Context-Free Languages

The Limits of Regular Languages

- The **pumping lemma for regular languages** can be used to establish limits on what languages are regular.
- If we want to describe more complex languages, we need a more powerful formalism.

Context-Free Grammars

- A **context-free grammar** (or **CFG**) is an entirely different formalism for defining certain languages.
- CFGs are best explained by example...

Arithmetic Expressions

- Suppose we want to describe all legal arithmetic expressions using addition, subtraction, multiplication, and division.
- Here is one possible CFG:

E → `int`
E → **E Op E**
E → `(`**E**`)`
Op → `+`
Op → `-`
Op → `*`
Op → `/`

E
⇒ **E Op E**
⇒ **E Op** `(`**E**`)`
⇒ **E Op** `(`**E Op E**`)`
⇒ **E** `*` `(`**E Op E**`)`
⇒ `int * (`**E Op E**`)`
⇒ `int * (int` **Op E**`)`
⇒ `int * (int` **Op** `int)`
⇒ `int * (int + int)`

Arithmetic Expressions

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- Here is one possible CFG:

E → `int`

E → **E Op E**

E → `(`**E**`)`

Op → `+`

Op → `-`

Op → `*`

Op → `/`

E
⇒ **E Op E**
⇒ **E Op** `int`
⇒ `int` **Op** `int`
⇒ `int / int`

Context-Free Grammars

- Formally, a context-free grammar is a collection of four objects:
    - A set of **nonterminal symbols** (also called **variables**),
    - A set of **terminal symbols** (the **alphabet** of the CFG)
    - A set of **production rules** saying how each nonterminal can be converted by a string of terminals and nonterminals, and
    - A **start symbol** (which must be a nonterminal) that begins the derivation.

E → `int`

E → **E Op E**

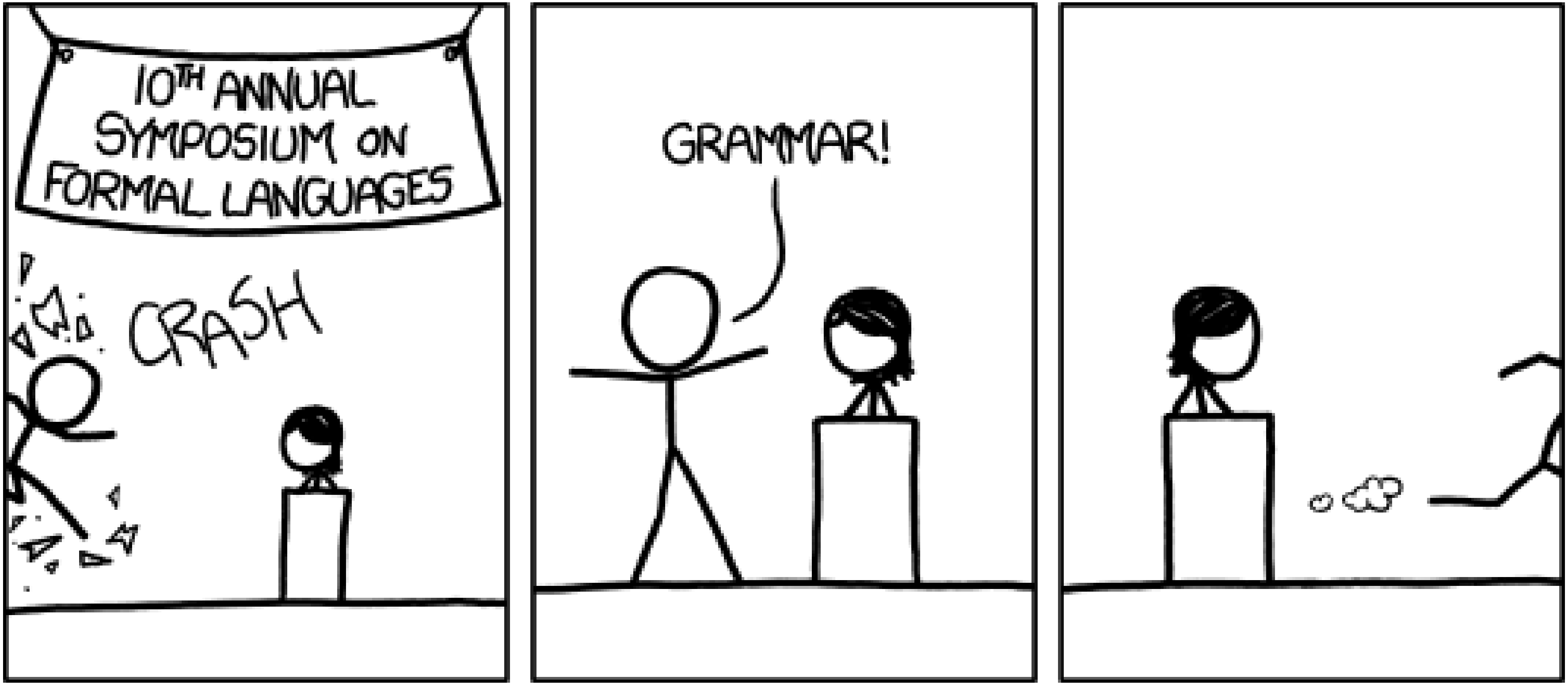
E → `(`**E**`)`

Op → `+`

Op → `-`

Op → `*`

Op → `/`



http://xkcd.com/1090/

A Notational Shorthand

E → `int`

E → **E Op E**

E → `(`**E**`)`

Op → `+`

Op → `-`

Op → `*`

Op → `/`

A Notational Shorthand

E → `int` | **E Op E** | `(`**E**`)`

Op → `+` | `-` | `*` | `/`

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

S → **`a*b`**

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

$$\mathbf{S} \rightarrow \mathbf{A}\mathtt{b}$$

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

$$\mathbf{S} \rightarrow \mathbf{A}\mathtt{b}$$

$$\mathbf{A} \rightarrow \mathbf{A}\mathtt{a} \mid \varepsilon$$

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

S → `a(b|c*)`

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

$$\mathbf{S} \rightarrow \texttt{a}\mathbf{X}$$

$$\mathbf{X} \rightarrow \texttt{(b|c*)}$$

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

$$\mathbf{S} \rightarrow \texttt{a}\mathbf{X}$$
$$\mathbf{X} \rightarrow \texttt{b} \mid \texttt{c*}$$

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

S → aX
X → b | C

Not Notational Shorthand

- The syntax for regular expressions does not carry over to CFGs.
- Cannot use *, |, or parentheses.

$$
\begin{aligned}
\mathbf{S} &\rightarrow \mathtt{a}\mathbf{X} \\
\mathbf{X} &\rightarrow \mathtt{b} \mid \mathbf{C} \\
\mathbf{C} &\rightarrow \mathbf{C}\mathtt{c} \mid \varepsilon
\end{aligned}
$$

More Context-Free Grammars

- Chemicals!

  $C_{19}H_{14}O_5S$

  $Cu_3(CO_3)_2(OH)_2$

  MnO_4^-

  S^{2-}

Form → **Cmp** | **Cmp Ion**

Cmp → **Term** | **Term Num** | **Cmp Cmp**

Term → **Elem** | `(`**Cmp**`)`

Elem → `H` | `He` | `Li` | `Be` | `B` | `C` | …

Ion → `+` | `-` | **IonNum** `+` | **IonNum** `-`

IonNum → `2` | `3` | `4` | ...

Num → `1` | **IonNum**

CFGs for Chemistry

Form → **Cmp** | **Cmp Ion**

Cmp → **Term** | **Term Num** | **Cmp Cmp**

Term → **Elem** | `(`**Cmp**`)`

Elem → `H` | `He` | `Li` | `Be` | `B` | `C` | …

Ion → `+` | `-` | **IonNum** `+` | **IonNum** `-`

IonNum → `2` | `3` | `4` | ...

Num → `1` | **IonNum**

Form
⇒ **Cmp Ion**
⇒ **Cmp Cmp Ion**
⇒ **Cmp Term Num Ion**
⇒ **Term Term Num Ion**
⇒ **Elem Term Num Ion**
⇒ `Mn` **Term Num Ion**
⇒ `Mn` **Elem Num Ion**
⇒ `MnO` **Num Ion**
⇒ `MnO` **IonNum Ion**
⇒ MnO_4 **Ion**
⇒ MnO_4^-

CFGs for Programming Languages

$$
\begin{array}{lcl}
\textbf{BLOCK} & \rightarrow & \textbf{STMT} \\
 & | & \texttt{\{}\ \textbf{STMTS}\ \texttt{\}} \\
 & & \\
\textbf{STMTS} & \rightarrow & \varepsilon \\
 & | & \textbf{STMT STMTS} \\
 & & \\
\textbf{STMT} & \rightarrow & \textbf{EXPR}\texttt{;} \\
 & | & \texttt{if (}\textbf{EXPR}\texttt{)}\ \textbf{BLOCK} \\
 & | & \texttt{while (}\textbf{EXPR}\texttt{)}\ \textbf{BLOCK} \\
 & | & \texttt{do}\ \textbf{BLOCK}\ \texttt{while (}\textbf{EXPR}\texttt{);} \\
 & | & \textbf{BLOCK} \\
 & | & \ldots \\
 & & \\
\textbf{EXPR} & \rightarrow & \texttt{identifier} \\
 & | & \texttt{constant} \\
 & | & \textbf{EXPR}\ \texttt{+}\ \textbf{EXPR} \\
 & | & \textbf{EXPR}\ \texttt{-}\ \textbf{EXPR} \\
 & | & \textbf{EXPR}\ \texttt{*}\ \textbf{EXPR} \\
 & | & \ldots
\end{array}
$$

Some CFG Notation

- Capital letters in **Bold Red Uppercase** will represent nonterminals.
  - i.e. **A**, **B**, **C**, **D**
- Lowercase letters in **`blue monospace`** will represent terminals.
  - i.e. **`t`**, **`u`**, **`v`**, **`w`**
- Lowercase Greek letters in ***gray italics*** will represent arbitrary strings of terminals and nonterminals.
  - i.e. α, γ, ω

Examples

- We might write an arbitrary production as

$$\mathbf{A} \to \boldsymbol{\omega}$$

- We might write a string of a nonterminal followed by a terminal as

$$\mathbf{A}\mathtt{t}$$

- We might write an arbitrary production containing a nonterminal followed by a terminal as

$$\mathbf{B} \to \boldsymbol{\alpha}\mathbf{A}\mathtt{t}\boldsymbol{\omega}$$

Derivations

E
⇒ **E Op E**
⇒ **E Op (E)**
⇒ **E Op (E Op E)**
⇒ **E * (E Op E)**
⇒ **int * (E Op E)**
⇒ **int * (int Op E)**
⇒ **int * (int Op int)**
⇒ **int * (int + int)**

- This sequence of steps is called a **derivation**.
- A string $\alpha A\omega$ **yields** string $\alpha\gamma\omega$ iff $A \to \gamma$ is a production.
- If α yields β, we write $\alpha \Rightarrow \beta$.
- We say that α **derives** β iff there is a sequence of strings where

$$\alpha \Rightarrow \alpha_1 \Rightarrow \alpha_2 \Rightarrow \dots \Rightarrow \beta$$

- If α derives β, we write $\alpha \Rightarrow^* \beta$.

The Language of a Grammar

- If *G* is a CFG with alphabet Σ and start symbol **S**, then the **language of *G*** is the set

$$\mathscr{L}(G) = \{ \boldsymbol{\omega} \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \boldsymbol{\omega} \}$$

- That is, the set of strings derivable from the start symbol.
- If *L* is a language and there is some CFG *G* such that *L* = $\mathscr{L}$(G), then we say that *L* is a **context-free language** (or **CFL**).

The Language of a Grammar

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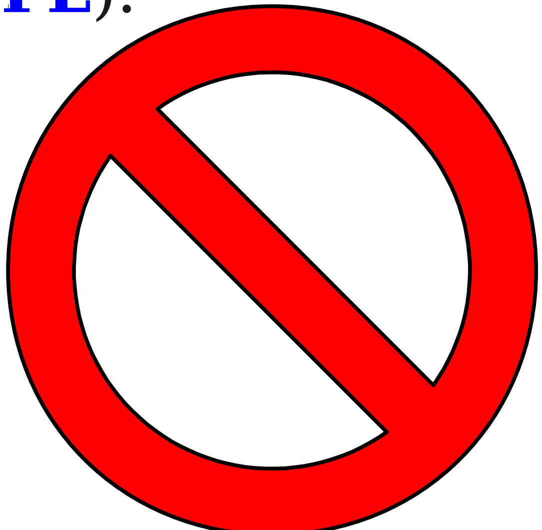
$$\mathscr{L}(G) = \{\ \boldsymbol{\omega} \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \boldsymbol{\omega}\ \}$$

- That is, the set of strings derivable from the start symbol.
- If L is a language and there is some CFG G such that $L = \mathscr{L}(\text{G})$, then we say that L is a **context-free language** (or **CFL**).



Regular and Context-Free Languages

- Consider the following CFG *G*:

$$\mathbf{S} \rightarrow \mathtt{0}\mathbf{S}\mathtt{1} \mid \varepsilon$$

- What strings can this generate?

0	0	0	0	0	0	**S**	1	1	1	1	1	1

Regular and Context-Free Languages

- Consider the following CFG *G*:

$$S \rightarrow 0S1 \mid \varepsilon$$

- What strings can this generate?

0	0	0	0	0	0	1	1	1	1	1	1

Regular and Context-Free Languages

- Consider the following CFG G:

$$\mathbf{S} \rightarrow \mathtt{0}\mathbf{S}\mathtt{1} \mid \varepsilon$$

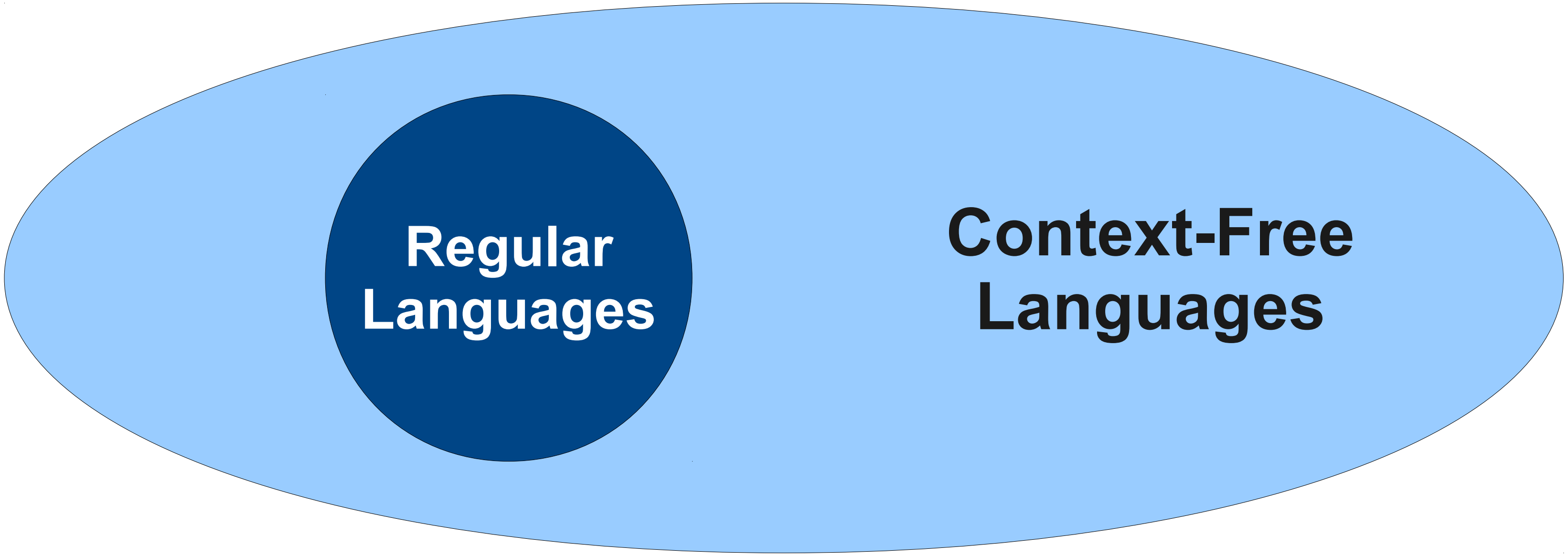
- What strings can this generate?

0	0	0	0	0	0	1	1	1	1	1	1

$$\mathscr{L}(G) = \{\ \mathtt{0}^{\mathrm{n}}\mathtt{1}^{\mathrm{n}} \mid n \in \mathbb{N}\ \}$$

Regular and Context-Free Languages

- Context-free languages are a **strict superset** of the regular languages.
- Every regular language is context-free, but not necessarily the other way around.
- We'll see a proof of this next time.



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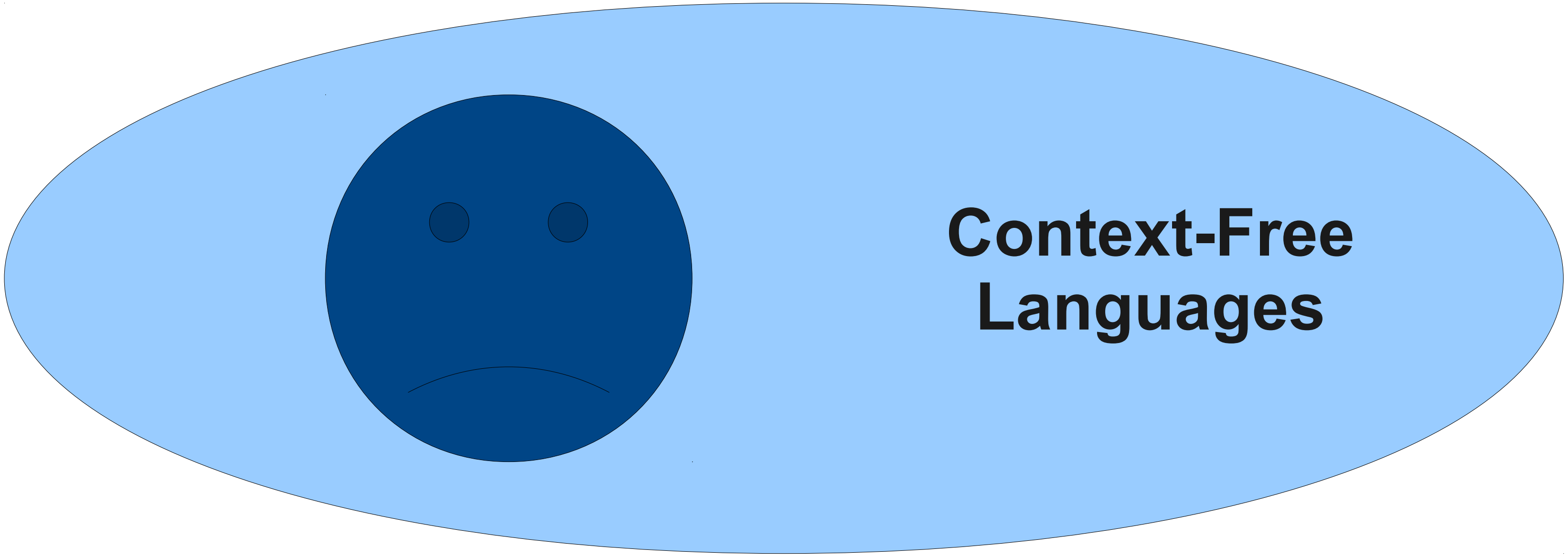


Leftmost Derivations

BLOCK → STMT
| { STMTS }

STMTS → ε
| STMT STMTS

STMT → EXPR;
| if (EXPR) BLOCK
| while (EXPR) BLOCK
| do BLOCK while (EXPR);
| BLOCK
| …

EXPR → identifier
| constant
| EXPR + EXPR
| EXPR – EXPR
| EXPR * EXPR
| EXPR = EXPR
| …

STMTS
⇒ STMT STMTS
⇒ EXPR; STMTS
⇒ EXPR = EXPR; STMTS
⇒ id = EXPR; STMTS
⇒ id = EXPR + EXPR; STMTS
⇒ id = id + EXPR; STMTS
⇒ id = id + constant; STMTS
⇒ id = id + constant;

Leftmost Derivations

- A **leftmost derivation** is a derivation in which each step expands the leftmost nonterminal.
- A **rightmost derivation** is a derivation in which each step expands the rightmost nonterminal.
- These will be of great importance next lecture when we discuss *pushdown automata*.

Related Derivations

E	E
⇒ E Op E	⇒ E Op E
⇒ int Op E	⇒ E Op (E)
⇒ int * E	⇒ E Op (E Op E)
⇒ int * (E)	⇒ E Op (E Op int)
⇒ int * (E Op E)	⇒ E Op (E + int)
⇒ int * (int Op E)	⇒ E Op (int + int)
⇒ int * (int + E)	⇒ E * (int + int)
⇒ int * (int + int)	⇒ int * (int + int)

Derivations Revisited

- A derivation encodes two pieces of information:
    - What productions were applied produce the resulting string from the start symbol?
    - In what order were they applied?
- Multiple derivations might use the same productions, but apply them in a different order.

Parse Trees

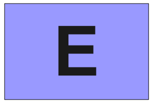
E

Parse Trees

E

E

Parse Trees

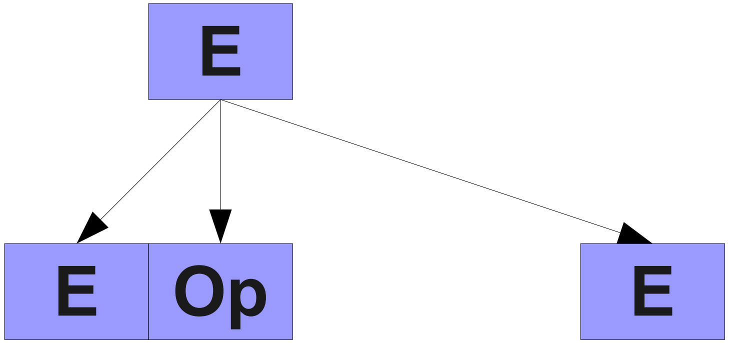


E

E

⇒ **E Op E**

Parse Trees



E

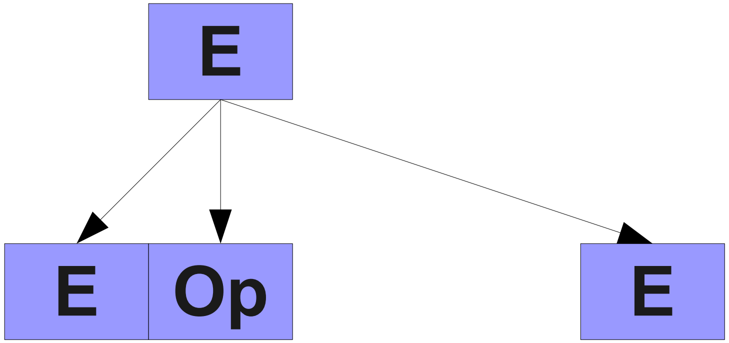
⇒ **E Op E**

Parse Trees

E

⇒ **E Op E**

⇒ **`int` Op E**

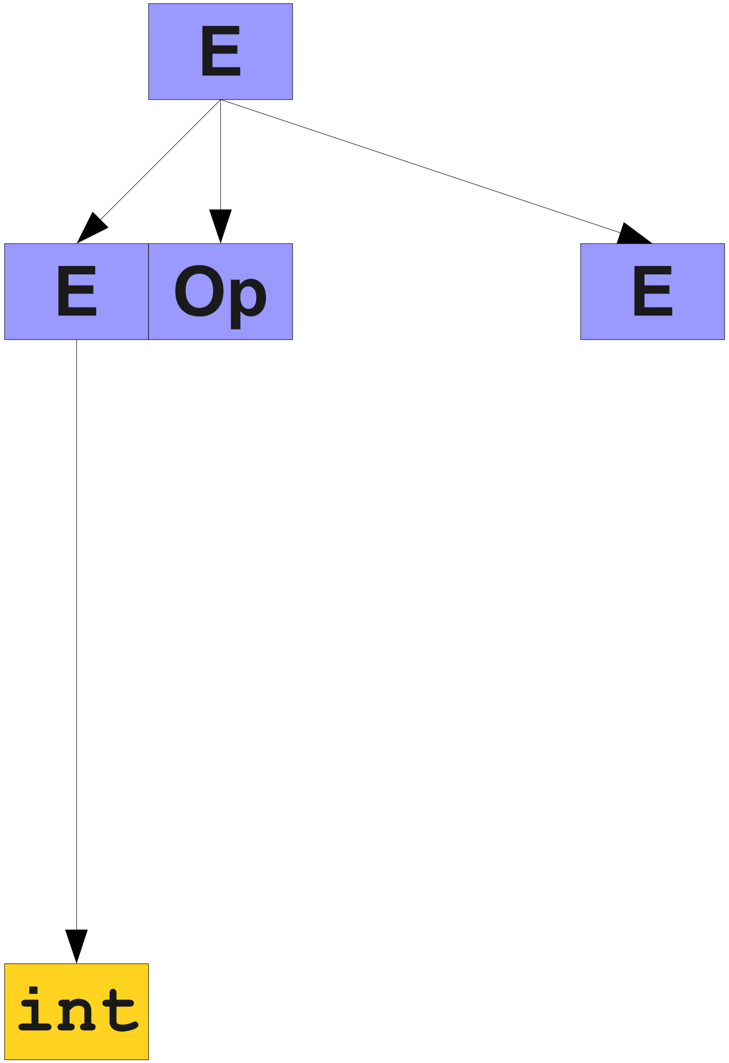


Parse Trees

E

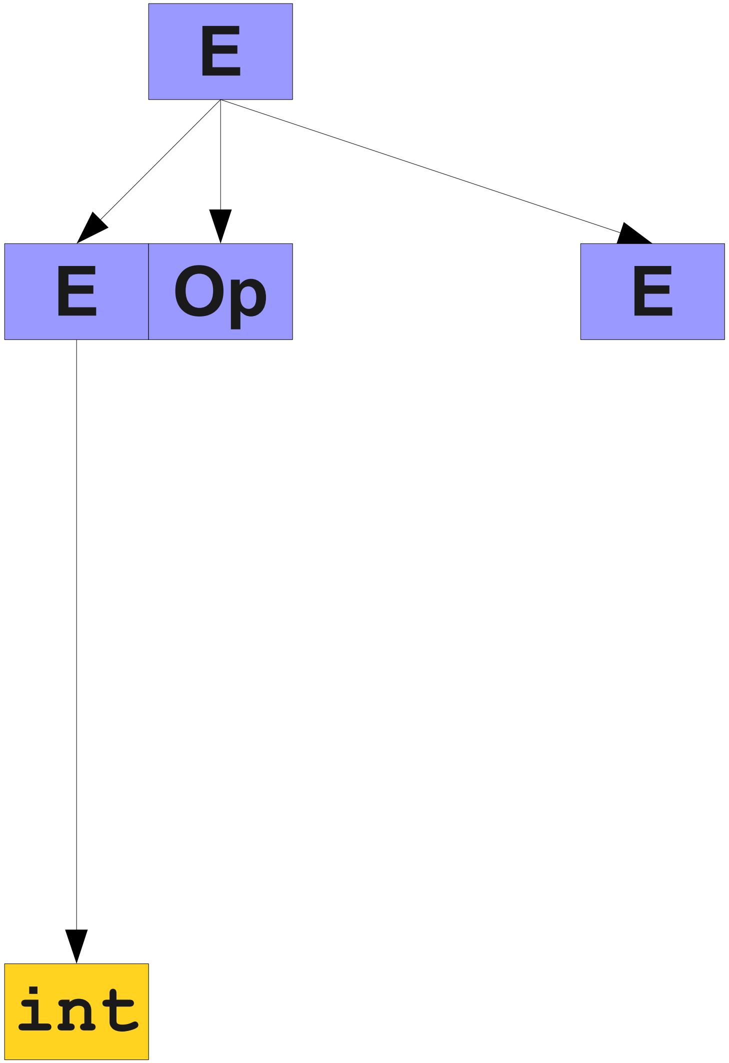
⇒ **E Op E**

⇒ `int` **Op E**



Parse Trees

E
⇒ **E Op E**
⇒ `int` **Op E**
⇒ `int * ` **E**



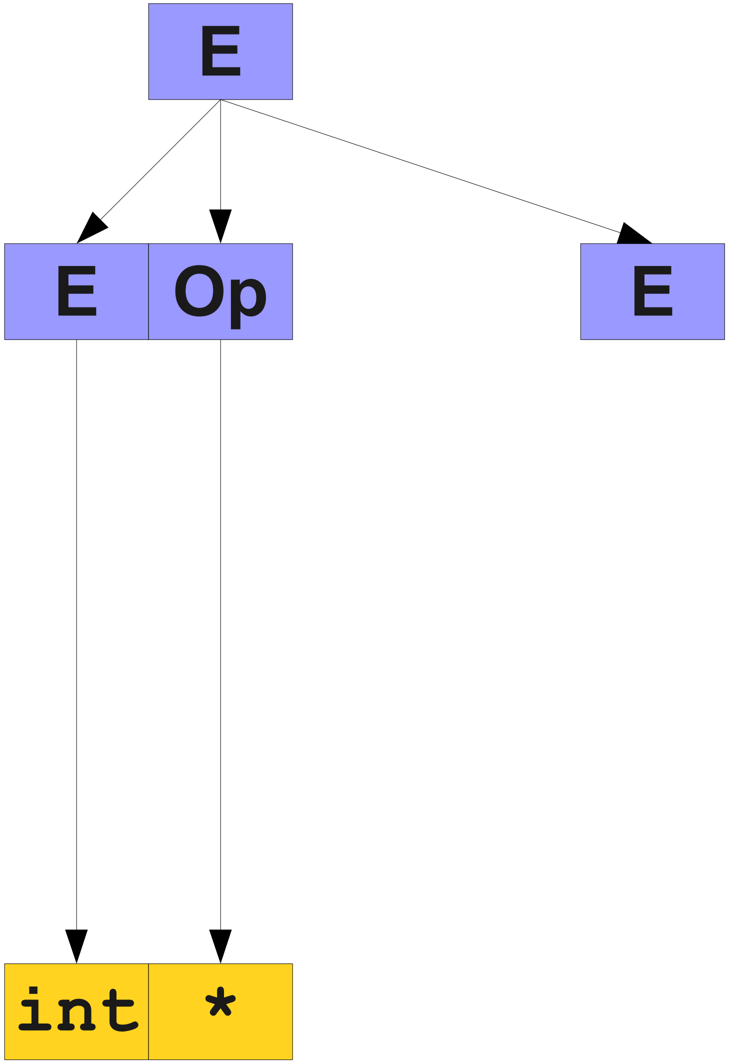
Parse Trees

E

⇒ **E Op E**

⇒ `int` **Op E**

⇒ `int * ` **E**



Parse Trees

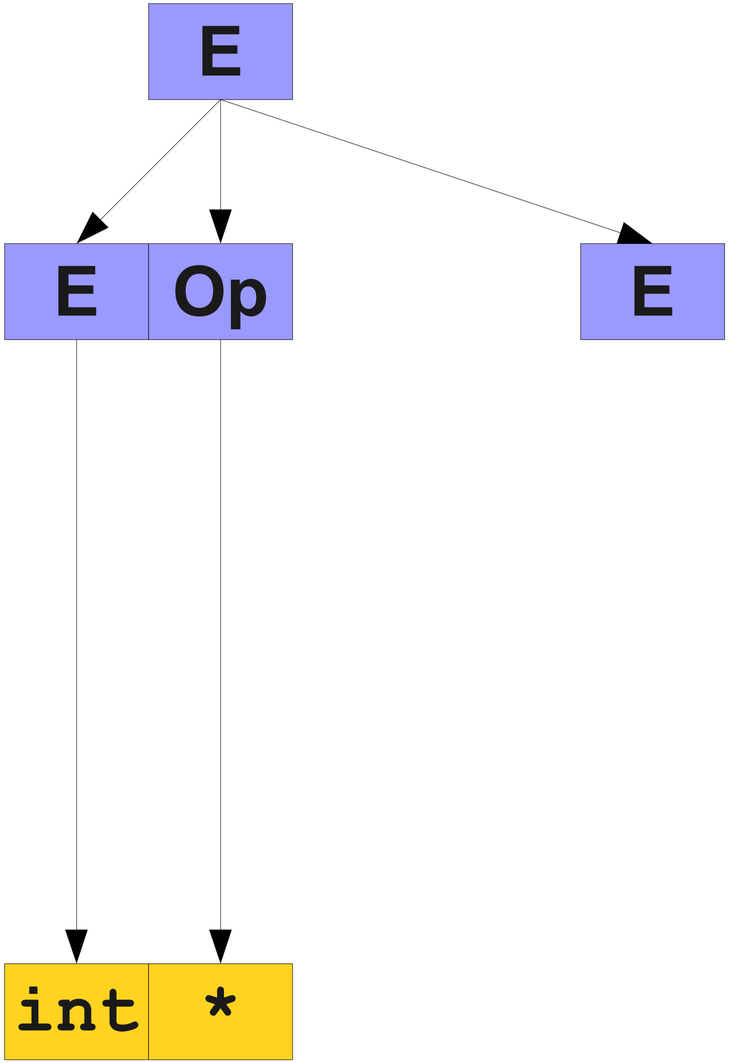
E

⇒ **E Op E**

⇒ `int` **Op E**

⇒ `int * ` **E**

⇒ `int * (`**E**`)`



Parse Trees

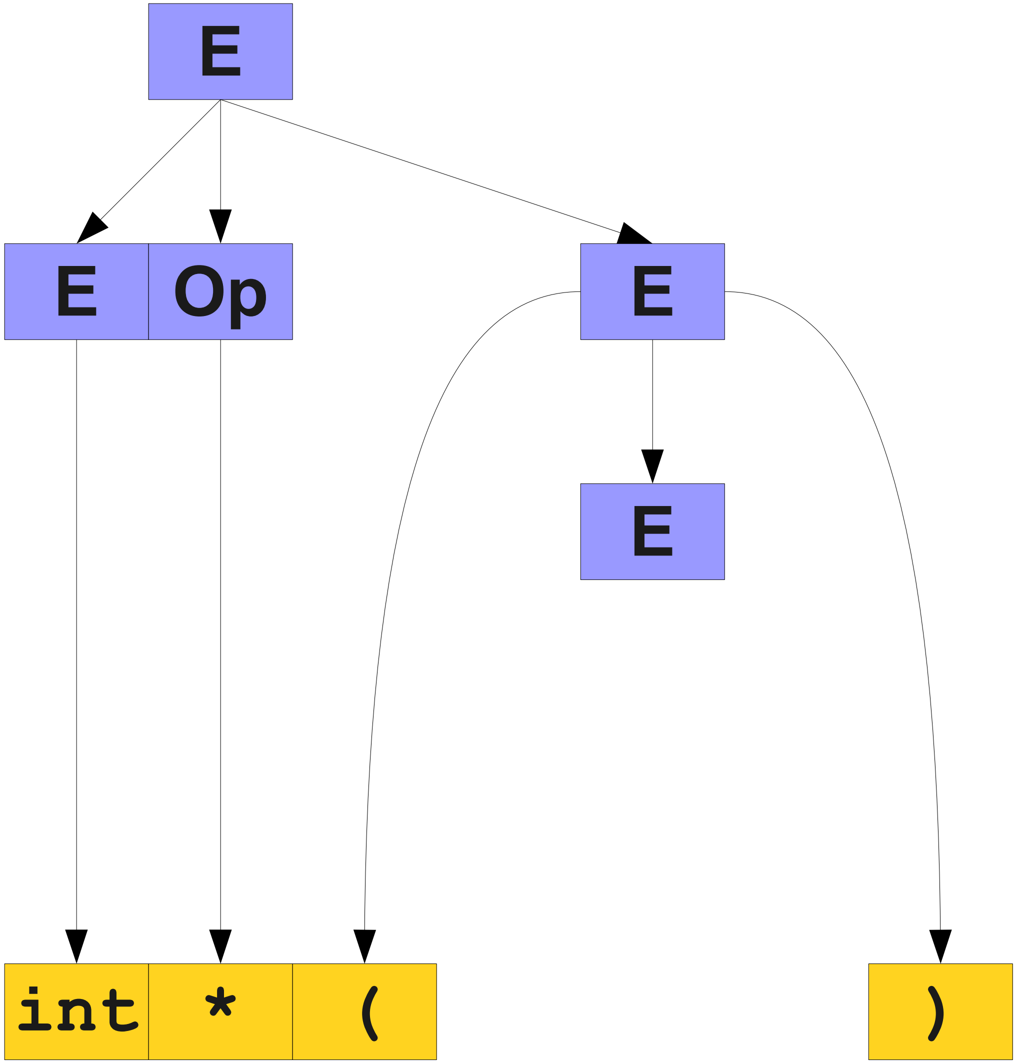
E

⇒ **E Op E**

⇒ `int` **Op E**

⇒ `int * `**E**

⇒ `int * (`**E**`)`



Parse Trees

E

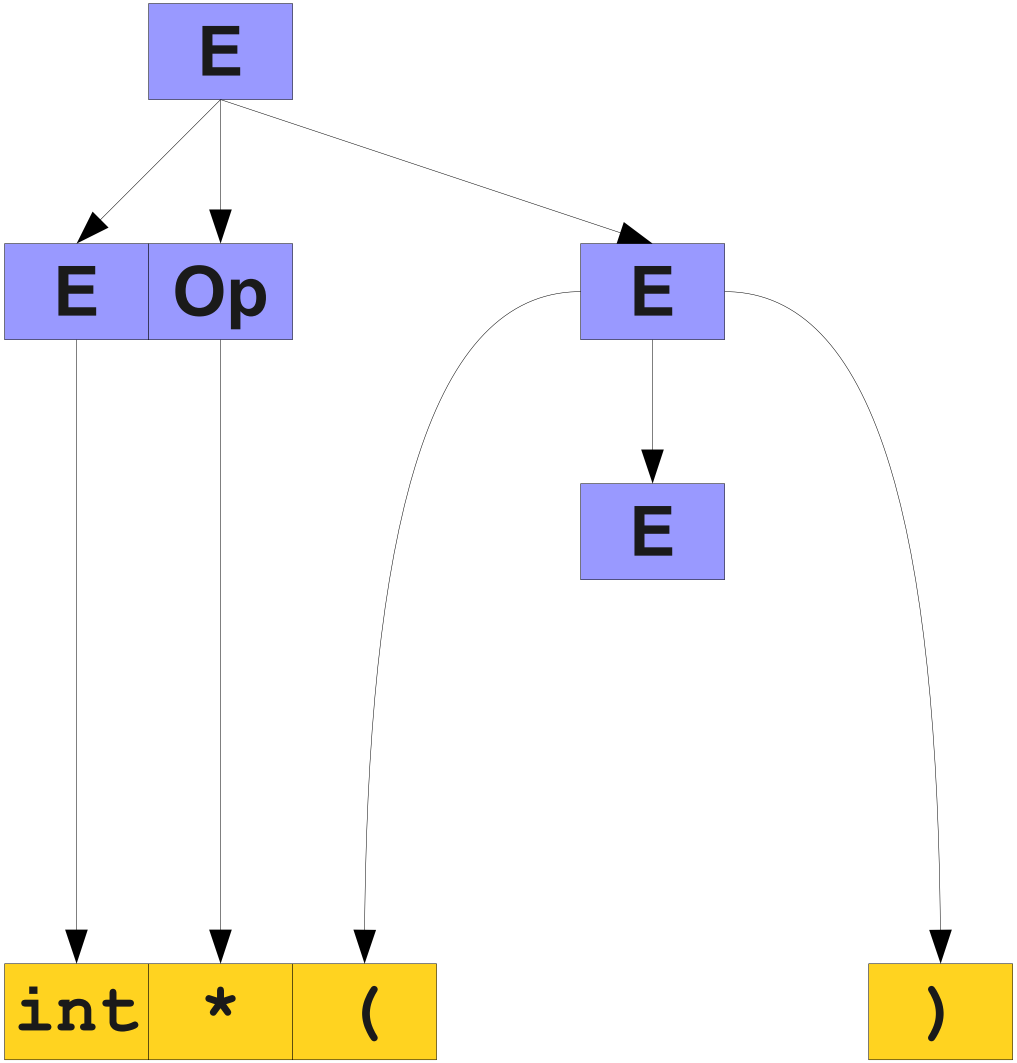
⇒ **E Op E**

⇒ `int` **Op E**

⇒ `int * `**E**

⇒ `int * (`**E**`)`

⇒ `int * (`**E Op E**`)`



Parse Trees

E

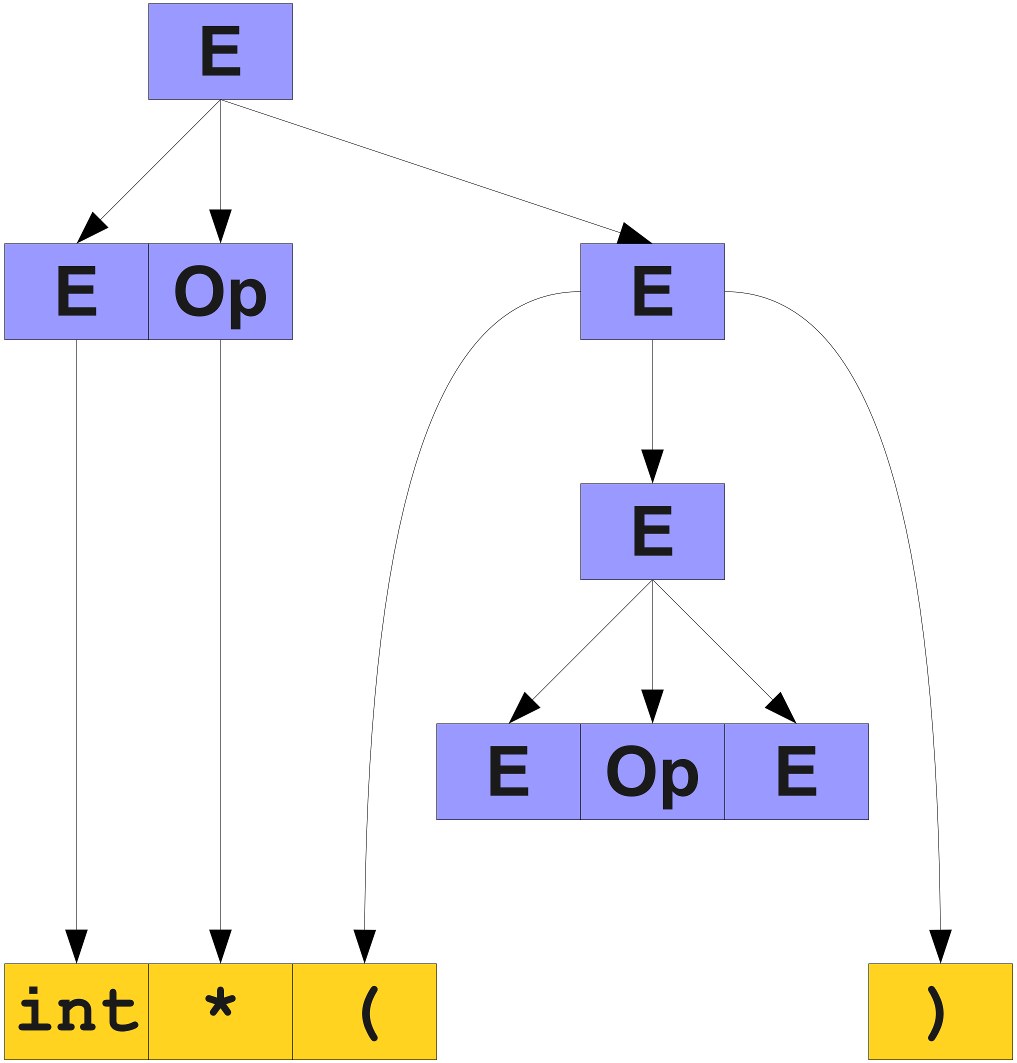
⇒ E Op E

⇒ int Op E

⇒ int * E

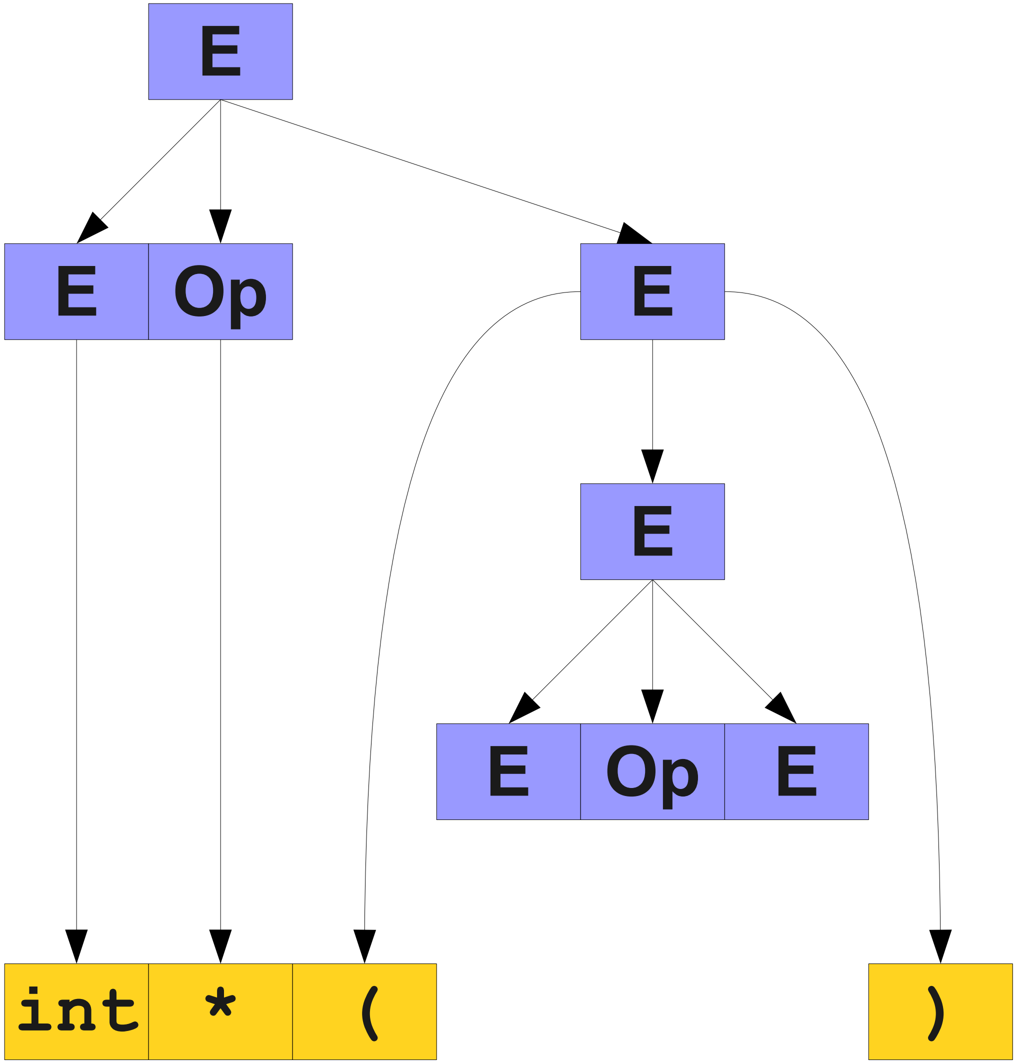
⇒ int * (E)

⇒ int * (E Op E)



Parse Trees

E
⇒ **E Op E**
⇒ int **Op E**
⇒ int * **E**
⇒ int * (**E**)
⇒ int * (**E Op E**)
⇒ int * (int **Op E**)



Parse Trees

E

⇒ E Op E

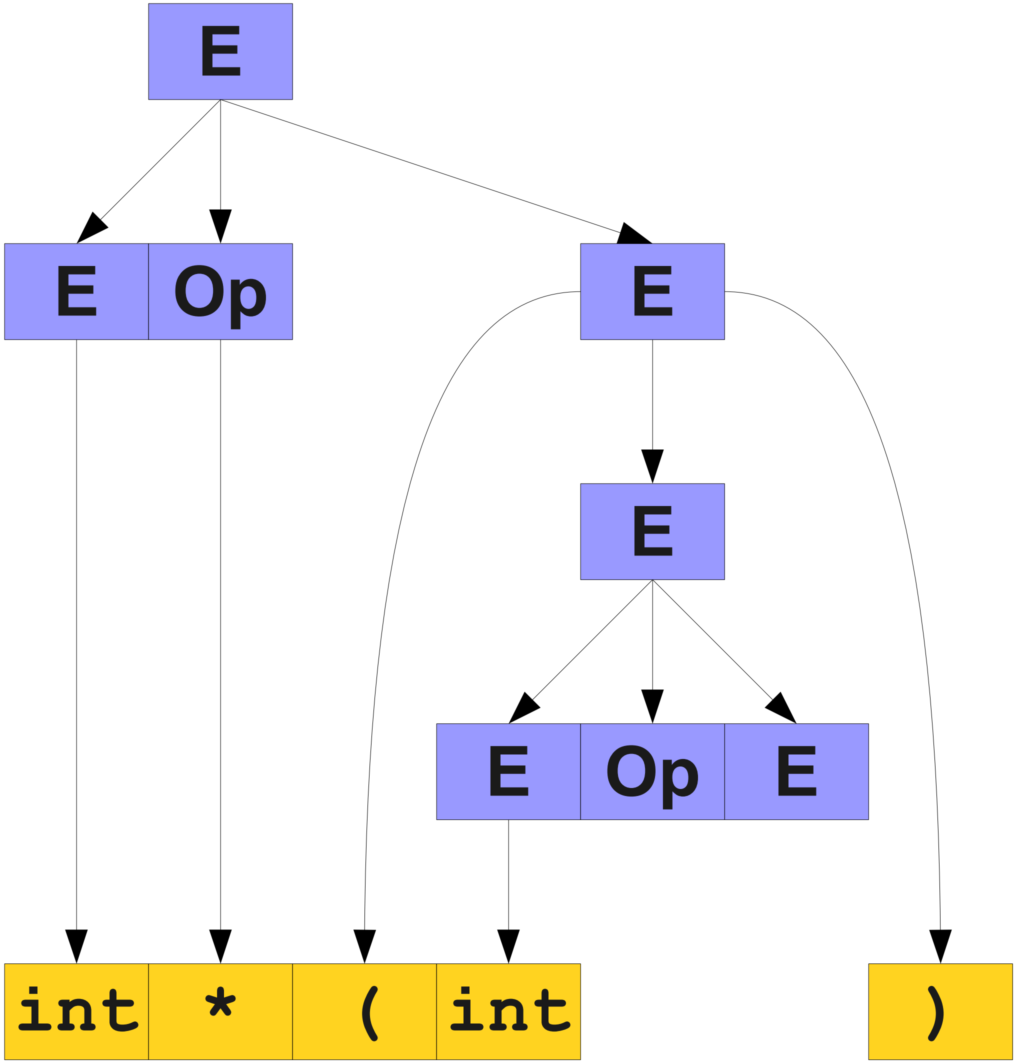
⇒ int Op E

⇒ int * E

⇒ int * (E)

⇒ int * (E Op E)

⇒ int * (int Op E)



Parse Trees

E

⇒ E Op E

⇒ int Op E

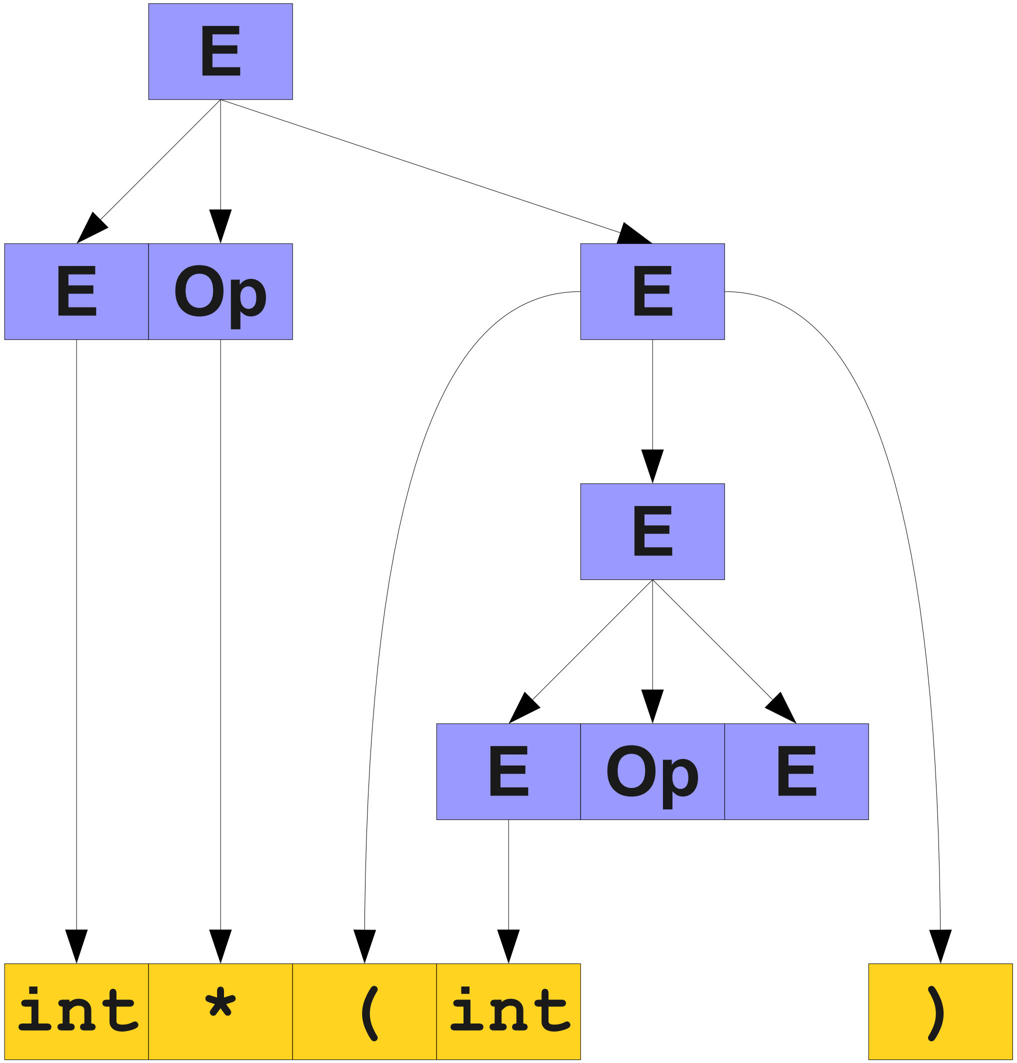
⇒ int * E

⇒ int * (E)

⇒ int * (E Op E)

⇒ int * (int Op E)

⇒ int * (int + E)



Parse Trees

E

⇒ **E Op E**

⇒ `int` **Op E**

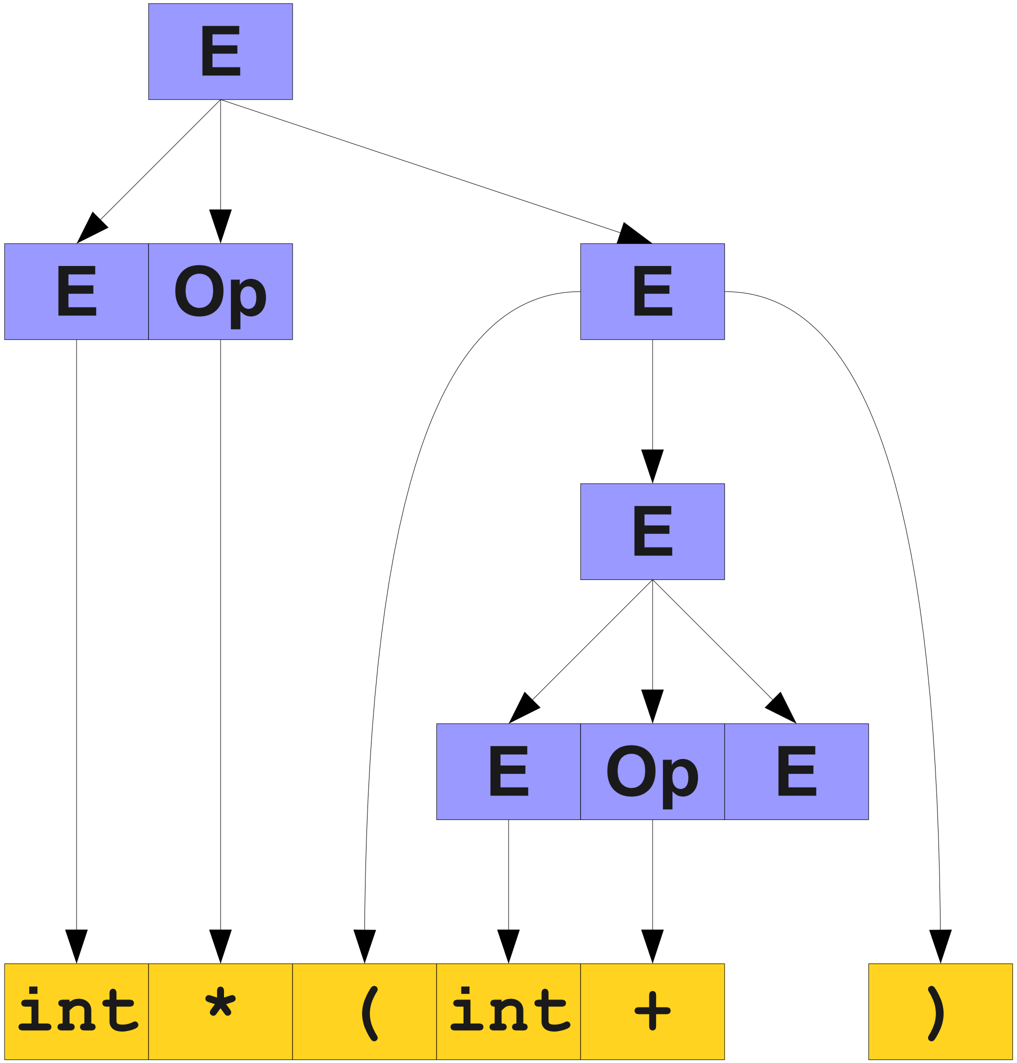
⇒ `int *` **E**

⇒ `int * (`**E**`)`

⇒ `int * (`**E Op E**`)`

⇒ `int * (int` **Op E**`)`

⇒ `int * (int +` **E**`)`



Parse Trees

E

⇒ **E Op E**

⇒ `int` **Op E**

⇒ `int * `**E**

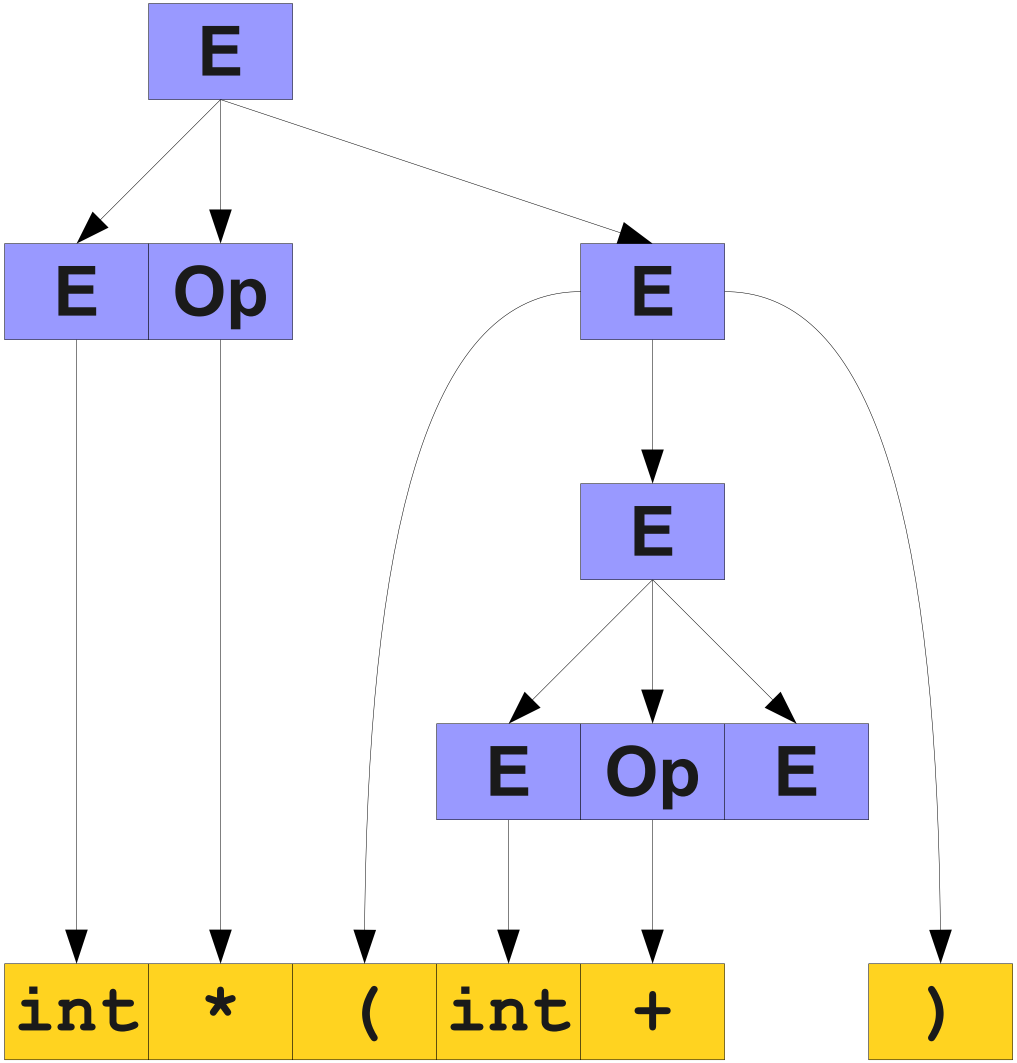
⇒ `int * (`**E**`)`

⇒ `int * (`**E Op E**`)`

⇒ `int * (int `**Op E**`)`

⇒ `int * (int + `**E**`)`

⇒ `int * (int + int)`



Parse Trees

E

⇒ E Op E

⇒ `int` Op E

⇒ `int * ` E

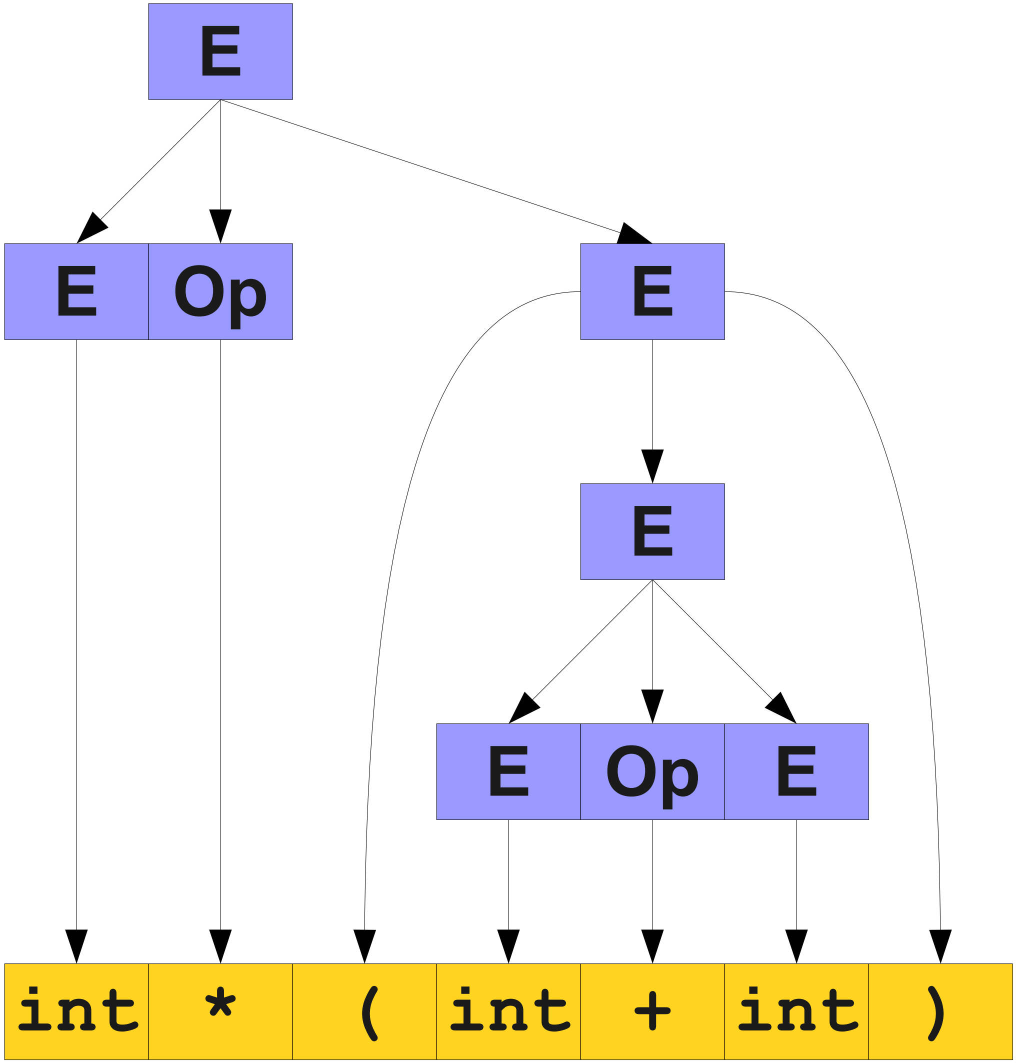
⇒ `int * (`E`)`

⇒ `int * (`E Op E`)`

⇒ `int * (int ` Op E`)`

⇒ `int * (int + `E`)`

⇒ `int * (int + int)`



Parse Trees

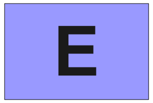
E

Parse Trees

E

E

Parse Trees



E

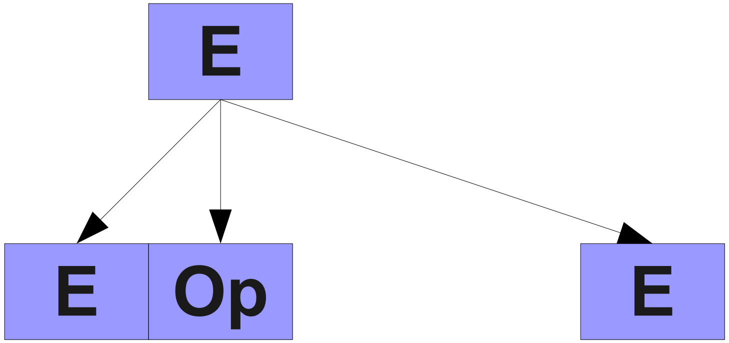
E

⇒ **E Op E**

Parse Trees

E

⇒ **E Op E**

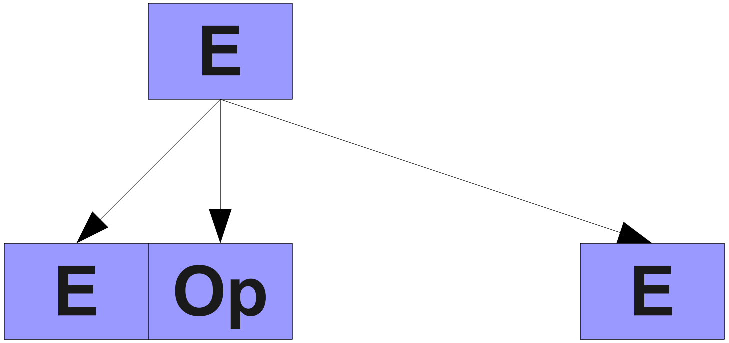


Parse Trees

E

⇒ **E Op E**

⇒ **E Op (E)**

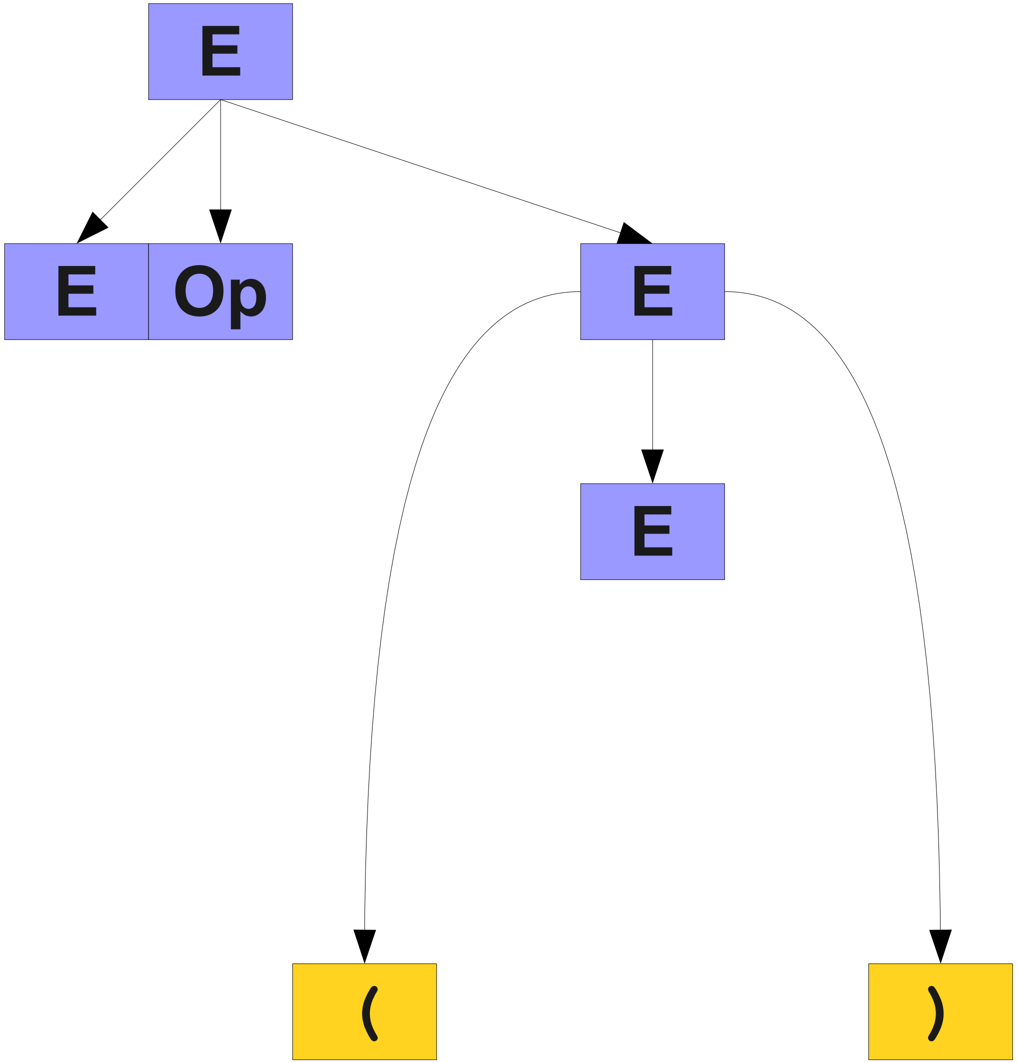


Parse Trees

E

⇒ E Op E

⇒ E Op (E)



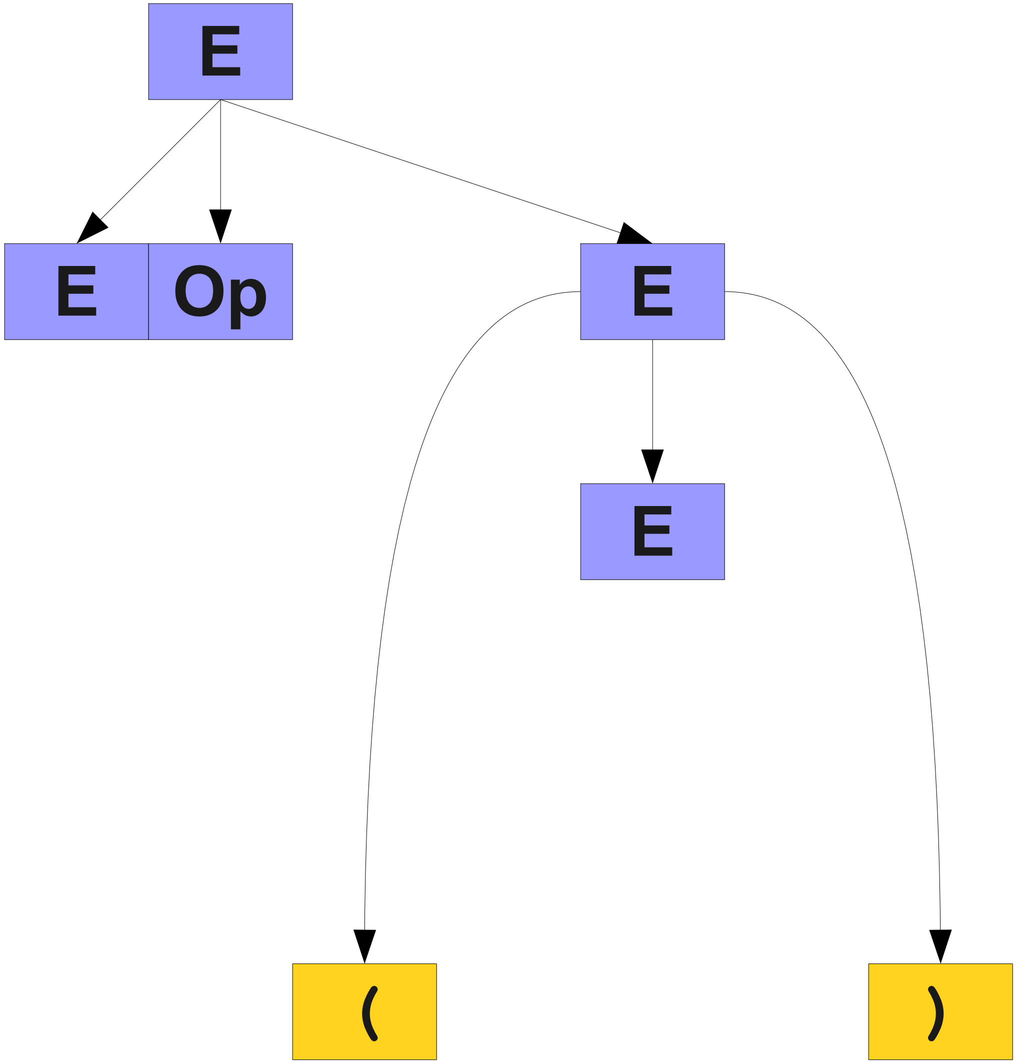
Parse Trees

E

⇒ E Op E

⇒ E Op (E)

⇒ E Op (E Op E)



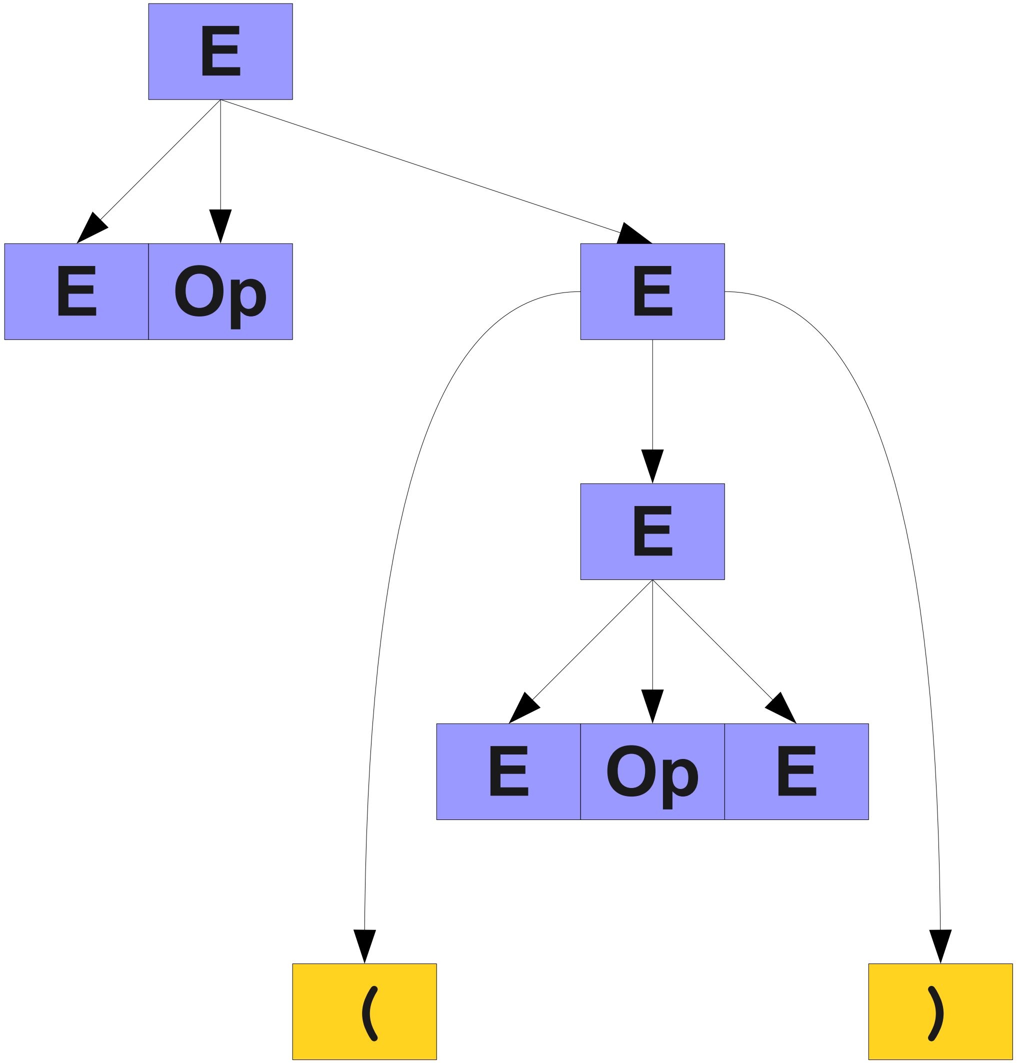
Parse Trees

E

⇒ **E Op E**

⇒ **E Op (E)**

⇒ **E Op (E Op E)**



Parse Trees

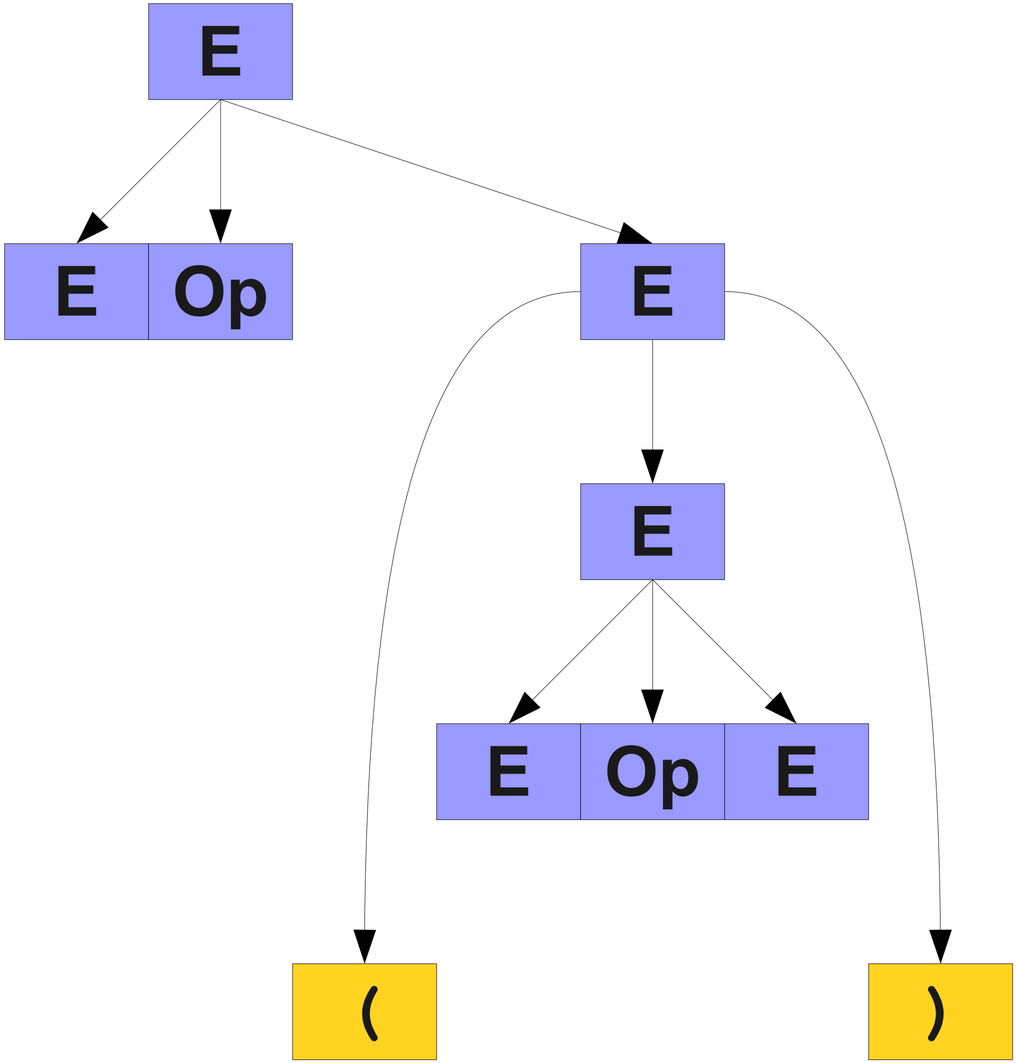
E

⇒ **E Op E**

⇒ **E Op** `(`**E**`)`

⇒ **E Op** `(`**E Op E**`)`

⇒ **E Op** `(`**E Op** `int)`



Parse Trees

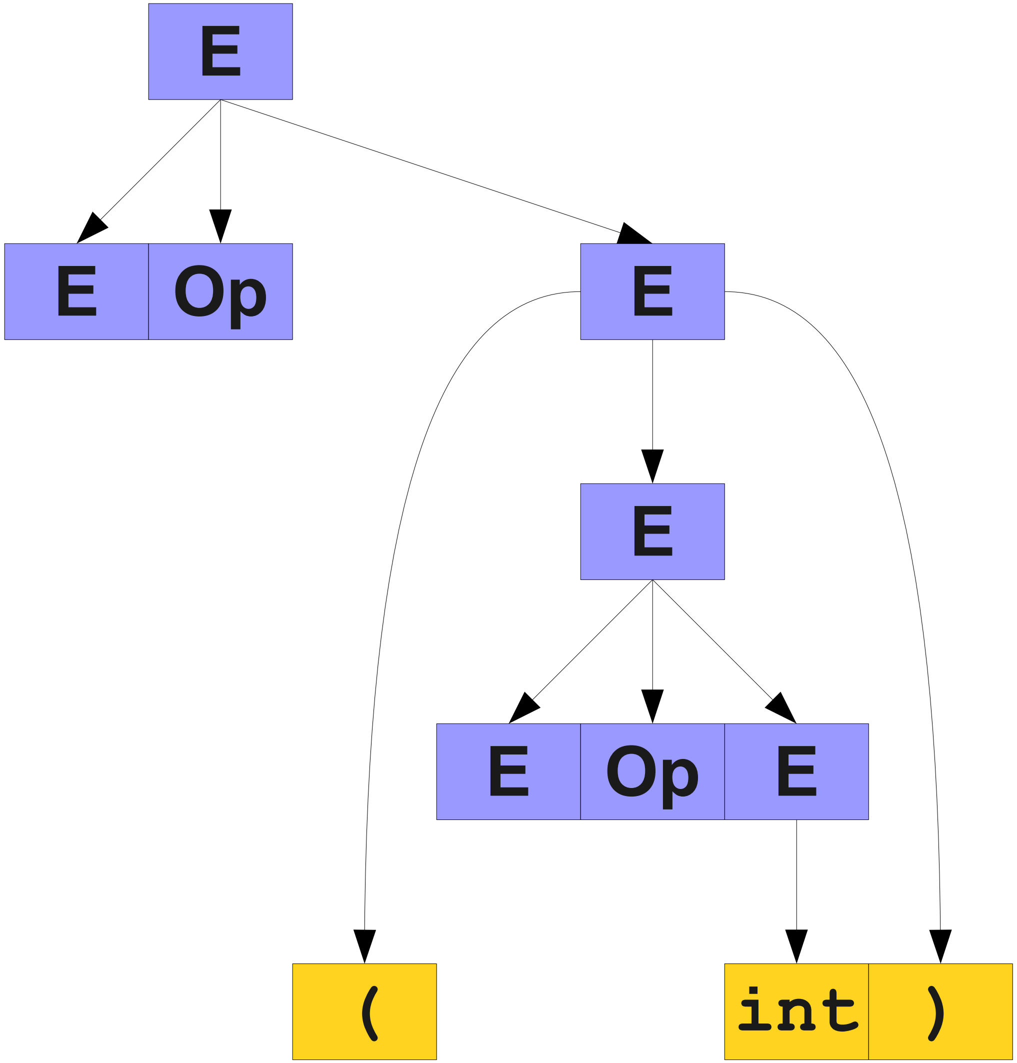
E

⇒ **E Op E**

⇒ **E Op** `(`**E**`)`

⇒ **E Op** `(`**E Op E**`)`

⇒ **E Op** `(`**E Op** `int)`



Parse Trees

E

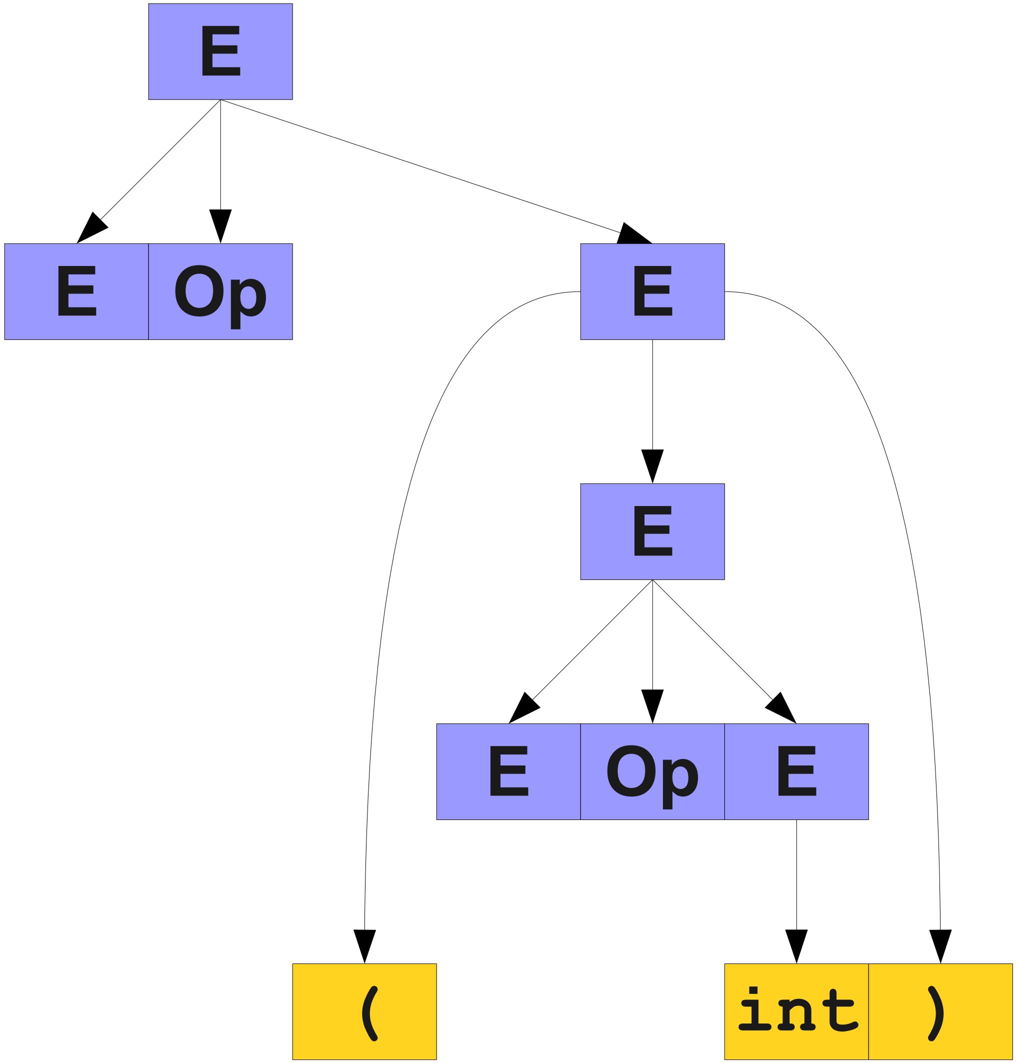
⇒ **E Op E**

⇒ **E Op** `(`**E**`)`

⇒ **E Op** `(`**E Op E**`)`

⇒ **E Op** `(`**E Op** `int)`

⇒ **E Op** `(`**E** `+ int)`



Parse Trees

E

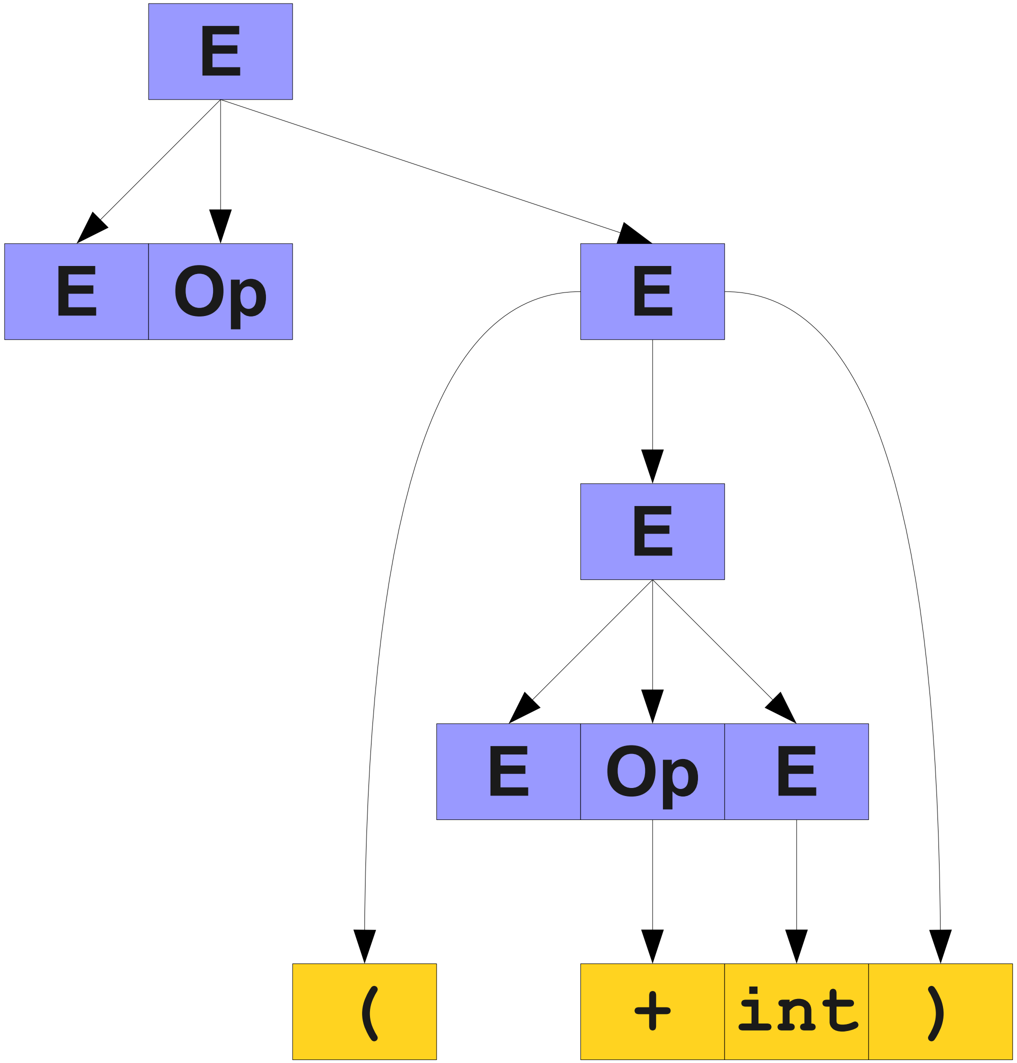
⇒ **E Op E**

⇒ **E Op** `(`**E**`)`

⇒ **E Op** `(`**E Op E**`)`

⇒ **E Op** `(`**E Op** `int)`

⇒ **E Op** `(`**E** `+ int)`



Parse Trees

E

⇒ **E Op E**

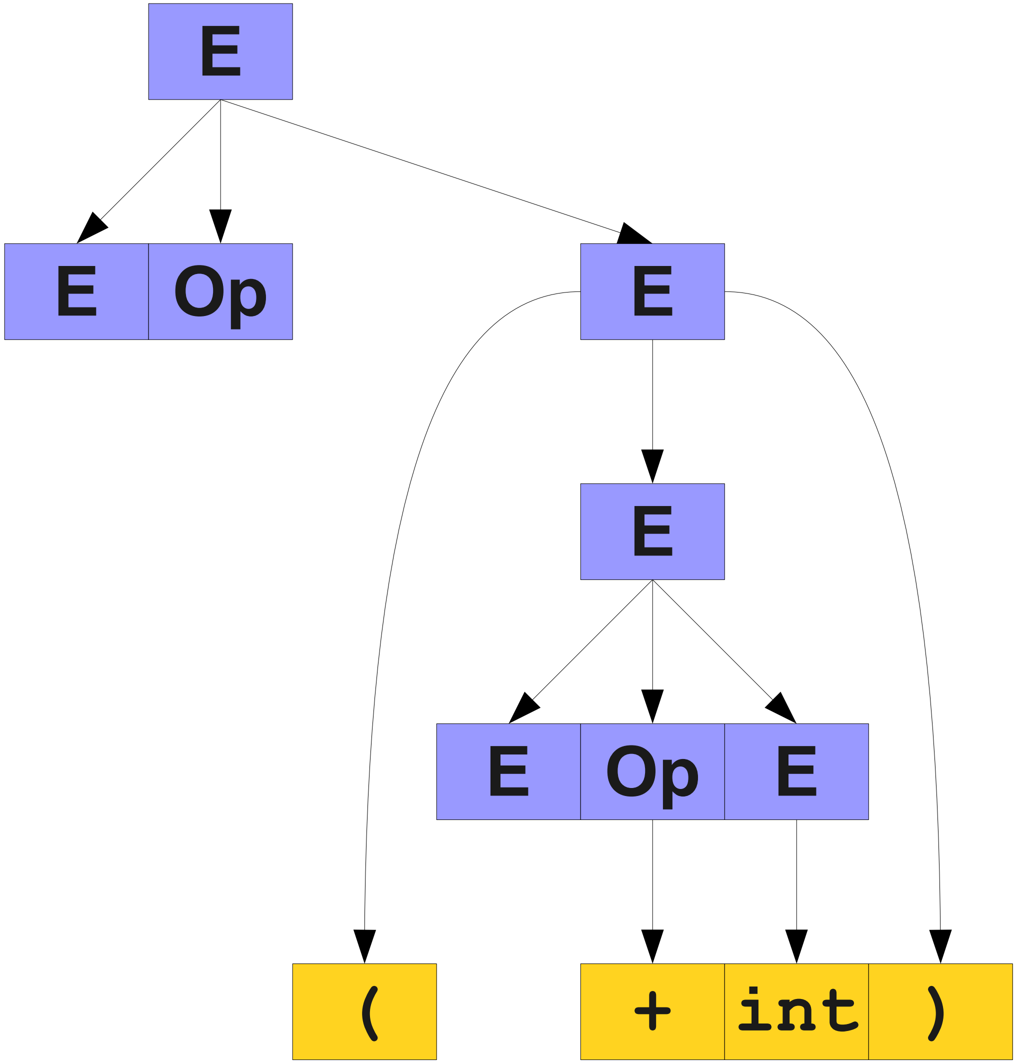
⇒ **E Op** `(`**E**`)`

⇒ **E Op** `(`**E Op E**`)`

⇒ **E Op** `(`**E Op** `int)`

⇒ **E Op** `(`**E** `+ int)`

⇒ **E Op** `(int + int)`



Parse Trees

E

⇒ **E Op E**

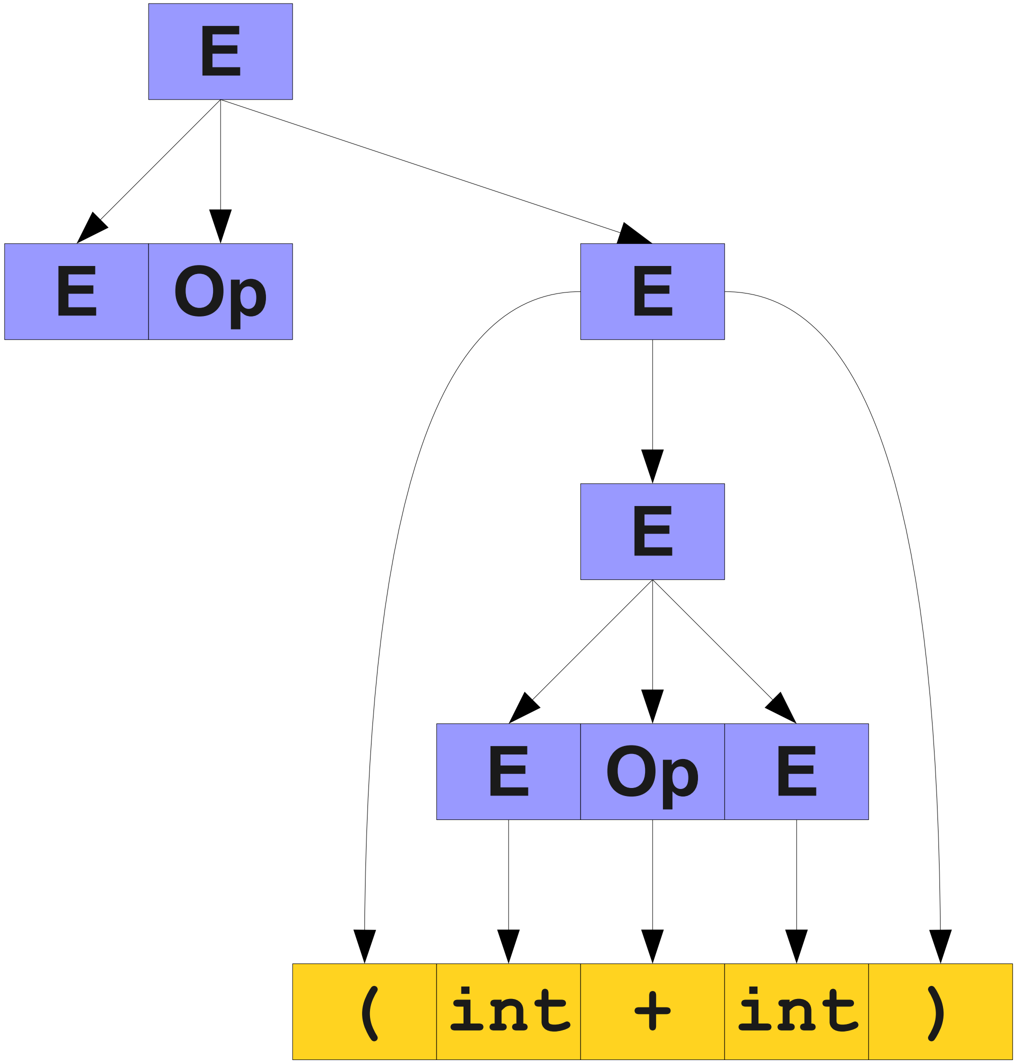
⇒ **E Op** `(`**E**`)`

⇒ **E Op** `(`**E Op E**`)`

⇒ **E Op** `(`**E Op** `int)`

⇒ **E Op** `(`**E** `+ int)`

⇒ **E Op** `(int + int)`



Parse Trees

E

⇒ **E Op E**

⇒ **E Op** `(`**E**`)`

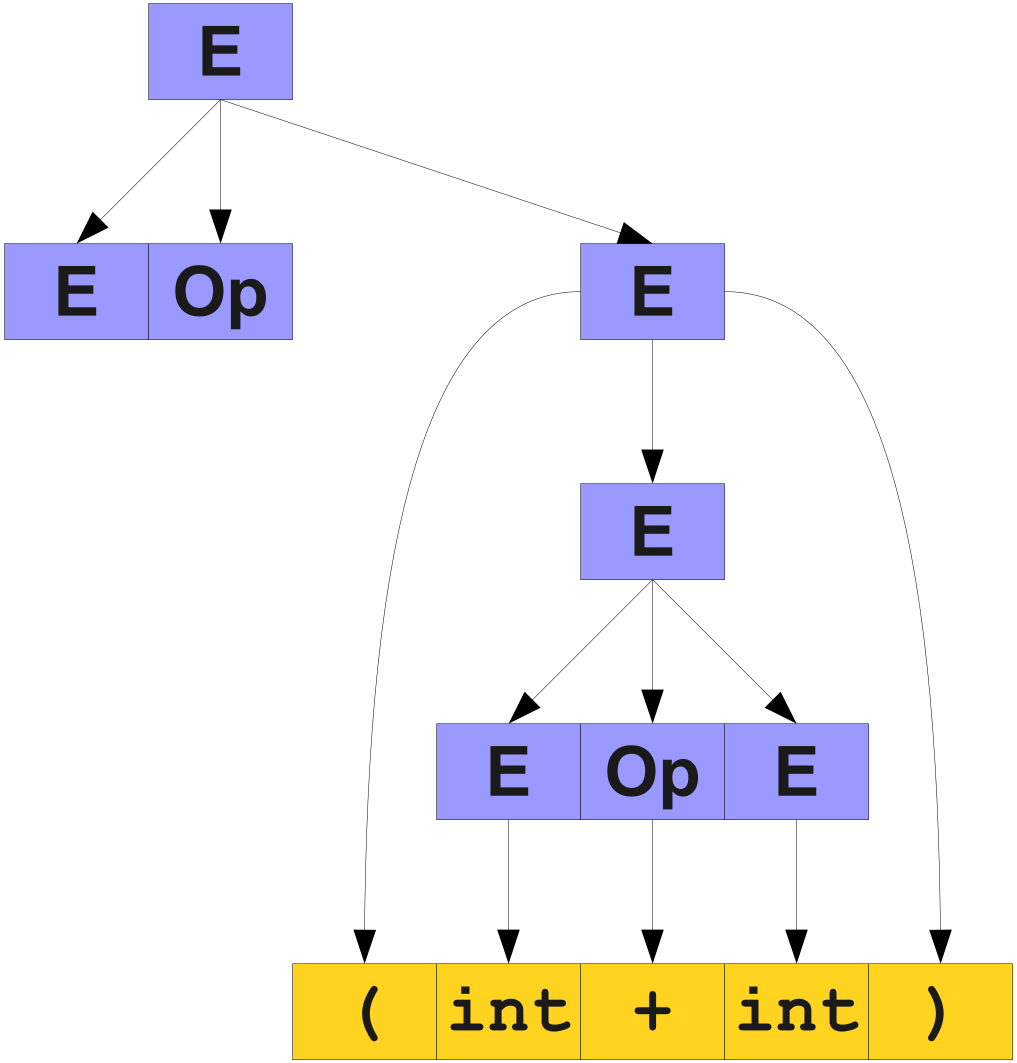
⇒ **E Op** `(`**E Op E**`)`

⇒ **E Op** `(`**E Op** `int)`

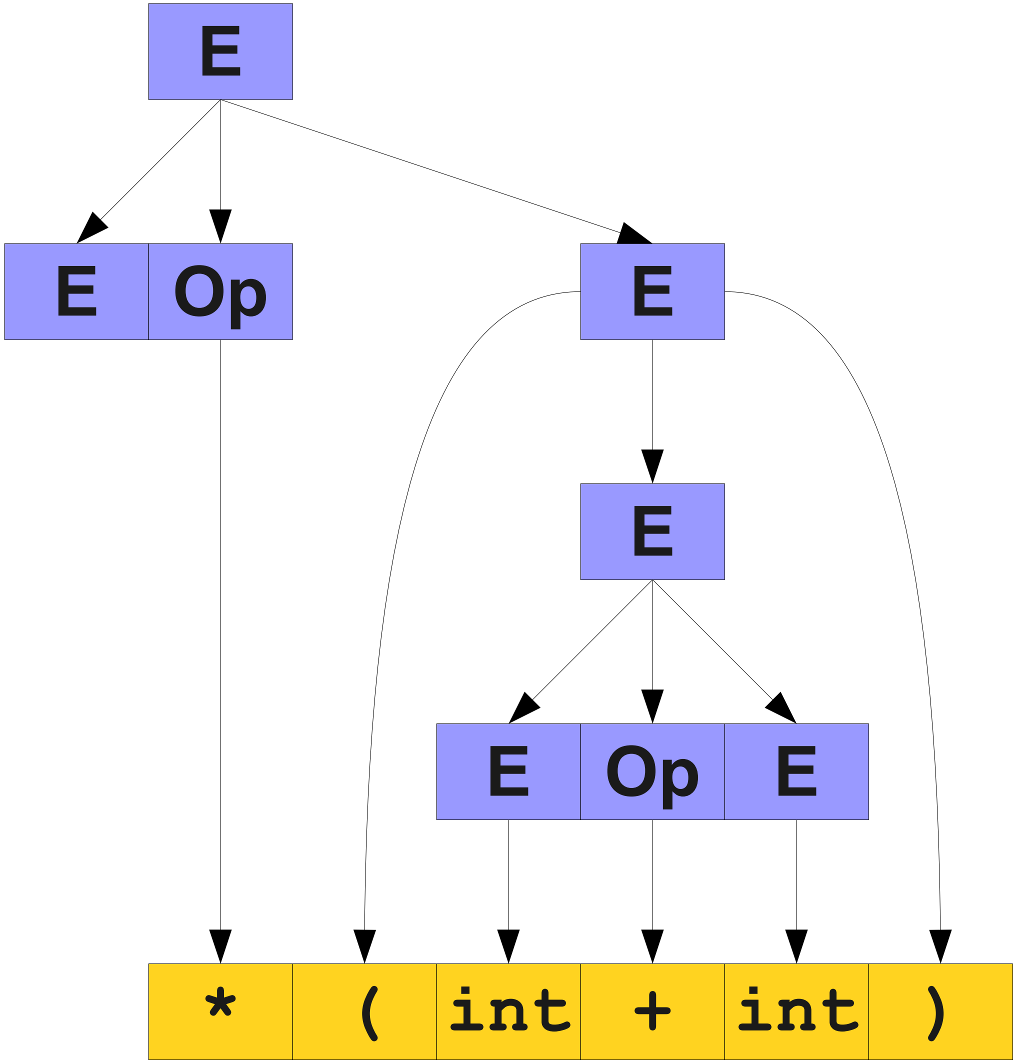
⇒ **E Op** `(`**E** `+ int)`

⇒ **E Op** `(int + int)`

⇒ **E** `* (int + int)`



Parse Trees



E
⇒ E Op E
⇒ E Op (E)
⇒ E Op (E Op E)
⇒ E Op (E Op int)
⇒ E Op (E + int)
⇒ E Op (int + int)
⇒ E * (int + int)

Parse Trees

E

⇒ E Op E

⇒ E Op (E)

⇒ E Op (E Op E)

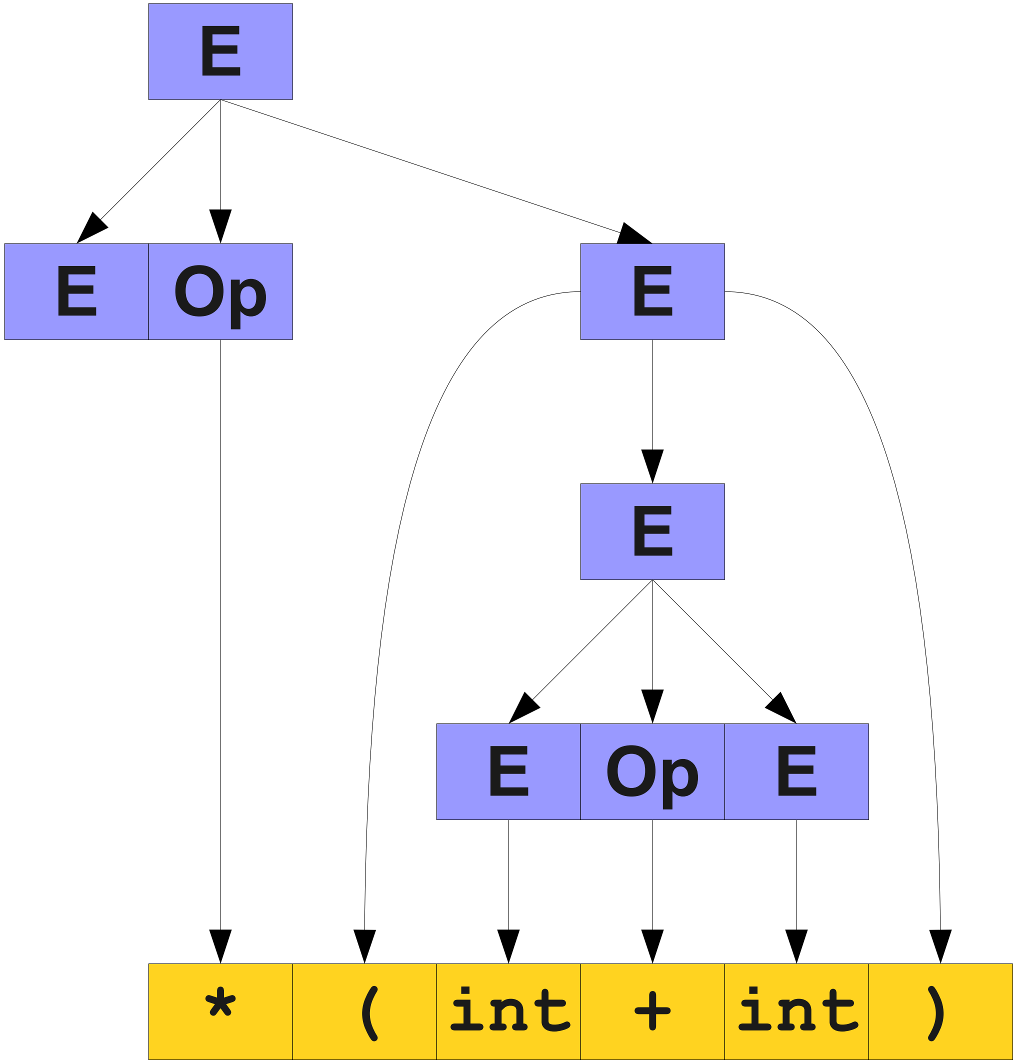
⇒ E Op (E Op int)

⇒ E Op (E + int)

⇒ E Op (int + int)

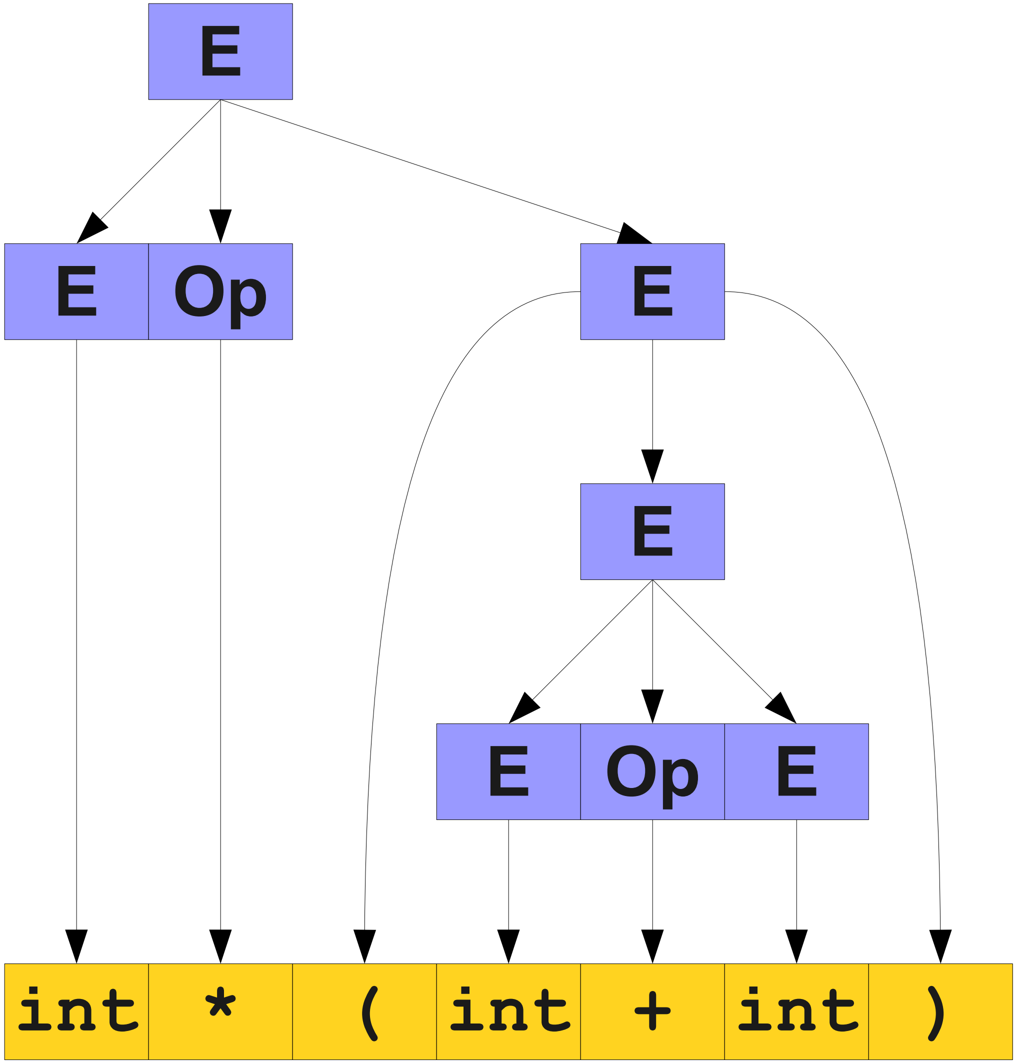
⇒ E * (int + int)

⇒ int * (int + int)

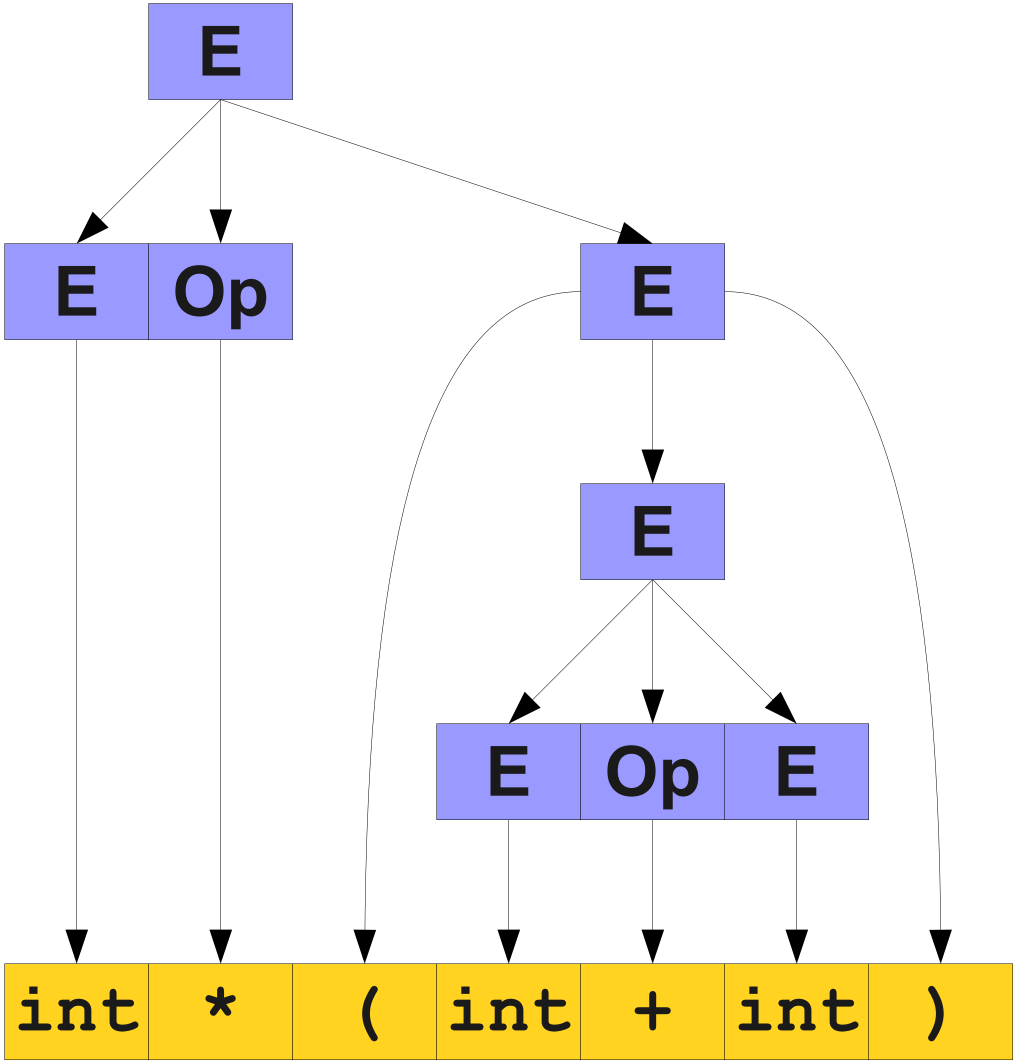


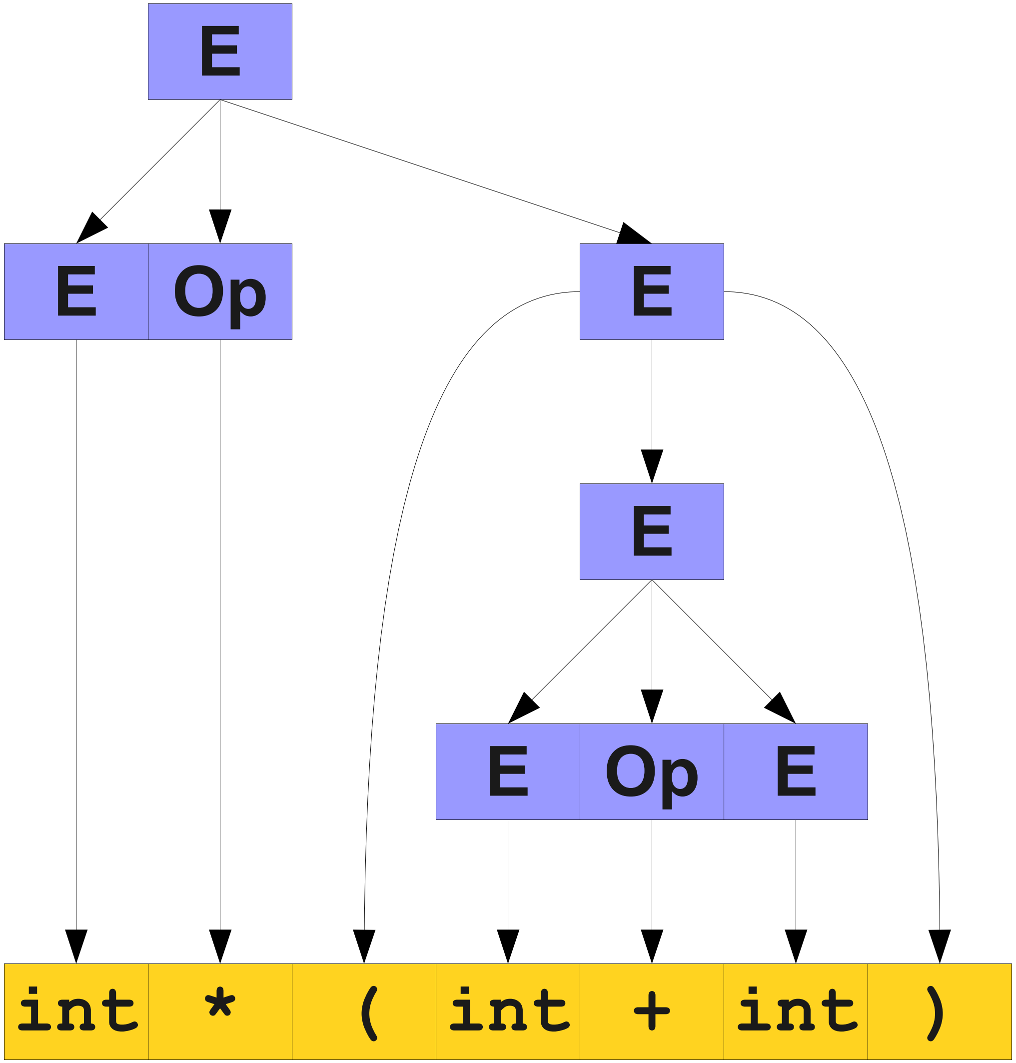
Parse Trees

E
⇒ E Op E
⇒ E Op (E)
⇒ E Op (E Op E)
⇒ E Op (E Op int)
⇒ E Op (E + int)
⇒ E Op (int + int)
⇒ E * (int + int)
⇒ int * (int + int)



For Comparison

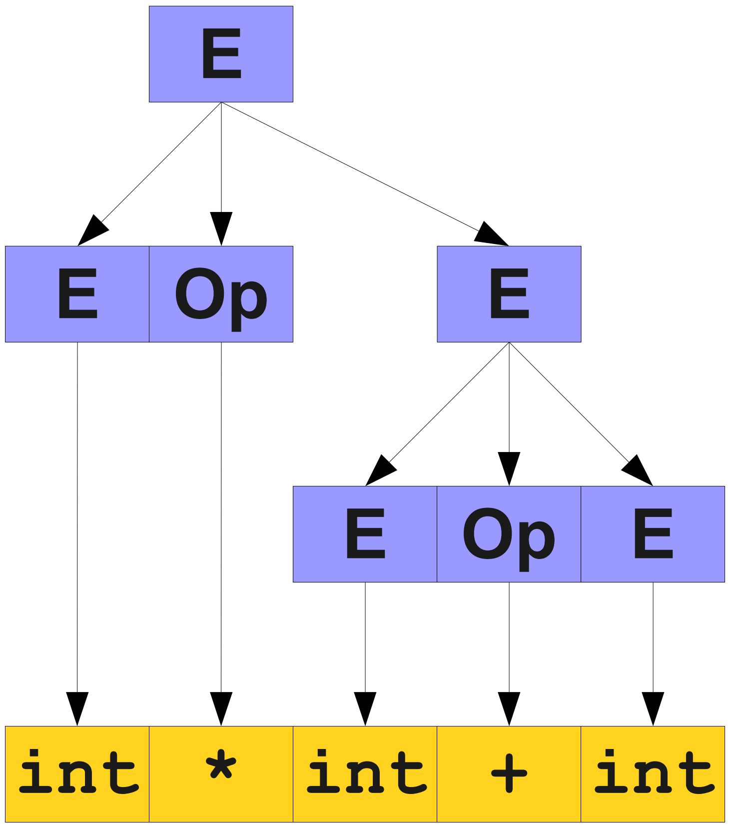




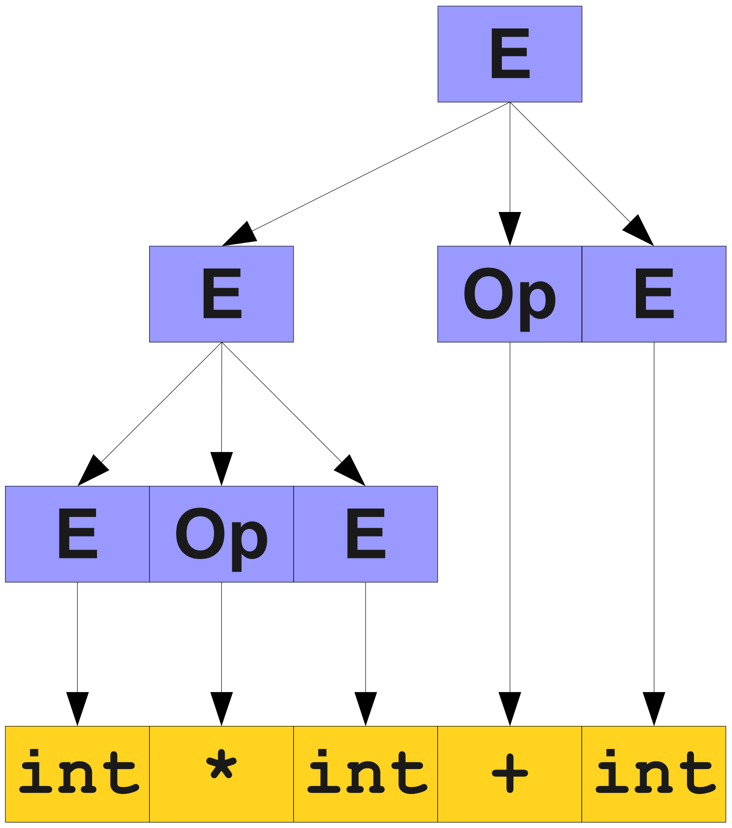
Parse Trees

- A **parse tree** is a tree encoding the steps in a derivation.
- Internal nodes represent nonterminal symbols used in the production.
- Walking the leaves in order gives the generated string.
- Encodes what productions are used, not the order in which those productions are applied.

A Serious Problem



`int * (int + int)`



`(int * int) + int`

Ambiguity

- A CFG is said to be **ambiguous** if there is at least one string with two or more parse trees.
- There is no algorithm for converting an arbitrary ambiguous grammar into an unambiguous one.
  - Some languages are **inherently ambiguous**, meaning that no unambiguous grammar exists for them.
- There is no algorithm for detecting whether an arbitrary grammar is ambiguous.

Is Ambiguity a Problem?

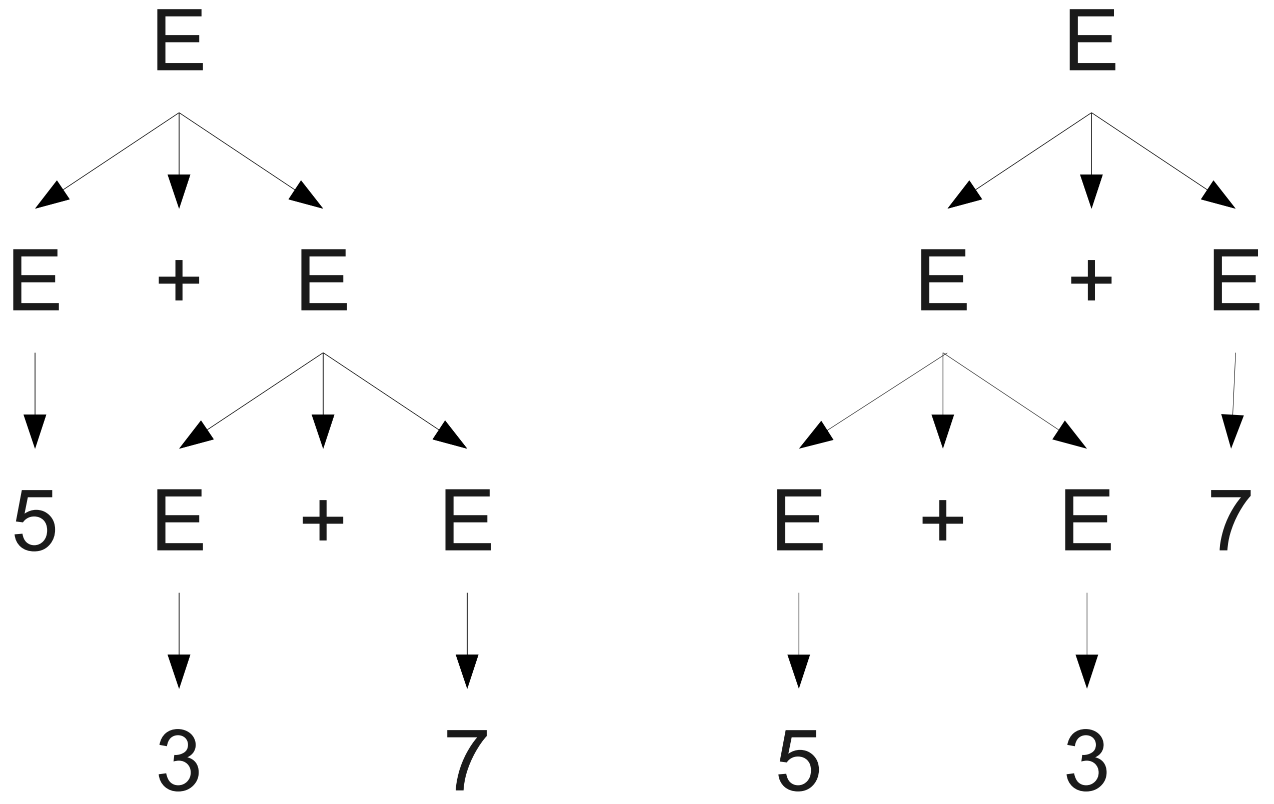
- Depends on **semantics**.

$$\mathbf{E} \rightarrow \texttt{int} \mid \mathbf{E} \mathbin{\texttt{+}} \mathbf{E}$$

Is Ambiguity a Problem?

- Depends on **semantics**.

E → **`int`** | **E** **+** **E**



Is Ambiguity a Problem?

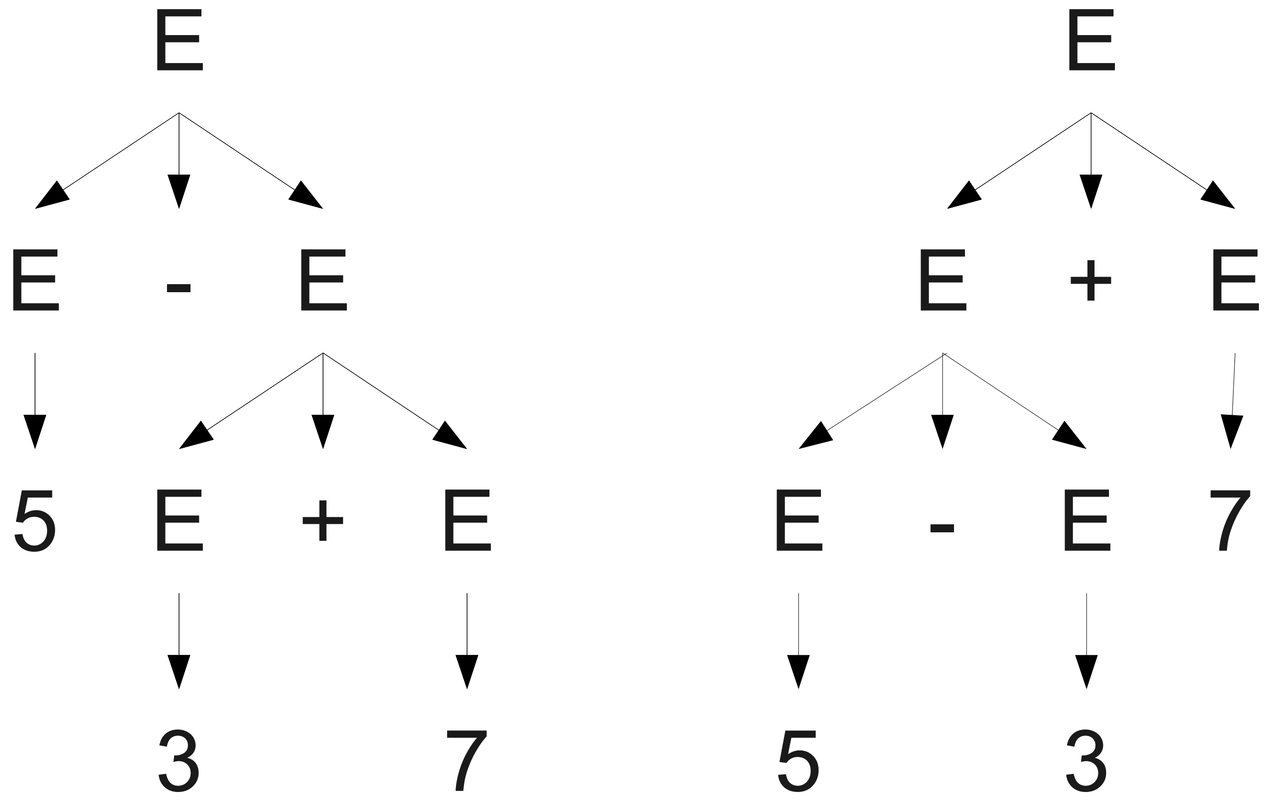
- Depends on **semantics**.

$$\mathbf{E} \rightarrow \texttt{int} \mid \mathbf{E} \texttt{ + } \mathbf{E} \mid \mathbf{E} \texttt{ - } \mathbf{E}$$

Is Ambiguity a Problem?

- Depends on **semantics**.

E → **`int`** | **E + E** | **E - E**



Resolving Ambiguity

- If a grammar can be made unambiguous at all, it is usually made unambiguous through **layering**.
- Have exactly one way to build each piece of the string.
- Have exactly one way of combining those pieces back together.

Example: Balanced Parentheses

- Consider the language of all strings of balanced parentheses.
- Examples:
  - ε
  - `()`
  - `(()())`
  - `((()))(())()`
- Here is one possible grammar for balanced parentheses:

P → **ε** | **PP** | **(P)**

Balanced Parentheses

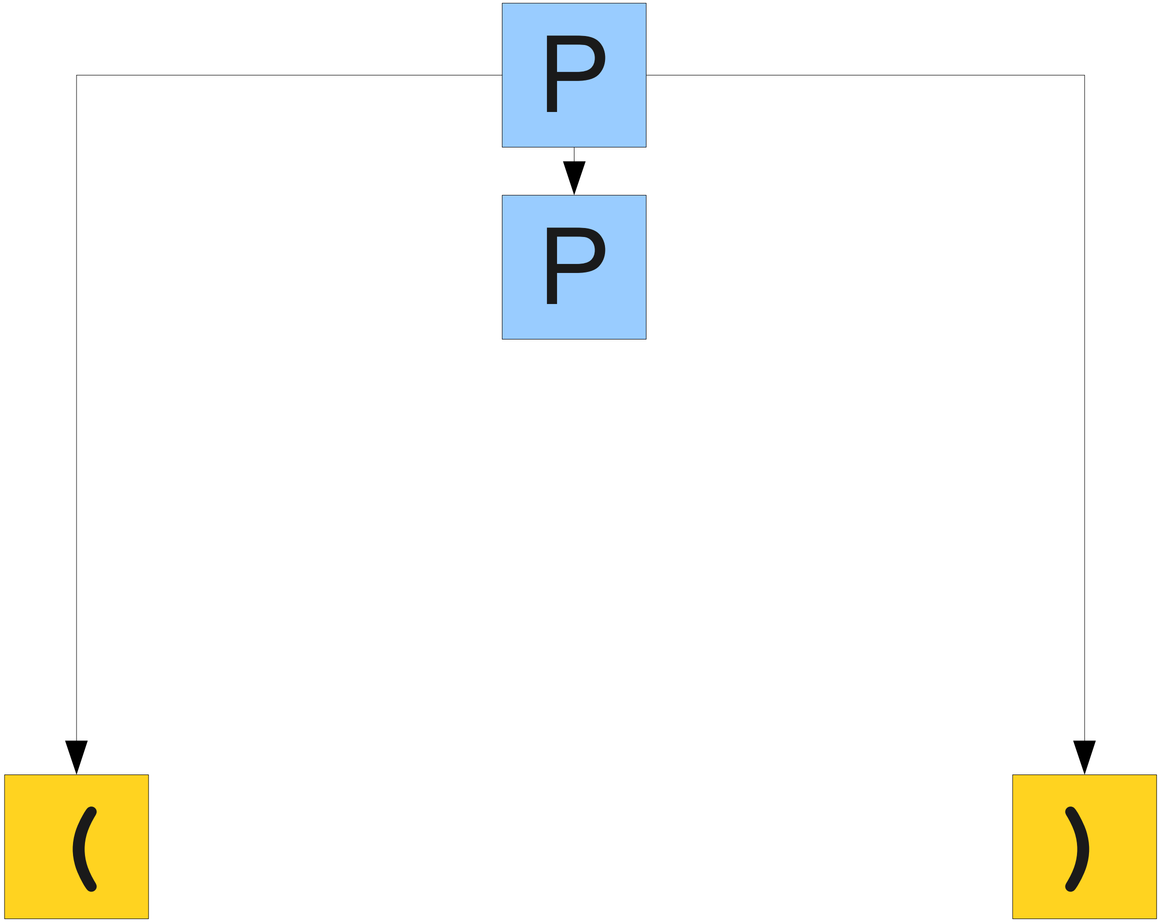
- Given the grammar **P** → **ε** | **PP** | **`(`P`)`**
- How might we generate the string `(()())`?

Balanced Parentheses

- Given the grammar **P** → **ε** | **PP** | **(P)**
- How might we generate the string `(()())`?

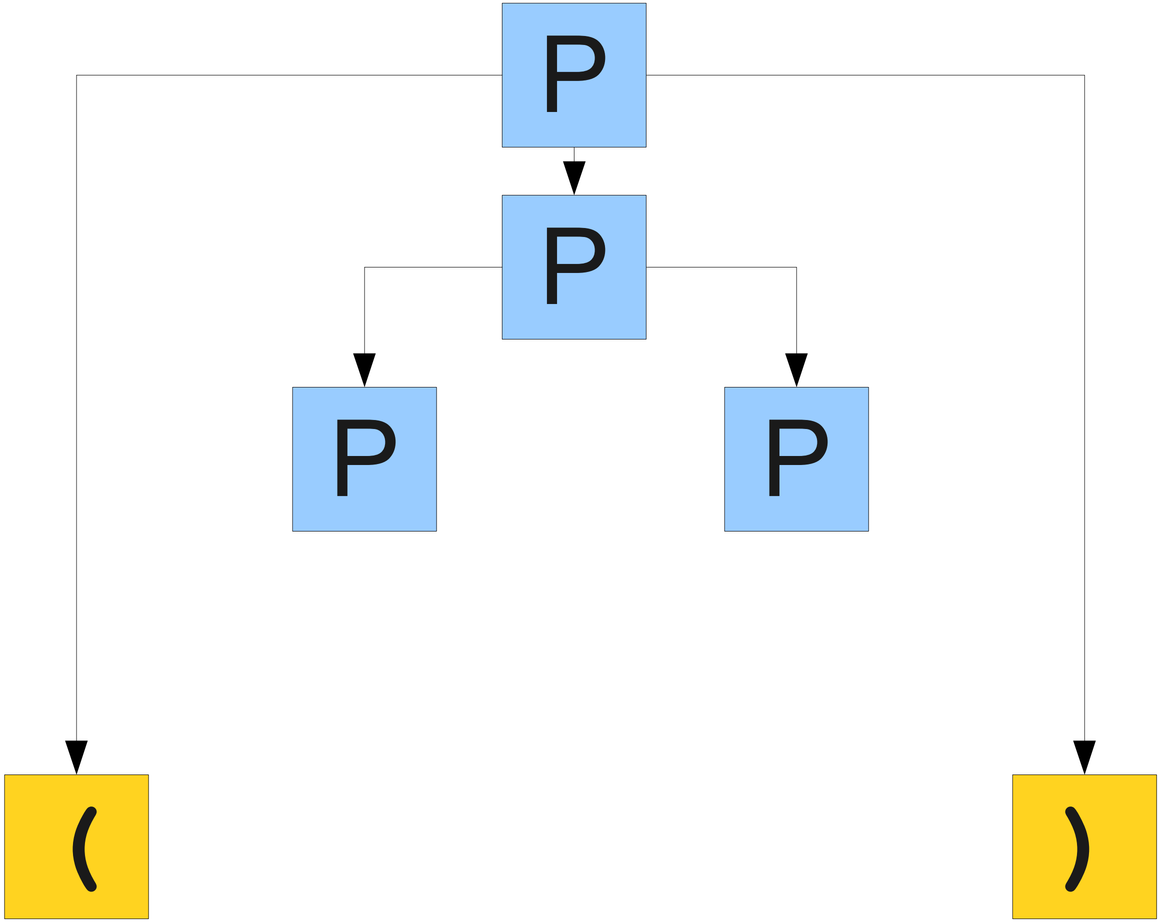
Balanced Parentheses

- Given the grammar **P** → **ε** | **PP** | **(P)**
- How might we generate the string `(()())`?



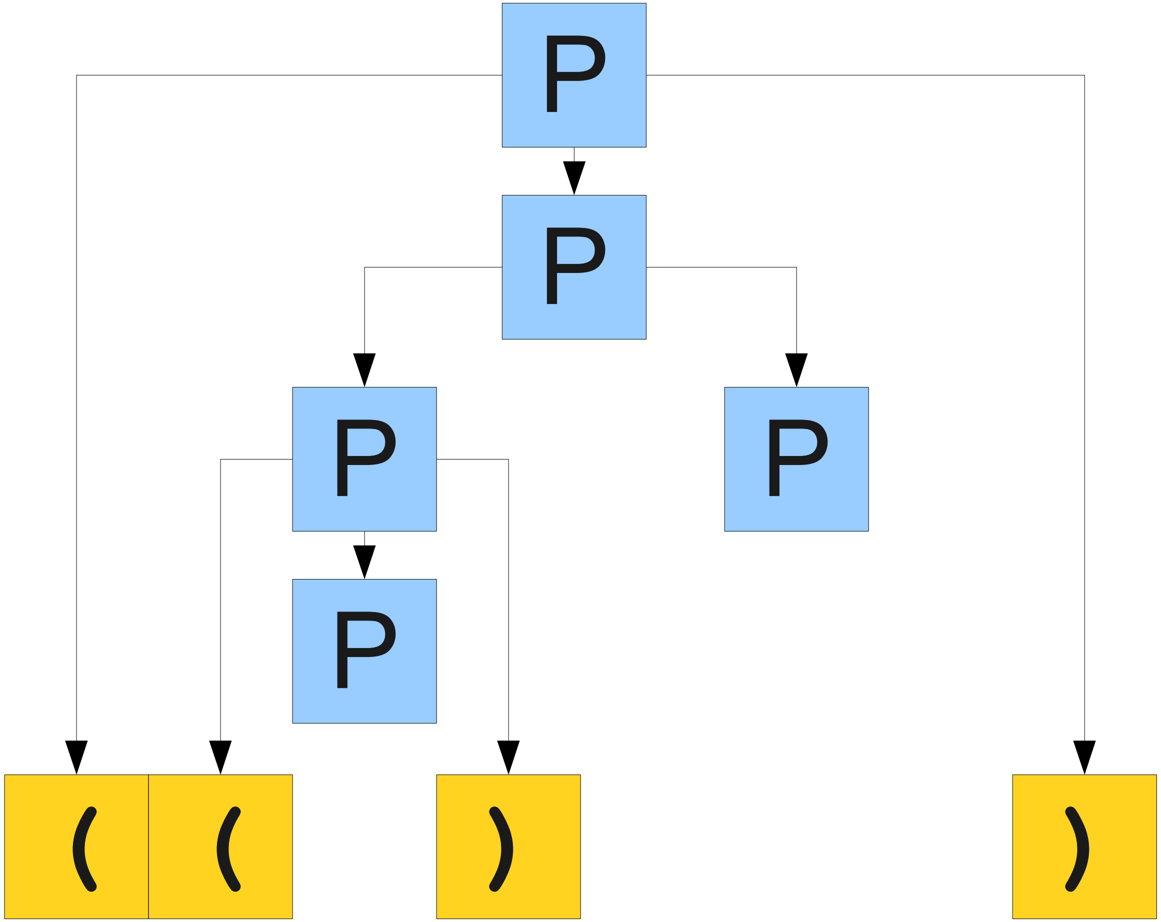
Balanced Parentheses

- Given the grammar **P** → **ε** | **PP** | **(P)**
- How might we generate the string `(()())`?



Balanced Parentheses

- Given the grammar **P** → **ε** | **PP** | **(P)**
- How might we generate the string `(()())`?



Balanced Parentheses

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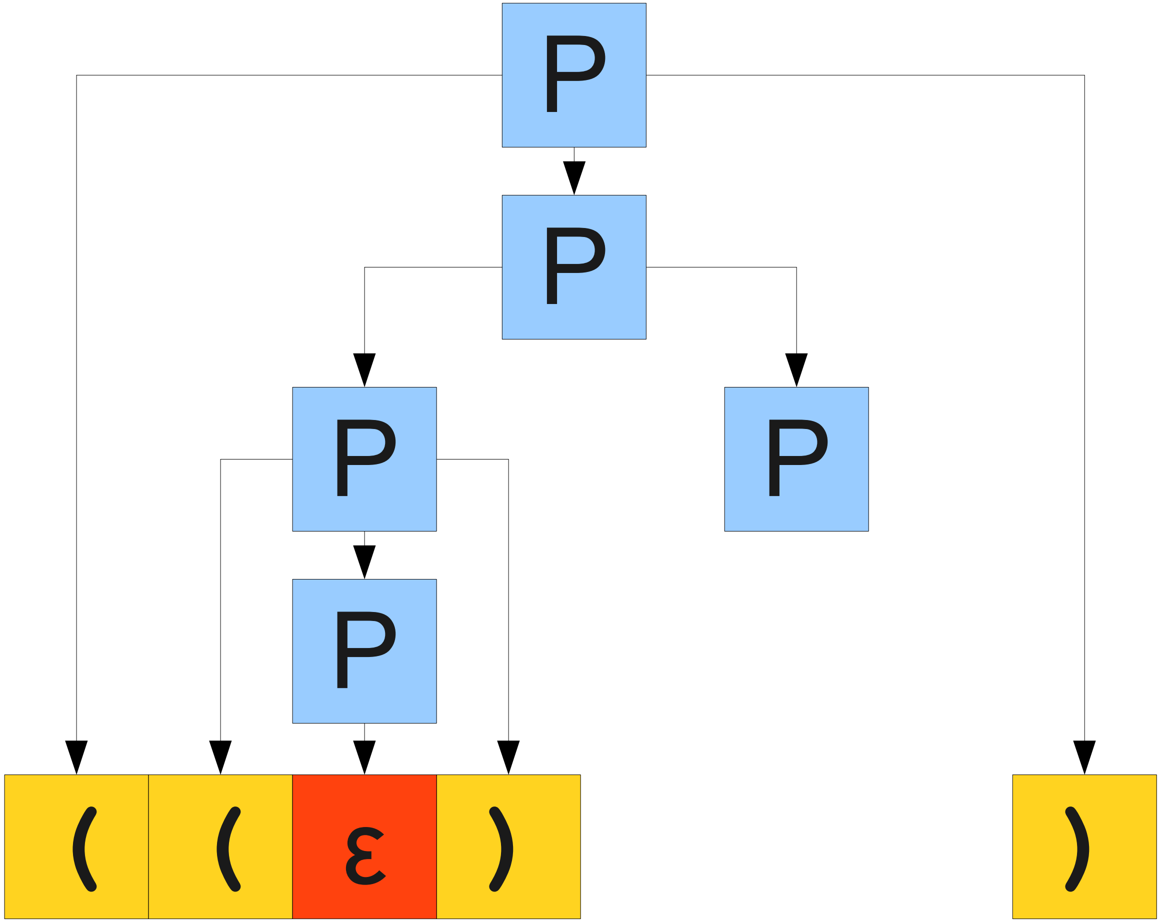

Balanced Parentheses

- Given the grammar **P** → **ε** | **PP** | **(P)**
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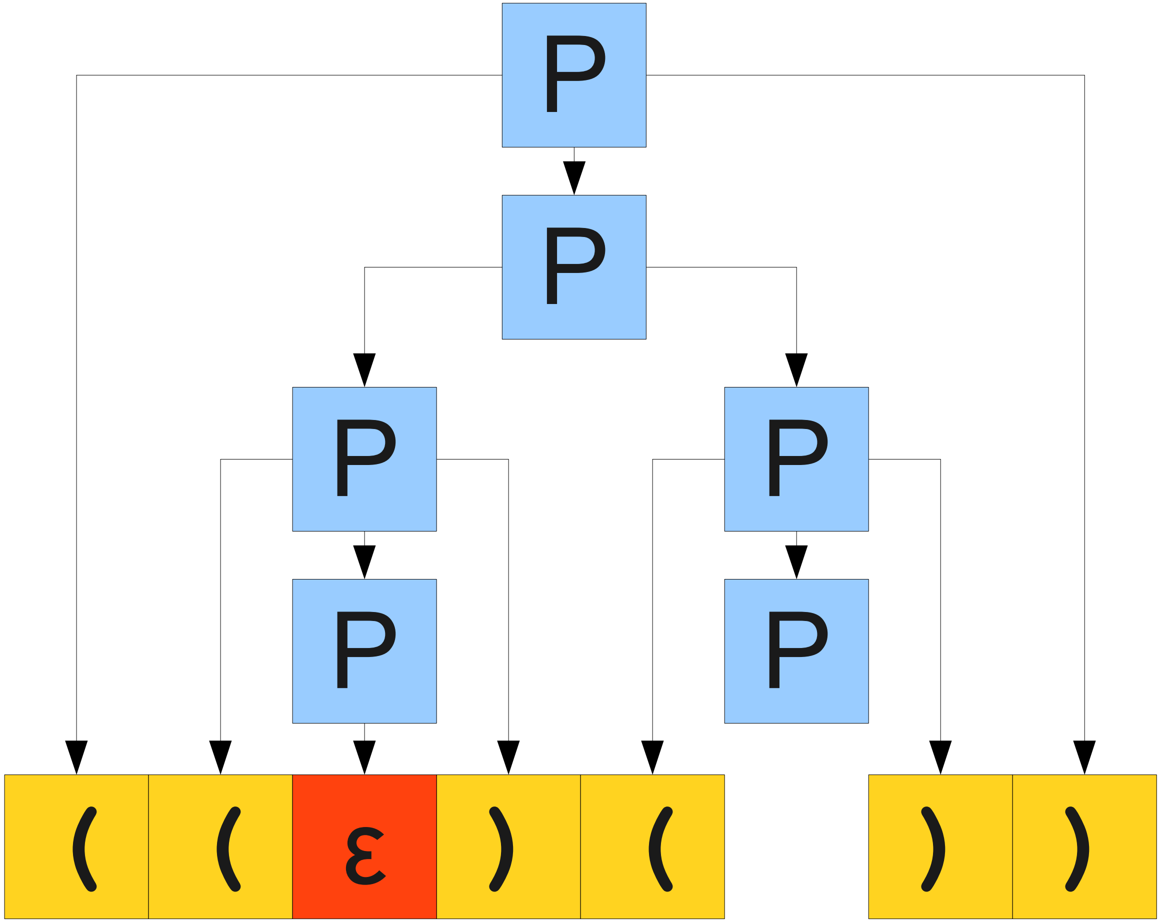
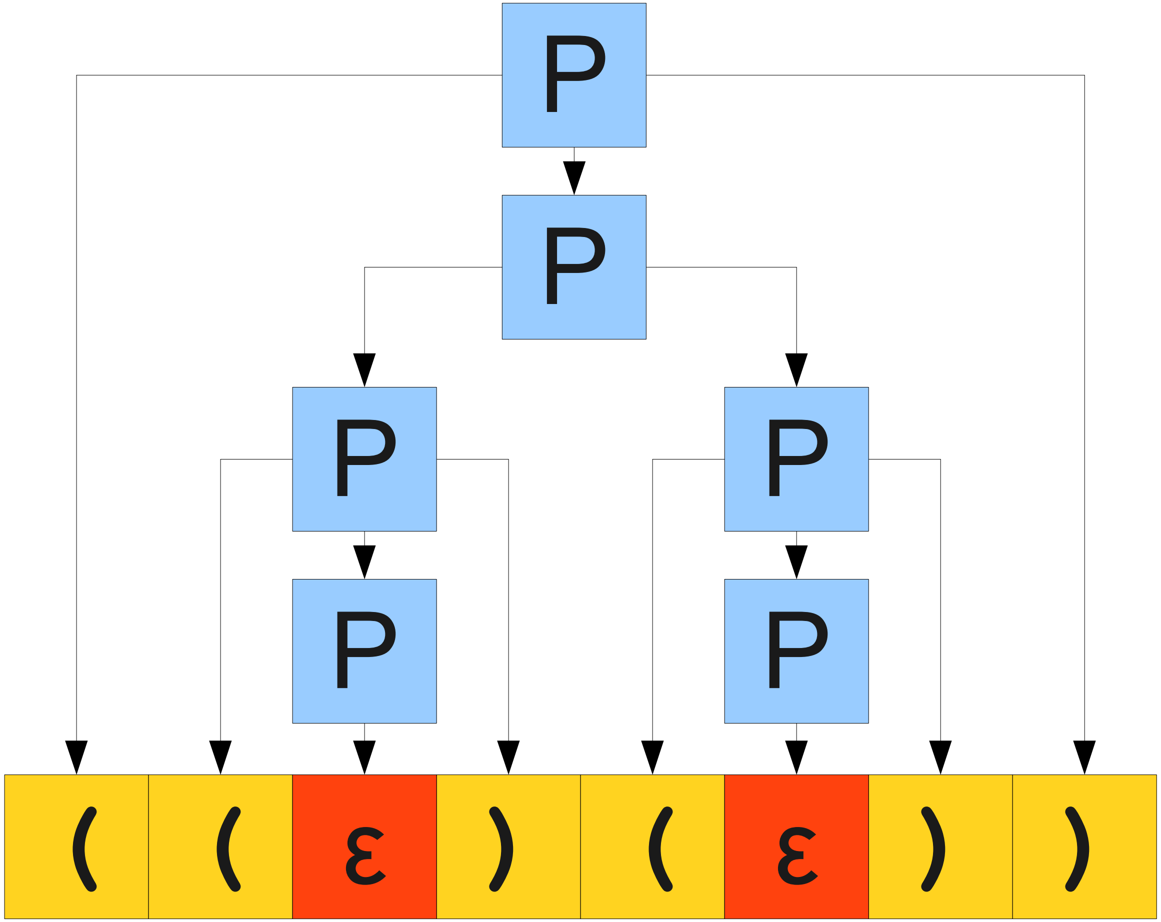
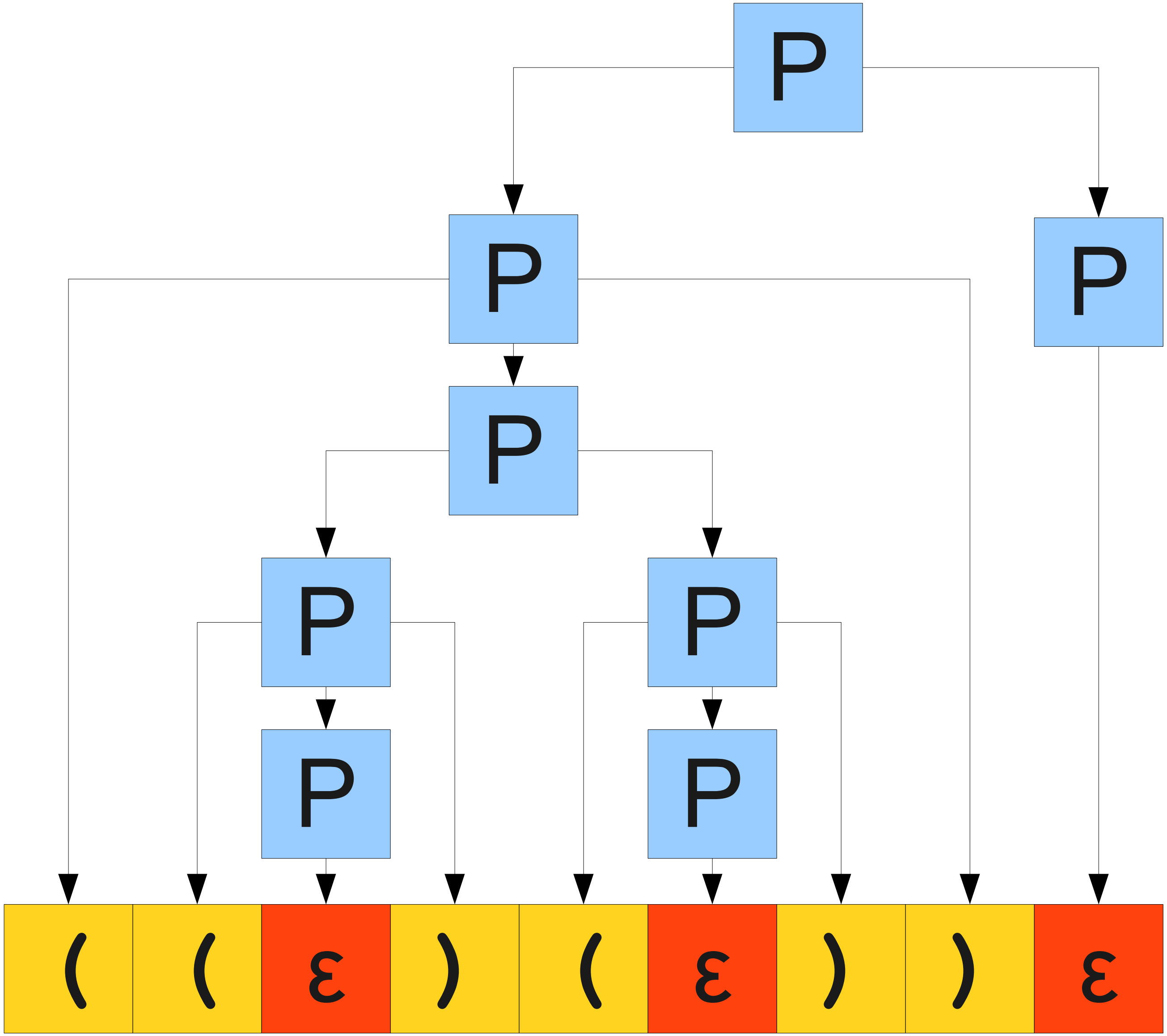


Balanced Parentheses

- Given the grammar **P** → **ε** | **PP** | **(P)**
- How might we generate the string `(()())`?

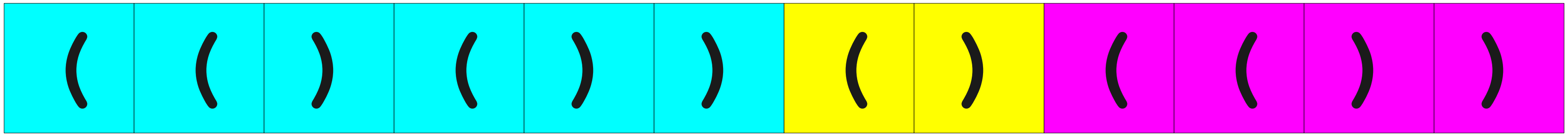


Balanced Parentheses

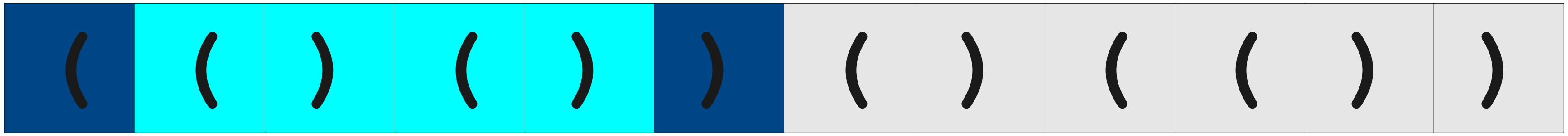


How to resolve this ambiguity?

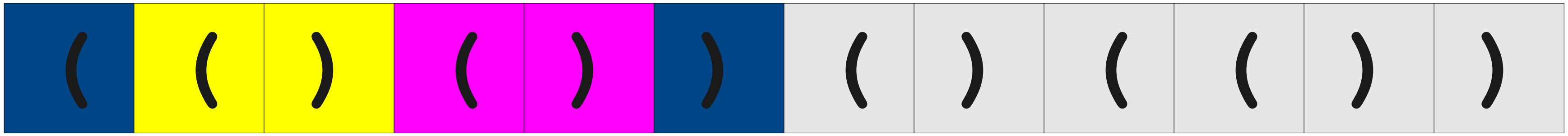
(	(	)	(	)	)	(	)	(	(	)	)



(	(	)	(	)	)	(	)	(	(	)	)



(	(	)	(	)	)	(	)	(	(	)	)



(( ) ( )) ( ) (( ))

Rethinking Parentheses

- A string of balanced parentheses is a sequence of strings that are themselves balanced parentheses.
- To avoid ambiguity, we can build the string in two steps:
  - Decide how many different substrings we will glue together.
  - Build each substring independently.

Let's ask the Internet for help!

Um... what?

- The way the nyancat flies across the sky is similar to how we can build up strings of balanced parentheses.

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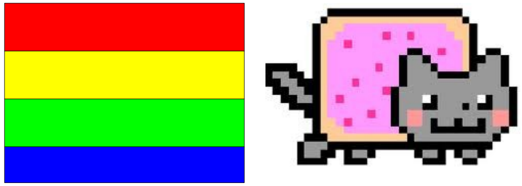

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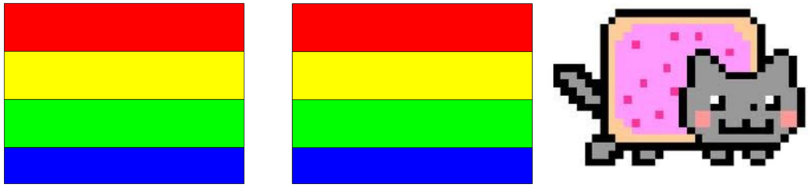

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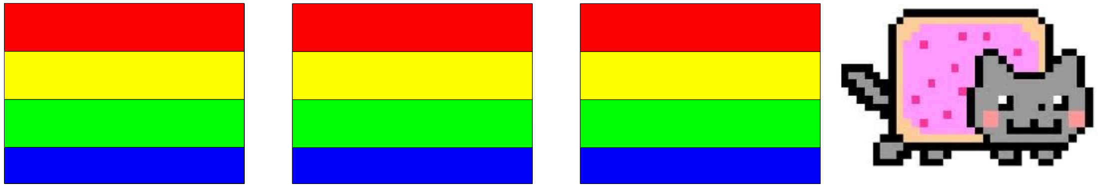

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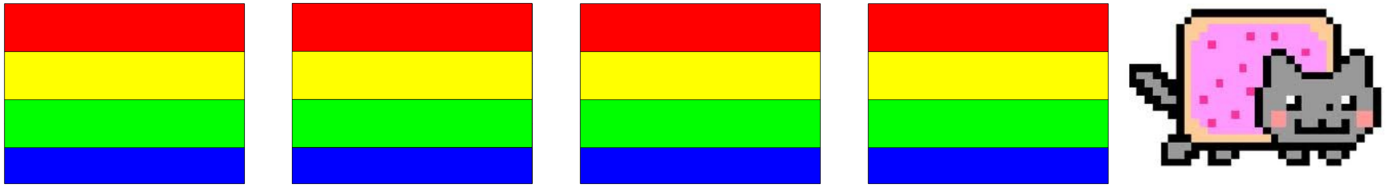

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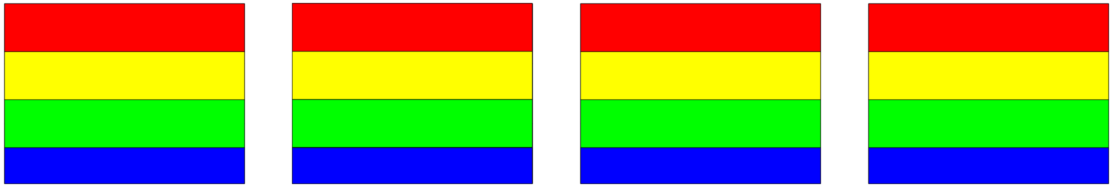

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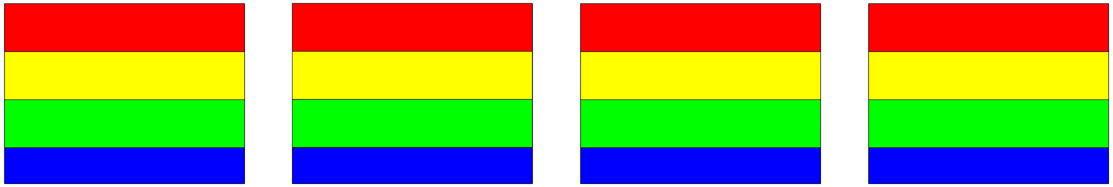
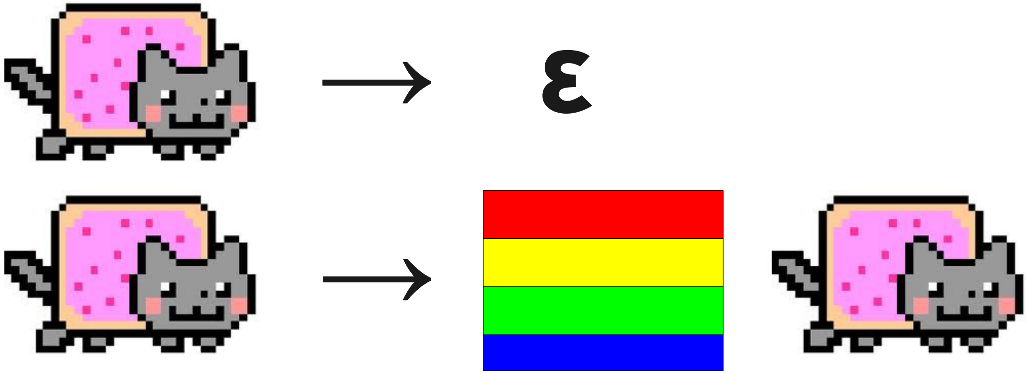




Building Parentheses

- Spread a string of parentheses across the string. There is exactly one way to do this for any number of parentheses.
- Expand out each substring by adding in parentheses and repeating.

$$\mathbf{S} \rightarrow \mathbf{P}\ \mathbf{S} \mid \varepsilon$$

$$\mathbf{P} \rightarrow (\ \mathbf{S}\ )$$

Building Parentheses

$$S \rightarrow P\ S \mid \varepsilon$$

$$P \rightarrow (\ S\ )$$

$$\begin{aligned}
& S \\
\Rightarrow\ & PS \\
\Rightarrow\ & PPS \\
\Rightarrow\ & PP \\
\Rightarrow\ & (S)P \\
\Rightarrow\ & (S)(S) \\
\Rightarrow\ & (PS)(S) \\
\Rightarrow\ & (P)(S) \\
\Rightarrow\ & ((S))(S) \\
\Rightarrow\ & (())(S) \\
\Rightarrow\ & (())()
\end{aligned}$$

Context-Free Grammars

- A regular expression can be
  - Any letter
  - ε
  - Ø
  - The concatenation of regular expressions.
  - The union of regular expressions.
  - The Kleene closure of a regular expression.
  - A parenthesized regular expression.

Context-Free Grammars

- This gives us the following CFG:

R → `a` | `b` | `c` | …

R → “`ε`”

R → `Ø`

R → **RR**

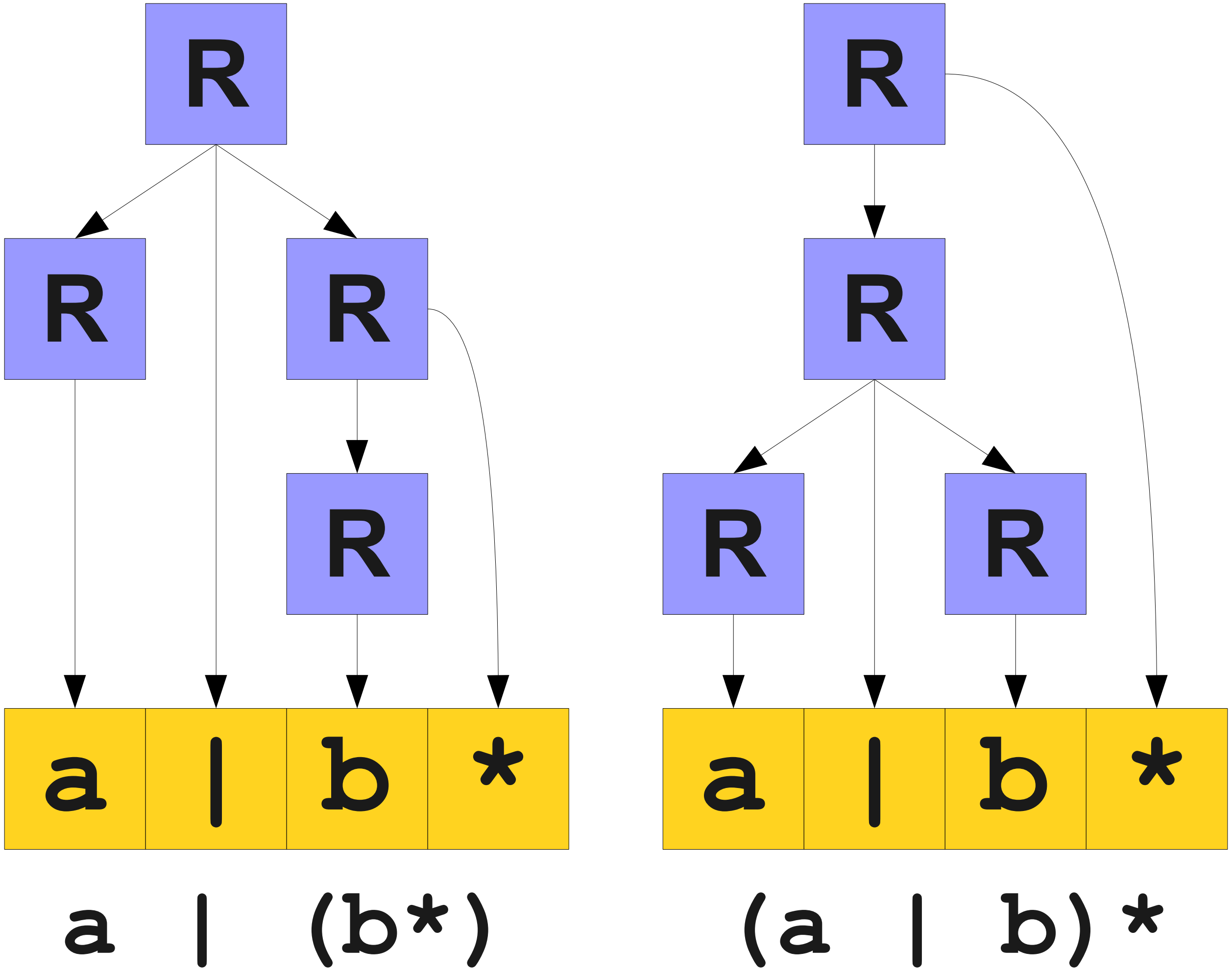
R → **R** “`|`” **R**

R → **R**`*`

R → `(`**R**`)`

An Ambiguous Grammar

R → `a` | `b` | `c` | …
R → "`ε`"
R → `Ø`
R → **RR**
R → **R** "`|`" **R**
R → **R**`*`
R → `(`**R**`)`



Resolving Ambiguity

- We can try to resolve the ambiguity via layering:

R → `a` | `b` | `c` | …

R → "`ε`"

R → `Ø`

R → **RR**

R → **R** "`|`" **R**

R → **R**`*`

R → `(`**R**`)`

a	a	|	b	*

Resolving Ambiguity

- We can try to resolve the ambiguity via layering:

R → `a` | `b` | `c` | …

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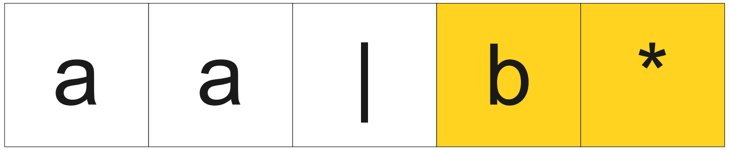
R → `∅`

R → **RR**

R → **R** "`|`" **R**

R → **R**`*`

R → `(`**R**`)`



a	a	|	b	*

Resolving Ambiguity

- We can try to resolve the ambiguity via layering:

R → a | b | c | …

R → "ε"

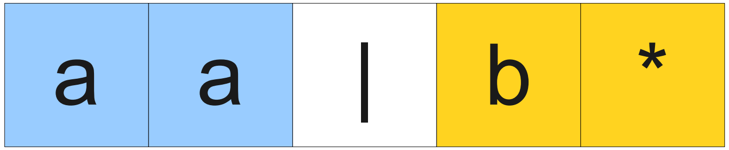
R → ∅

R → RR

R → R "|" R

R → R*

R → (R)



a	a	|	b	*

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R → `a` | `b` | `c` | …

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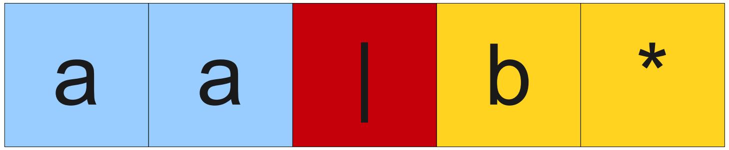
R → `(`**R**`)`



a	a	|	b	*

Resolving Ambiguity

- We can try to resolve the ambiguity via layering:

R → a | b | c | …

R → “ε”

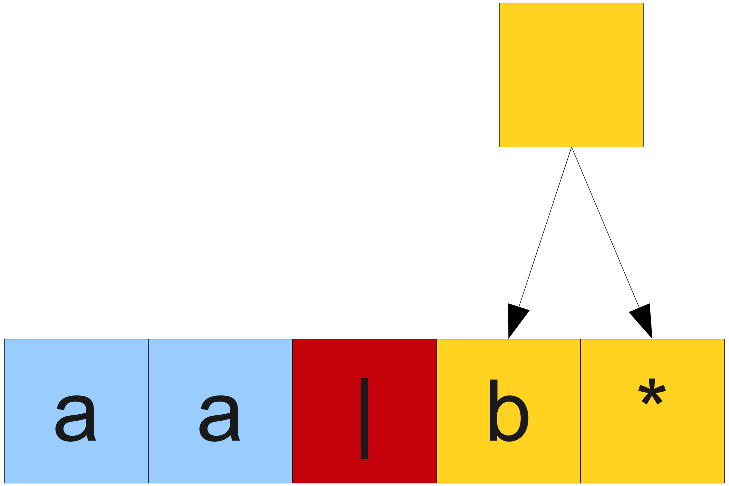
R → Ø

R → RR

R → R “|” R

R → R*

R → (R)



Resolving Ambiguity

- We can try to resolve the ambiguity via layering:

R → `a` | `b` | `c` | …

R → "`ε`"

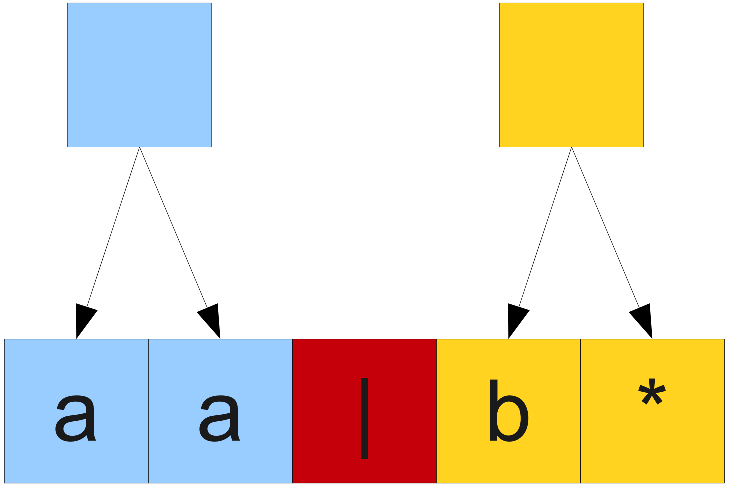
R → `Ø`

R → **RR**

R → **R** "`|`" **R**

R → **R**`*`

R → `(`**R**`)`



Resolving Ambiguity

- We can try to resolve the ambiguity via layering:

R → `a` | `b` | `c` | …

R → "`ε`"

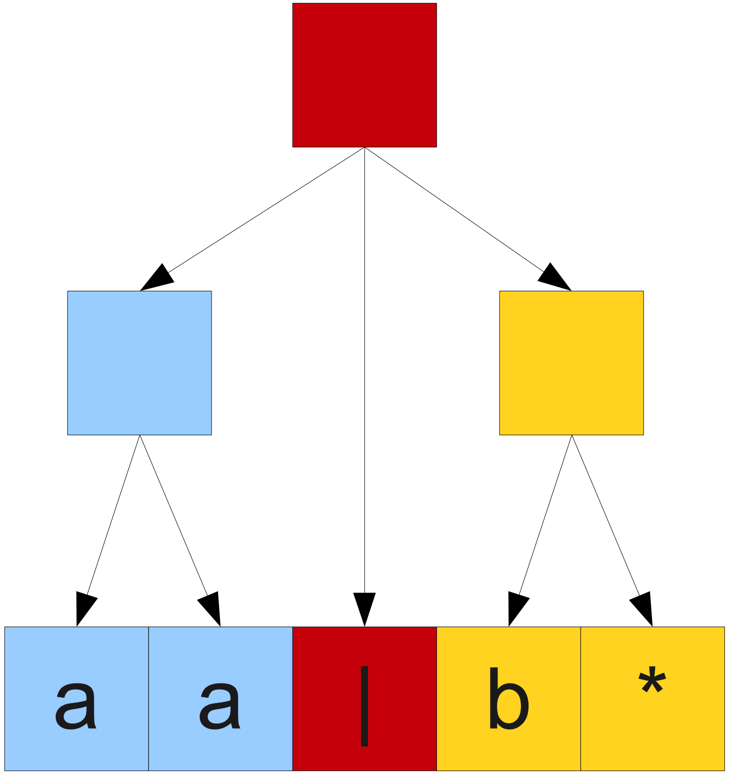
R → `Ø`

R → **RR**

R → **R** "`|`" **R**

R → **R**`*`

R → `(`**R**`)`



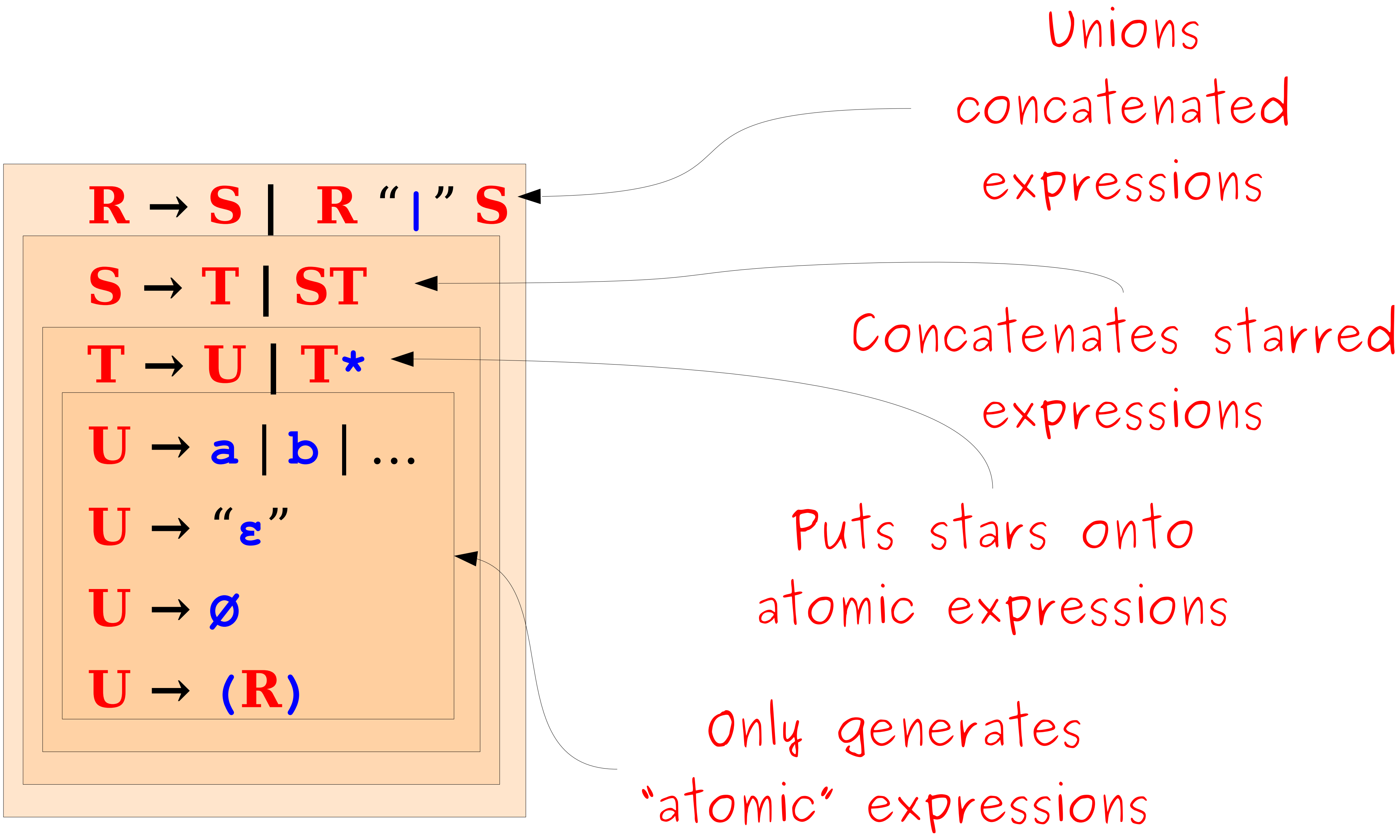
Resolving Ambiguity

- We can try to resolve the ambiguity via layering:

R → a | b | c | …
R → “ε”
R → Ø
R → RR
R → R “|” R
R → R*
R → (R)

R → S | R “|” S
S → T | ST
T → U | T*
U → a | b | c | …
U → “ε”
U → Ø
U → (R)

Why is this unambiguous?



R → S | R "|" S

S → T | ST

T → U | T*

U → a | b | …

U → "ε"

U → Ø

U → (R)

Unions concatenated expressions

Concatenates starred expressions

Puts stars onto atomic expressions

Only generates "atomic" expressions

R

R → S | R "|" S

S → T | ST

T → U | T*

U → a | b | c | ...

U → "ε"

U → ø

U → (R)

a	b	|	c	|	a	*

R → **S** | **R** “`|`” **S**

S → **T** | **ST**

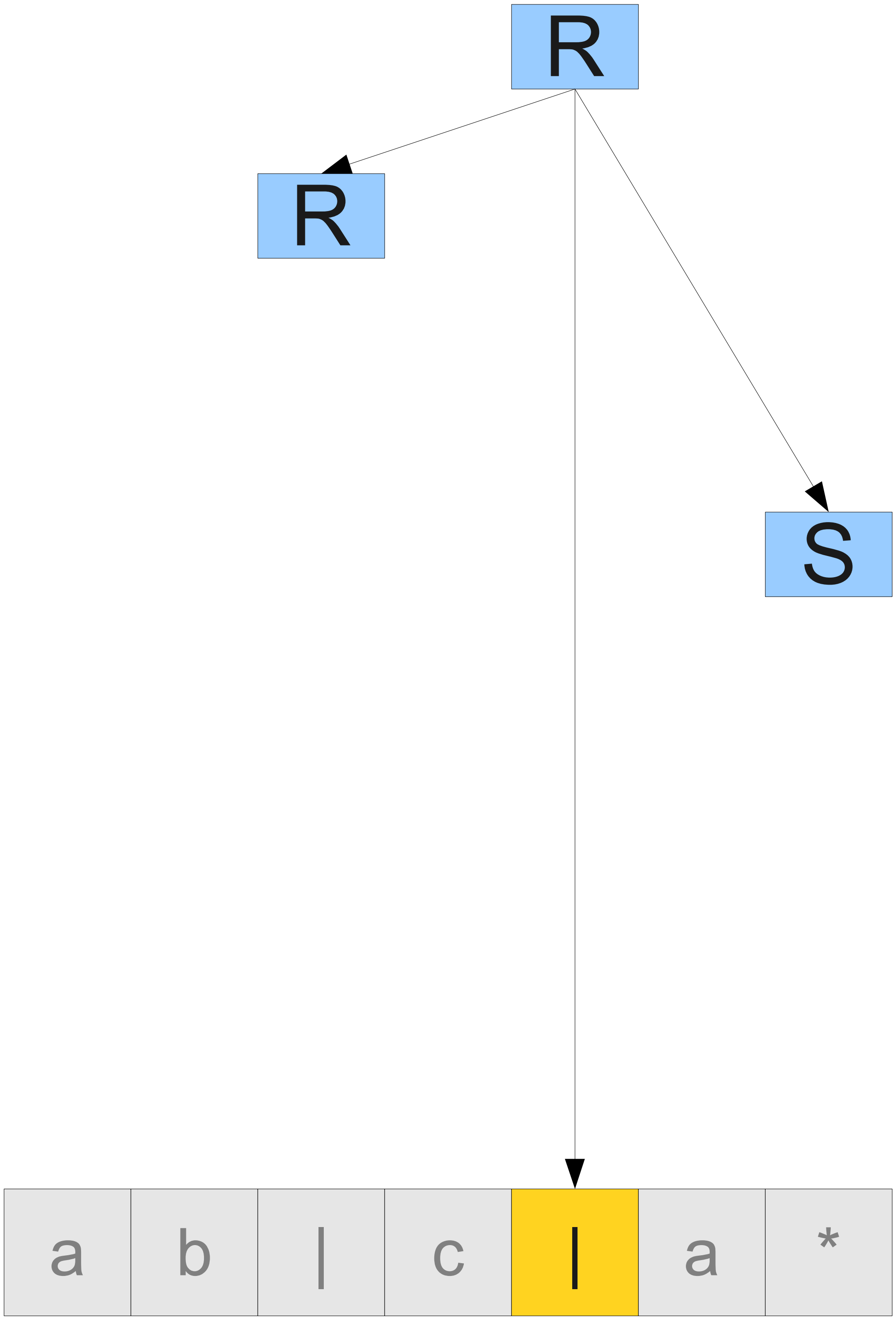
T → **U** | **T**`*`

U → `a` | `b` | `c` | …

U → “`ε`”

U → `Ø`

U → `(`**R**`)`



R → **S** | **R** “|” **S**

S → **T** | **ST**

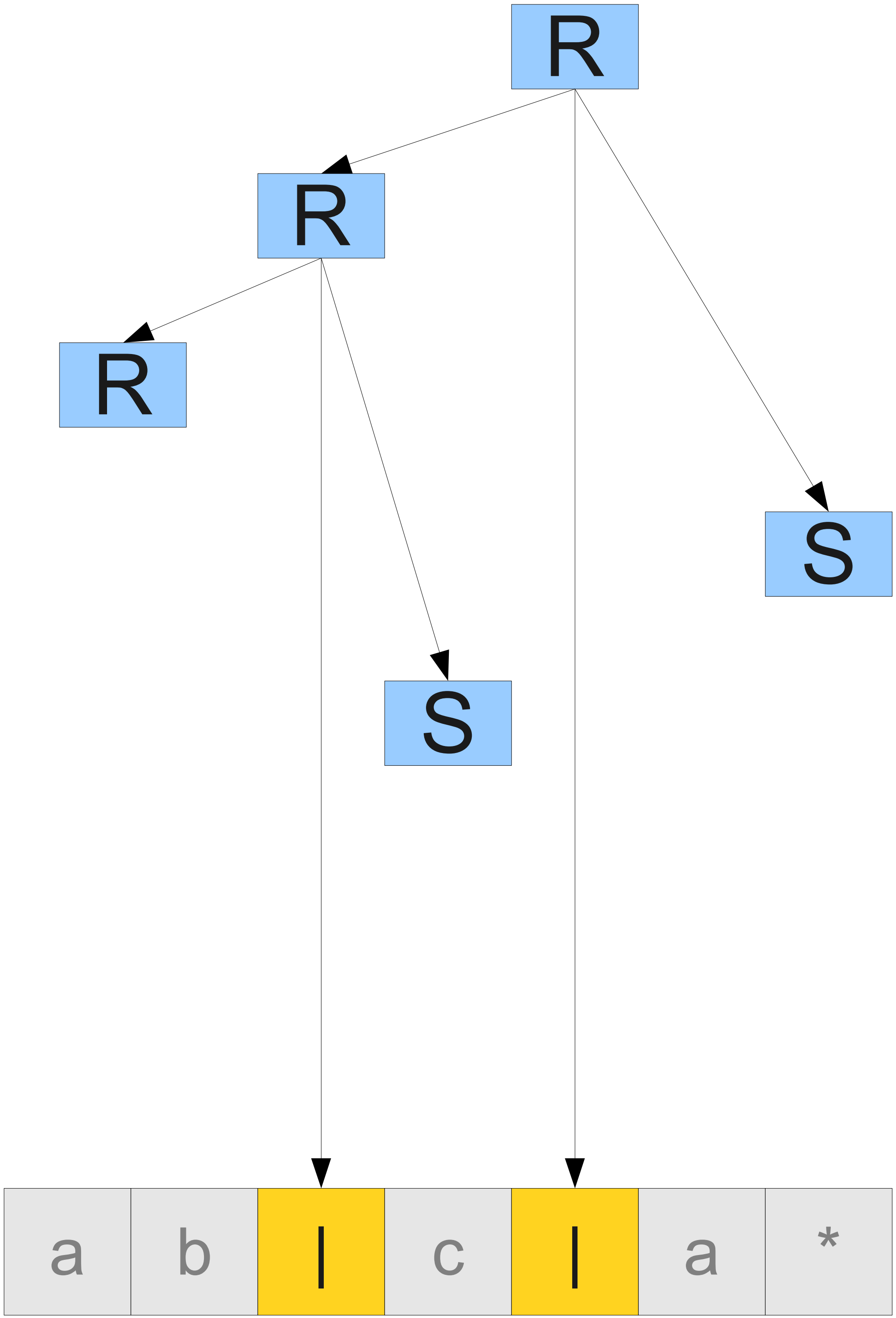
T → **U** | **T***

U → a | b | c | …

U → “ε”

U → ∅

U → (**R**)



R → **S** | **R** “|” **S**

S → **T** | **ST**

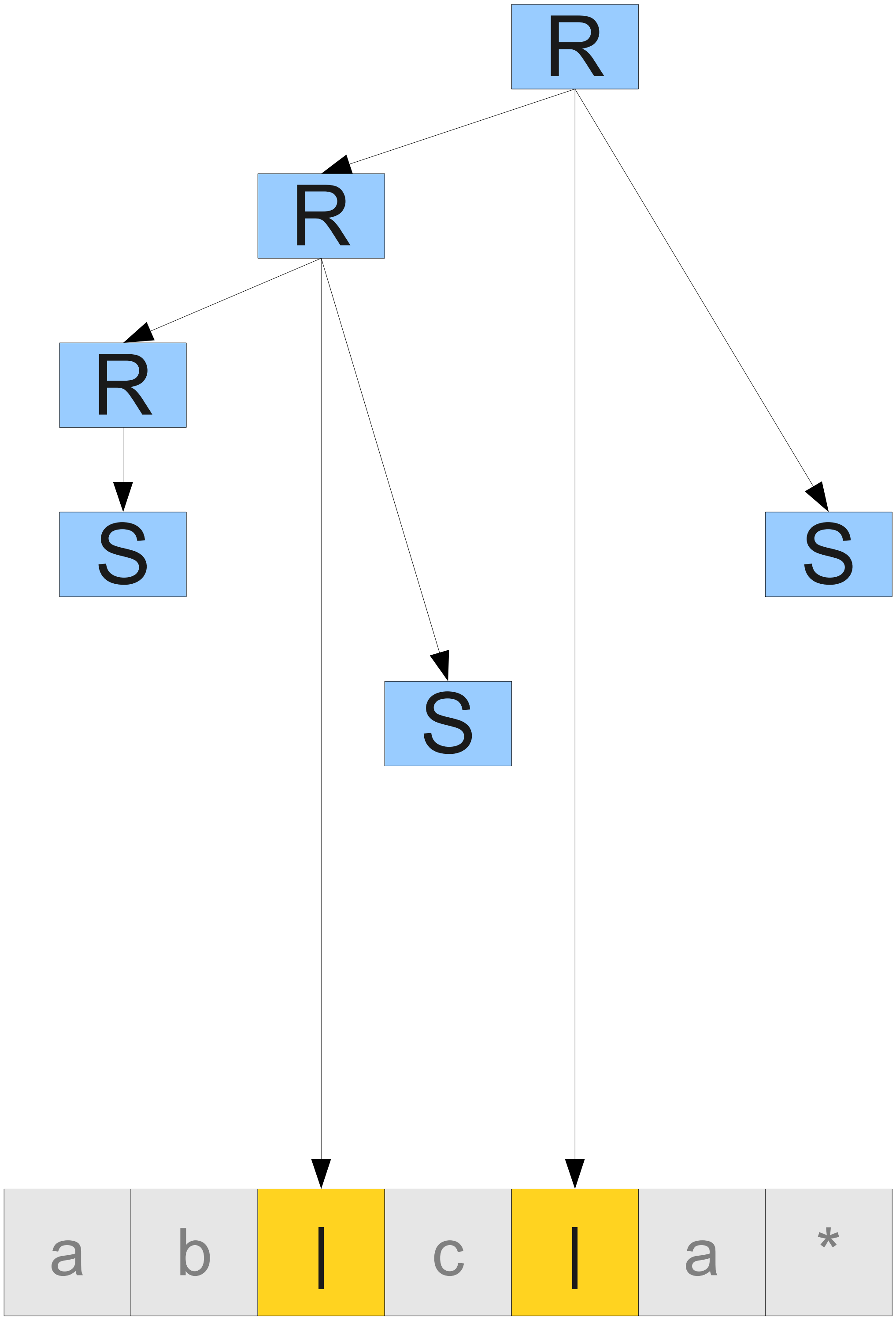
T → **U** | **T***

U → a | b | c | …

U → “ε”

U → Ø

U → (**R**)



R → **S** | **R** “|” **S**

S → **T** | **ST**

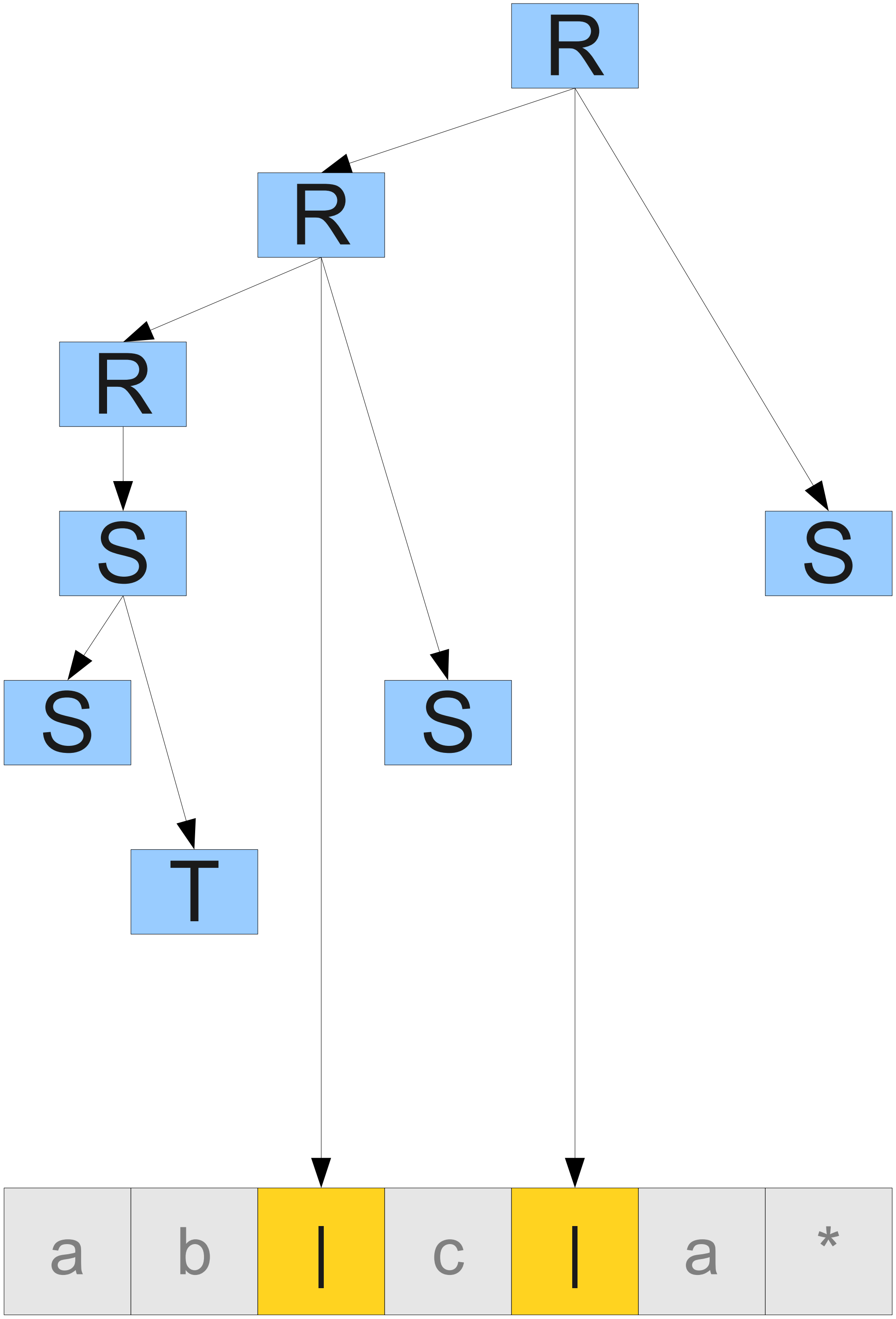
T → **U** | **T***

U → a | b | c | …

U → “ε”

U → ∅

U → (**R**)



R → **S** | **R** “|” **S**

S → **T** | **ST**

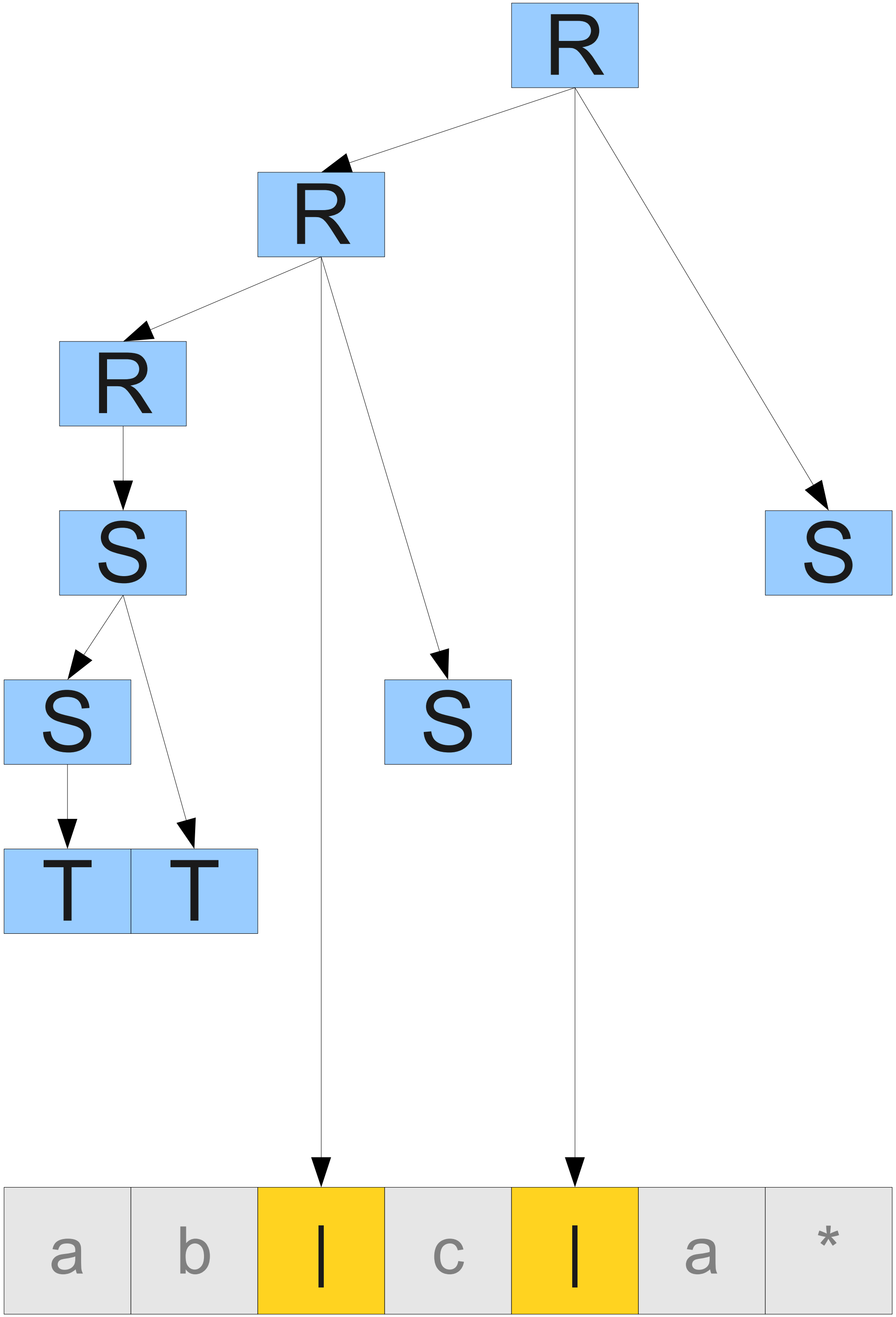
T → **U** | **T**`*`

U → `a` | `b` | `c` | …

U → “`ε`”

U → `Ø`

U → `(`**R**`)`



R → **S** | **R** “|” **S**

S → **T** | **ST**

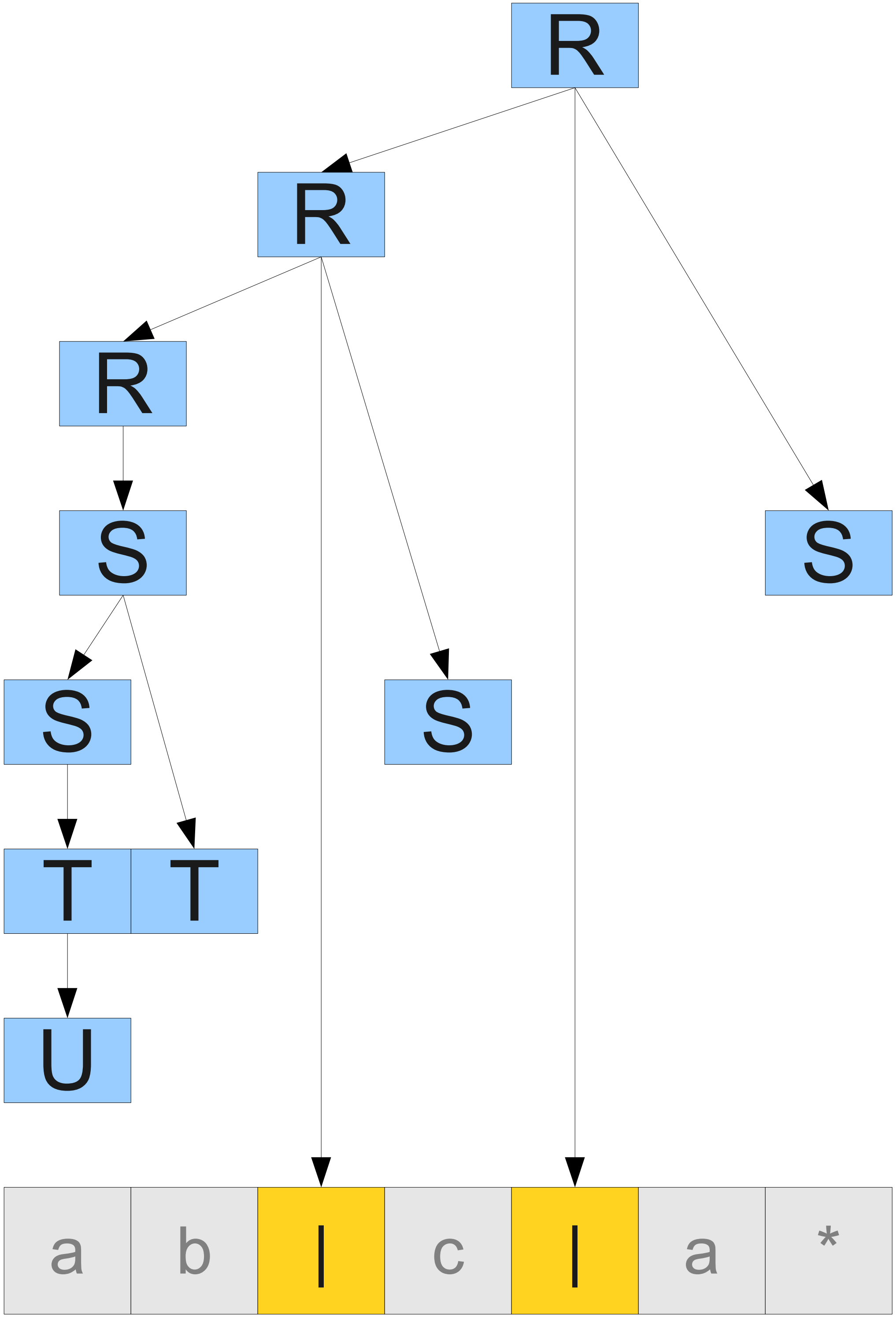
T → **U** | **T**`*`

U → `a` | `b` | `c` | …

U → “`ε`”

U → `ø`

U → `(`**R**`)`



R → S | R “|” S

S → T | ST

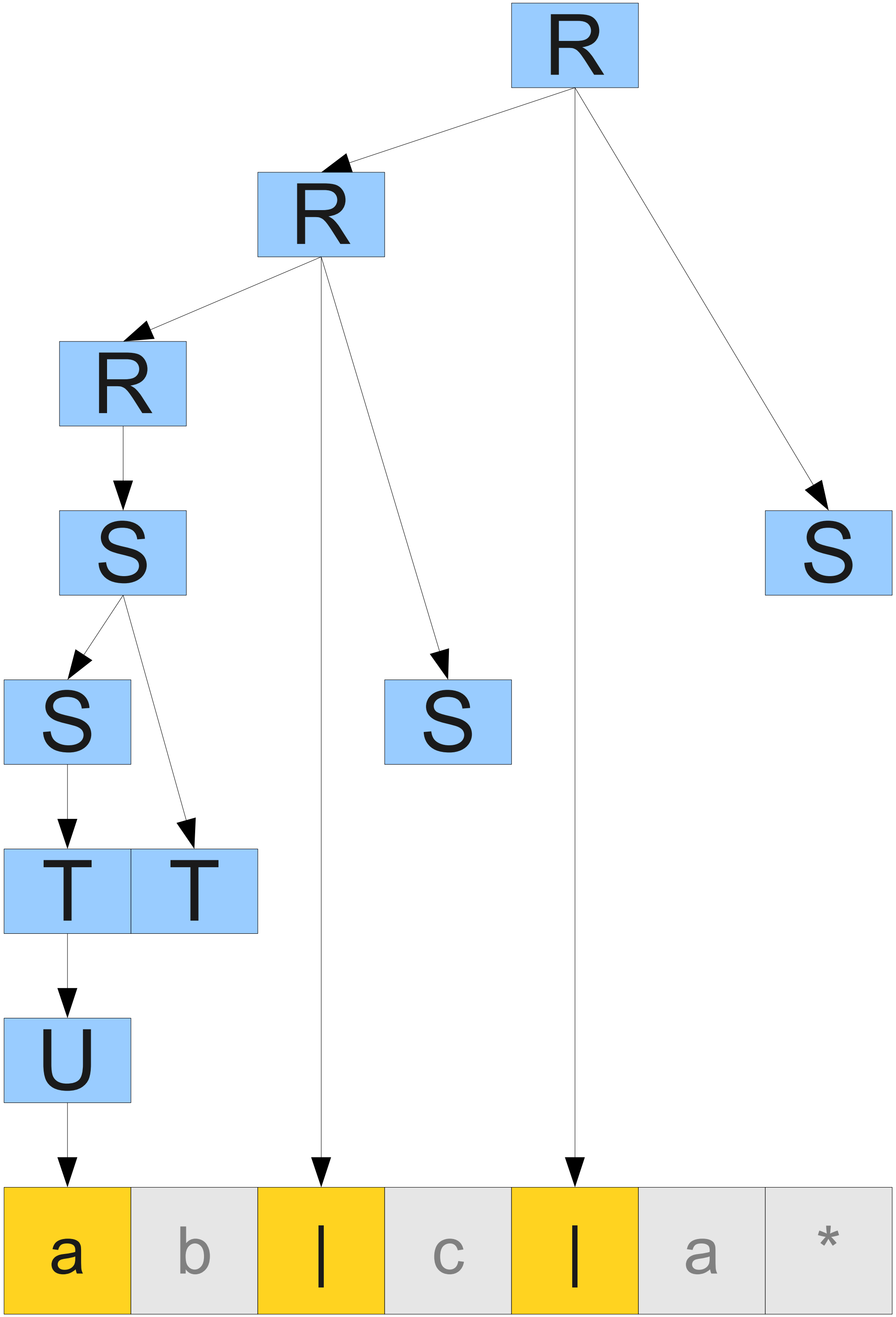
T → U | T*

U → a | b | c | …

U → “ε”

U → Ø

U → (R)



R → **S** | **R** “|” **S**

S → **T** | **ST**

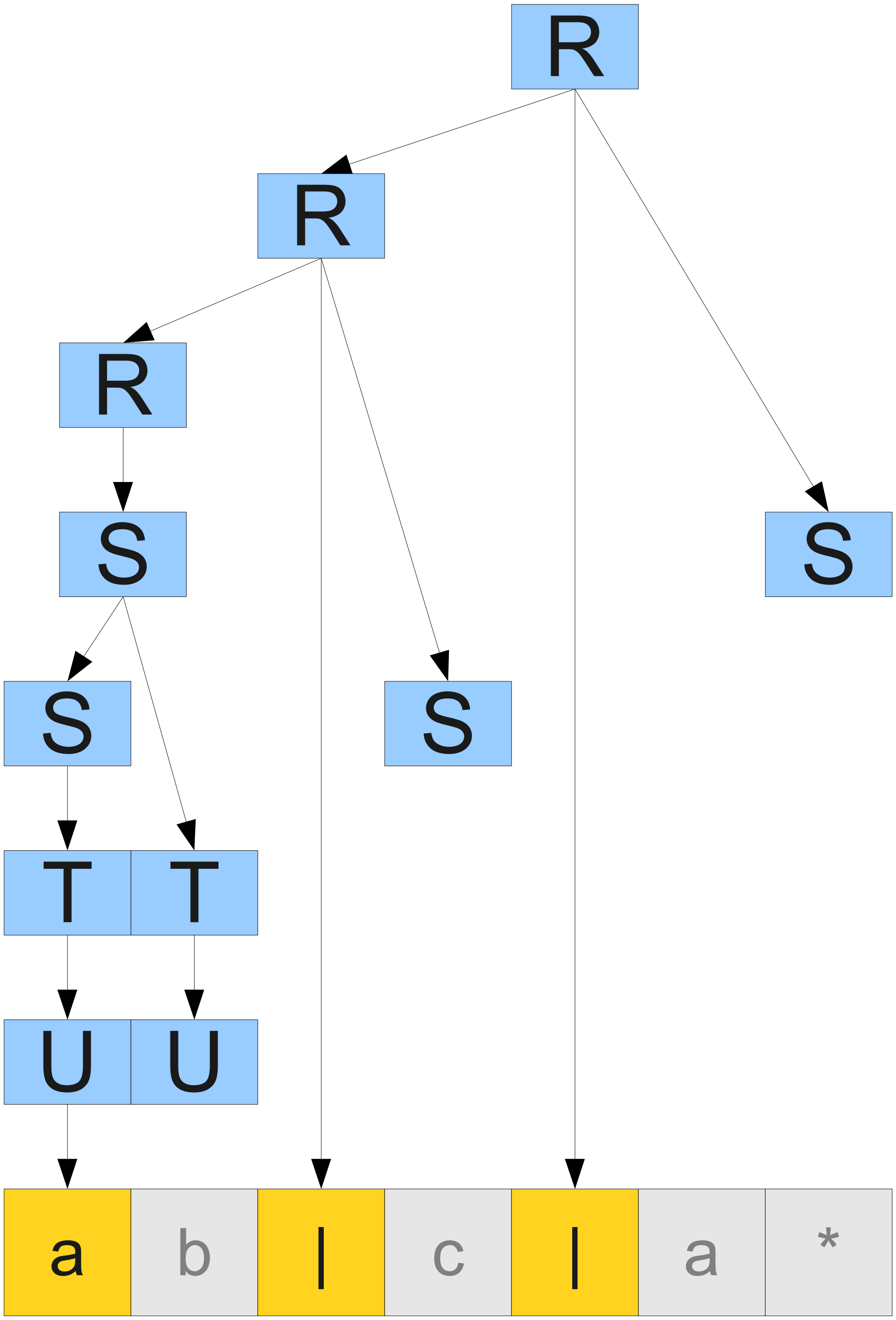
T → **U** | **T***

U → a | b | c | …

U → “ε”

U → ∅

U → (**R**)



R → **S** | **R** “|” **S**

S → **T** | **ST**

T → **U** | **T***

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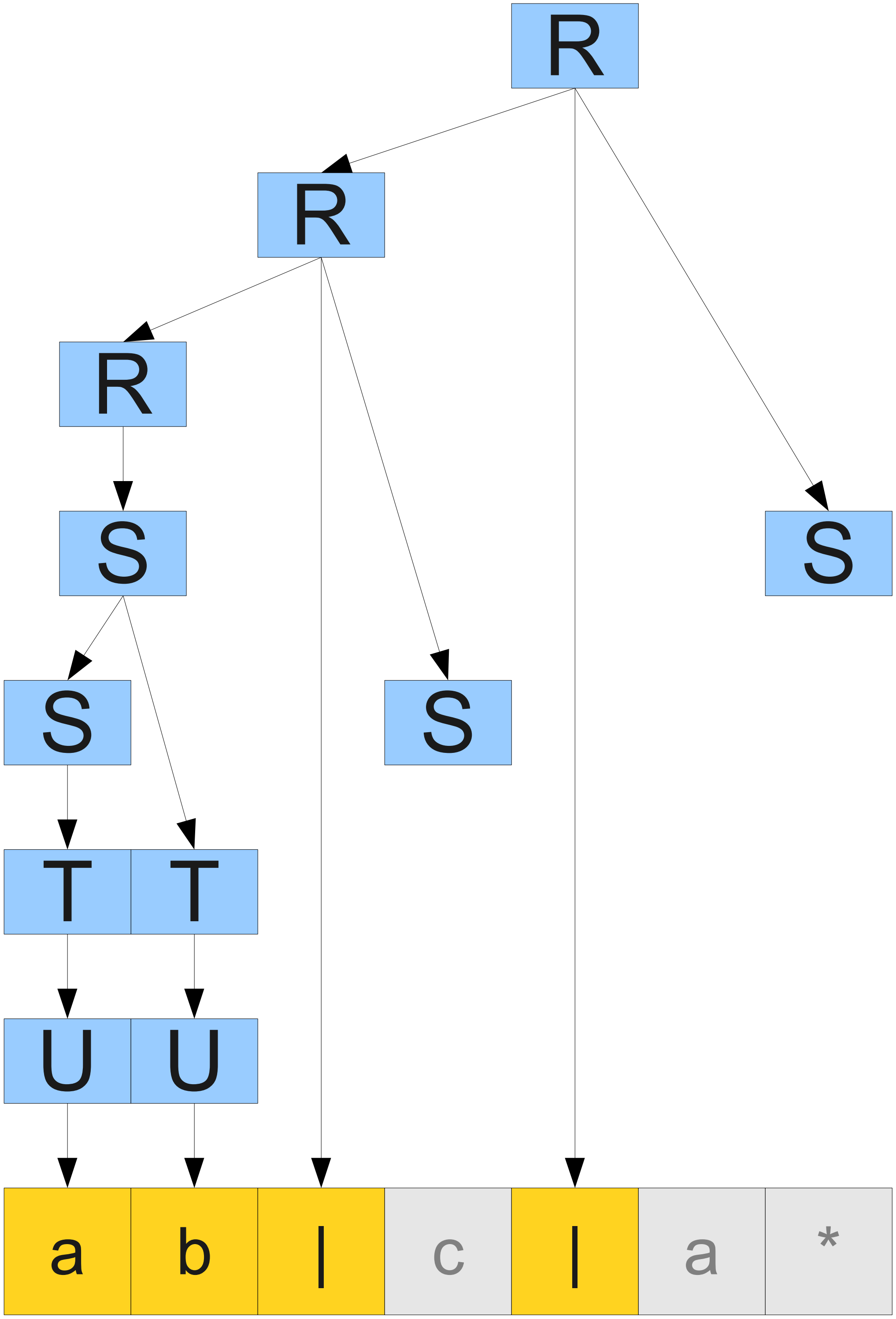
U → (**R**)



R → **S** | **R** “|” **S**

S → **T** | **ST**

T → **U** | **T**`*`

U → `a` | `b` | `c` | ...

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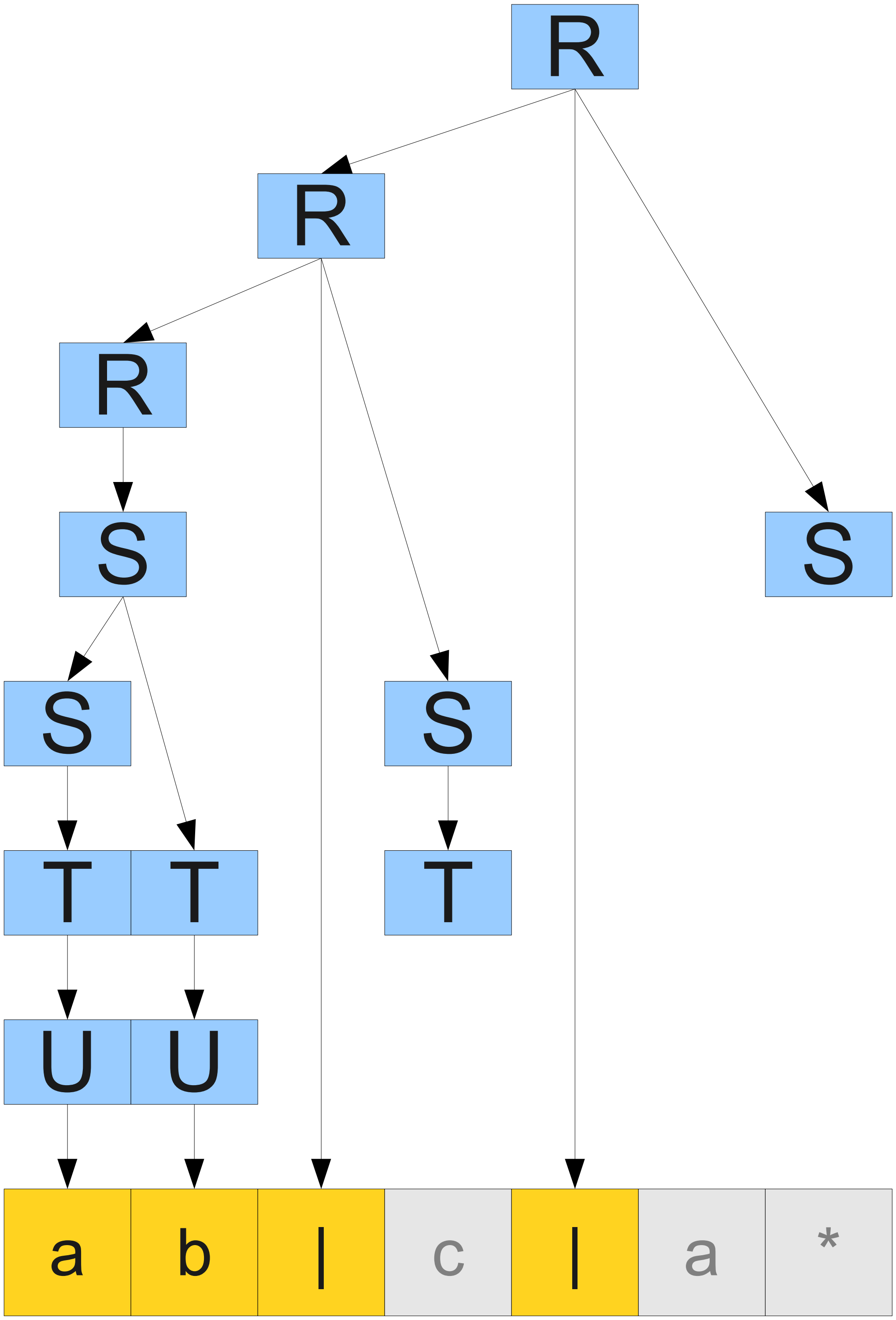
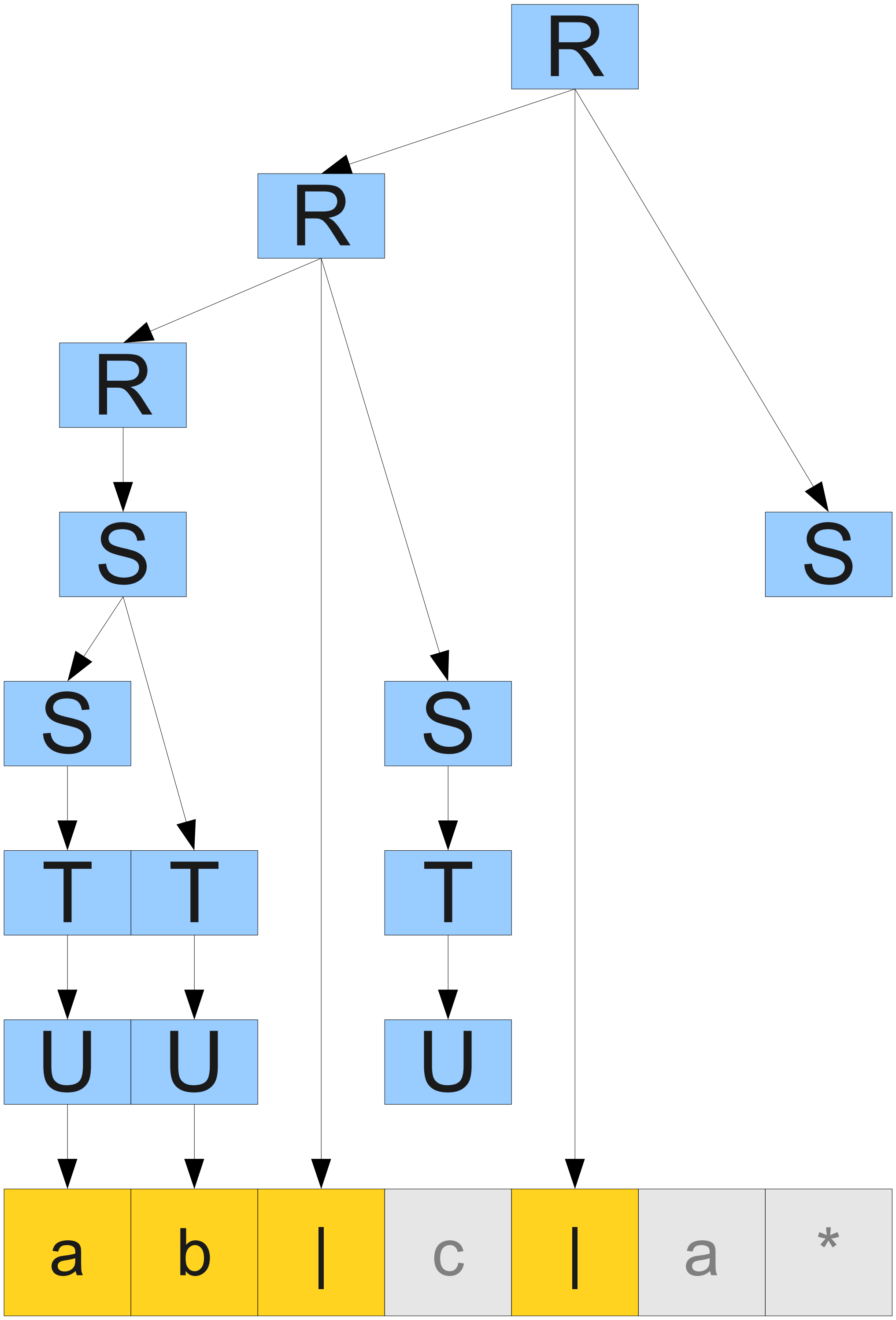
U → `Ø`

U → `(`**R**`)`



R → **S** | **R** "|" **S**
S → **T** | **ST**
T → **U** | **T***
U → a | b | c | …
U → "ε"
U → Ø
U → (**R**)



R → S | R "|" S

S → T | ST

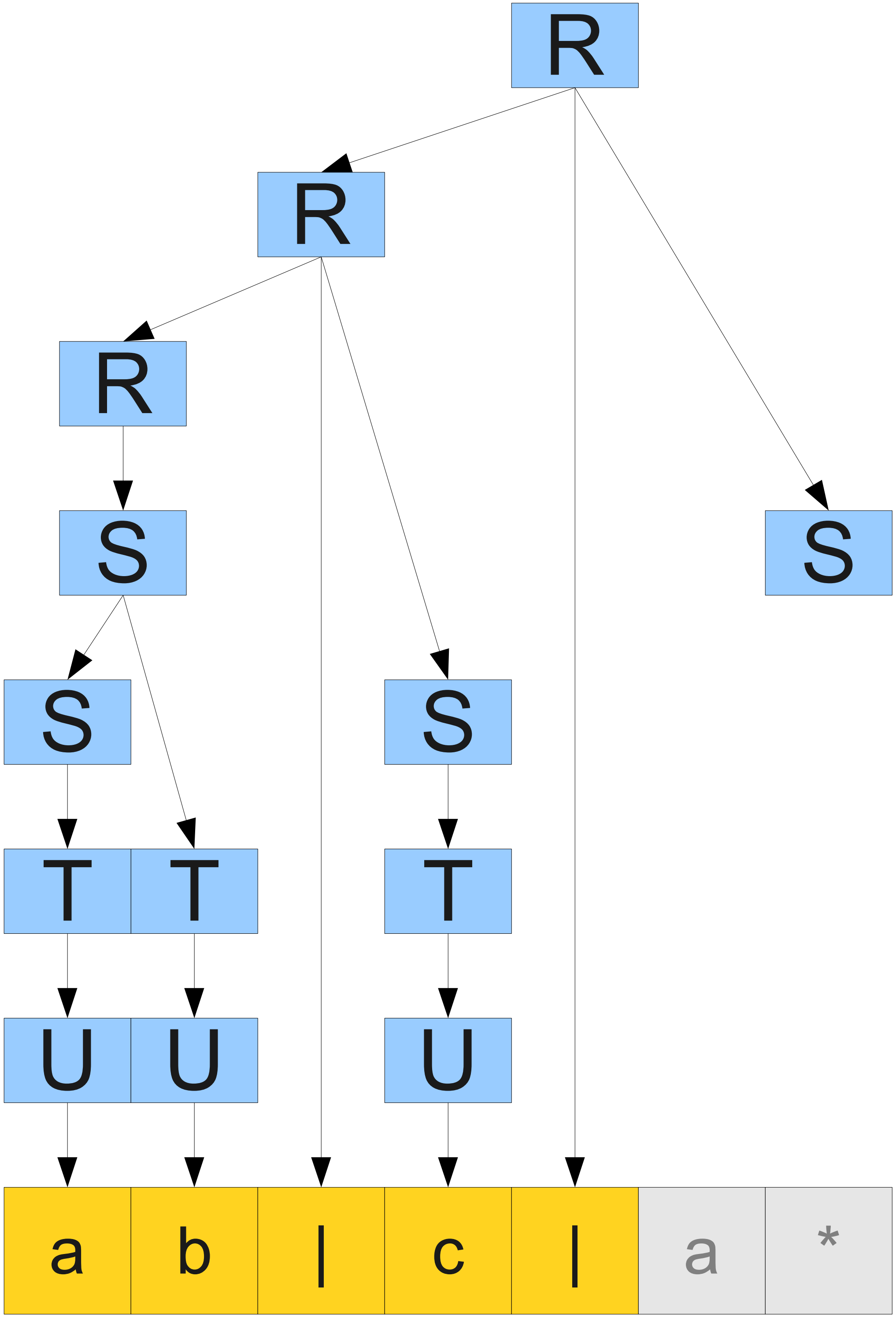
T → U | T*

U → a | b | c | …

U → "ε"

U → Ø

U → (R)



R → **S** | **R** “|” **S**

S → **T** | **ST**

T → **U** | **T***

U → a | b | c | …

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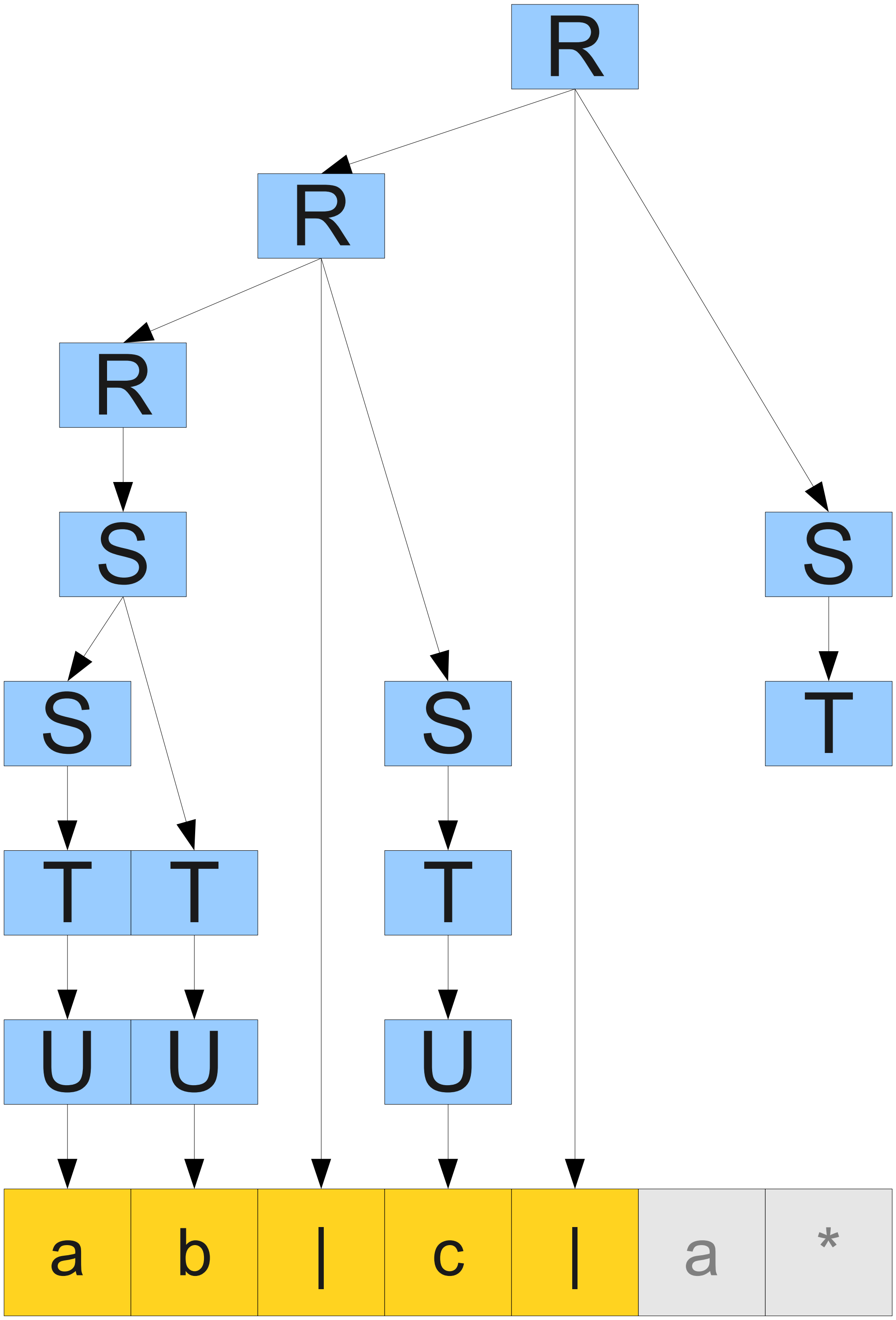
U → (**R**)



R → **S** | **R** “|” **S**

S → **T** | **ST**

T → **U** | **T***

U → a | b | c | …

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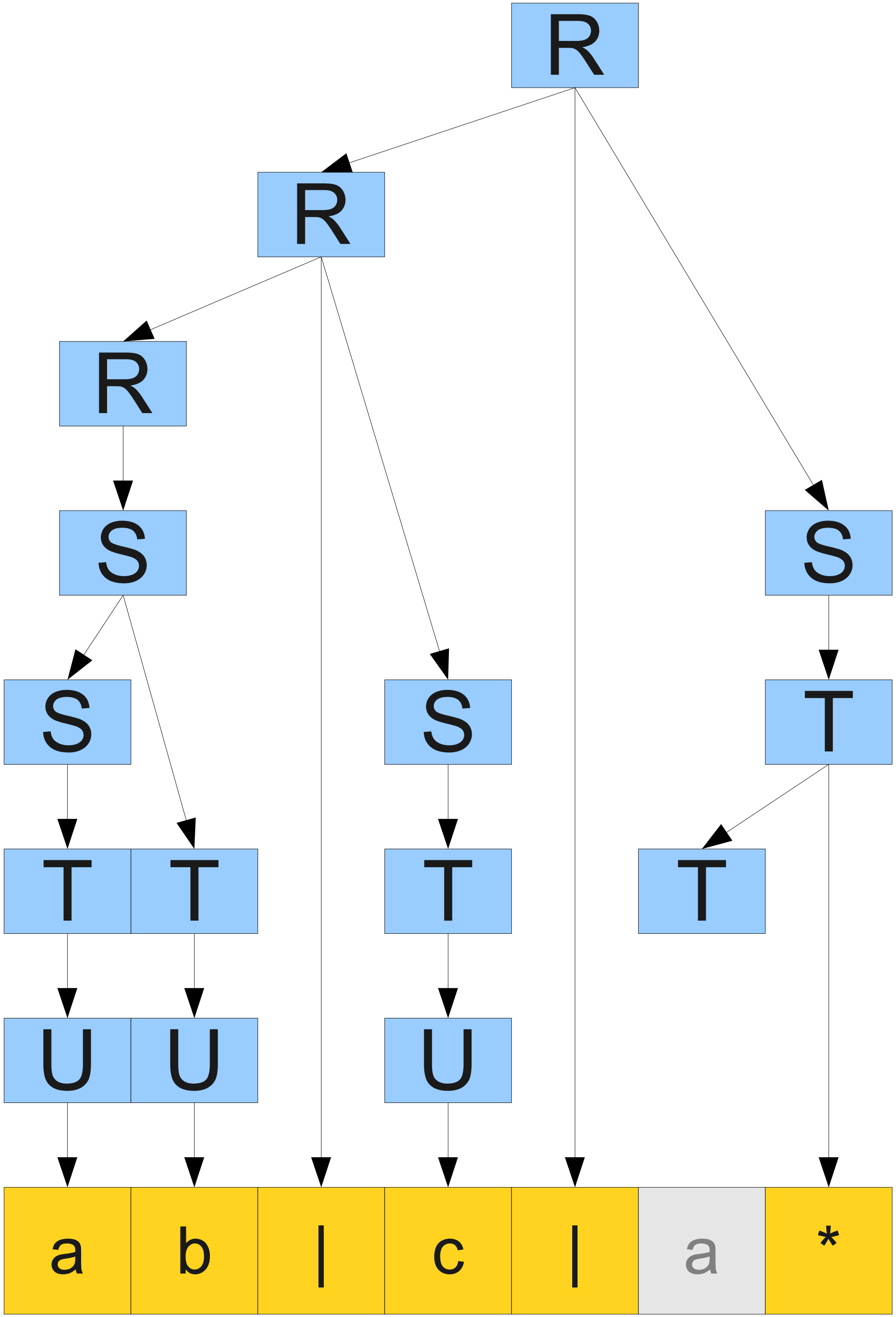
U → ∅

U → (**R**)



R → **S** | **R** “|” **S**
S → **T** | **ST**
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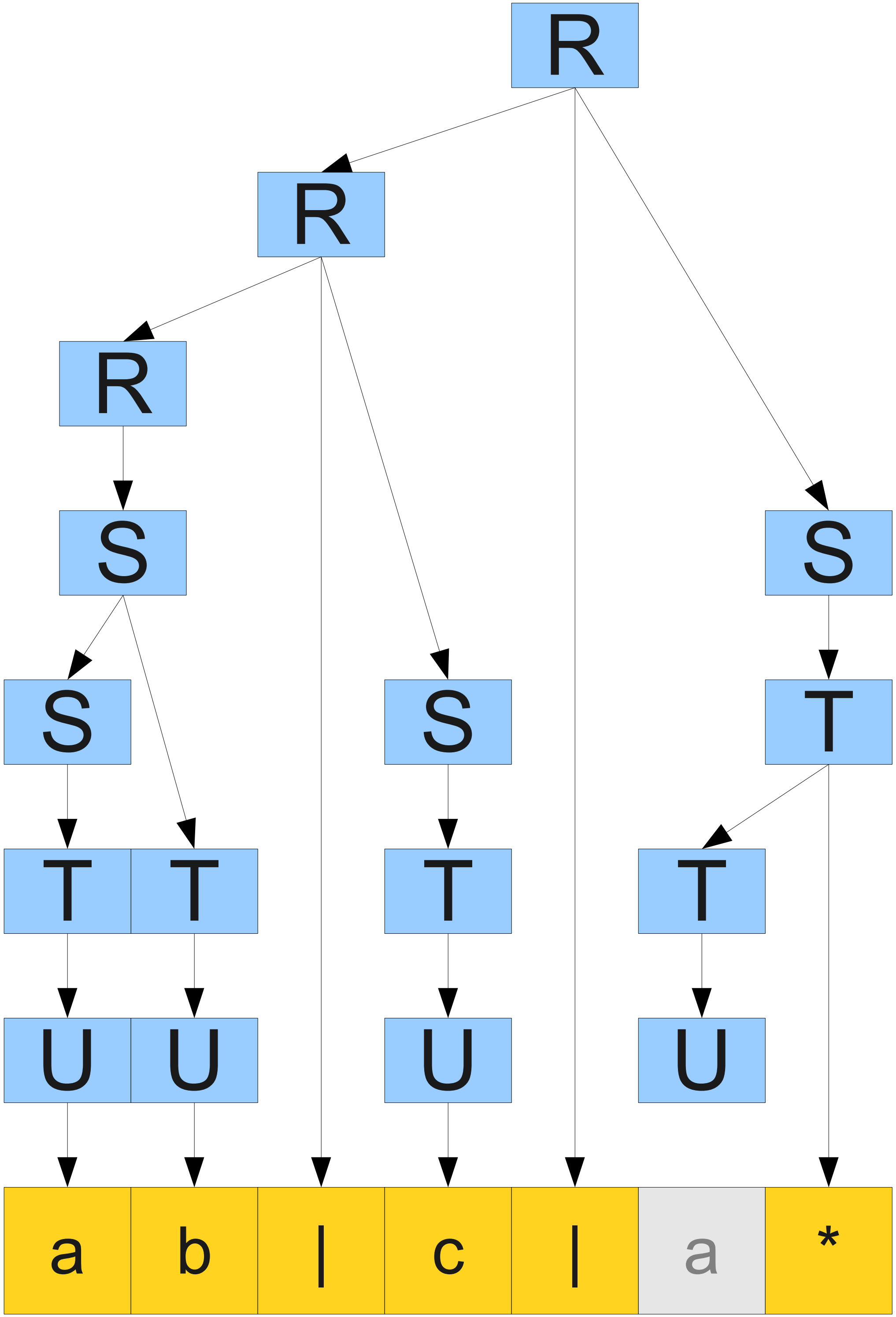

R → **S** | **R** “|” **S**

S → **T** | **ST**

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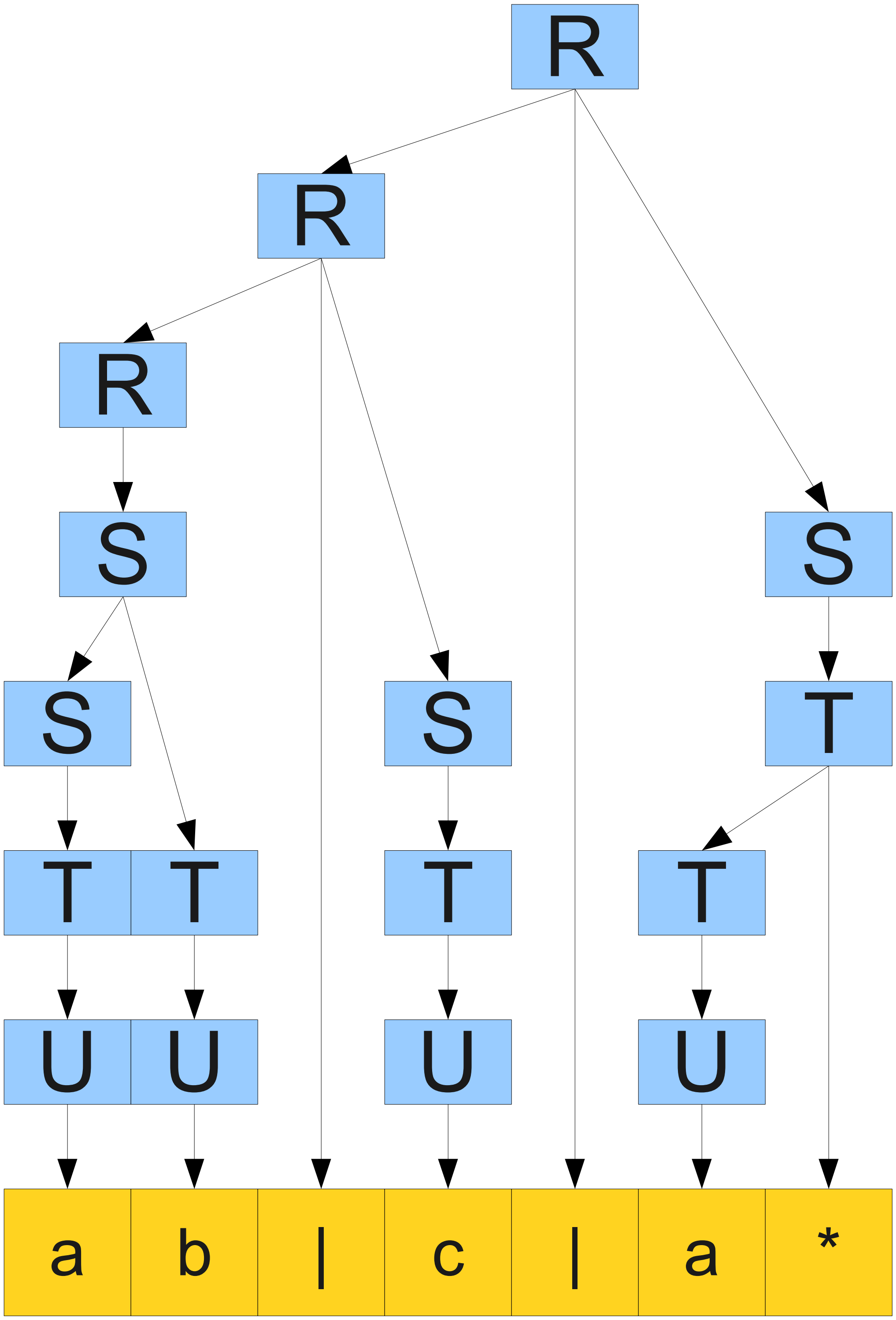
U → (**R**)



Summary

- **Context-free grammars** give a way to describe a class of formal languages (the **context-free languages**) that are strictly larger than the regular languages.
- A **parse tree** shows how a string can be **derived** from a grammar.
- A grammar is **ambiguous** if it can derive the same string multiple ways.
- There is no algorithm for eliminating ambiguity; it must be done by hand.

Closure Properties of Context-Free Languages

Closure Properties

- If L_1 and L_2 are regular, then
  - $\overline{L}_1$ is regular.
  - $L_1 \cup L_2$ is regular.
  - $L_1 \cap L_2$ is regular.
  - L_1L_2 is regular.
  - L_1^* is regular.
  - $h^*(L_1)$ is regular.
- How many of these properties still hold for context-free languages?

The Union of CFLs

- Suppose that L_1 and L_2 are ***context-free*** languages.
- Is $L_1 \cup L_2$ a context-free language?

X → **S** | **E**

S → **PS** | **ε**
P → **(S)**

E → **E + E** | **`int`**

- ***Yes!*** Use the above construction.
  - Rename nonterminals in the two grammars if necessary.

The Concatenation of CFLs

- Suppose that L_1 and L_2 are ***context-free*** languages.
- Is $L_1 L_2$ a context-free language?

X → SE

S → PS | ε
P → (S)

E → E + E | int

- ***Yes!*** Use the above construction.
  - Rename nonterminals in the two grammars if necessary.

The Kleene Closure of CFLs

- Suppose that *L* is a ***context-free*** language.
- Is *L** a context-free language?

$$\mathbf{X} \rightarrow \mathbf{EX} \mid \varepsilon$$

$$\mathbf{E} \rightarrow \mathbf{E} + \mathbf{E} \mid \texttt{int}$$

- ***Yes!*** Use the above construction.

Closure Properties of CFLs

- If L_1 and L_2 are context-free languages, then
  - $L_1 \cup L_2$ is context-free.
  - $L_1 L_2$ is context-free.
  - L_1^* is context-free.
  - $h^*(L_1)$ is context-free.
- Do the other properties still hold?
- We'll see early next week...

Next Time

- **Pushdown Automata**
  - Automata for recognizing CFLs.
  - A beautiful generalization of DFAs and NFAs.
  - An easy proof that any regular language is context-free.