Finite Automata

Part Two

Announcements

- Practice midterm solutions available.
- Second practice midterm available.
- Midterm review session this **Saturday, October 27** at **2PM** in **Gates 104**.
  - Come with questions!
  - Leave with answers!
- Problem Set 3 and Problem Set 4 Checkpoints graded; will be returned at end of lecture.

A Friendly Reminder

∀ goes with →

∃ goes with ∧

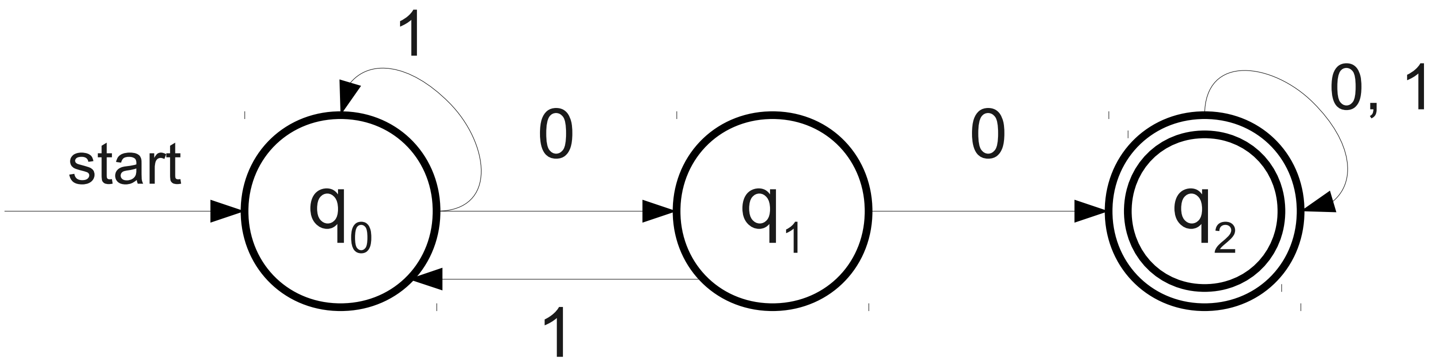
Finite Automata

DFAs, Informally

- A DFA is defined relative to some alphabet Σ.
- For each state in the DFA, there must be **exactly one** transition defined for each symbol in the alphabet.
  - This is the “deterministic” part of DFA.
- There is a **unique** start state.
- There may be multiple accepting states.

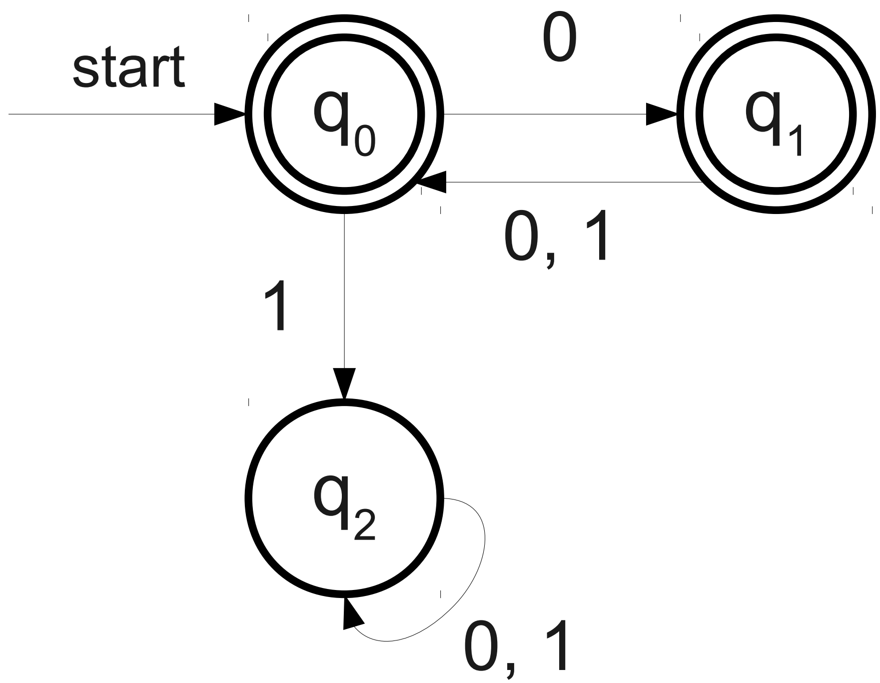
Recognizing Languages with DFAs

$$L = \{\ w \in \{0, 1\}^* \mid w \text{ contains } 00 \text{ as a substring }\}$$



Recognizing Languages with DFAs

$L = \{ w \in \{0, 1\}^* |$ all even-numbered characters of w are 0 $\}$



More Elaborate DFAs

$$L = \{ w \mid w \text{ is a C-style comment } \}$$

Suppose the alphabet is

$$\Sigma = \{ \texttt{a}, \texttt{*}, \texttt{/} \}$$

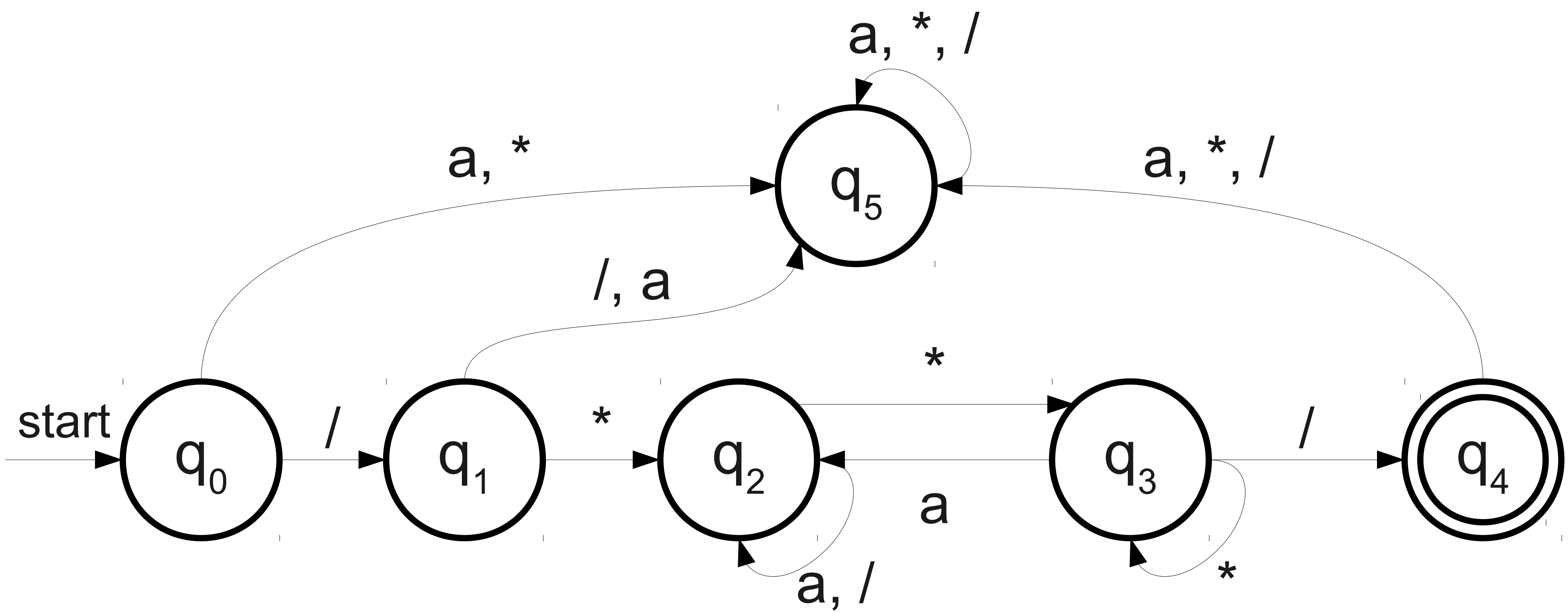
Try designing a DFA for comments!

Some test cases:

ACCEPTED	REJECTED
`/*a*/`	`/**`
`/**/`	`/**/a`
`/***/`	`aaa/**/`
`/*aaa*aaa*/`	`/*/`

More Elaborate DFAs

$L = \{ \ w \mid w \text{ is a C-style comment } \}$



More Elaborate DFAs

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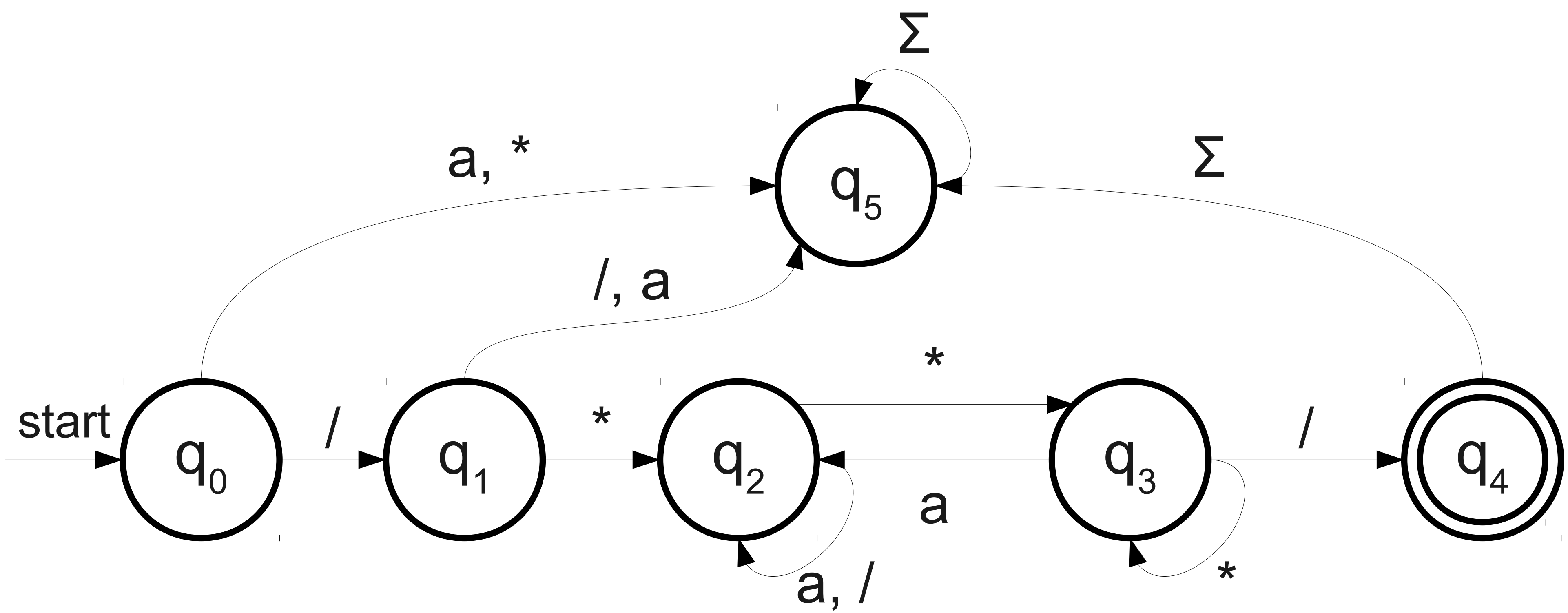
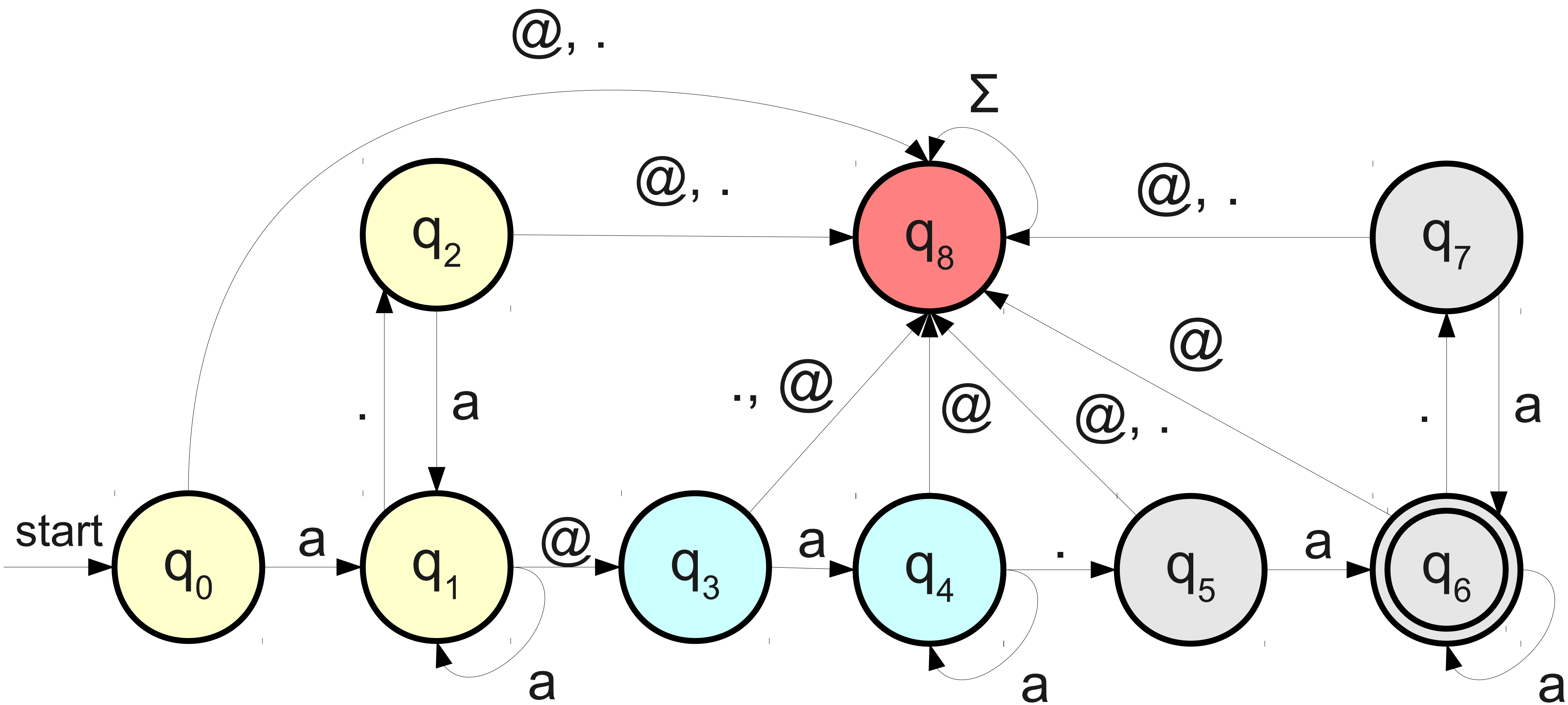
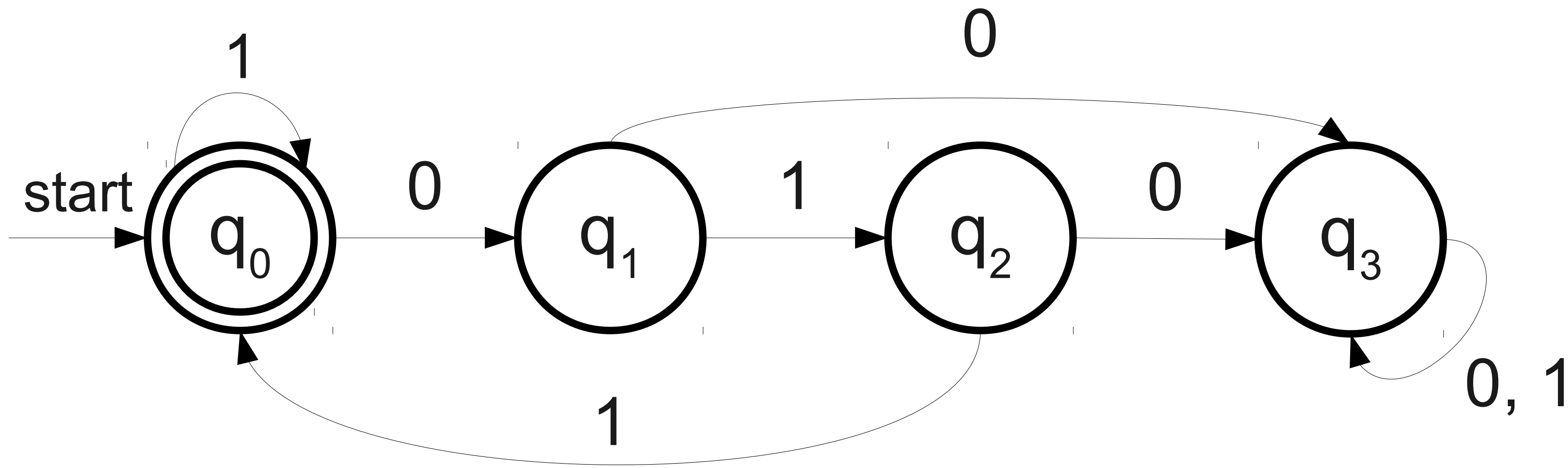


More Elaborate DFAs

$L = \{ w \mid w \text{ is a legal email address } \}$



Tabular DFAs



	0	1
*q_0	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
q_3	q_3	q_3

The star indicates that this is an accepting state.

Code? In a Theory Course?

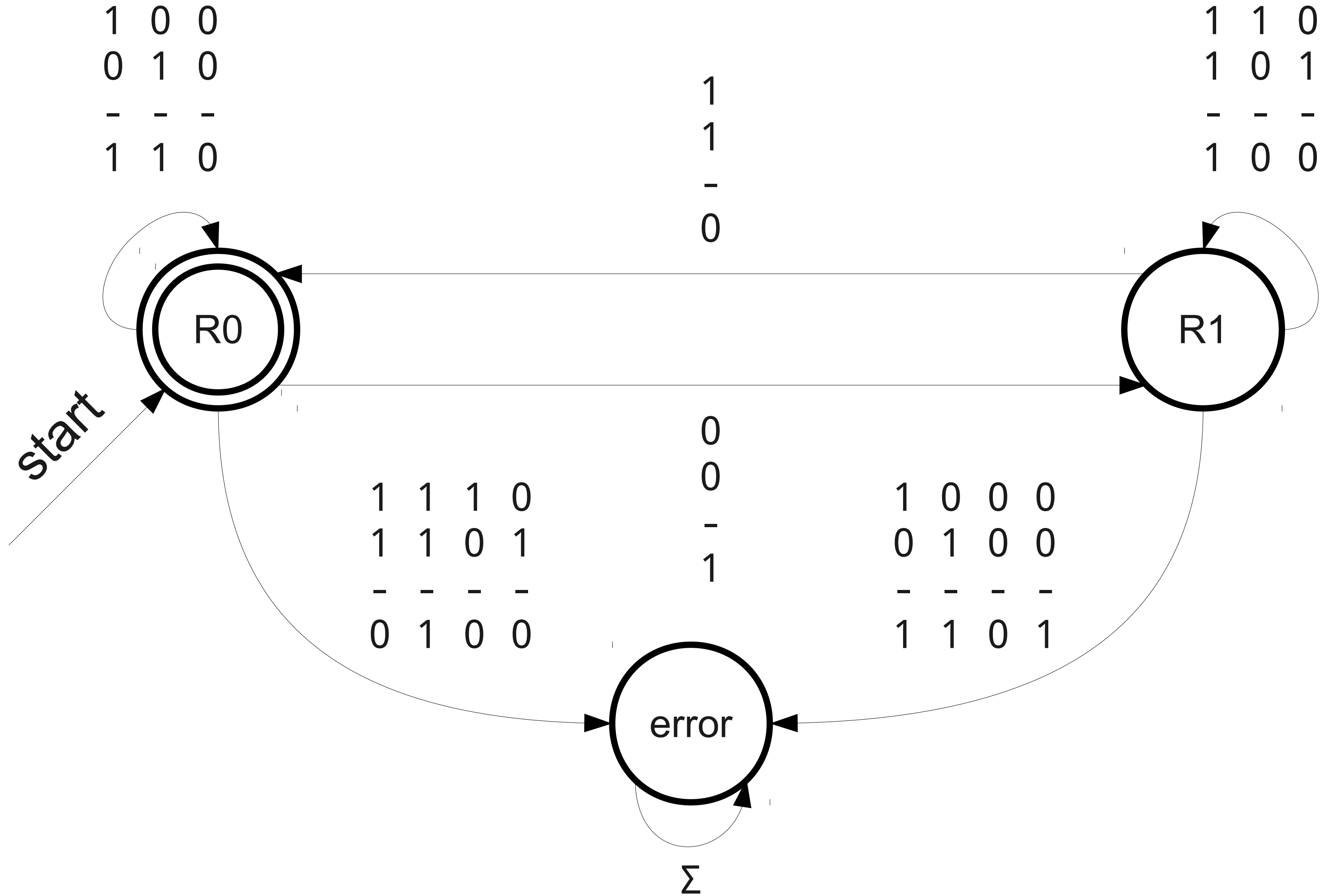
```
int kTransitionTable[kNumStates][kNumSymbols] = {
     {0, 0, 1, 3, 7, 1, …},
      …
};
bool kAcceptTable[kNumStates] = {
    false,
    true,
    true,
    …
};
bool SimulateDFA(string input) {
    int state = 0;
    for (char ch: input)
        state = kTransitionTable[state][ch];
    return kAcceptTable[state];
}
```

The Complexity of Addition

$$
\begin{array}{cccccc}
 & & 1 & 0 & 0 & 1 \\
 + & & 1 & 1 & 0 & 1 \\
\hline
 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

Our Alphabet

$$\left\{ \frac{\begin{matrix}0\\0\end{matrix}}{0}, \frac{\begin{matrix}0\\0\end{matrix}}{1}, \frac{\begin{matrix}1\\0\end{matrix}}{0}, \frac{\begin{matrix}1\\0\end{matrix}}{1}, \frac{\begin{matrix}0\\1\end{matrix}}{0}, \frac{\begin{matrix}0\\1\end{matrix}}{1}, \frac{\begin{matrix}1\\1\end{matrix}}{0}, \frac{\begin{matrix}1\\1\end{matrix}}{1} \right\}$$


1 0 0
0 1 0
- - -
1 1 0
1
1
-
0
1 1 0
1 0 1
- - -
1 0 0
R0
R1
start
0
0
-
1
1 1 1 0
1 1 0 1
- - - -
0 1 0 0
1 0 0 0
0 1 0 0
- - - -
1 1 0 1
error
Σ

A Formal Definition of DFAs

- Formally, a DFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
  - Q is a set of states.
  - Σ is an alphabet.
  - $\delta : Q \times \Sigma \to Q$ is the **transition function**.
  - $q_0 \in Q$ is the **start state**.
  - $F \subseteq Q$ is a set of **accepting states**.

A Formal Definition of Acceptance

- Given a DFA $D = (Q, \Sigma, \delta, q_0, F)$, we can formally define what it means for D to accept a string $w \in \Sigma^*$.
- **Idea**: Define a function $\delta^* : \Sigma^* \to Q$ that says what state we end up in if we run the DFA on a given string.
- This function represents the effect of running the computer on a given input.

A Formal Definition of Acceptance

- **Notation**: If $\boldsymbol{\omega}$ is a string and **a** is a character, then $\boldsymbol{\omega}$**a** is the string formed by appending **a** to $\boldsymbol{\omega}$.
- Given a DFA $(Q, \Sigma, \delta, q_0, F)$, δ^* is defined recursively.
- $\delta^* (\varepsilon) = q_0$
  - Running the automaton on ε ends in the start state.
- $\delta^* (\omega \mathtt{a}) = \delta(\delta^*(\omega), \mathtt{a})$
  - Running on $\boldsymbol{\omega}$**a** is equal to running the automaton on $\boldsymbol{\omega}$, then following the transition for **a**.

A Formal Definition of Acceptance

- Using our δ^* function, we can formally define the language of a DFA.
- Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA.
- Define $\mathscr{L}(D) = \{ w \in \Sigma^* \mid \delta^*(w) \in F \}$
  - The set of strings w that cause the DFA to end up in an accepting state.

So What?

- We now have a mathematically rigorous way of defining whether a DFA accepts a string.
- We can try making changes to DFAs and can formally prove how those changes transform the language of the DFA.

A language L is called a **regular language** iff there exists a DFA D such that $\mathscr{L}(D) = L$.

The Complement of a Language

- Given a language $L \subseteq \Sigma^*$, the **complement** of that language (denoted $\overline{L}$) is the language of all strings in Σ^* not in L.
- Formally:

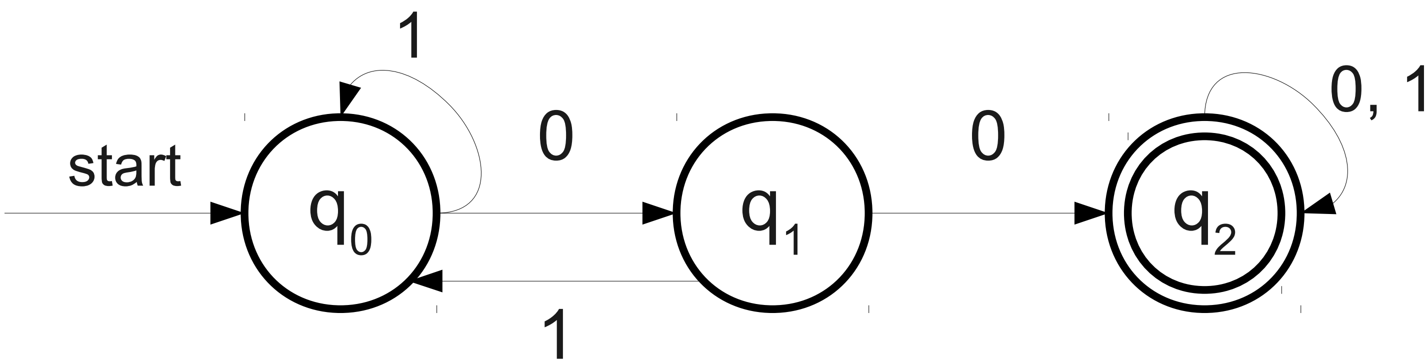
$$\overline{L} = \Sigma^* - L$$

Complementing Regular Languages

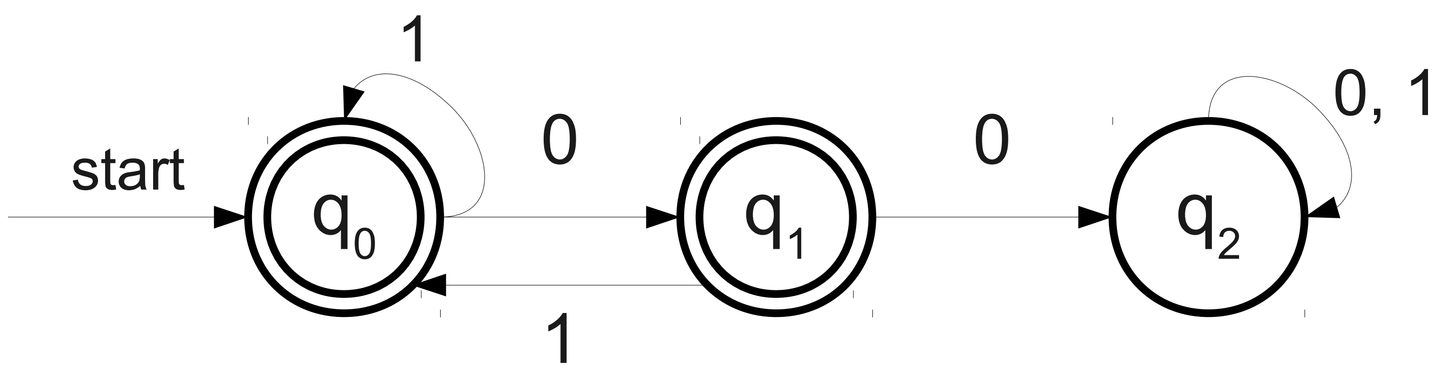
- Recall: A **regular language** is a language accepted by some DFA.
- **Question:** If L is a regular language, is $\overline{L}$ a regular language?
- If the answer is “yes,” then there must be some way to construct a DFA for $\overline{L}$.
- If the answer is “no,” then some language L can be accepted by a DFA, but $\overline{L}$ cannot be accepted by any DFA.

Complementing Regular Languages

$L = \{ \ w \in \{0, 1\}^* \mid w$ contains 00 as a substring $\}$

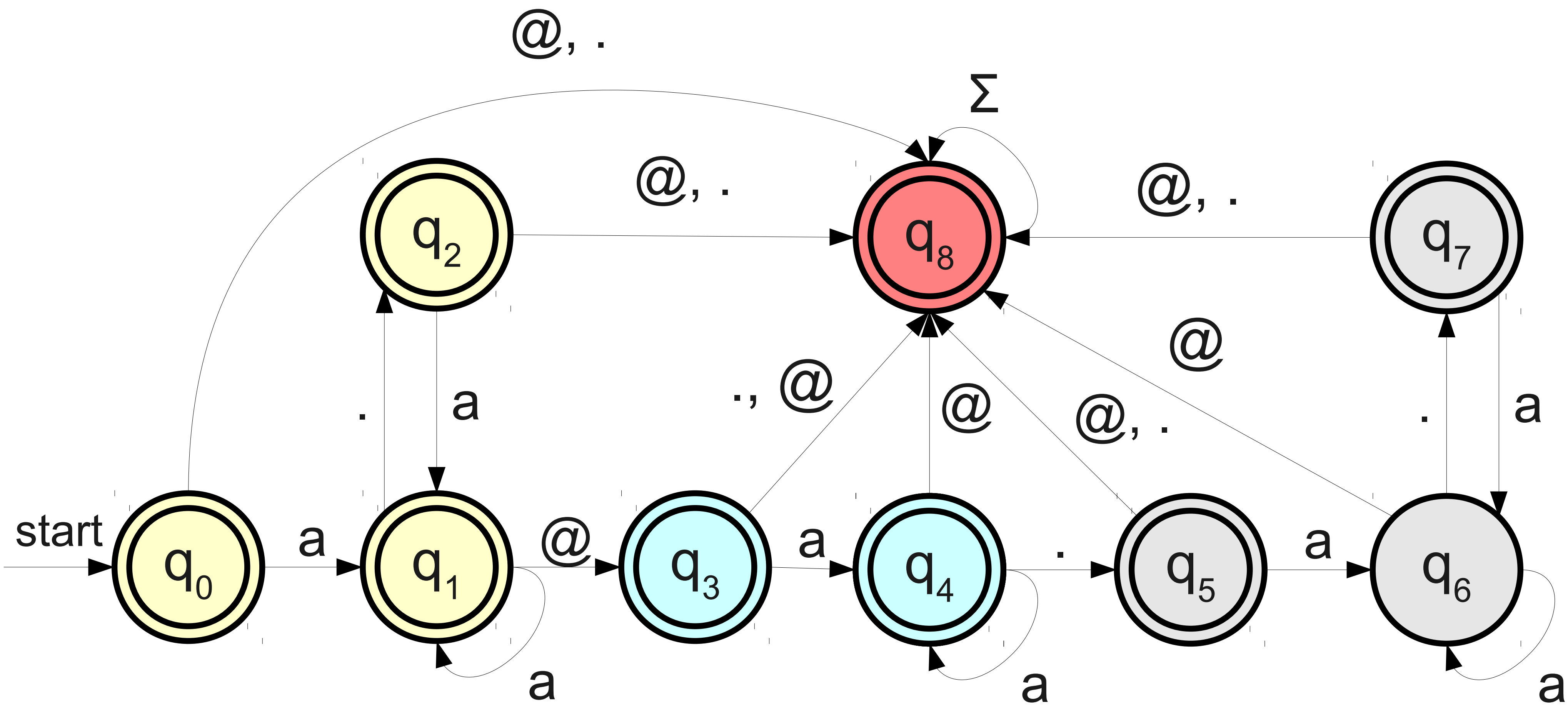


$\overline{L} = \{ \ w \in \{0, 1\}^* \mid w$ **does not contain** 00 as a substring $\}$



Complementing Regular Languages

$\overline{L}$ = { w | w is **not** a legal email address }



Constructions on Automata

- Much of our discussion of automata will consider constructions that transform one automaton into another.
- Exchanging accepting and rejecting states is a simple construction sometimes called the **complement construction**.
- Does this construction always work?
- How would we prove it?

Theorem: If $D = (Q, \Sigma, \delta, q_0, F)$ is a DFA with language $\mathscr{L}(D)$, then the DFA $D' = (Q, \Sigma, \delta, q_0, Q - F)$ has language $\overline{\mathscr{L}(D)}$.

Proof: By definition, $\mathscr{L}(D') = \{ w \in \Sigma^* \mid \delta^*(w) \in Q - F \}$. So

$$\mathscr{L}(D') = \{ w \in \Sigma^* \mid \delta^*(w) \in Q \land \delta^*(w) \notin F \}$$
$$\mathscr{L}(D') = \{ w \in \Sigma^* \mid \delta^*(w) \in Q \} - \{ w \in \Sigma^* \mid \delta^*(w) \in F \}$$

Since $\delta^* : \Sigma^* \to Q$, any string $w \in \Sigma^*$ satisfies $\delta^*(w) \in Q$. Thus

$$\mathscr{L}(D') = \{ w \in \Sigma^* \mid w \in \Sigma^* \} - \{ w \in \Sigma^* \mid \delta^*(w) \in F \}$$
$$\mathscr{L}(D') = \Sigma^* - \{ w \in \Sigma^* \mid \delta^*(w) \in F \}$$
$$\mathscr{L}(D') = \Sigma^* - \mathscr{L}(D)$$
$$\mathscr{L}(D') = \overline{\mathscr{L}(D)}. \blacksquare$$

Closure Properties

- If L is a regular language, $\overline{L}$ is a regular language.
- If we begin with a regular language and complement it, we end up with a regular language.
- This is an example of a **closure property of regular languages**.
  - The regular languages are **closed under complementation**.
  - We'll see more such properties later on.

NFAs

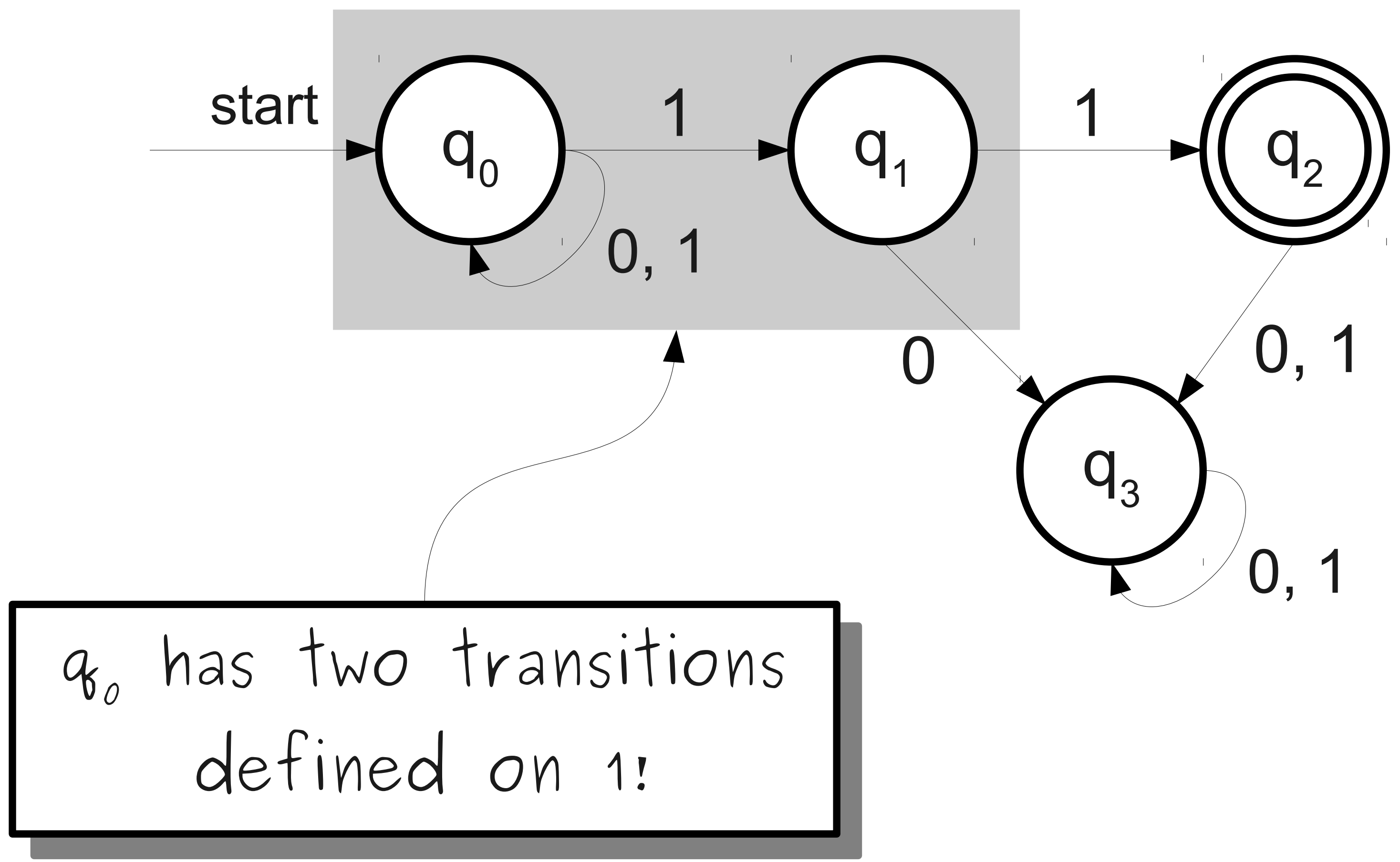
NFAs

- An **NFA** is a
  - **N**ondeterministic
  - **F**inite
  - **A**utomaton
- Conceptually similar to a DFA, but equipped with the vast power of **nondeterminism**.

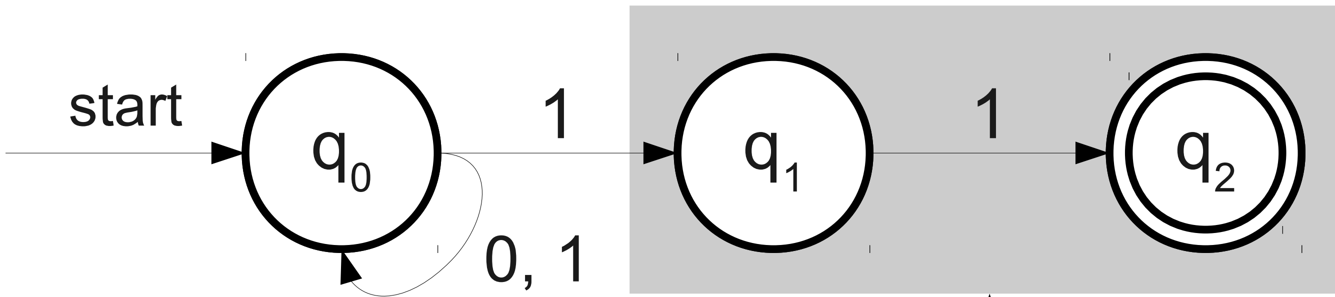
(Non)determinism

- A model of computation is **deterministic** if at every point in the computation, there is exactly one choice that can make.
- The machine accepts if that series of choices leads to an accepting state.
- A model of computation is **nondeterministic** if the computing machine may have multiple decisions that it can make at one point.
- The machine accepts if ***any*** series of choices leads to an accepting state.

A Simple NFA


start
q0
1
q1
1
q2
0, 1
0
0, 1
q3
0, 1
q0 has two transitions defined on 1!

A More Complex NFA

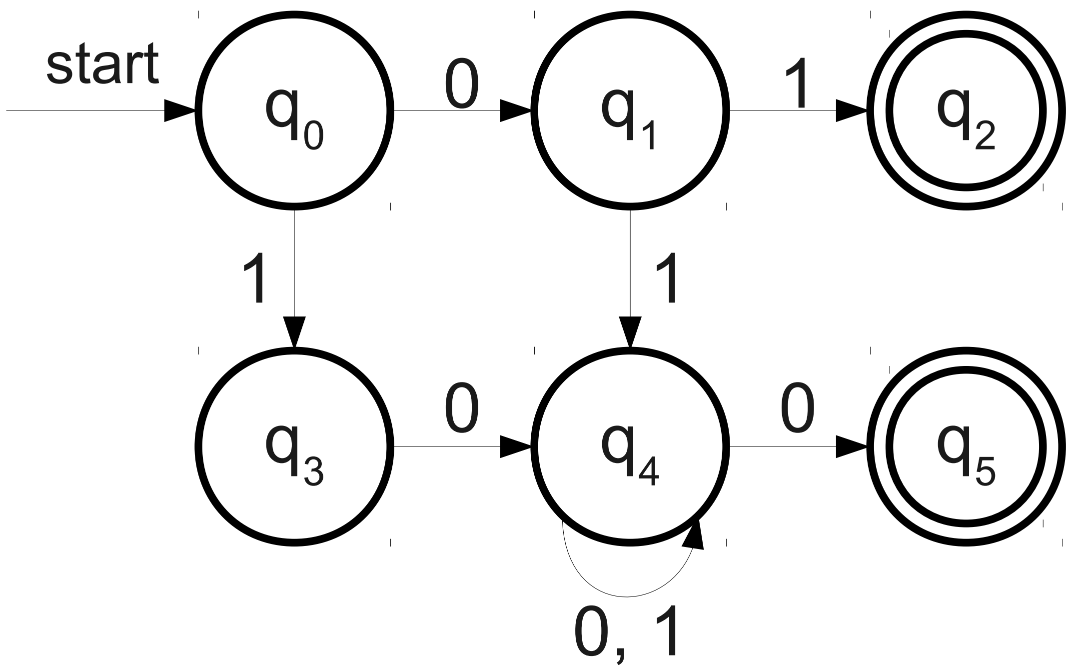


If a NFA needs to make a transition when no transition exists, the automaton dies and that particular path rejects.

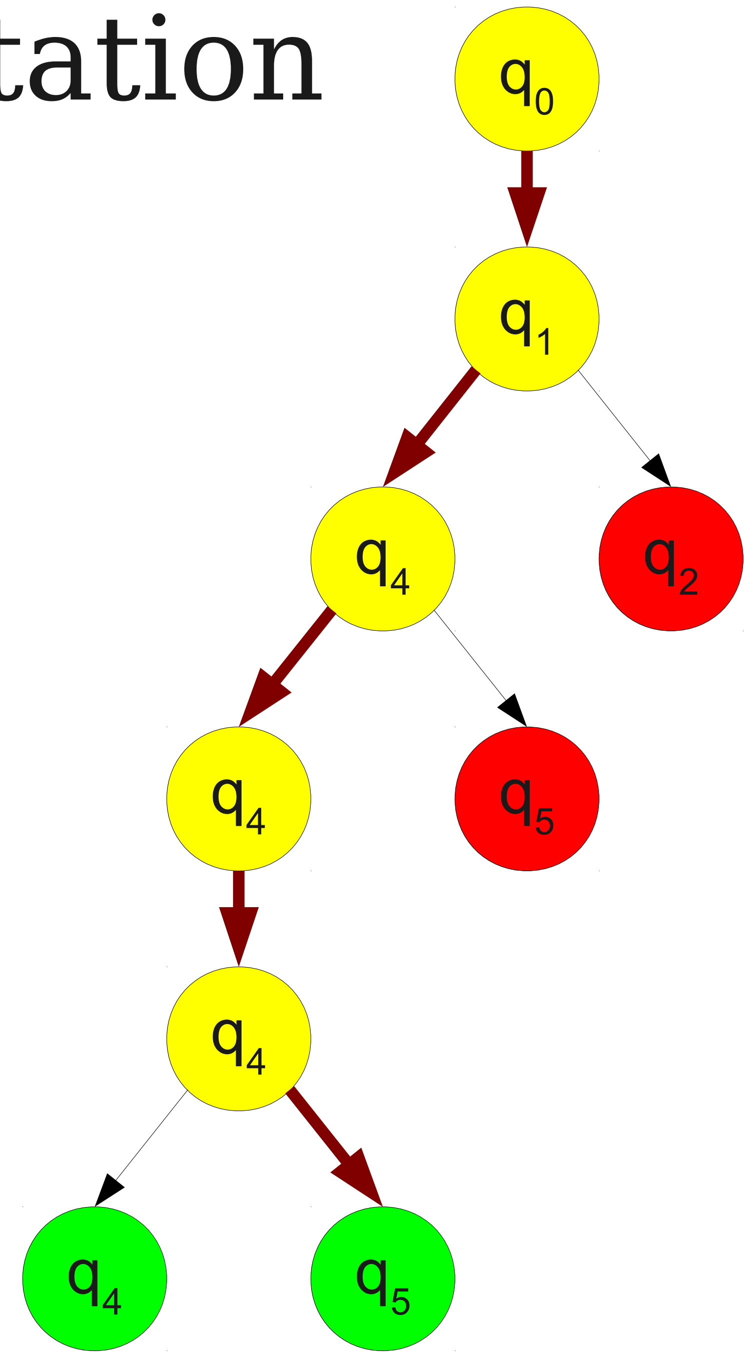
Intuiting Nondeterminism

- Nondeterministic machines are a serious departure from physical computers.
- How can we build up an intuition for them?
- Three approaches:
  - **Tree computation**
  - **Perfect guessing**
  - **Massive parallelism**

Tree Computation



0 1 0 1 0



Nondeterminism as a Tree

- At each decision point, the automaton clones itself for each possible decision.
- The series of choices forms a directed, rooted tree.
- At the end, if any active accepting states remain, we accept.

Perfect Guessing

- We can view nondeterministic machines as having **Magic Superpowers** that enable them to guess the correct choice of moves to make.
- Idea: Machine can always guess the right choice if one exists.
- No physical analog for something of this sort.
  - (Those of you thinking quantum computing – this is not the same thing. We actually don't fully know the relation between quantum and nondeterministic computation.)

Massive Parallelism

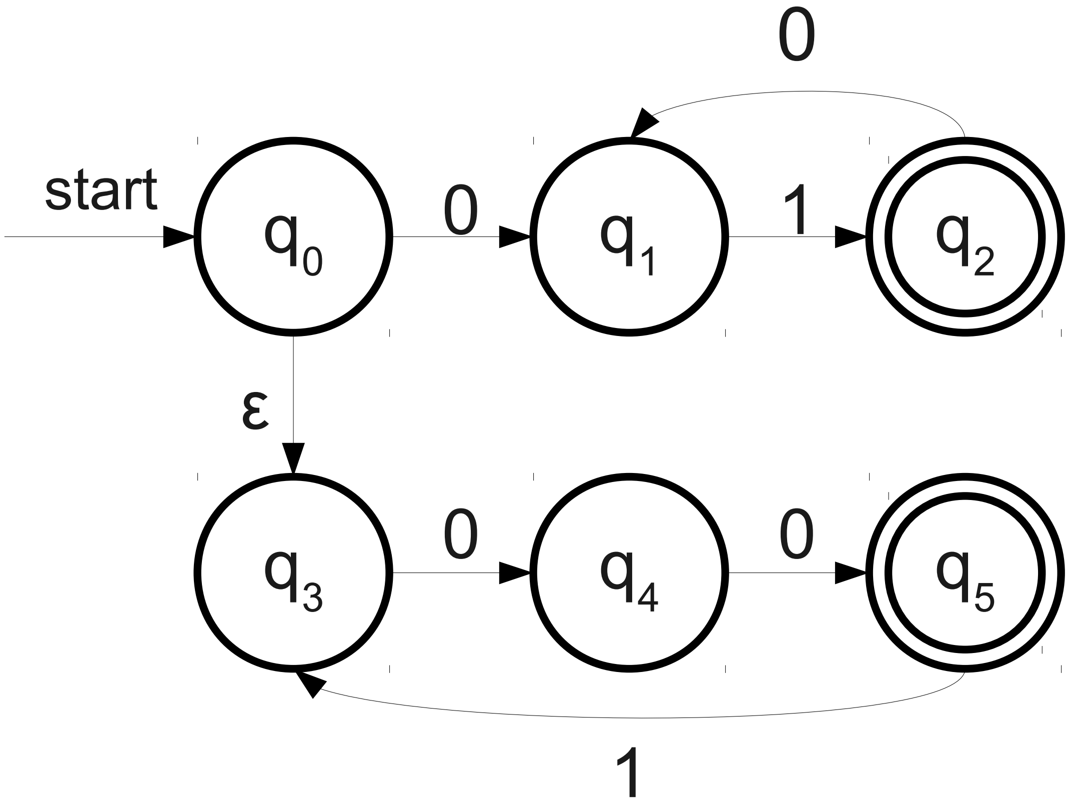
- An NFA can be thought of as a DFA that can be in many states at once.
- Each symbol read causes a transition on every active state into each potential state that could be visited.
- Nondeterministic machines can be thought of as machines that can try any number of options in parallel.
  - No fixed limit on processors; makes multicore machines look downright wimpy!

So What?

- We will turn to these three intuitions for nondeterminism more later in the quarter.
- Nondeterministic machines may not be feasible, but they give a great basis for interesting questions:
  - Can any problem that can be solved by a nondeterministic machine be solved by a deterministic machine?
  - Can any problem that can be solved by a nondeterministic machine be solved **efficiently** by a deterministic machine?
- The answers vary from automaton to automaton.

ε-Transitions

- NFAs have a special type of transition called the **ε-transition**.
- An NFA may follow any number of ε-transitions at any time without consuming any input.



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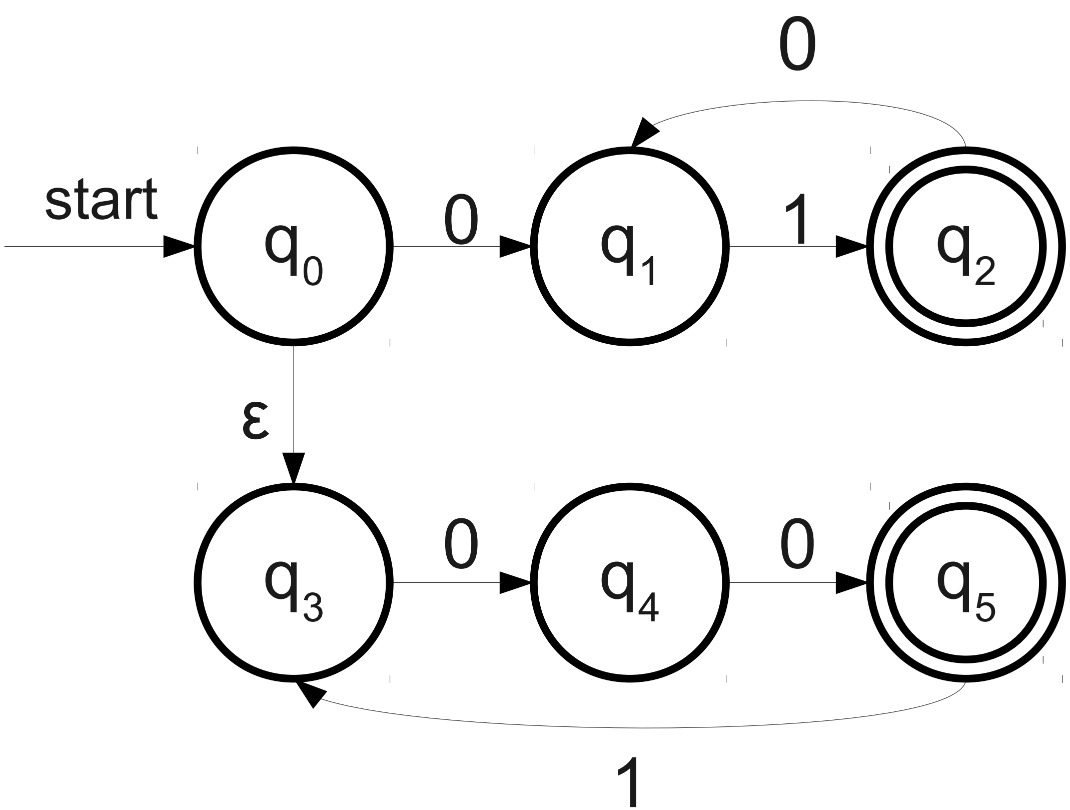

0 0 1 0 0

A Formal Definition of NFAs

- Formally, an NFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
  - Q is a set of states.
  - Σ is an alphabet.
  - $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to \wp(Q)$ is the **transition function**.
  - $q_0 \in Q$ is the **start state**.
  - $F \subseteq Q$ is a set of **accepting states**.