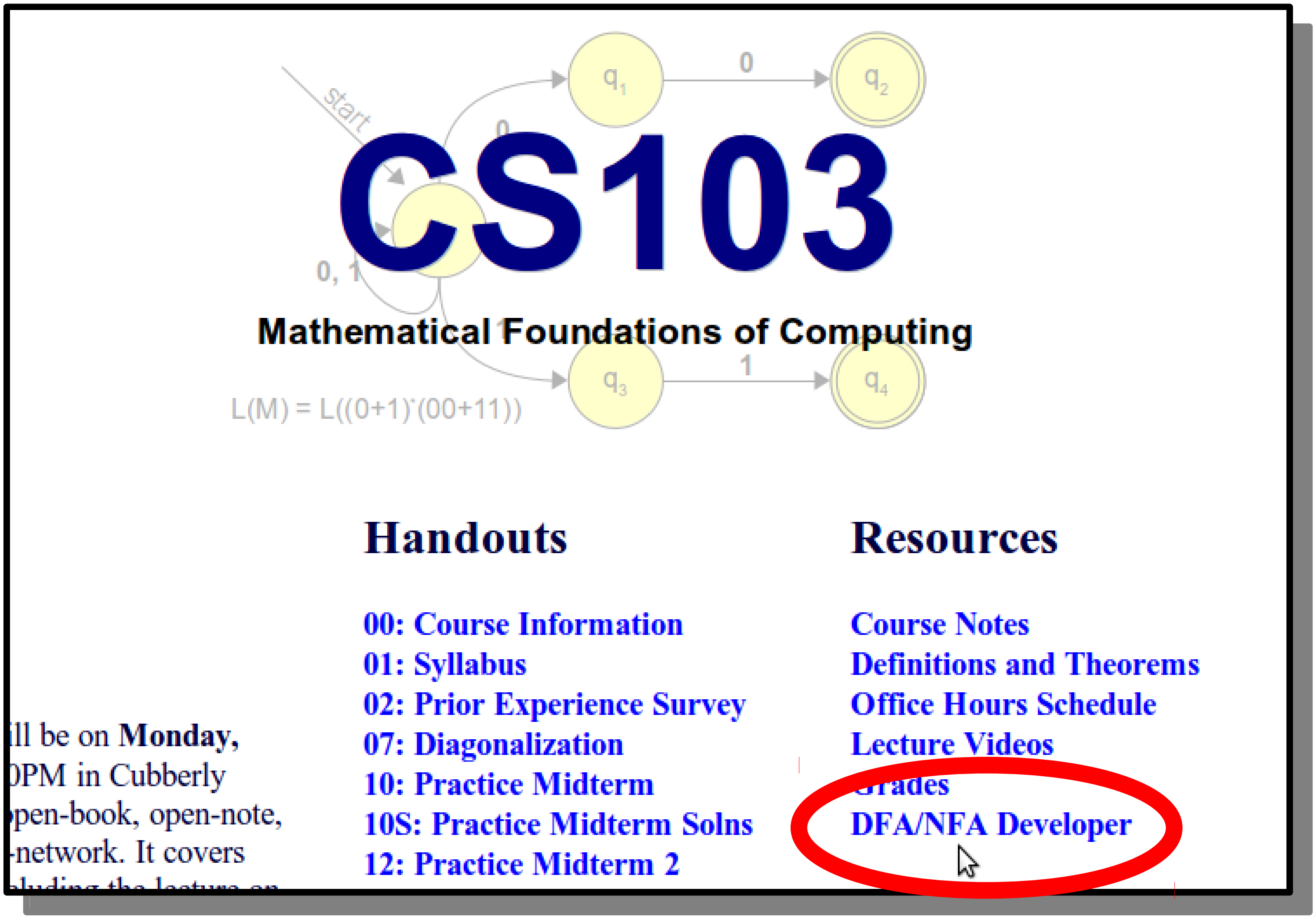
Finite Automata

Part Three

Friday Four Square!
Today at 4:15PM, Outside Gates.

Announcements

- Problem Set 4 due right now.
- Problem Set 5 out, due next Friday, November 2.
  - Play around with finite automata and regular languages.
  - **No checkpoint problems**.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q1
q2
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q4
0
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L(M) = L((0+1)*(00+11))
CS103
Mathematical Foundations of Computing
Handouts
00: Course Information
01: Syllabus
02: Prior Experience Survey
07: Diagonalization
10: Practice Midterm
10S: Practice Midterm Solns
12: Practice Midterm 2
Resources
Course Notes
Definitions and Theorems
Office Hours Schedule
Lecture Videos
Grades
DFA/NFA Developer
ill be on Monday,
PM in Cubberly
pen-book, open-note,
-network. It covers

Midterm

- Midterm is next **Monday, October 29** in **Cubberly Auditorium** from **7PM – 10PM**.
- Covers material up through and including this Monday's lecture on finite automata and DFAs.
- Review session this **Saturday, October 27** in **Gates 104** at **2PM**.

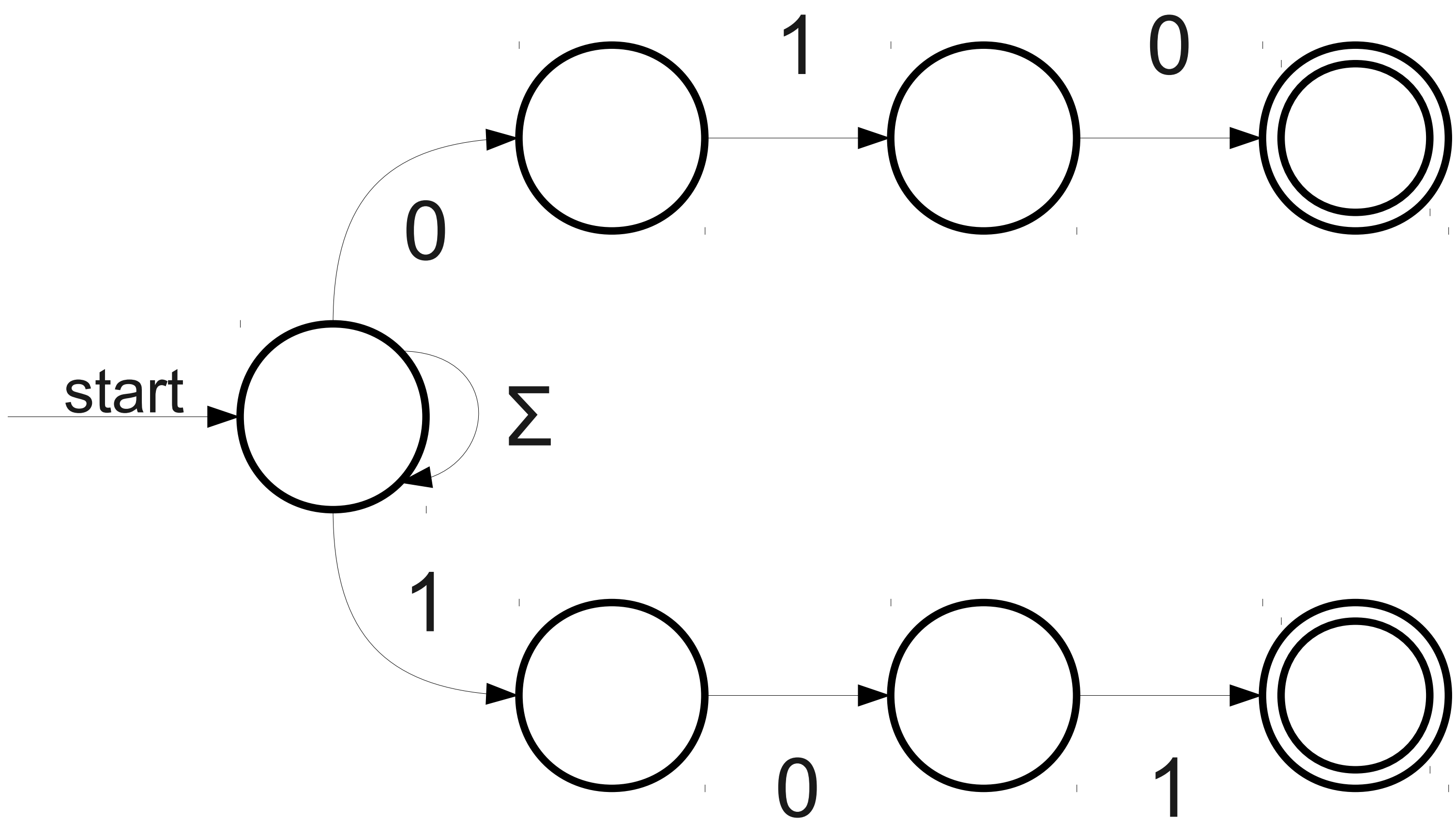
Designing NFAs

NFAs

- An **NFA** is a
  - **N**ondeterministic
  - **F**inite
  - **A**utomaton
- Conceptually similar to a DFA, but equipped with the vast power of **nondeterminism**.
- There can be many or no transitions defined on certain inputs.
- An NFA accepts a string if *any* series of choices causes the string to enter an accepting state.

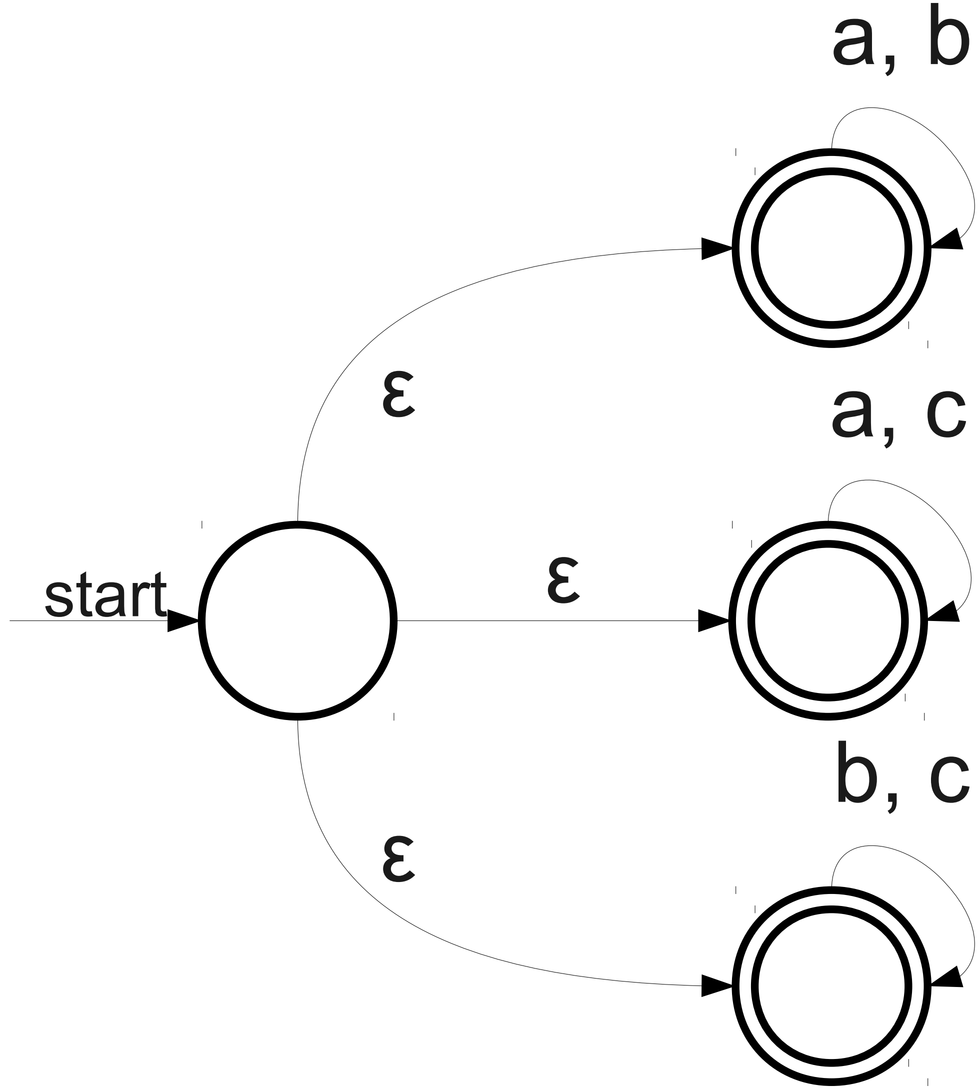
Guess-and-Check

$L = \{ w \in \{0, 1\}^* \mid w \text{ ends in } 010 \text{ or } 101 \}$



Guess-and-Check

$L = \{ w \in \{$**a**, **b**, **c**$\}^* \mid$ at least one of **a**, **b**, or **c** is not in w $\}$



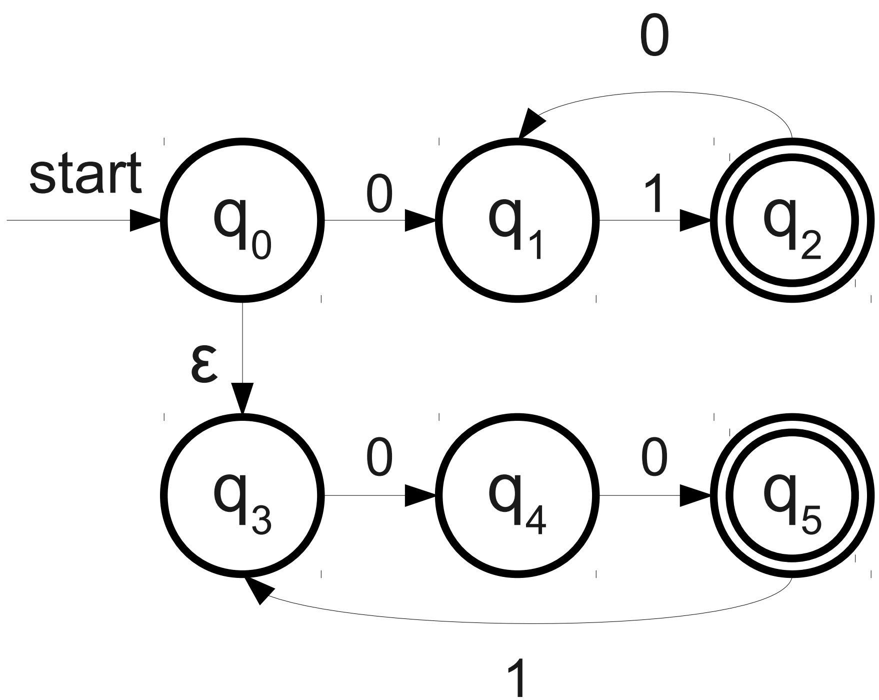
NFAs and DFAs

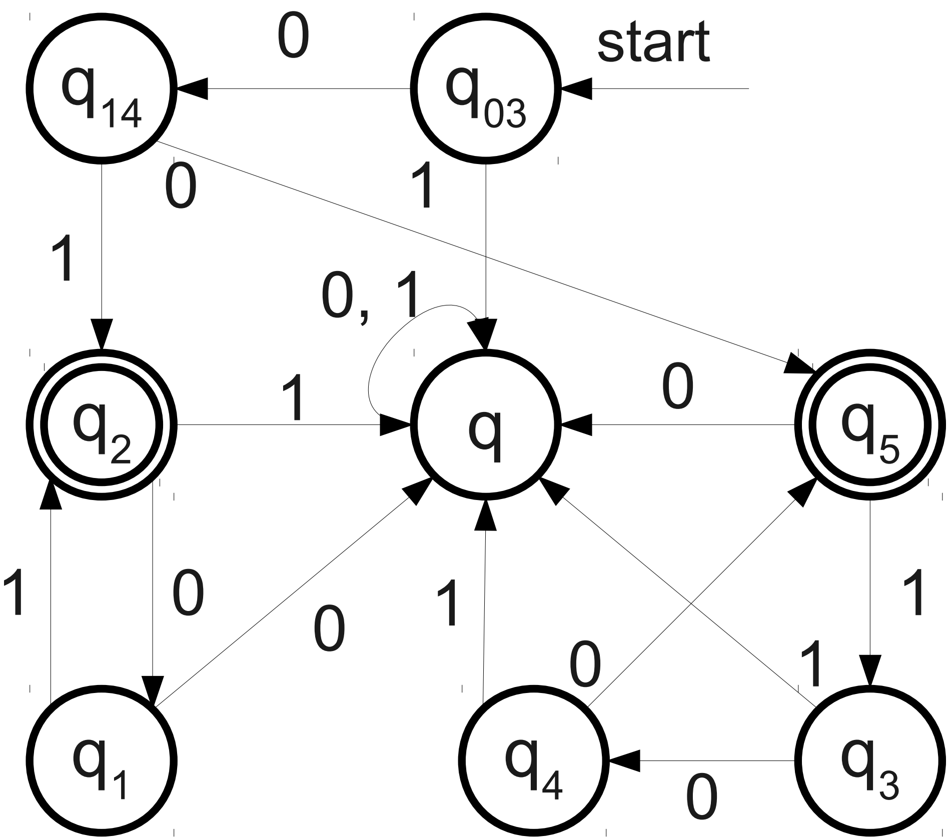
- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Just use the same set of transitions as before.
- **Question**: Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is **yes!**

Simulation

- **Simulation** is a key technique in computability theory.
- If we can build an automaton A' whose behavior ***simulates*** that of another automaton A, then we can make a connection between A and A'.
- To show that any language accepted by an NFA can be accepted by a DFA, we will show how to make a DFA that ***simulates*** the execution of an NFA.

Simulating an NFA with a DFA





The Subset Construction

- This construction for transforming an NFA into a DFA is called the **subset construction** (or sometimes the **powerset construction**).
- Intuitively:
  - States of the new DFA correspond to **sets of states** of the NFA.
  - The initial state is the start state, plus all states reachable from the start state via ε-transitions.
  - Transition on state S on character a is found by following all possible transitions on a for each state in S, then taking the set of states reachable from there by ε-transitions.
  - Accepting states are any set of states where *some* state in the set is an accepting state.
- **Read Sipser for a formal account.**

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- Fact: $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- Interesting challenge: Find a language for which this worst-case behavior occurs (there are infinitely many of them!)

A language L is called a **regular language** iff there exists a DFA D such that $\mathscr{L}(D) = L$.

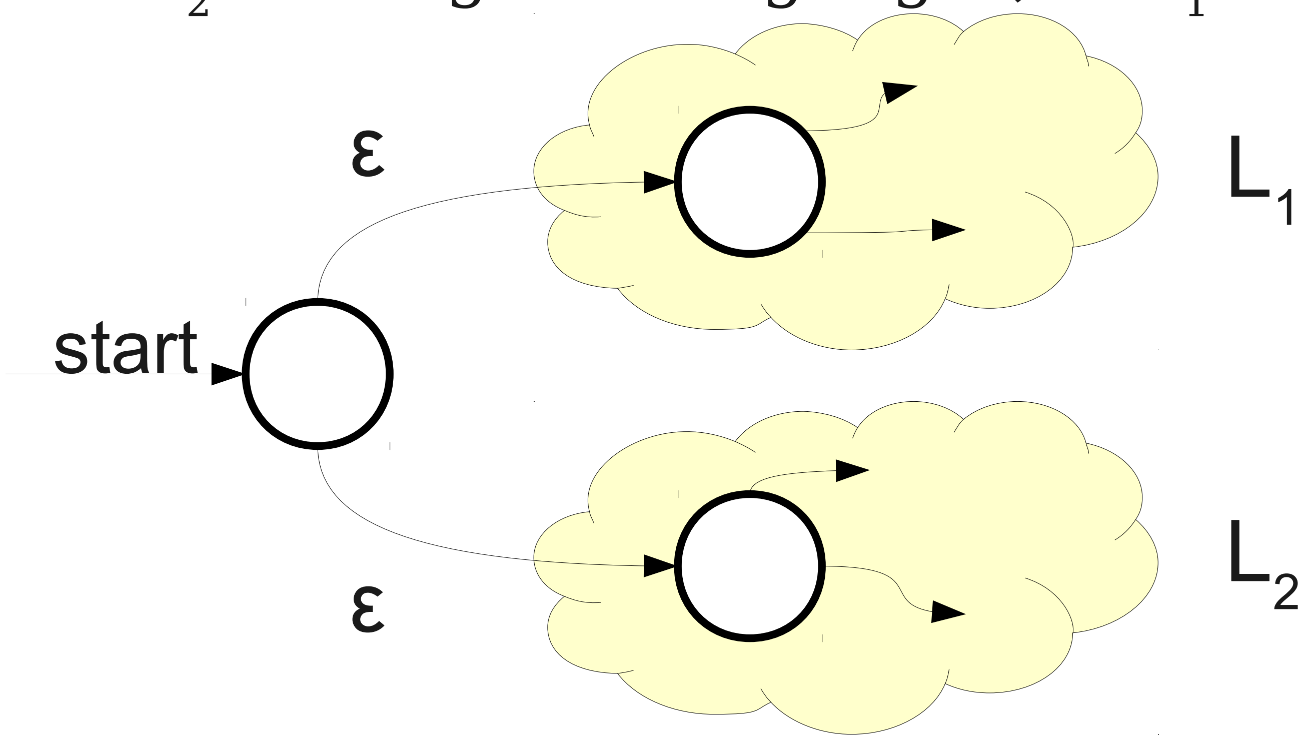
An Important Result

Theorem: A language L is regular iff there is some NFA N such that $\mathscr{L}(N) = L$.

Proof Sketch: If L is regular, there exists some DFA for it, which we can easily convert into an NFA. If L is accepted by some NFA, we can use the subset construction to convert it into a DFA that accepts the same language, so L is regular. ■

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?



The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Concatenation

- The **concatenation** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1 L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \land x \in L_2 \}$$

- The set of strings that can be split into two pieces: a string from L_1 and a string from L_2.

Concatenation Example

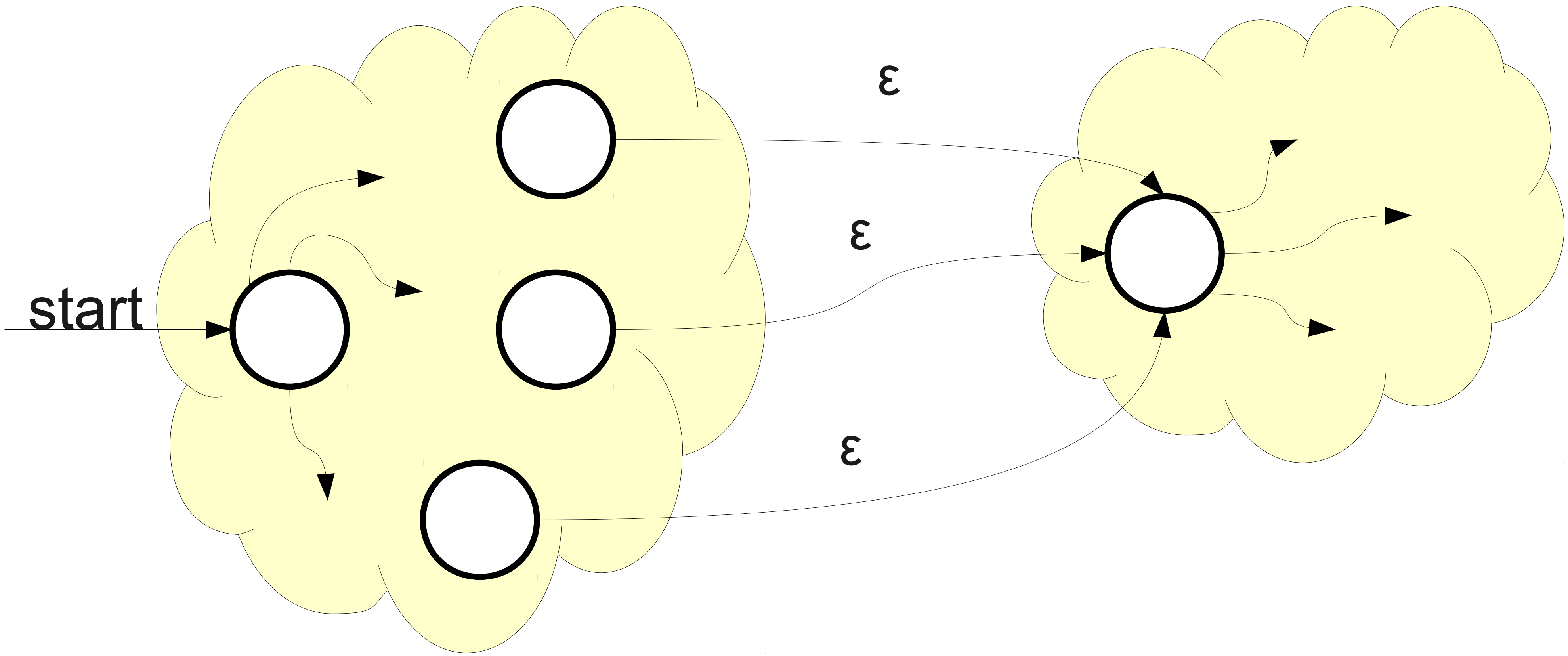
- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Velociraptor**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Eats**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is

  { **TheVelociraptorEatsTheWhale**,
  TheWhaleLovesTheRainbow,
  TheRainbowJugglesTheRainbow, … }

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such at $x \in L_1$ and $y \in L_2$?
- **Idea**: Run the automaton for L_1 on w, and whenever L_1 reaches an accepting state, optionally hand the rest off w to L_2.
  - If L_2 accepts the remainder, then L_1 accepted the first part and the string is in L_1L_2.
  - If L_2 rejects the remainder, then the split was incorrect.

Concatenating Regular Languages



Lots and Lots of Concatenation

- Consider the language *L* = { **aa**, **b** }
- *LL* is the set of strings formed by concatenating pairs of strings in *L*.
  - { **aaaa**, **aab**, **baa**, **bb** }
- *LLL* is the set of strings formed by concatenating triples of strings in *L*.
  - { **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}
- *LLLL* is the set of strings formed by concatenating quadruples of strings in *L*.
  - { **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:
- $L^0 = \{ \varepsilon \}$
  - The set containing just the empty string.
  - Idea: Any string formed by concatenating zero strings together is the empty string.
- $L^{n+1} = LL^n$
  - Idea: Concatenating $(n + 1)$ strings together works by concatenating n strings, then concatenating one more.

The Kleene Closure

- An important operation on languages is the **Kleene Closure**, which is defined as

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

- Intuitively, all possible ways of concatenating any number of copies of strings in *L* together.

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

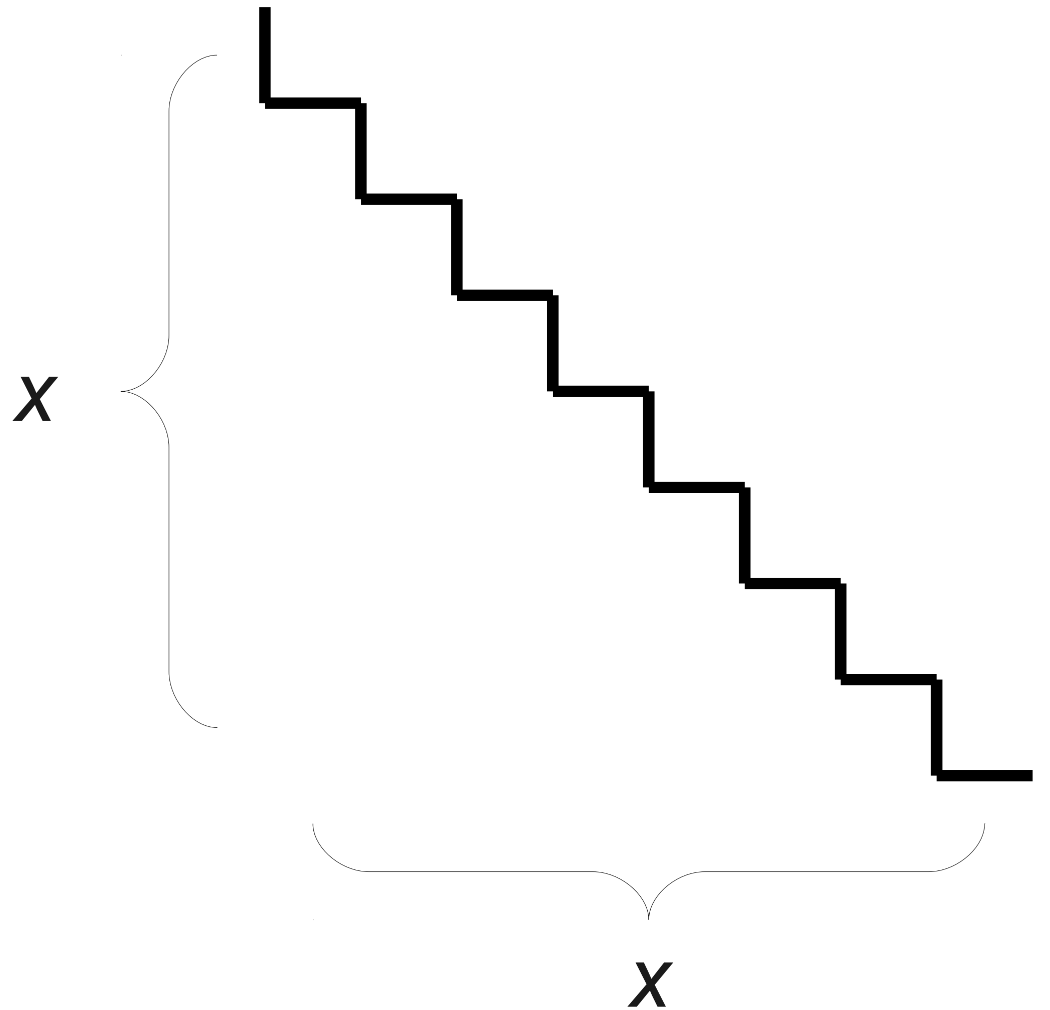
…

}

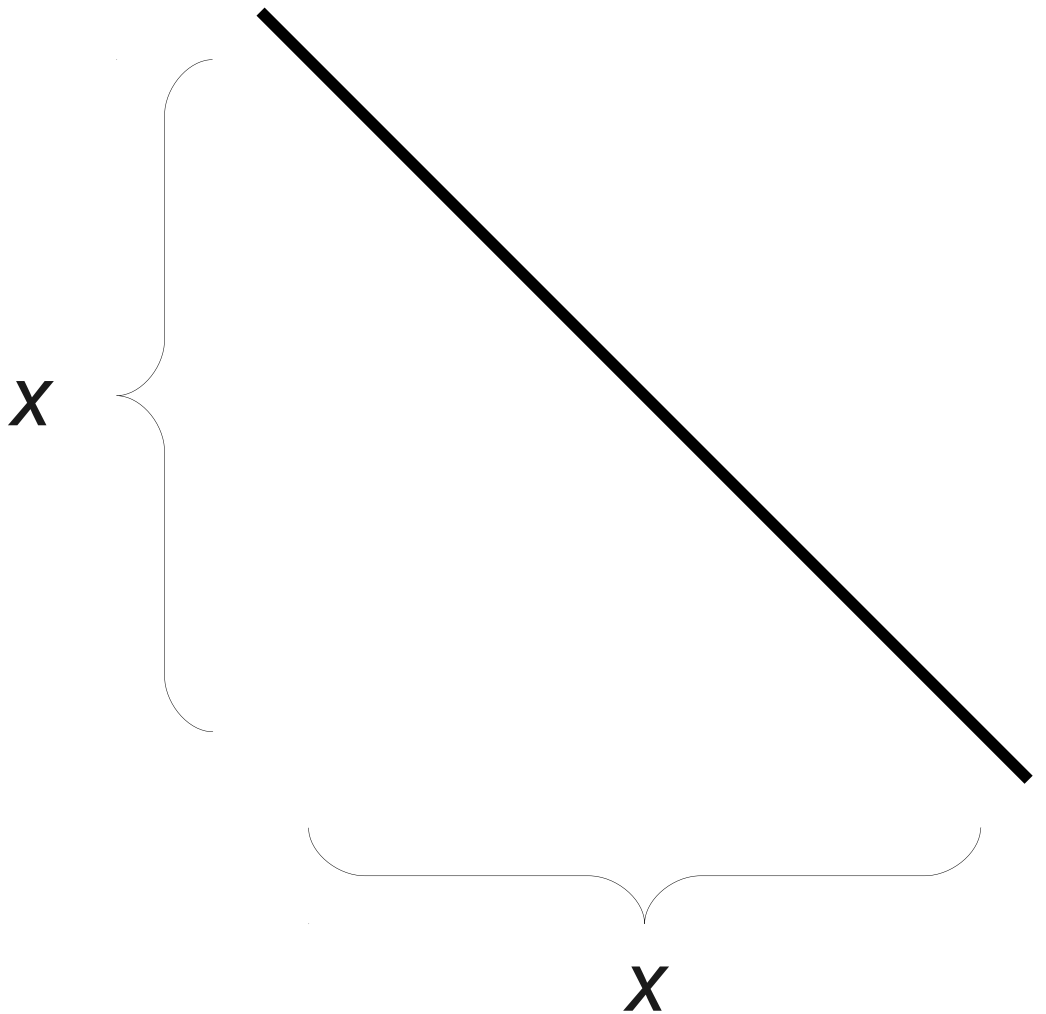
Reasoning about Infinity

- How do we prove properties of this infinite union?
- **A Bad Line of Reasoning**:
  - $L^0 = \{ \varepsilon \}$ is regular.
  - $L^1 = L$ is regular.
  - $L^2 = LL$ is regular
  - $L^3 = L(LL)$ is regular
  - …
  - So their infinite union is regular.

Reasoning about Infinity



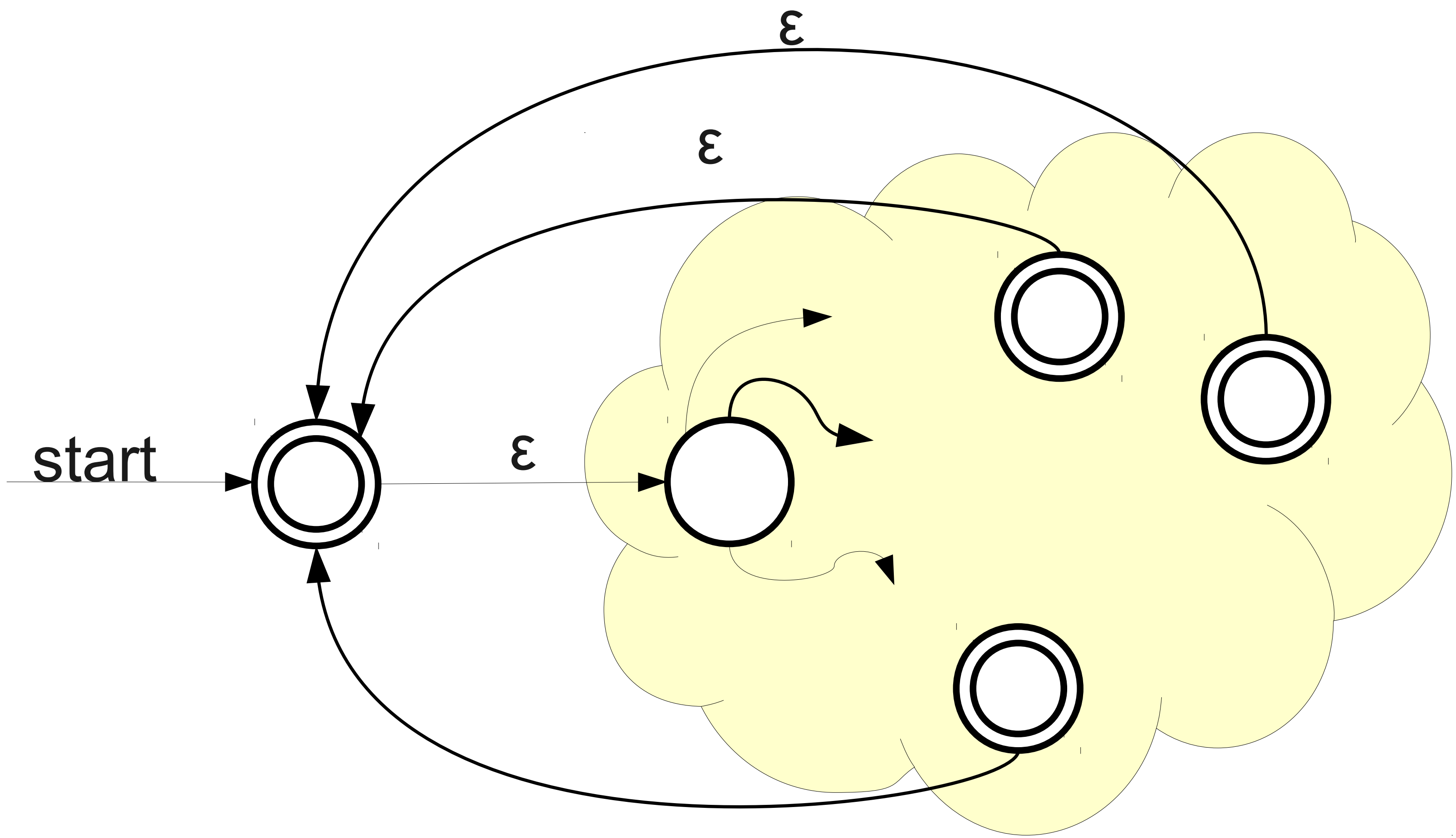
Reasoning about Infinity



Reasoning About the Infinite

- If a series of finite objects all have some property, their infinite union **does not** necessarily have that property!
  - No matter how many times we zigzag that line, it's never straight.
  - Concluding that it must be equal “in the limit” is not mathematically precise.
  - (This is why calculus is interesting).
- **A better intuition:** Can we convert an NFA for the language L to an NFA for the language L^*?

The Kleene Star



Summary

- NFAs are a powerful type of automaton that allows for **nondeterministic** choices.
- NFAs can also have **ε-transitions** that move from state to state without consuming any input.
- The **subset construction** shows that NFAs are not more powerful than DFAs, because any NFA can be converted into a DFA that accepts the same language.
- The union, intersection, difference, complement, concatenation, and Kleene closure of regular languages are all regular languages.

Another View of Regular Languages

Rethinking Regular Languages

- We currently have several tools for showing a language is regular.
  - Construct a DFA for it.
  - Construct an NFA for it.
  - Apply closure properties to existing languages.
- We have not spoken much of this last idea.

Constructing Regular Languages

- **Idea**: Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *A bottom-up approach to the regular languages.*

Regular Expressions

- **Regular expressions** are a family of descriptions that can be used to capture the regular languages.
- Often provide a compact and human-readable description of the language.
- Used as the basis for numerous software systems (Perl, **`flex`**, **`grep`**, etc.)

Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- The symbol **ε** is a regular expression that represents the language { ε }
  - This is not the same as Ø!
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language { **a** }

Compound Regular Expressions

- We can combine together existing regular expressions in four ways.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 R_2}$ is a regular expression represents the **concatenation** of the languages of R_1 and R_2.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 \mid R_2}$ is a regular expression representing the **union** of R_1 and R_2.
- If R is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the **Kleene closure** of R.
- If R is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as R.

Operator Precedence

- Regular expression operator precedence is

$$(R)$$

$$R^*$$

$$R_1 R_2$$

$$R_1 \mid R_2$$

- So `ab*c|d` is parsed as `((a(b*))c)|d`

Regular Expression Examples

- The regular expression `trick|treat` represents the regular language { `trick`, `treat` }
- The regular expression `booo*` represents the regular language { `boo`, `booo`, `boooo`, … }
- The regular expression `candy!(candy!)*` represents the regular language { `candy!`, `candy!candy!`, `candy!candy!candy!`, … }

Regular Expressions, Formally

- The **language of a regular expression** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\varepsilon) = \{\varepsilon\}$
  - $\mathscr{L}(\emptyset) = \emptyset$
  - $\mathscr{L}(\mathtt{a}) = \{\mathtt{a}\}$
  - $\mathscr{L}(R_1\ R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \mid R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^*) = \mathscr{L}(R)^*$
  - $\mathscr{L}((R)) = \mathscr{L}(R)$