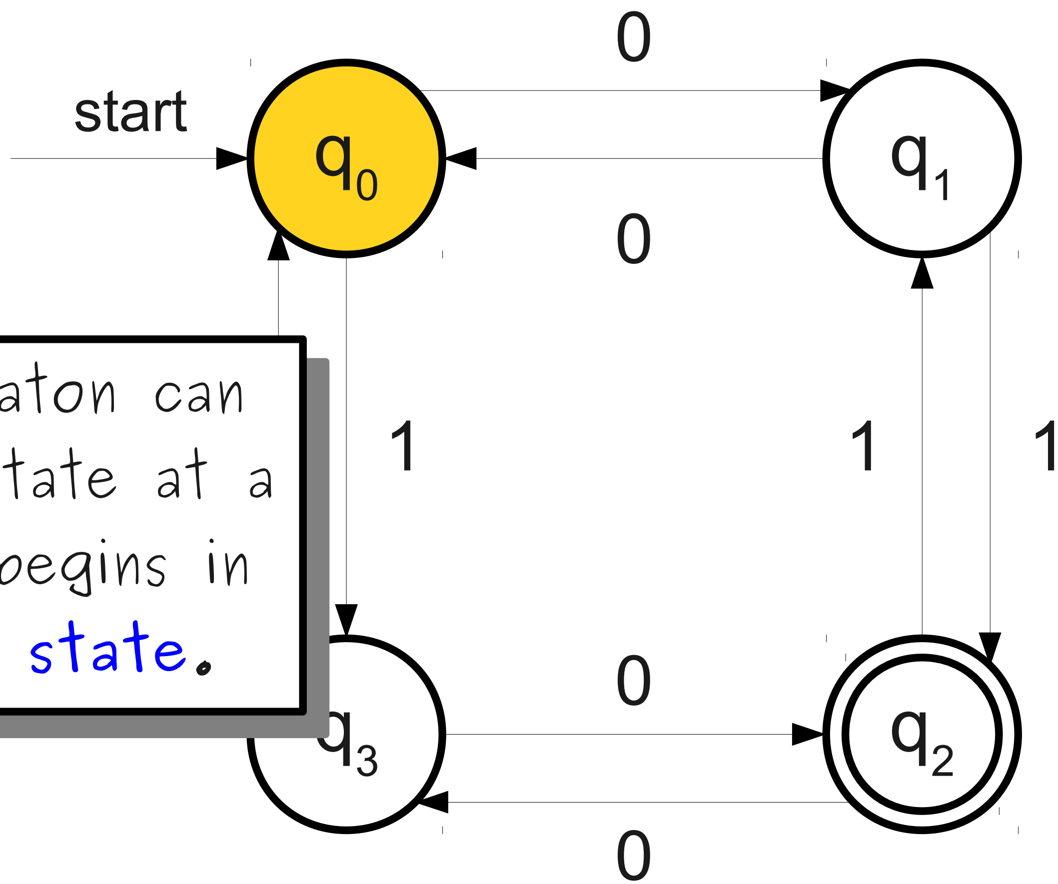
A Simple Finite Automaton



The automaton is run on an **input string** and answers "yes" or "no."

0 1 0 1 1 0

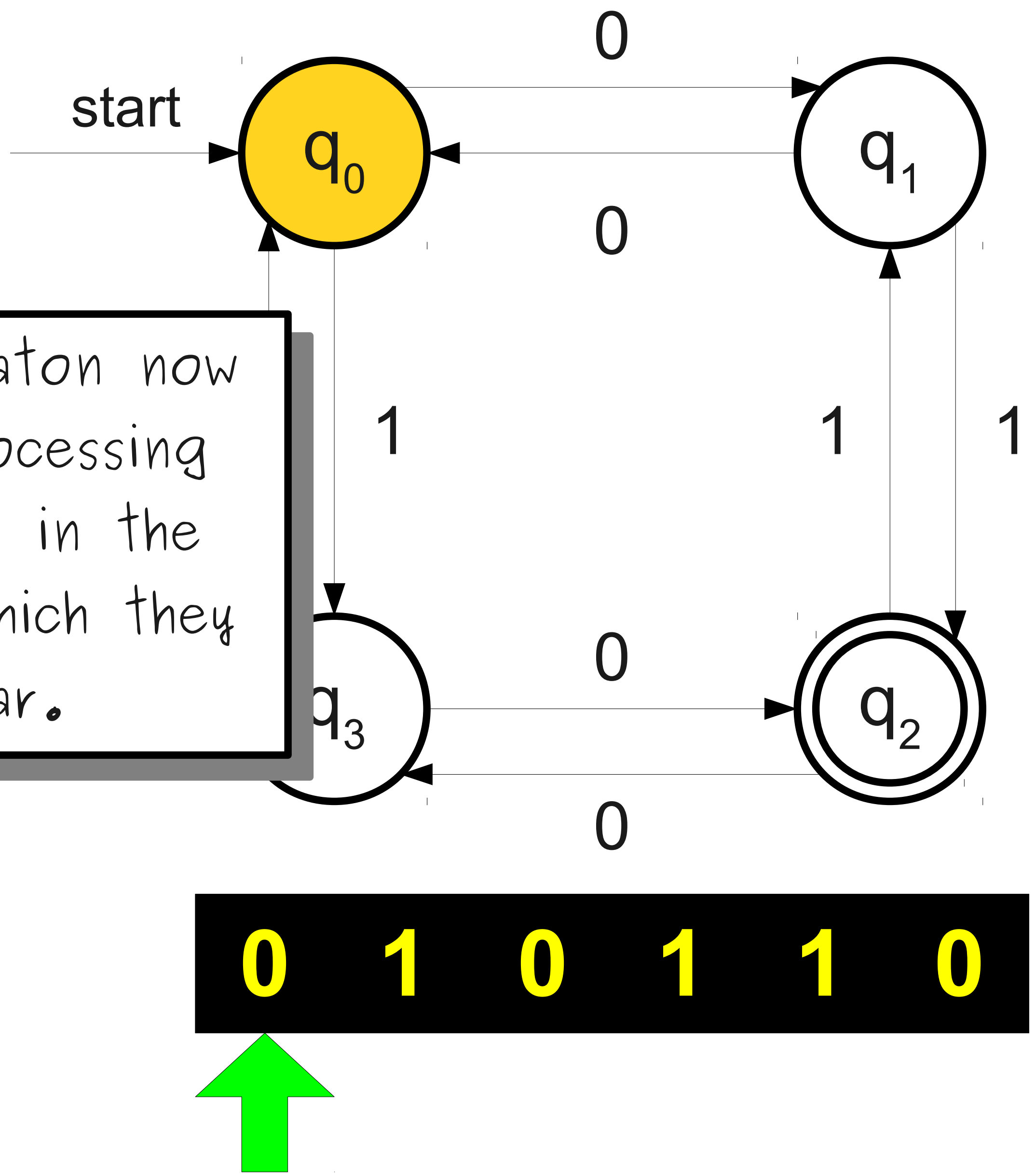
A Simple Finite Automaton



The automaton can be in one state at a time. It begins in the start state.

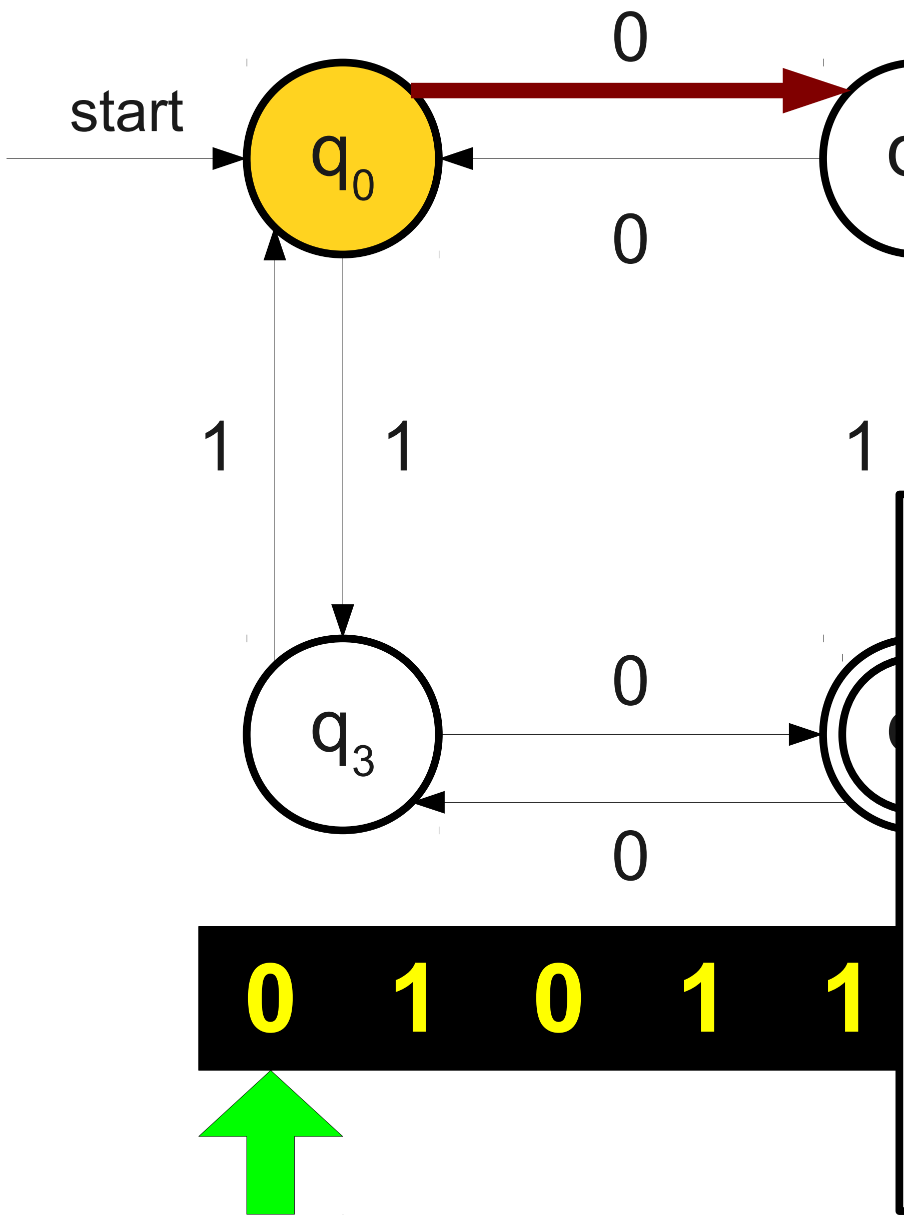
0 1 0 1 1 0

A Simple Finite Automaton



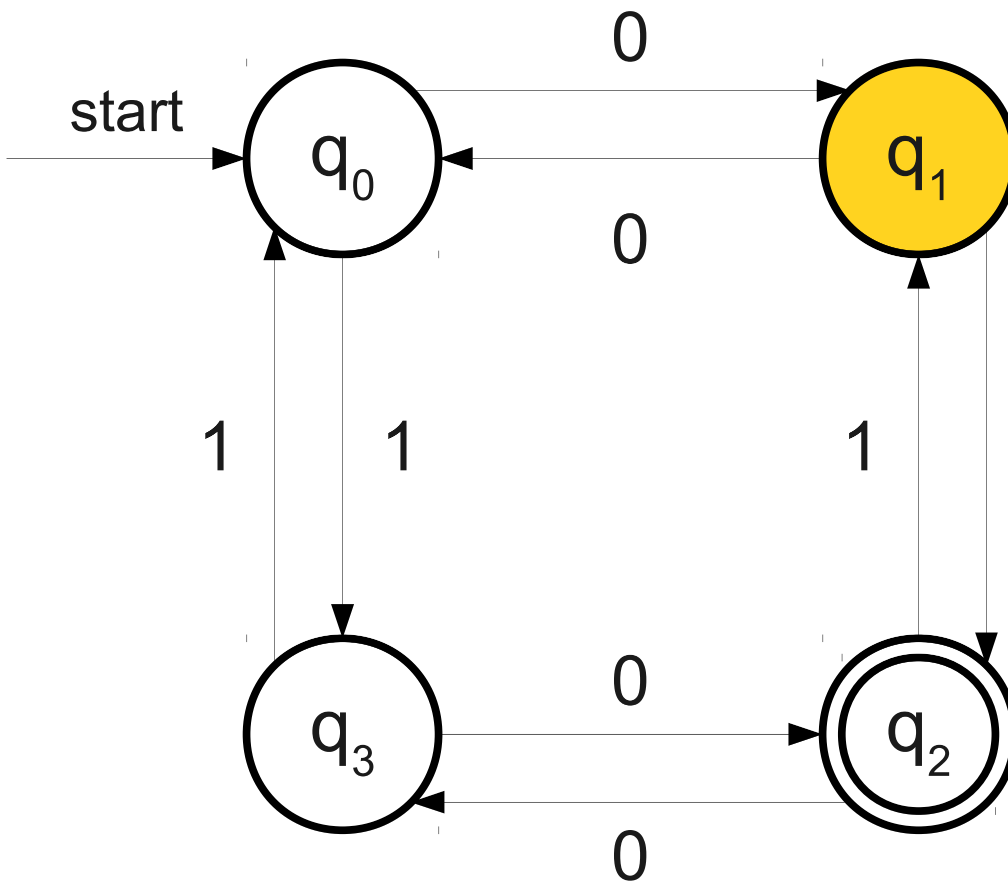
The automaton now begins processing characters in the order in which they appear.

A Simple Finite Automaton



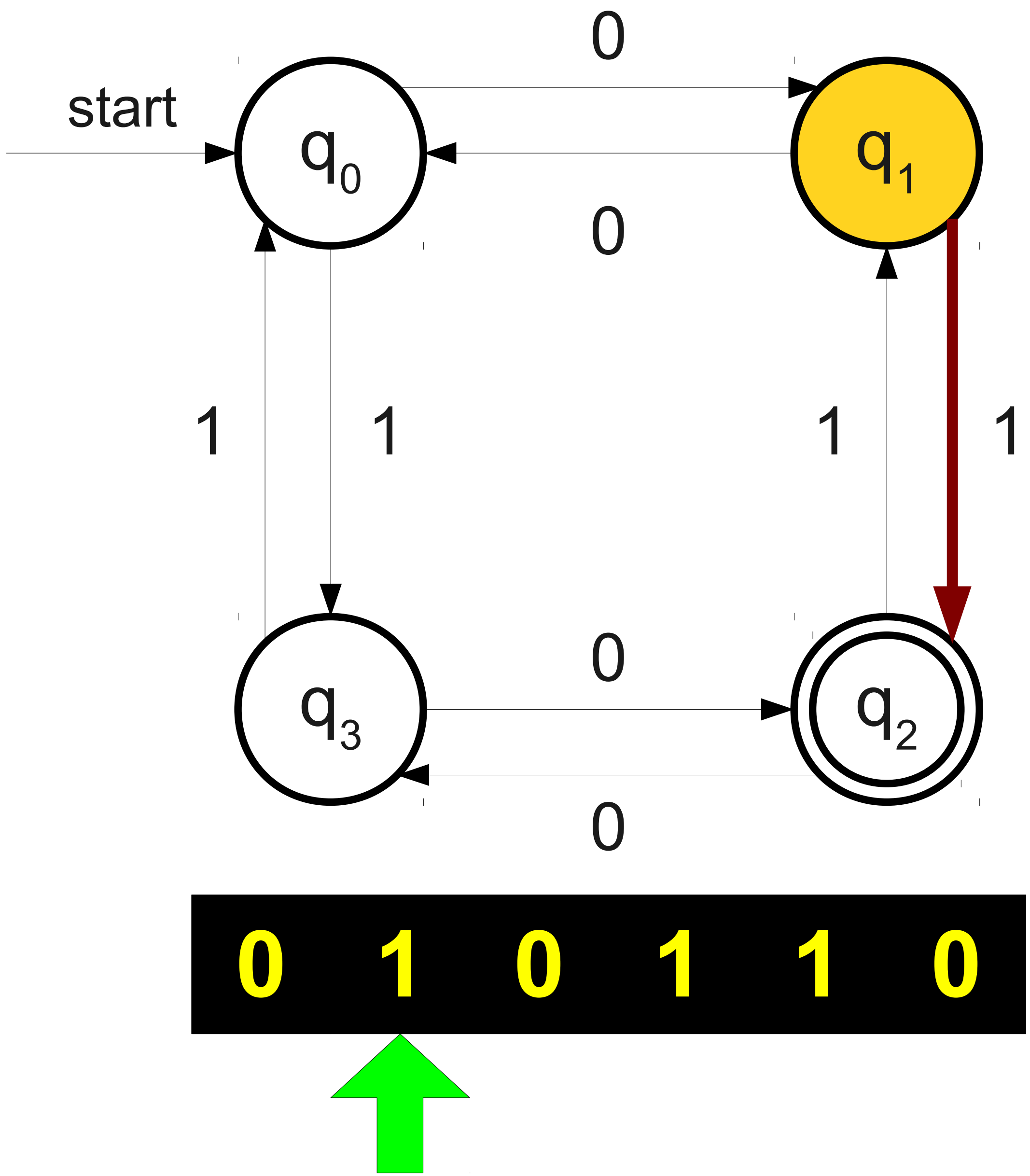
Each arrow in this diagram represents a **transition**. The automaton always follows the transition corresponding to the current symbol being read.

A Simple Finite Automaton

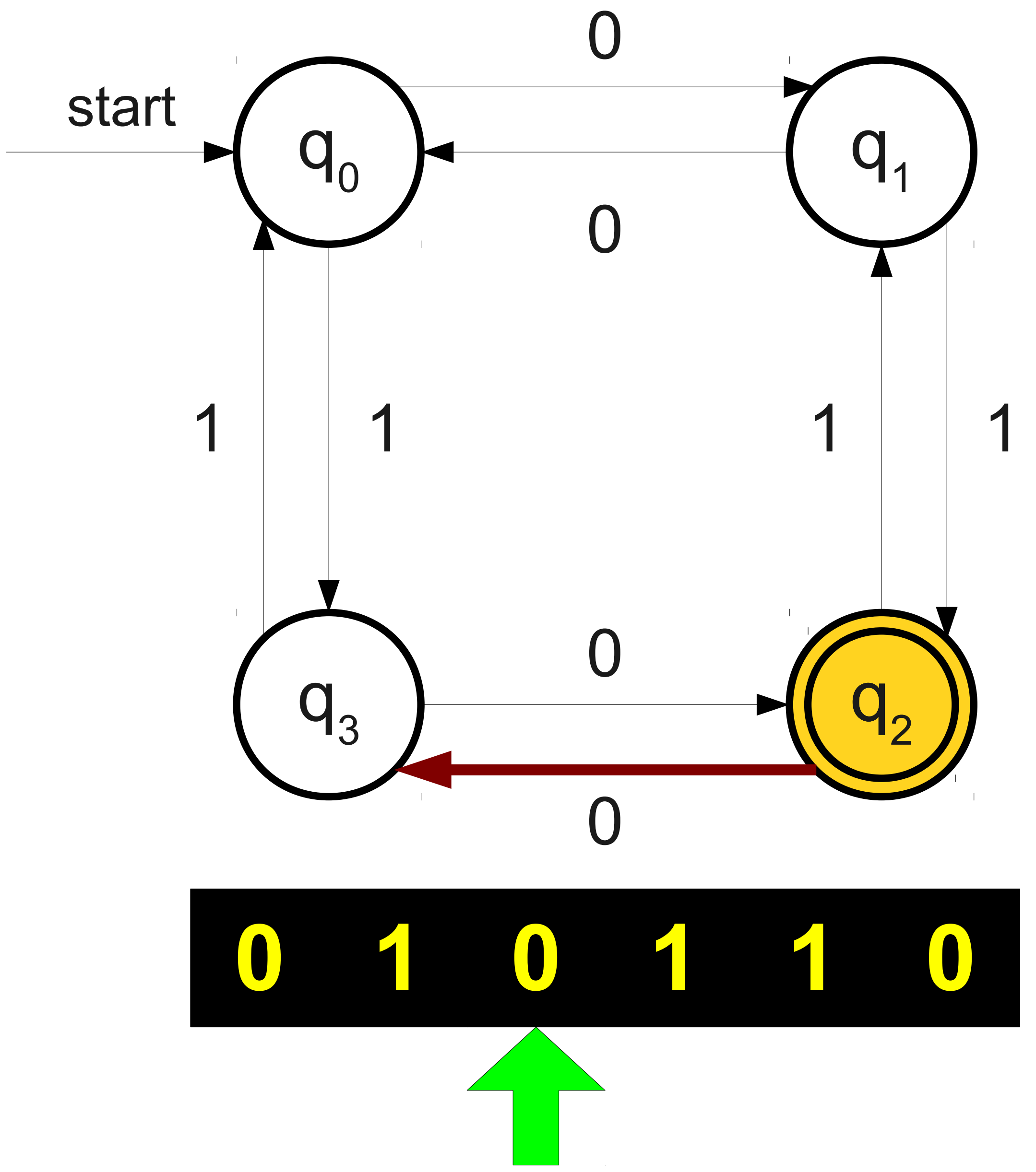


After transitioning, the automaton considers the next symbol in the input.

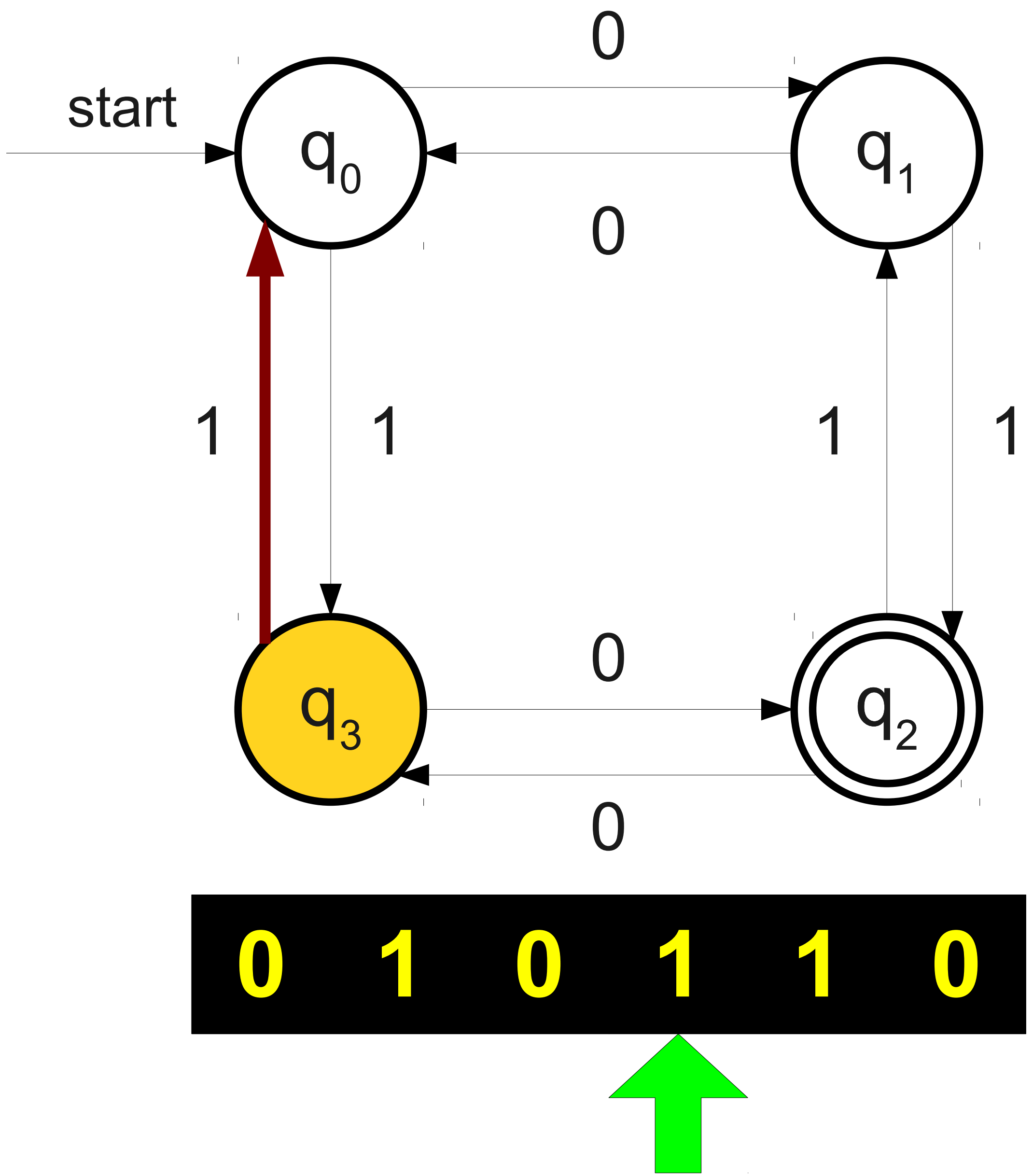
A Simple Finite Automaton



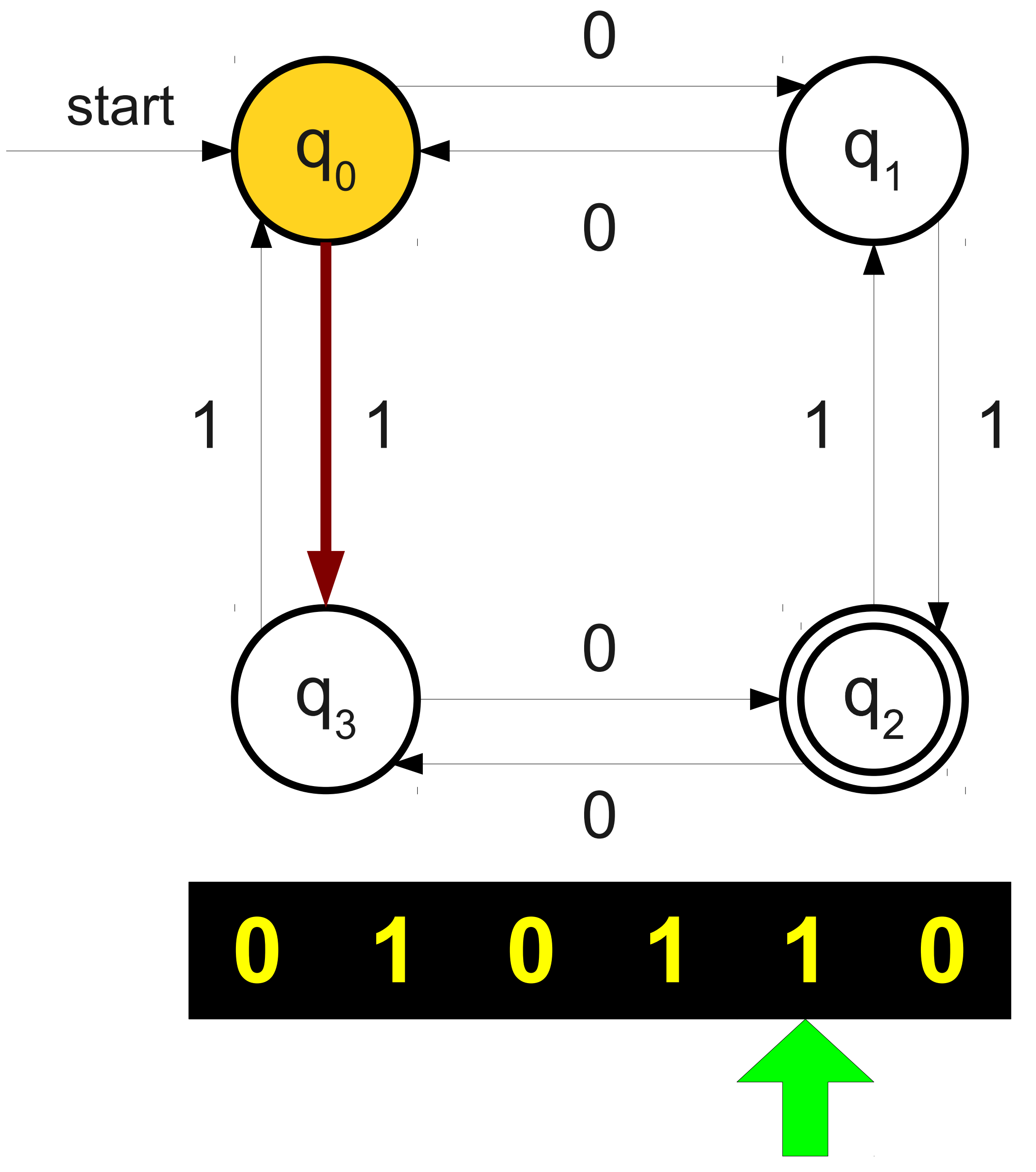
A Simple Finite Automaton



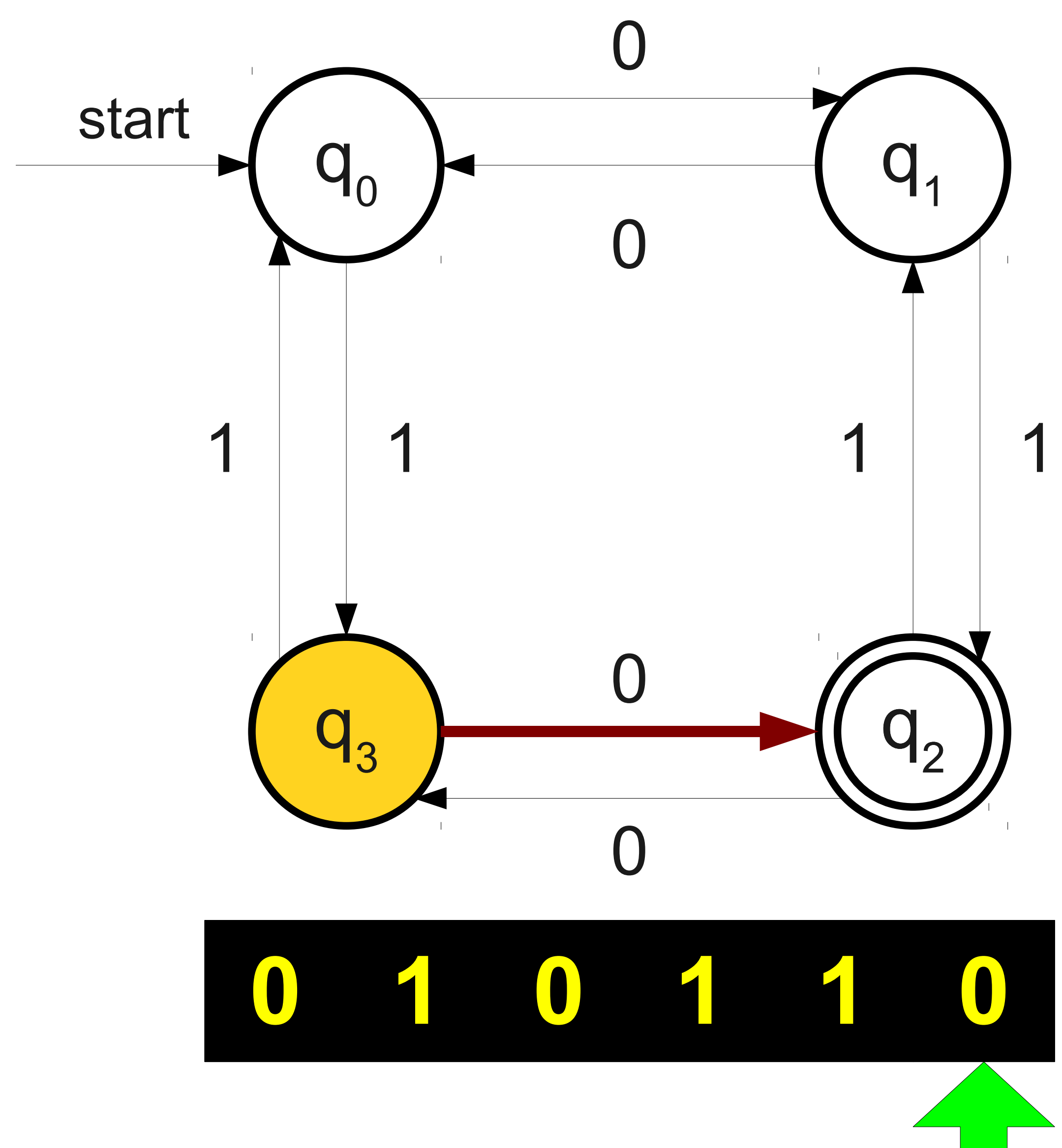
A Simple Finite Automaton



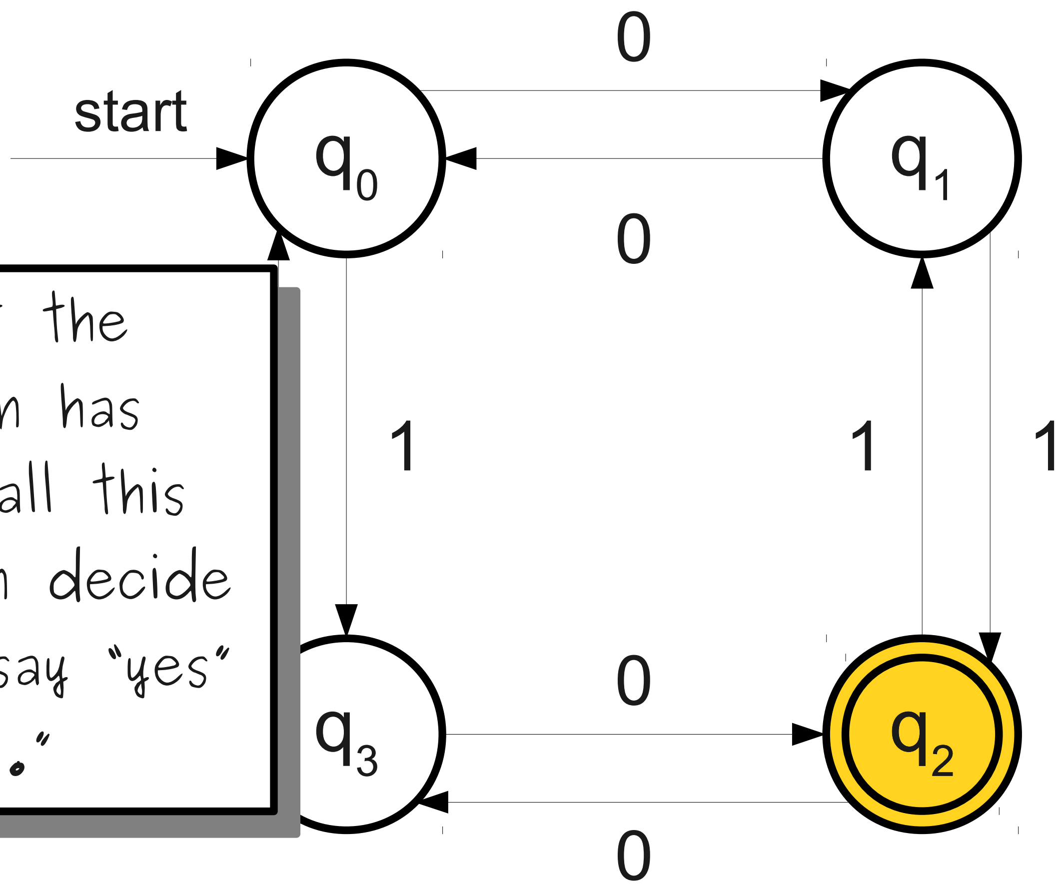
A Simple Finite Automaton



A Simple Finite Automaton



A Simple Finite Automaton



Now that the automaton has looked at all this input, it can decide whether to say "yes" or "no."

The double circle indicates that this state is an **accepting state**, so the automaton outputs "yes."

0 1 0 1 1 0

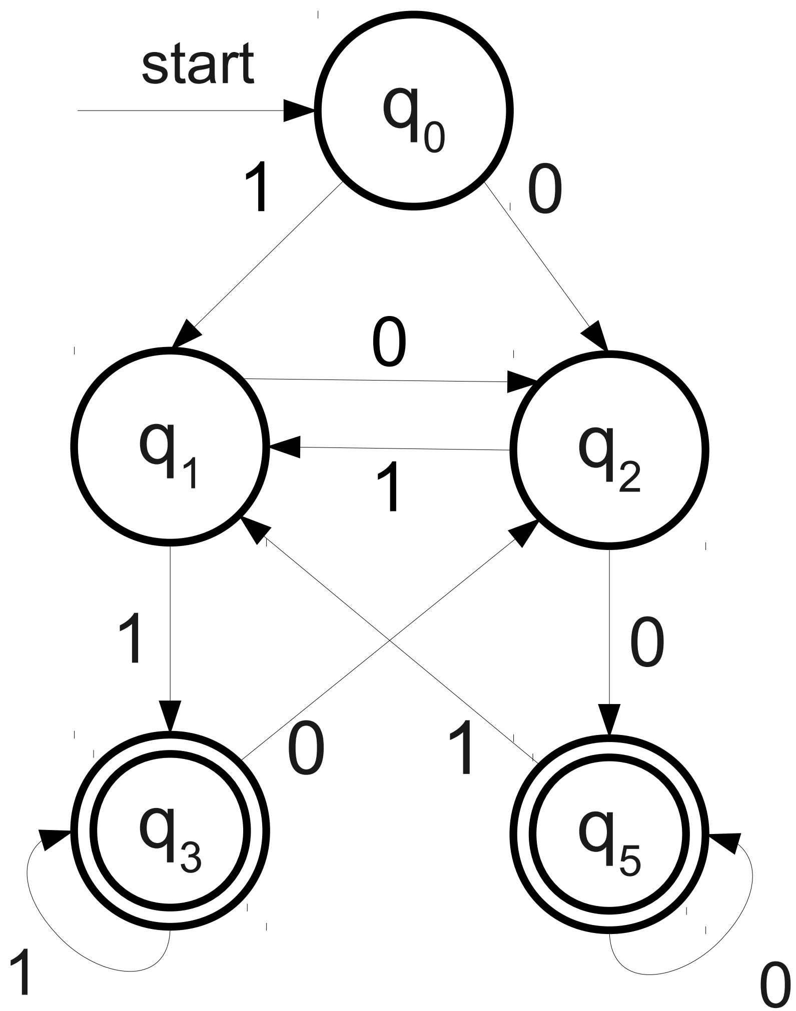
The Story So Far

- A **finite automaton** is a collection of **states** joined by **transitions**.
- Some state is designated as the **start state**.
- Some states are designated as **accepting states**.
- The automaton processes a string by beginning in the start state and following the indicated transitions.
- If the automaton ends in an accepting state, it **accepts** the input.
- Otherwise, the automaton **rejects** the input.

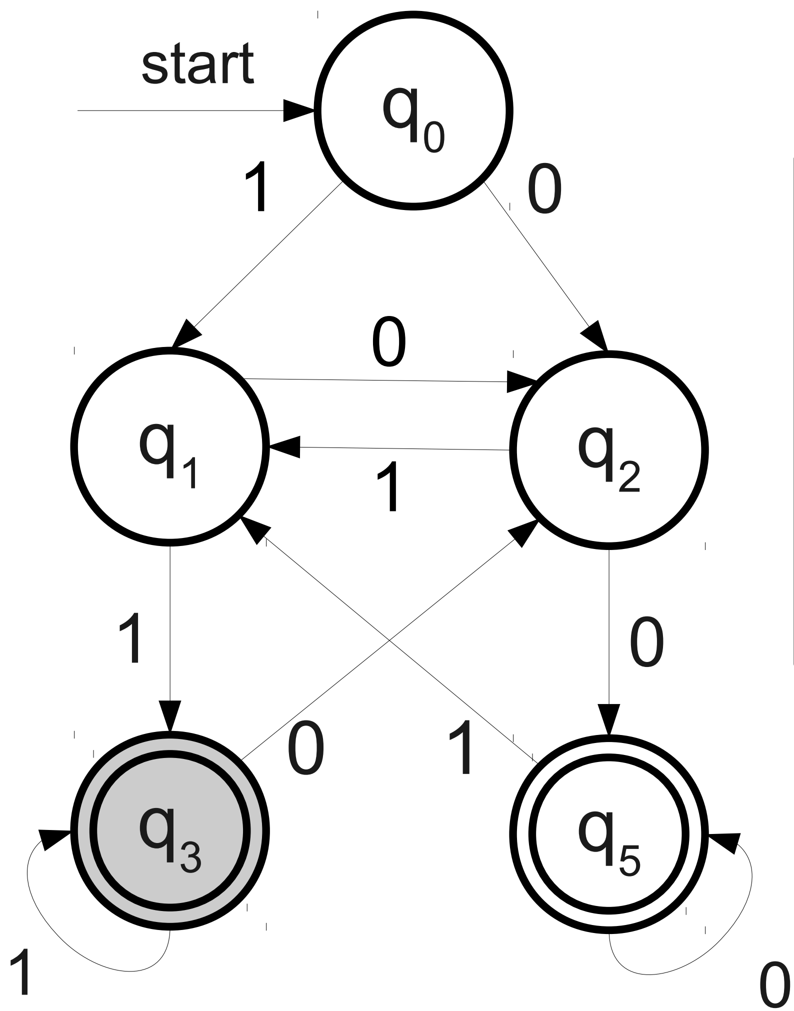
A finite automaton does **not** accept as soon as the input enters an accepting state.

A finite automaton accepts if it **ends** in an accepting state.

What Does This Accept?

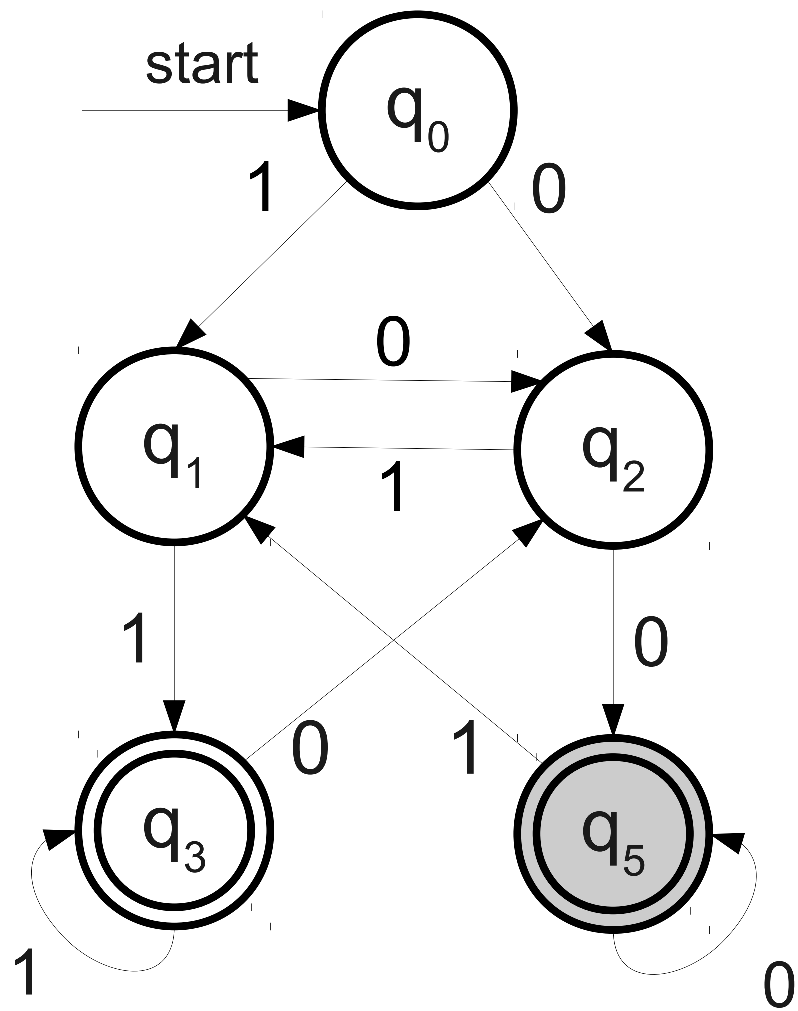


What Does This Accept?



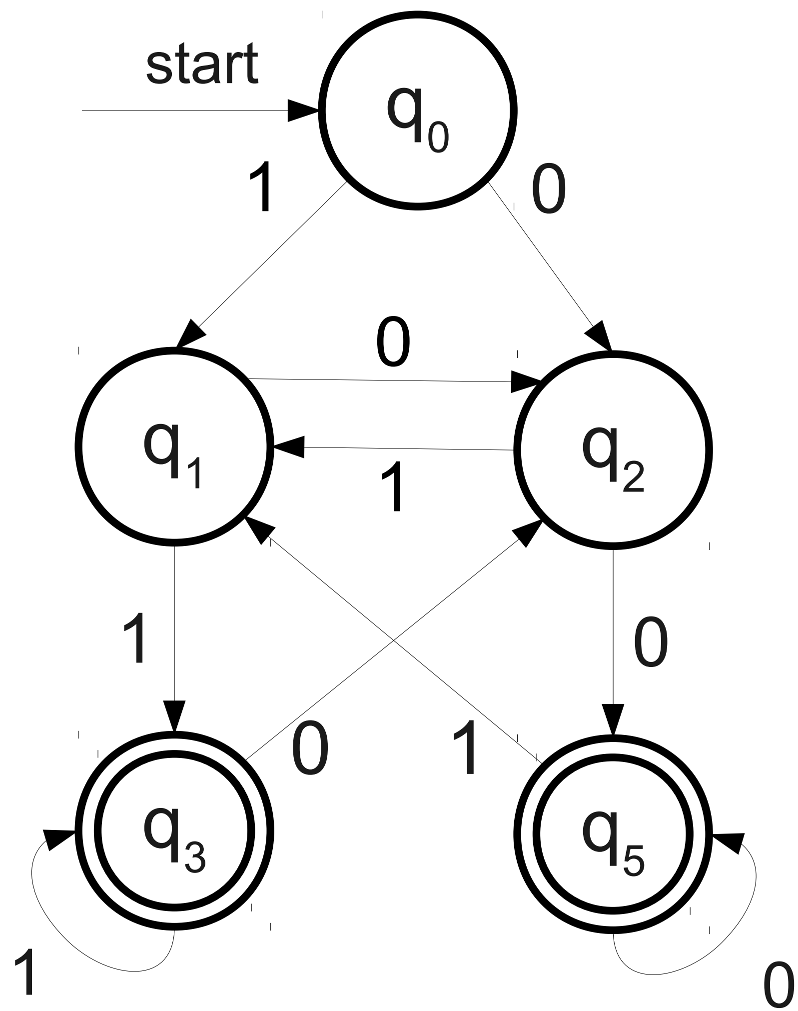
No matter where we start in the automaton, after seeing two 1's, we end up in accepting state q_3.

What Does This Accept?



No matter where we start in the automaton, after seeing two 0's, we end up in accepting state q_5.

What Does This Accept?



This automaton accepts a string iff it ends in 00 or 11.

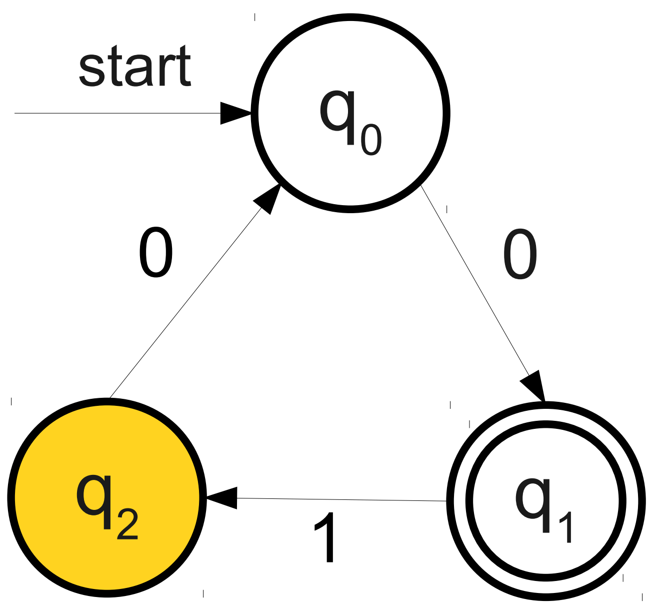
The **language of an automaton** is the set of strings that it accepts.

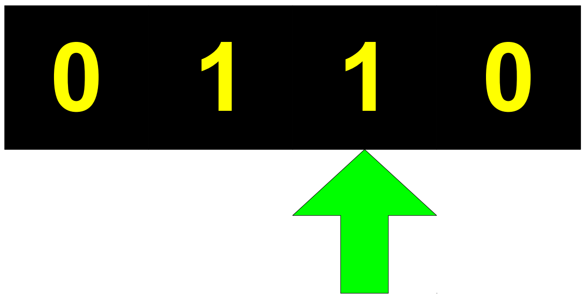
If A is an automaton, we denote the language of A as $\mathscr{L}(A)$.

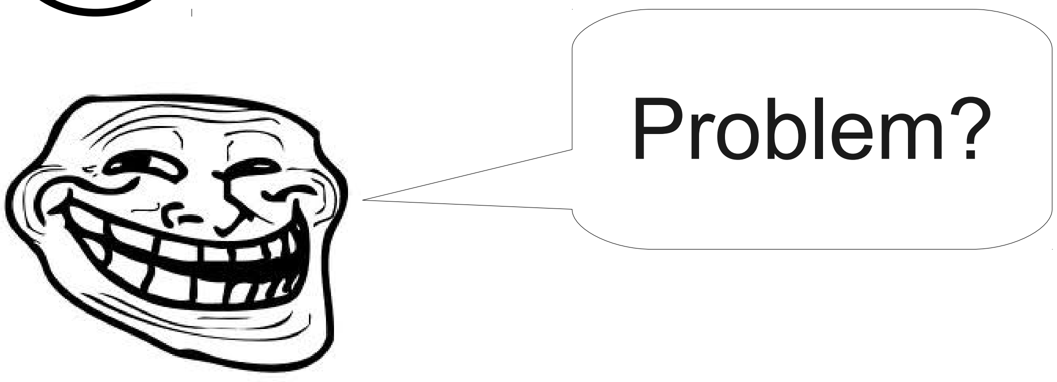
Intuitively:

$$\mathscr{L}(A) = \{ \ w \in \Sigma^* \mid A \text{ accepts } w \ \}$$

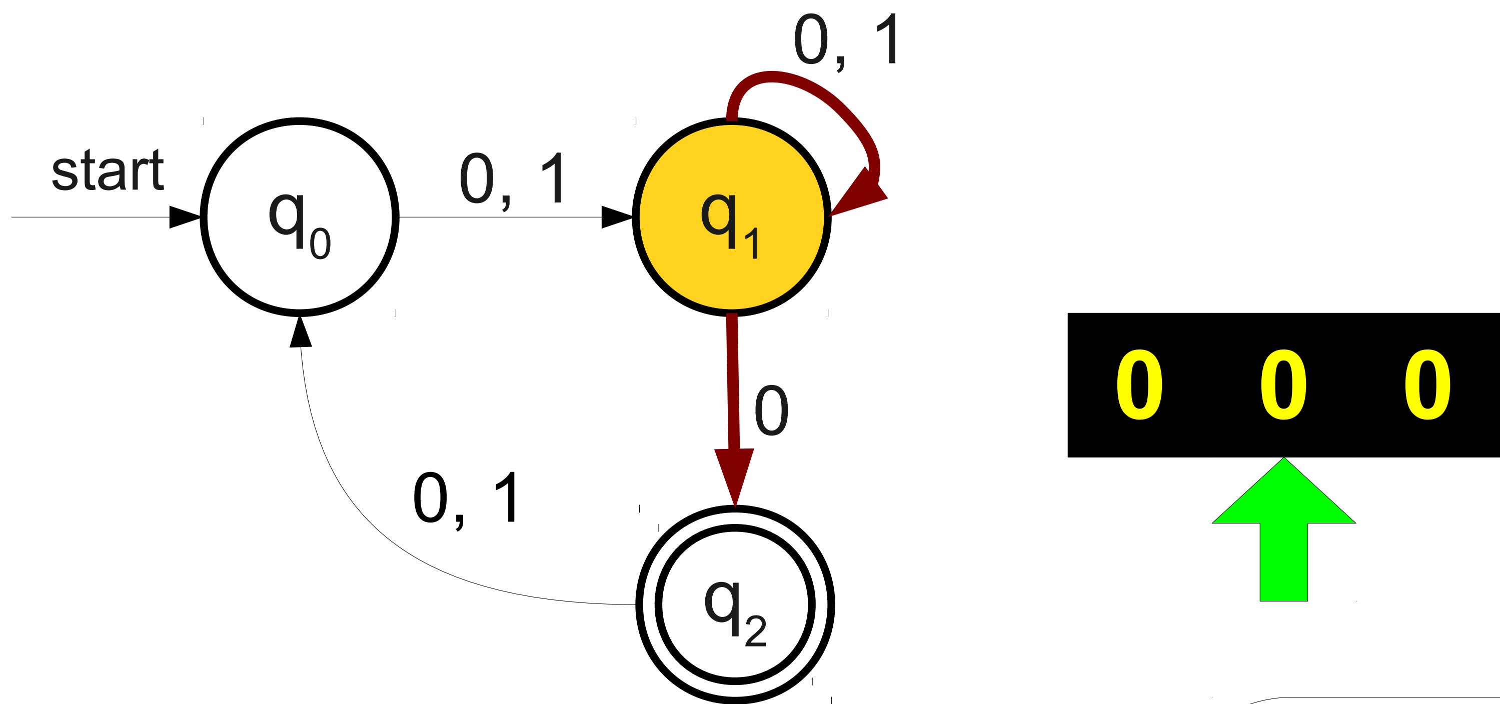
A Small Problem

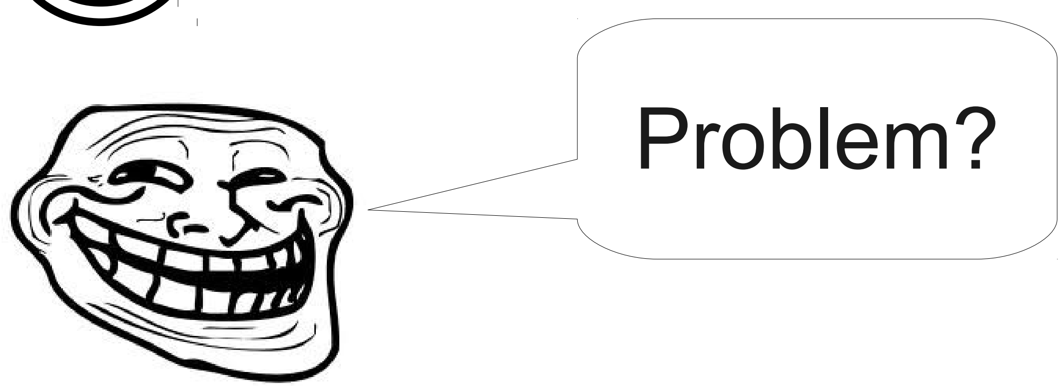






Another Small Problem





The Need for Formalism

- In order to reason about the limits of what finite automata can and cannot do, we need to formally specify their behavior in *all* cases.
- All of the following need to be defined or disallowed:
  - What happens if there is no transition out of a state on some input?
  - What happens if there are *multiple* transitions out of a state on some input?

DFAs

- A **DFA** is a
  - **D**eterministic
  - **F**inite
  - **A**utomaton
- DFAs are the simplest type of automaton that we will see in this course.

DFAs, Informally

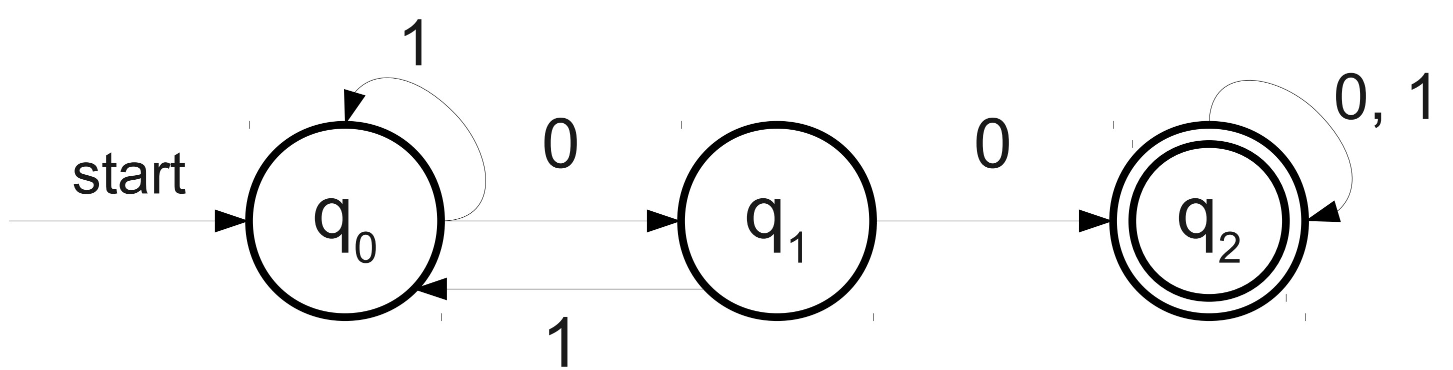
- A DFA is defined relative to some alphabet Σ.
- For each state in the DFA, there must be **exactly one** transition defined for each symbol in the alphabet.
  - This is the “deterministic” part of DFA.
- There is a **unique** start state.
- There may be multiple accepting states.

Designing DFAs

- At each point in its execution, the DFA can only remember what state it is in.
- A good way to design DFAs is to think about what information you would need to pick up where you left off.
  - Each state acts as a "memento" of what you're supposed to do next.

Recognizing Languages with DFAs

$L = \{ \, w \in \{0, 1\}^* \mid w \text{ contains } 00 \text{ as a substring } \}$



Recognizing Languages with DFAs

$L = \{ w \in \{0, 1\}^* | \text{ all even-numbered digits of } w \text{ are } 0 \}$

YES	**NO**
01	**1**
00**0**1	**0**0**1**
01**0**1**0**1**0**0**0**1	**0**0**0**0**1**

Recognizing Languages with DFAs

$L = \{ w \in \{0, 1\}^* \mid \text{all even-numbered digits of } w \text{ are } 0 \}$