Programming Turing Machines

Turing Machines are Hard

	1			×			=			B		
q_s	B	R	q_1	B	R	$q_{\times}$	B	R	$q_=$	B	R	q_{s2}
q_1	1	R	q_1	1	R	$q_{\times}$	1	R	$q_=$	1	L	q_R
$q_{\times}$	×	R	q_1	×	R	$q_{\times}$	×	R	$q_=$	×	L	q_R
$q_=$	=	R	q_1	=	R	$q_{\times}$	=	R	$q_=$	=	L	q_R
q_R	1	L	q_R	×	L	q_R	=	L	q_R	B	R	q_{s2}
q_{s2}	1	R	q_{s2}	×	R	$q_{\times 2}$	Reject			Reject		
$q_{\times 2}$	1	R	$q_{\times 2}$	Reject			=	R	$q_{=2}$	Reject		
$q_{=2}$	1	R	$q_{=2}$	Reject			Reject			B	L	q_{L2}
q_{L2}	1	L	q_{L2}	×	L	q_{L2}	=	L	q_{L2}	B	R	**D**

Outline for Today

- **A programming language for Turing machines**.
- Design a simple programming language that "compiles" down to Turing machines.
- Keep extending our language to see just how powerful the Turing machine is.

Our Initial Language: **WB**

- Programming language **WB** (“Wang B-machine”) controls a tape head over a singly-infinite tape, as in a normal Turing machine.
- Language has six commands:
  - **`Move`** ***direction***
    - Moves the tape head the specified direction (either left or right)
  - **`Write`** ***s***
    - Writes symbol ***s*** to the tape.
  - **`Go to`** ***N***
    - Jumps to instruction number *N* (all instructions are numbered)
  - **`If reading`** ***s*****`, go to`** ***N***
    - If the current tape symbol is ***s***, jump to the instruction numbered ***N***.
  - **`Accept`** and **`Reject`**
    - Ends the program.
- Statements in **WB** are executed in the order in which they appear, unless control flow changes.

A Simple Program in **WB**

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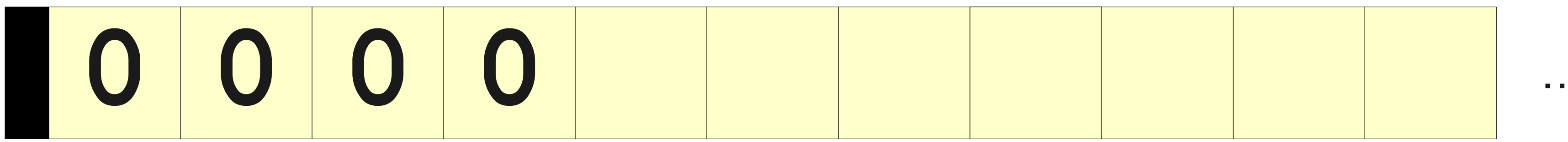


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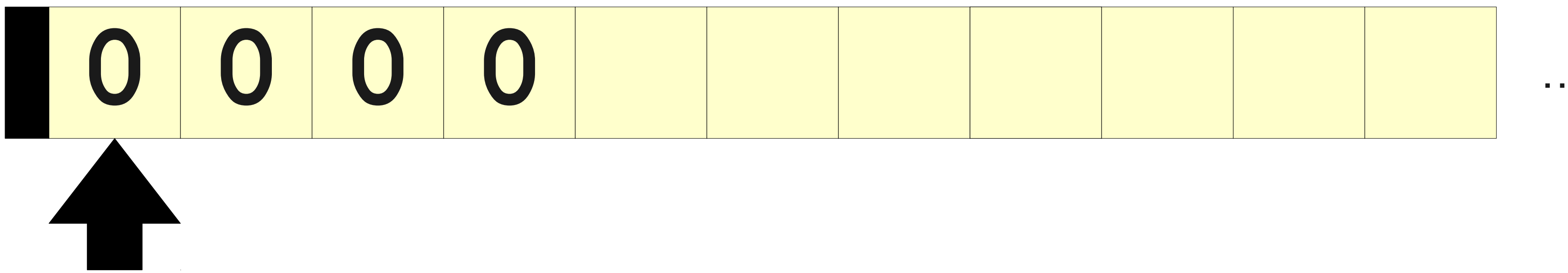


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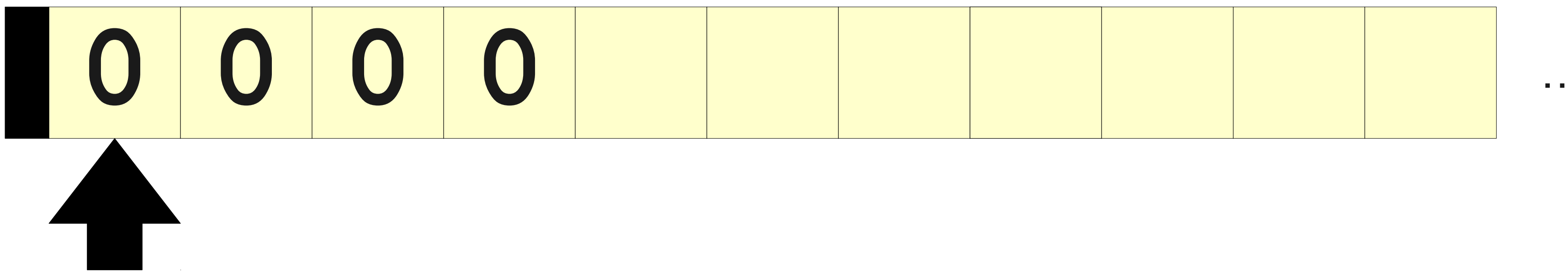


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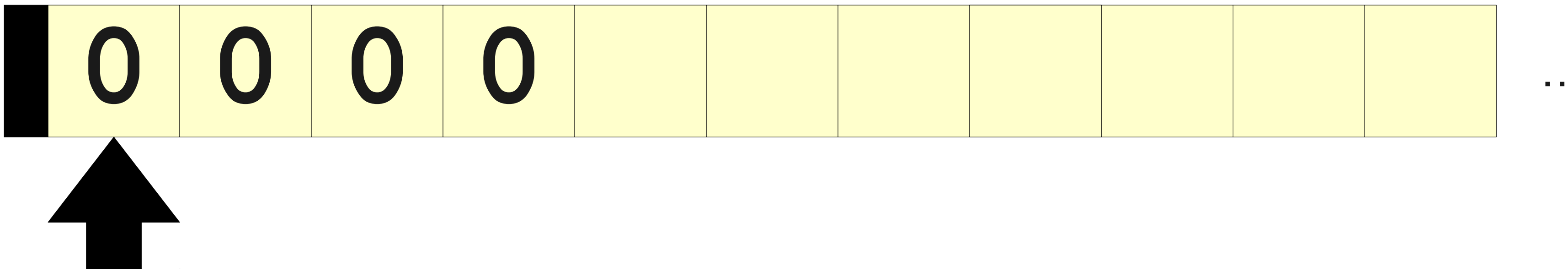


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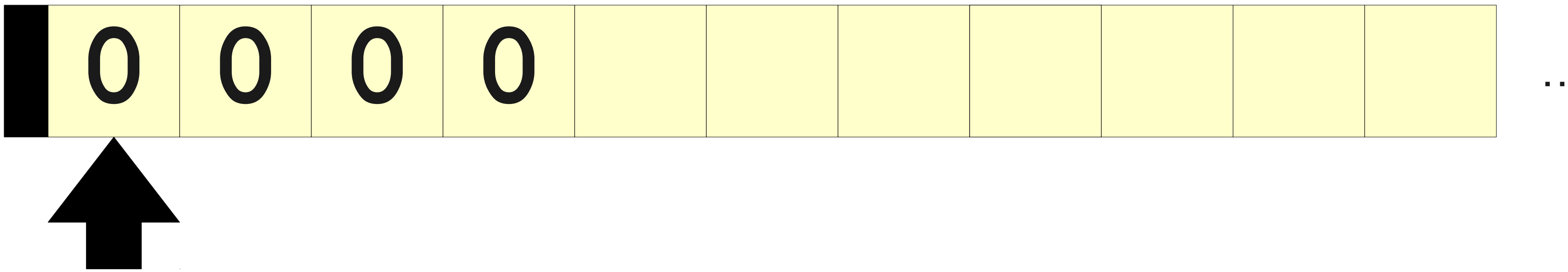


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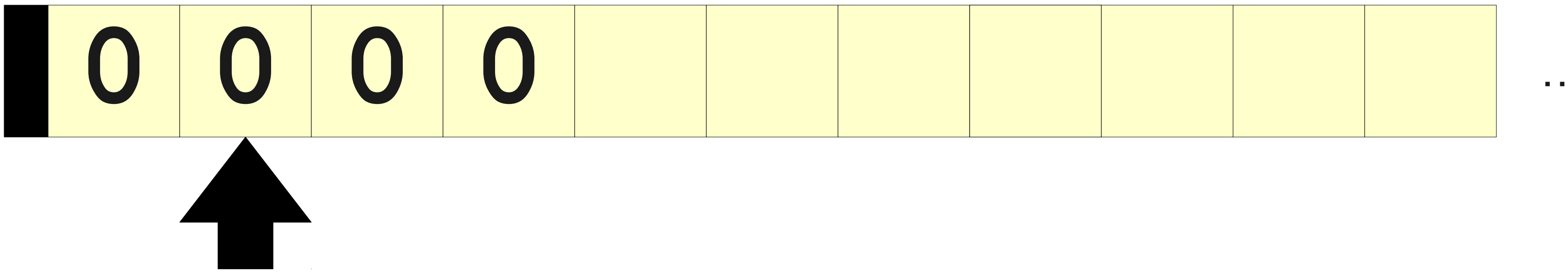


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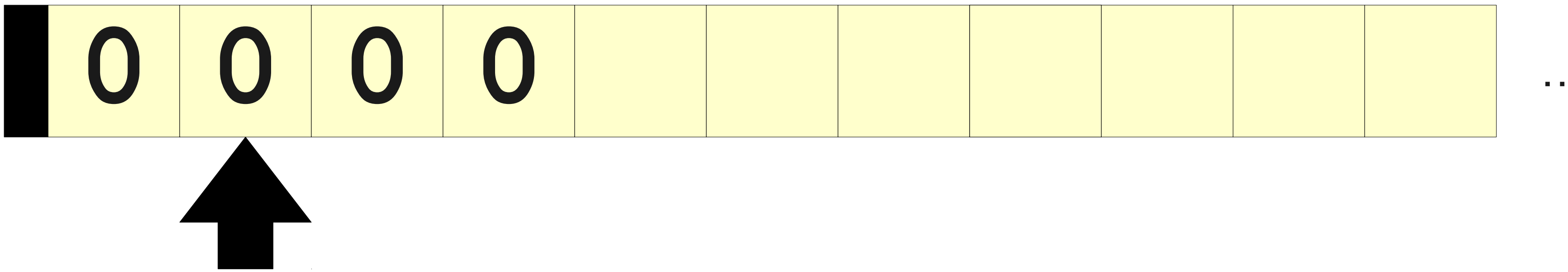


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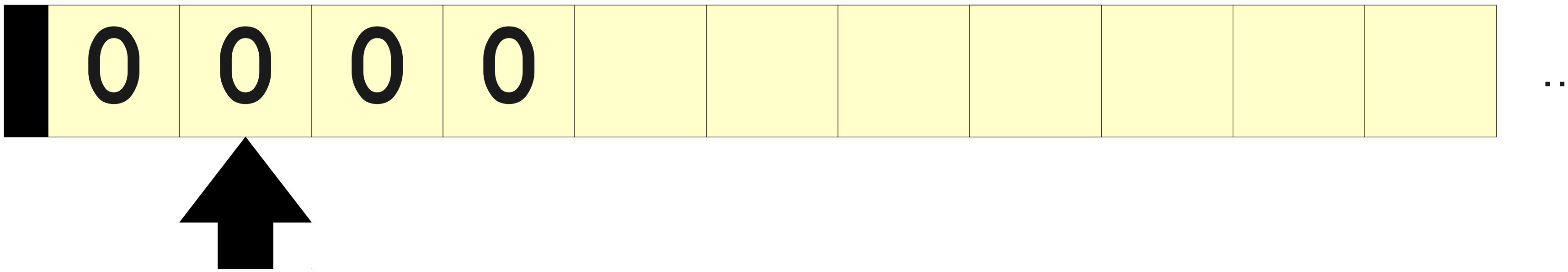


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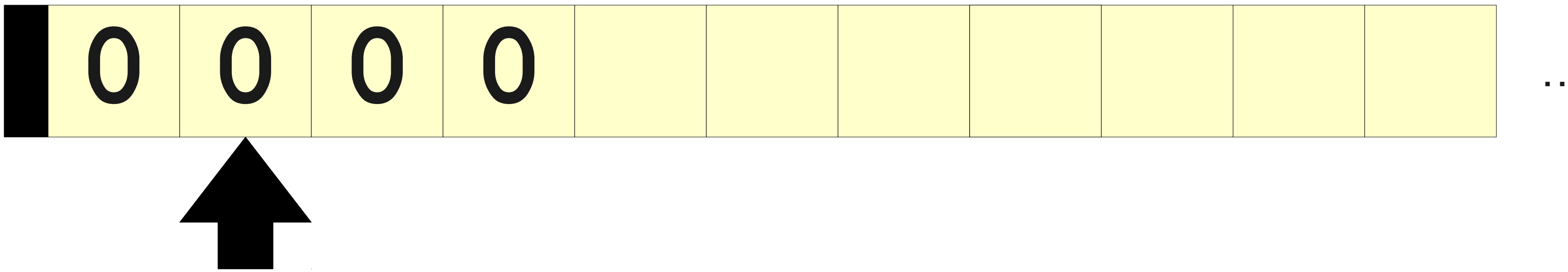


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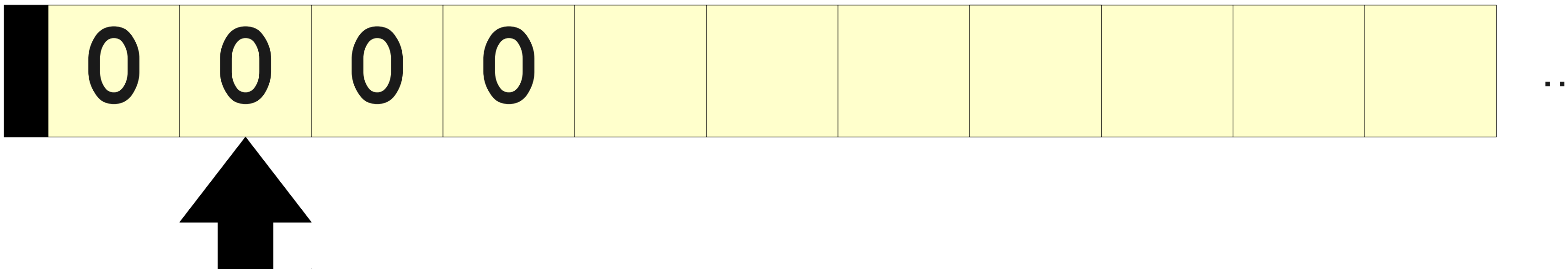


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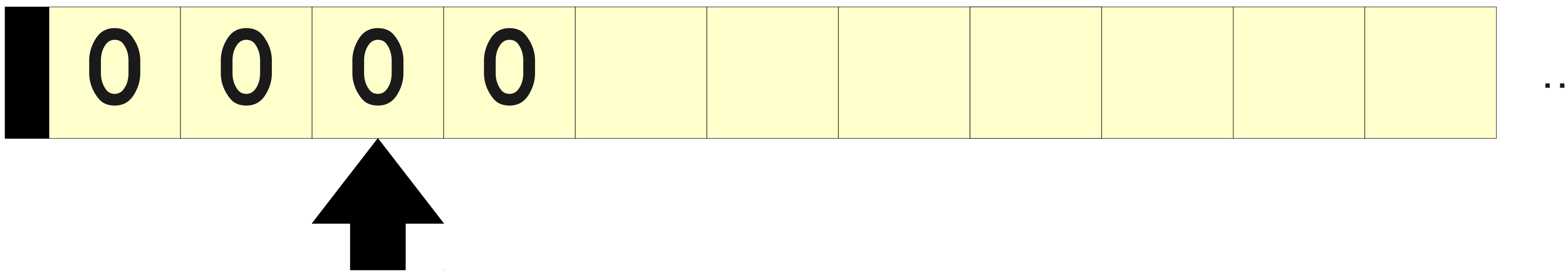


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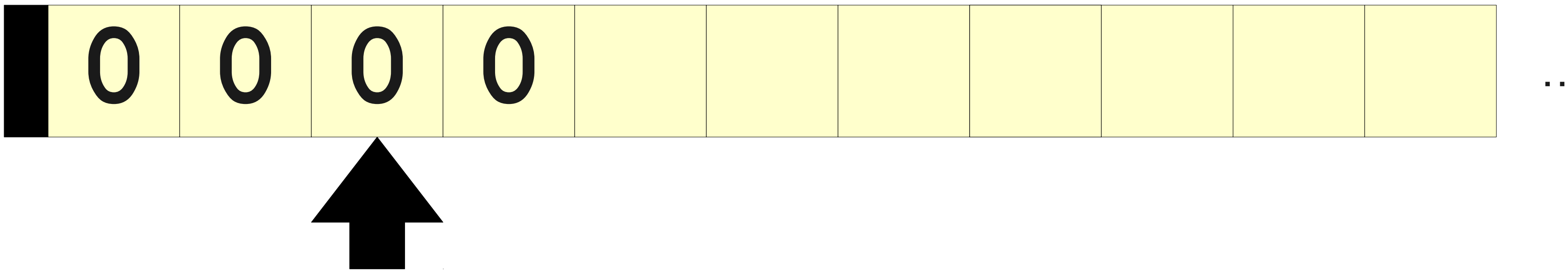


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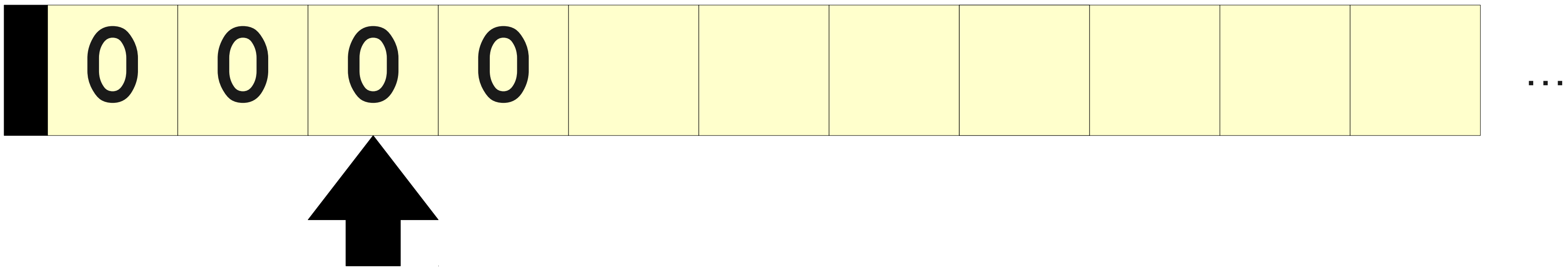


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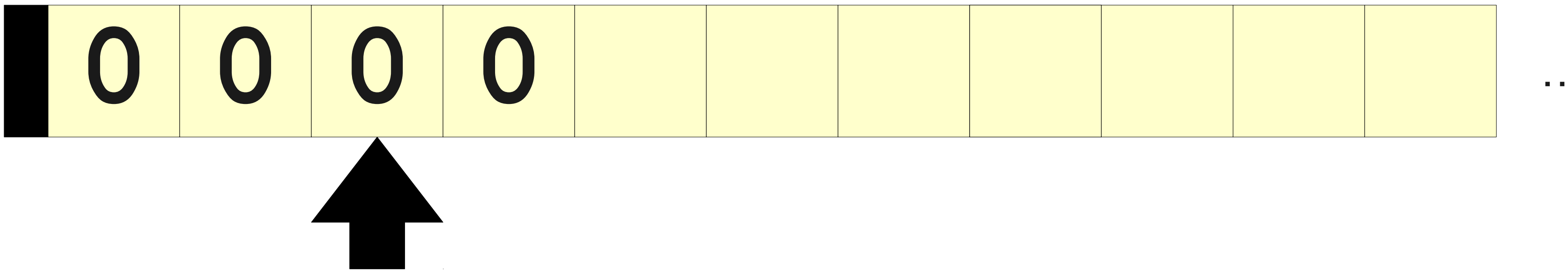


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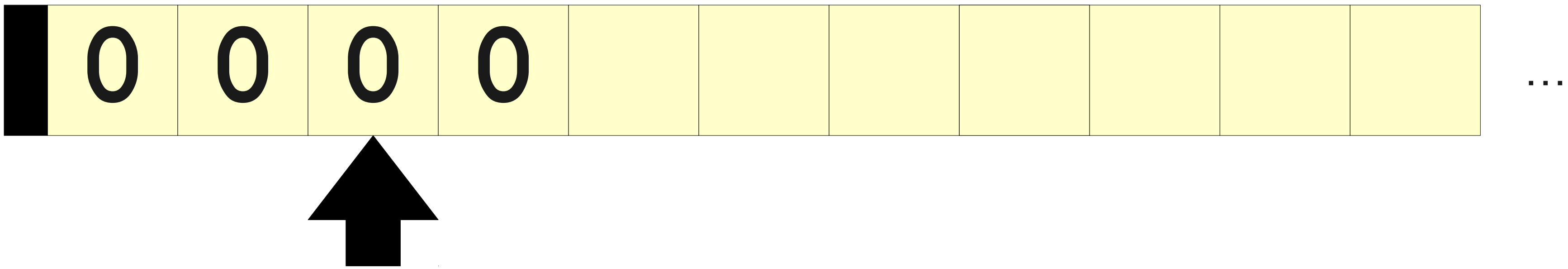


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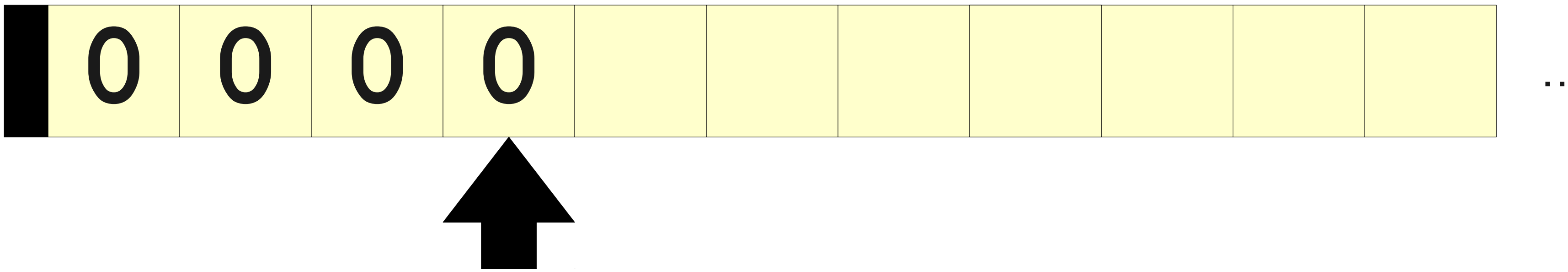


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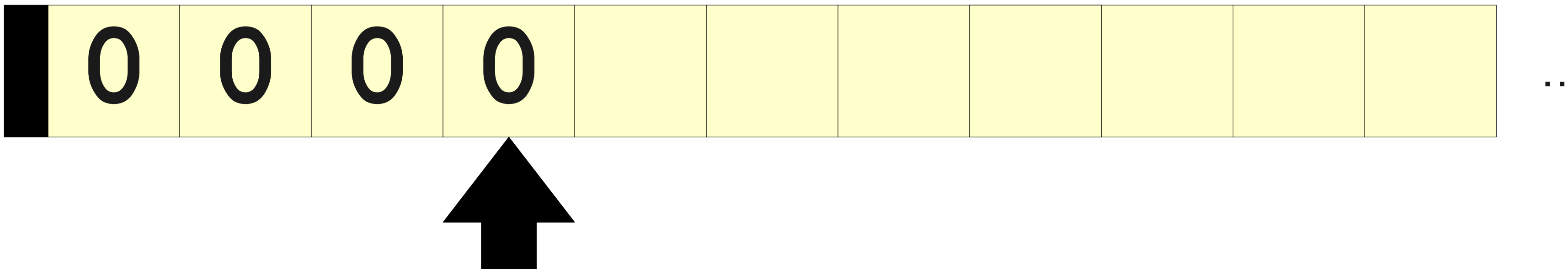


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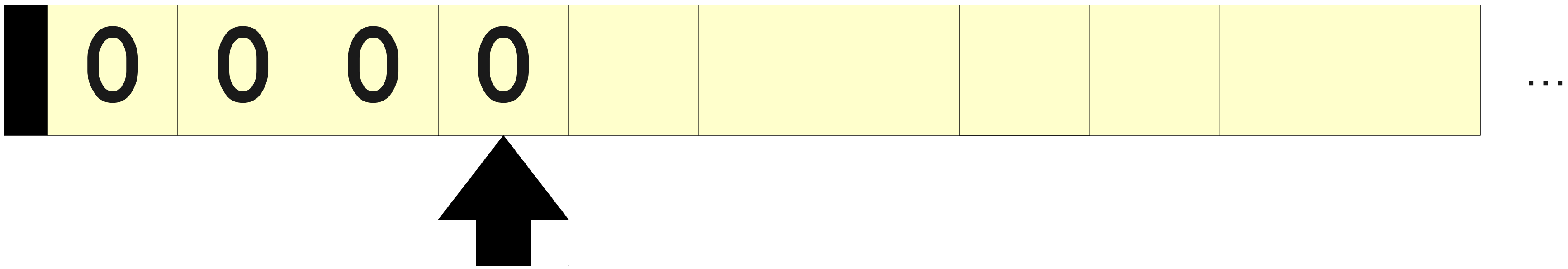


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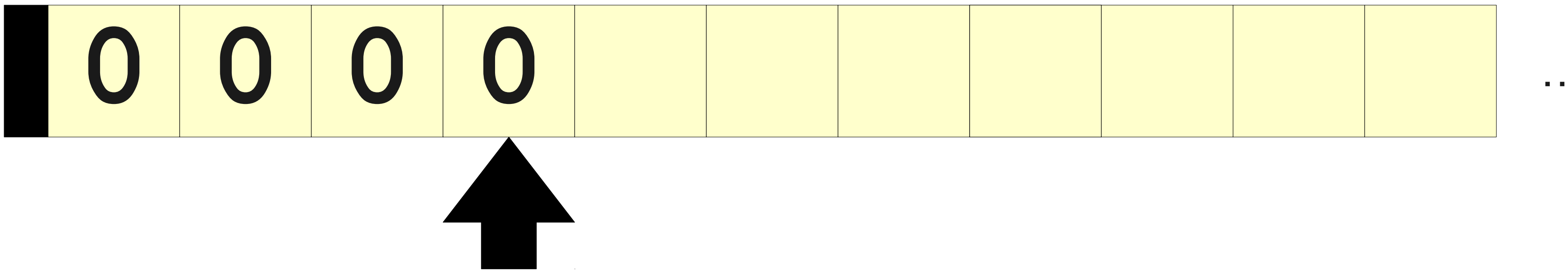


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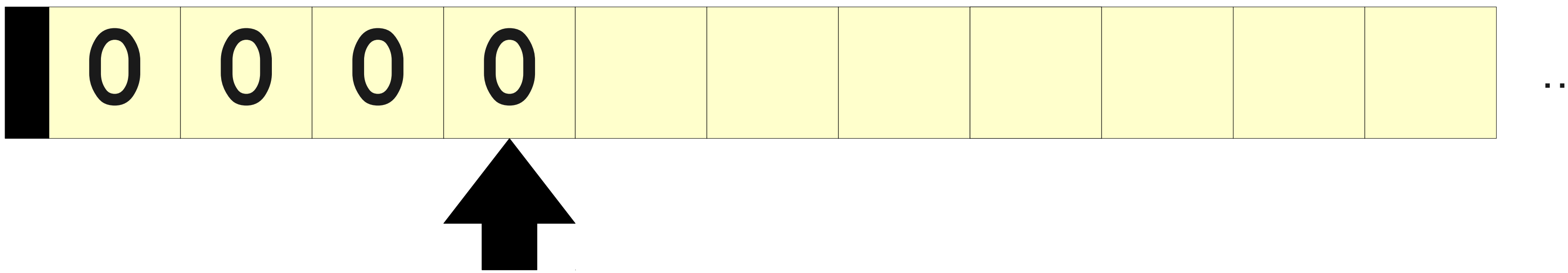


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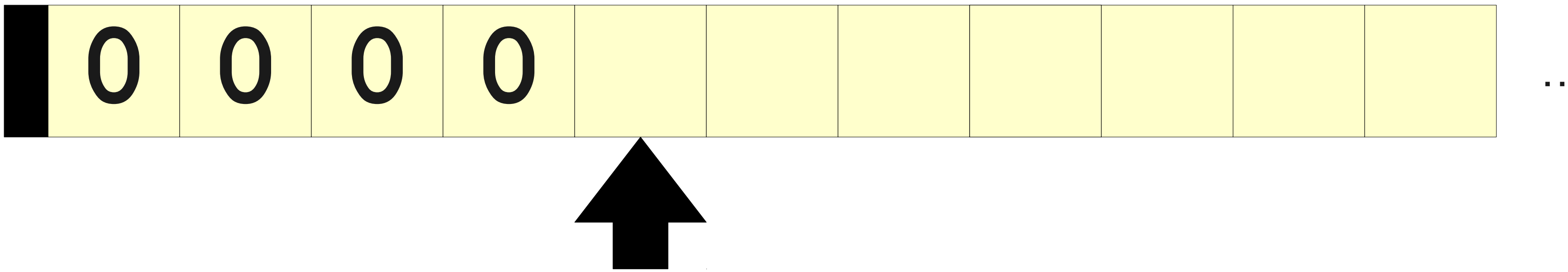


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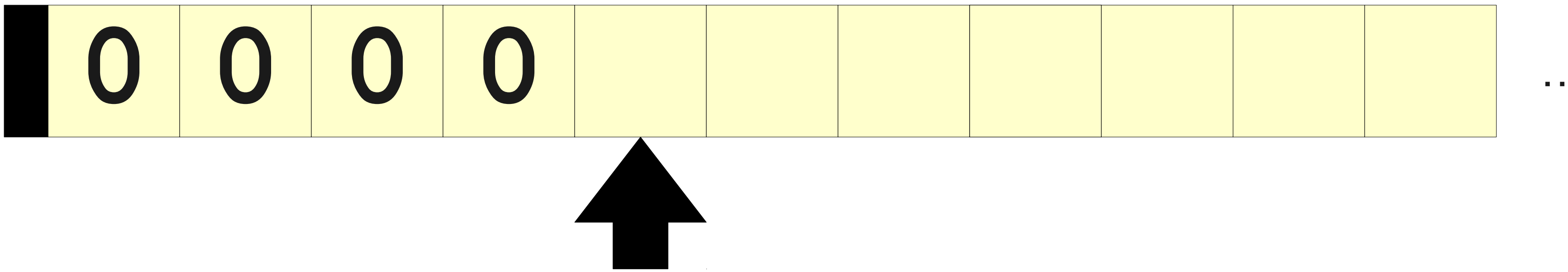


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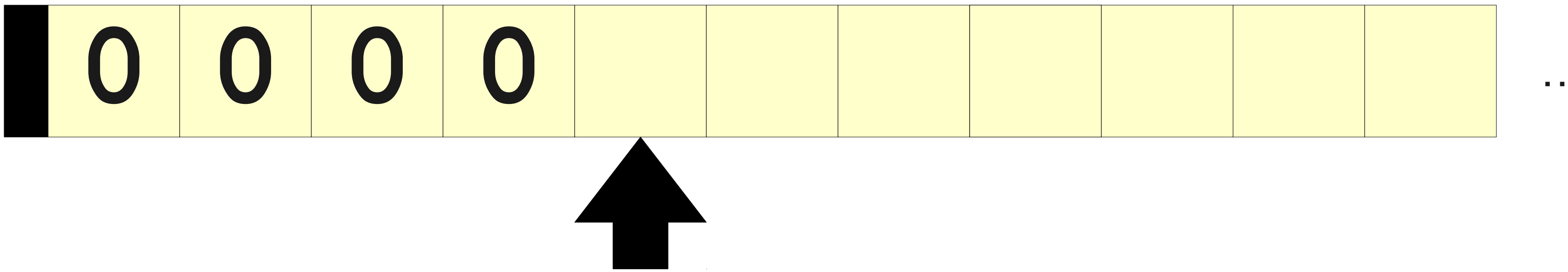


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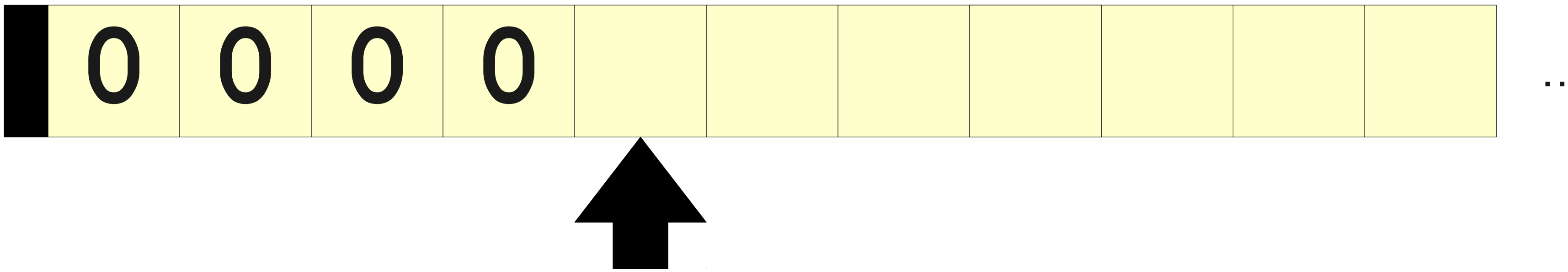


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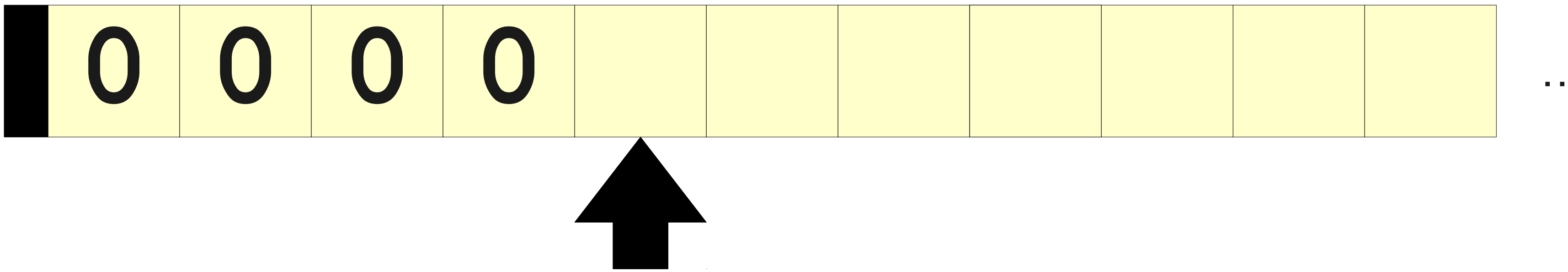


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A **WB** Program for Even Palindromes

- Suppose we want to test if a string is an even-length palindrome.
- Idea: Cross off the first symbol and match it with the symbol on the far side of the tape.
- If it matches, great! Repeat.
- Otherwise, we should reject.

A **WB** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.
```

```
// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
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15: If reading 1, go to Next.
16: Reject.

// Next
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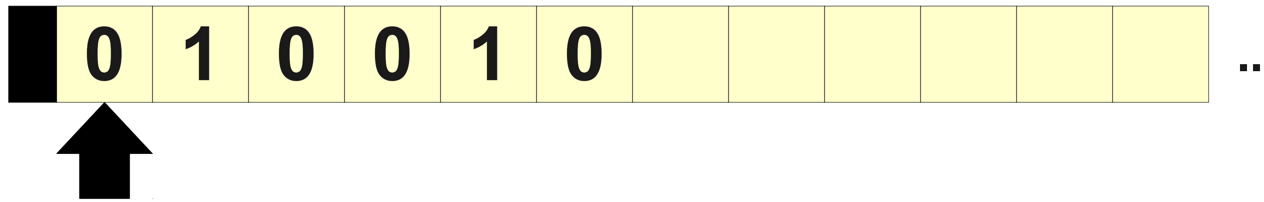


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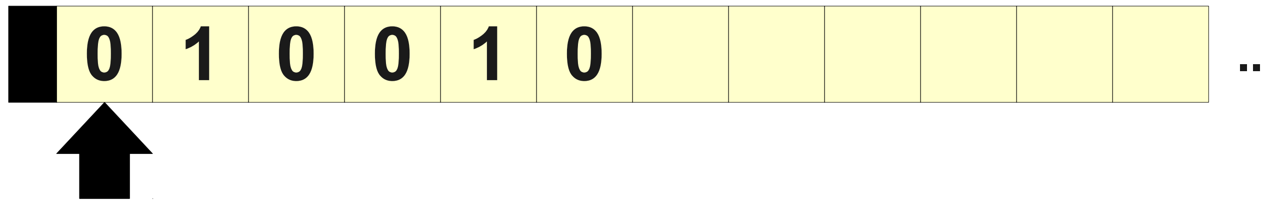


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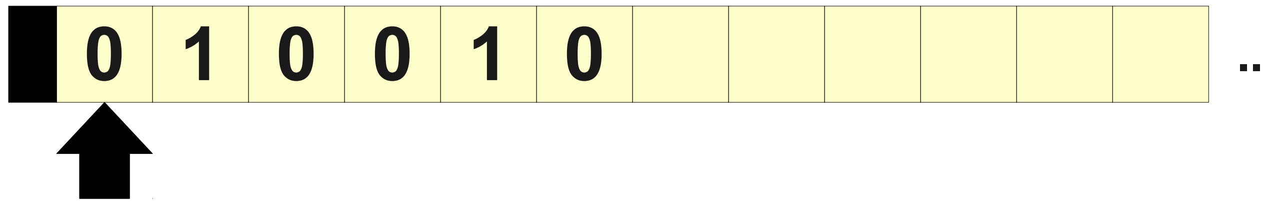


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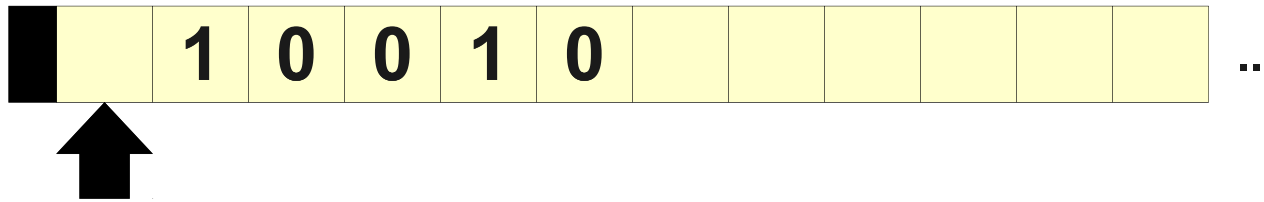


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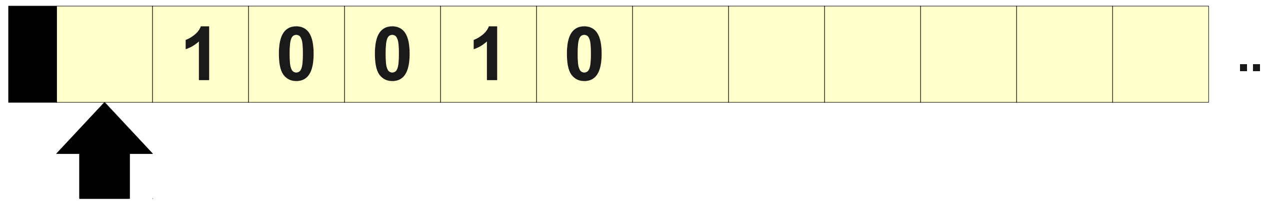


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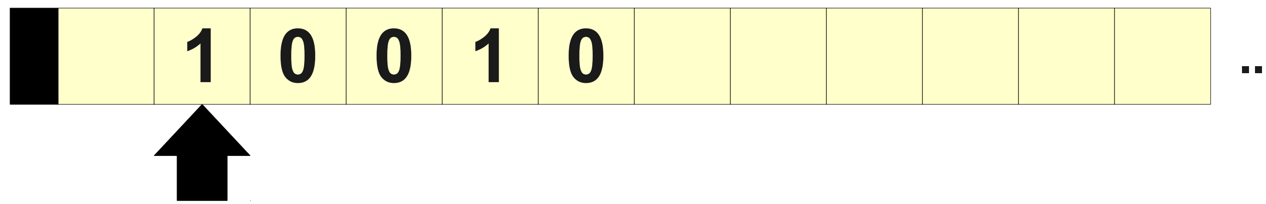


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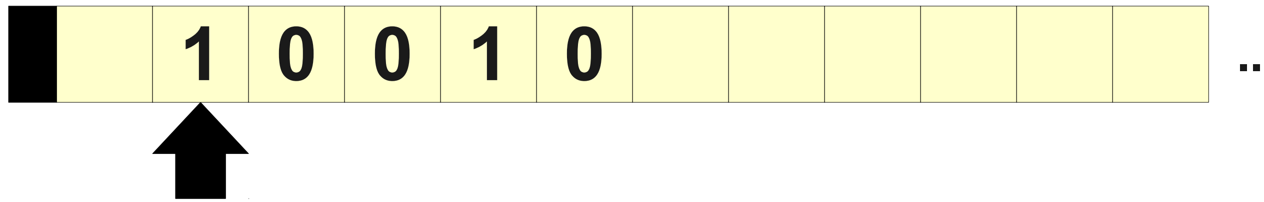


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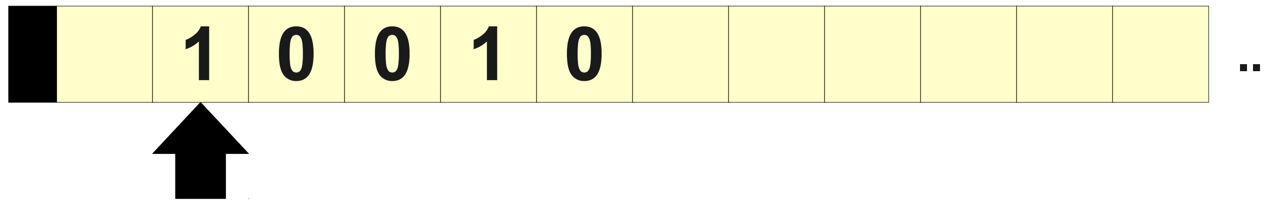


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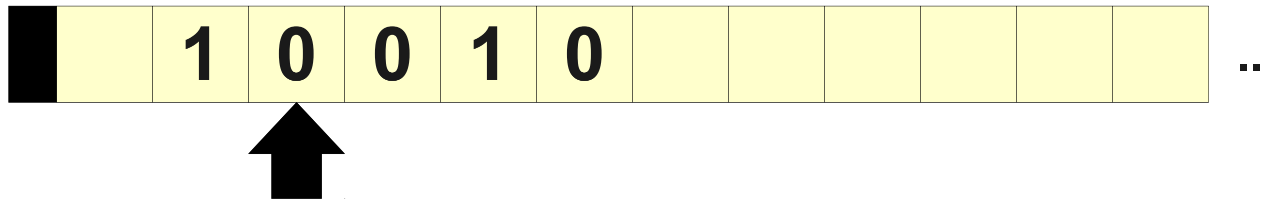


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 2: Accept

// M0
 3: Write B.
 4: Move right.
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 9: Reject.
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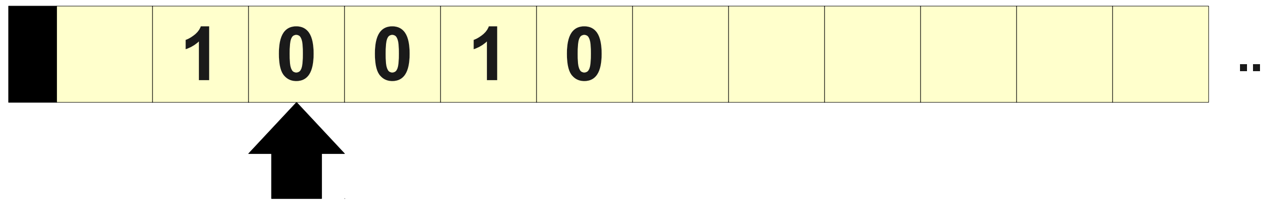


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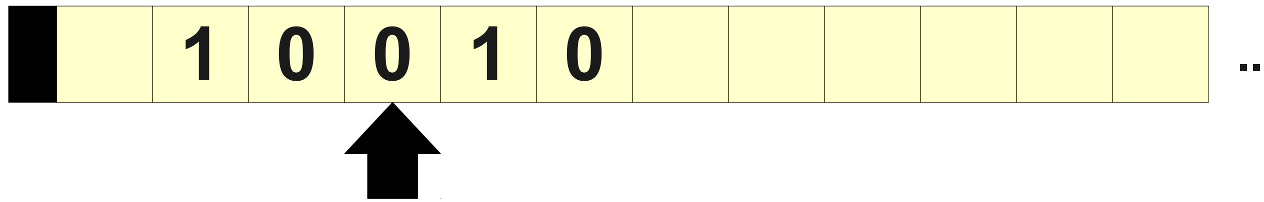


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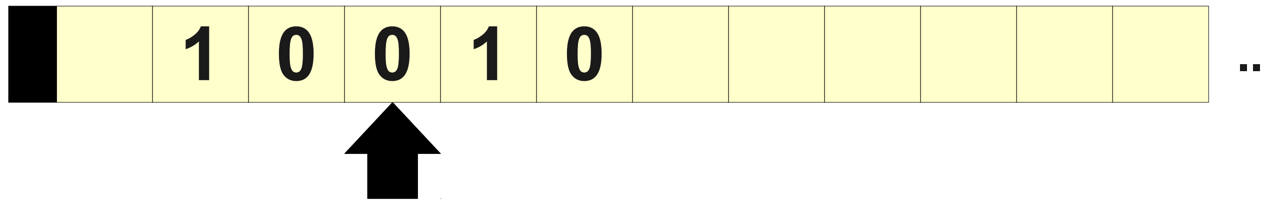


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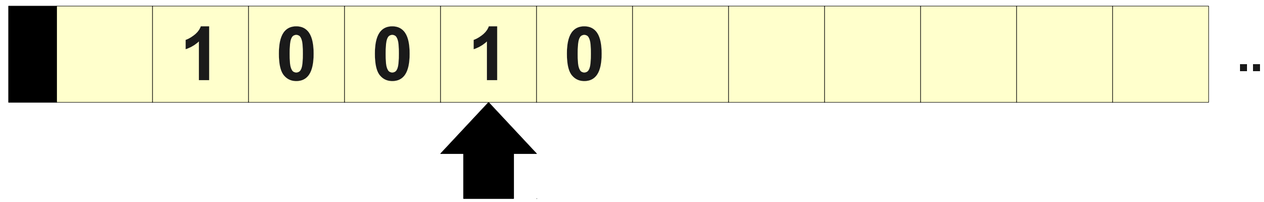


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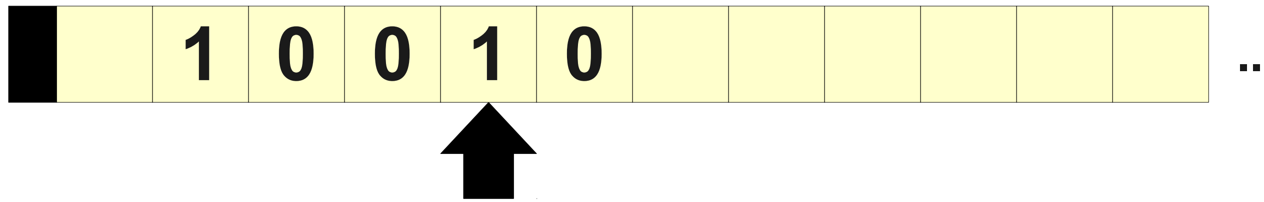


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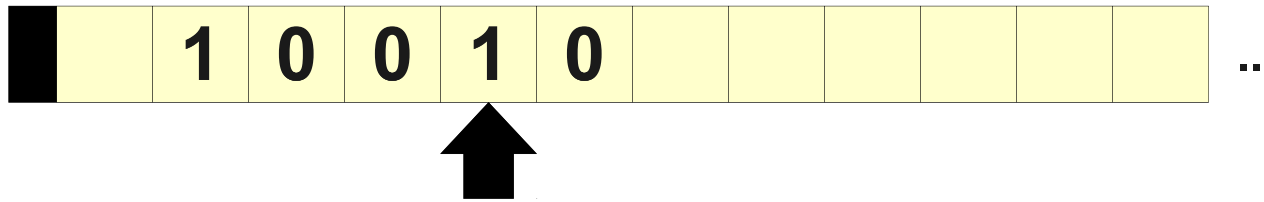


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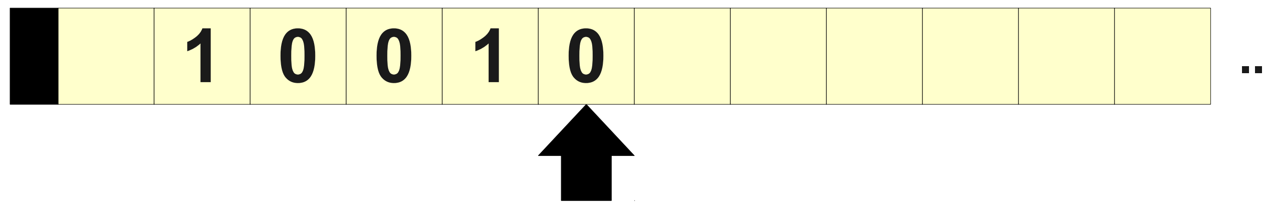


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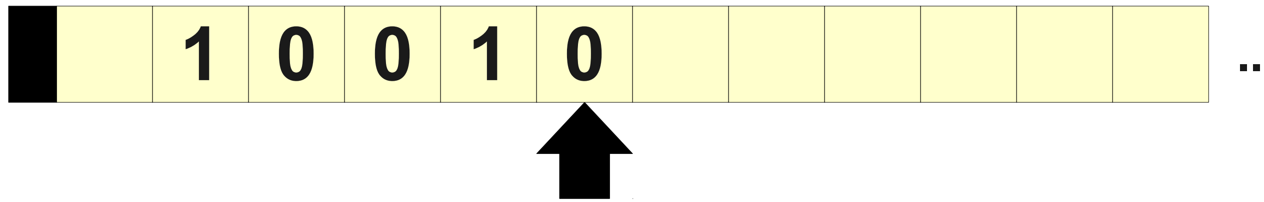


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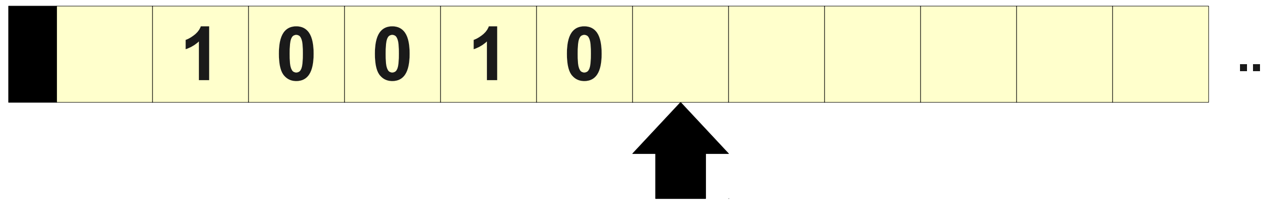


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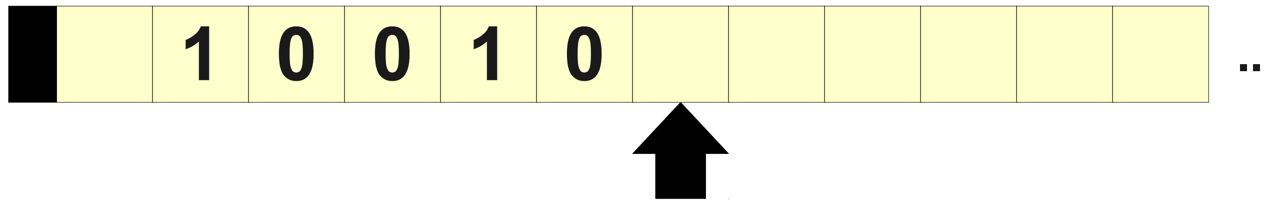


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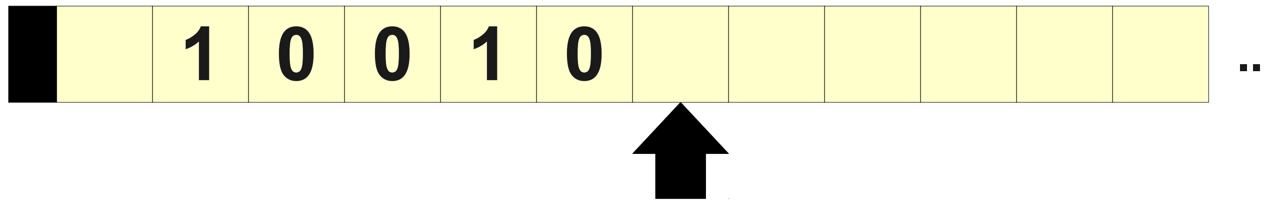


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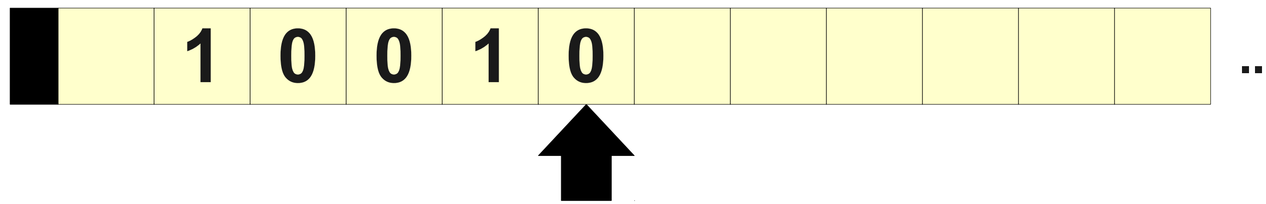


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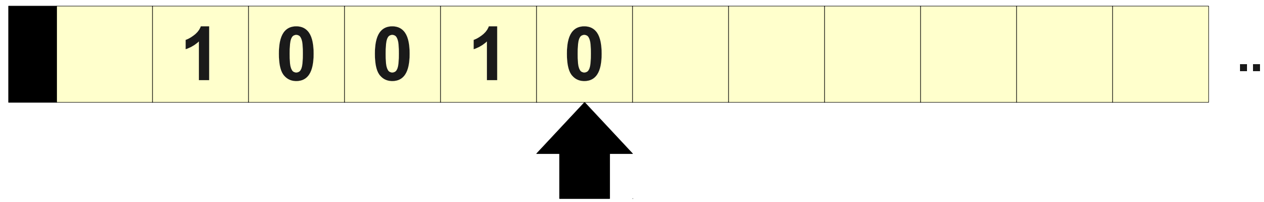


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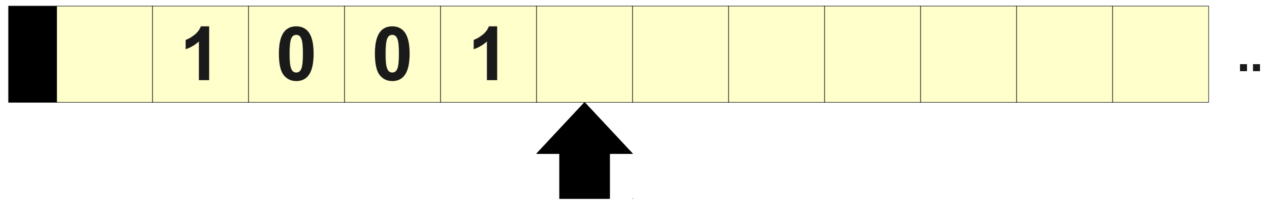


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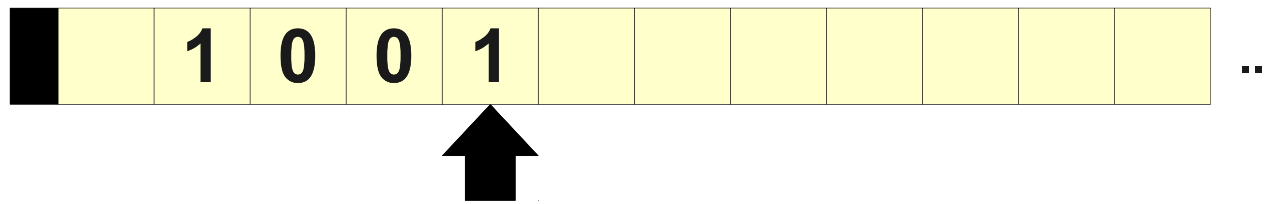


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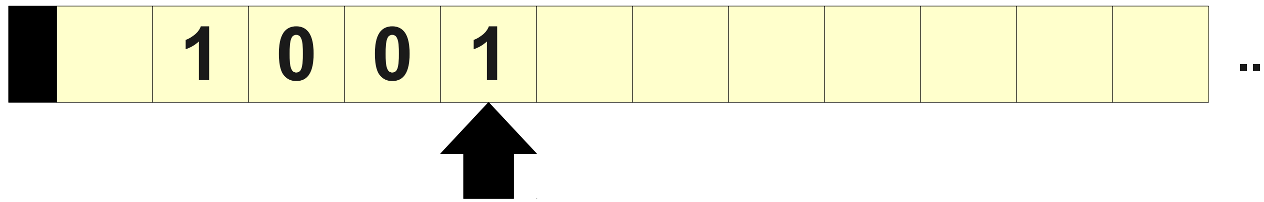


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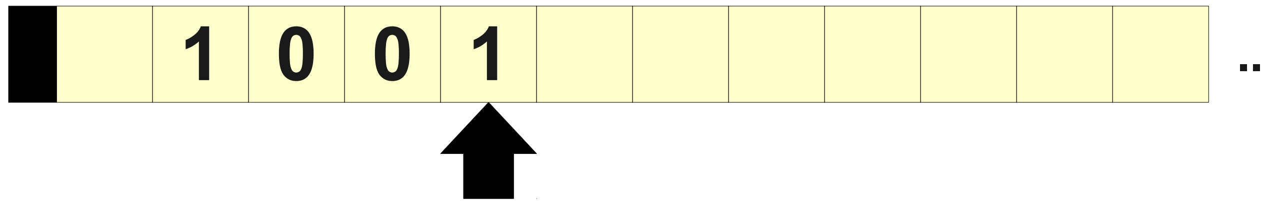


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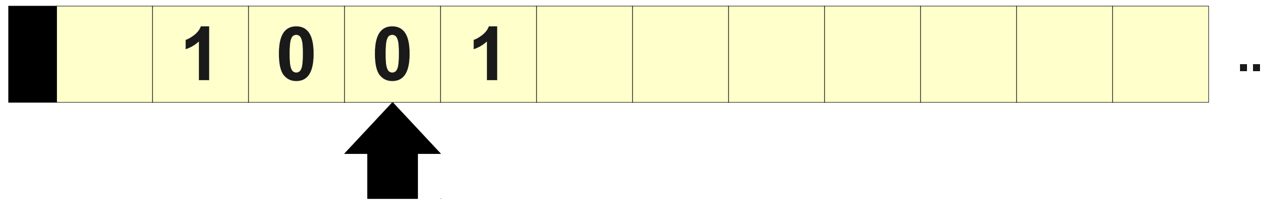


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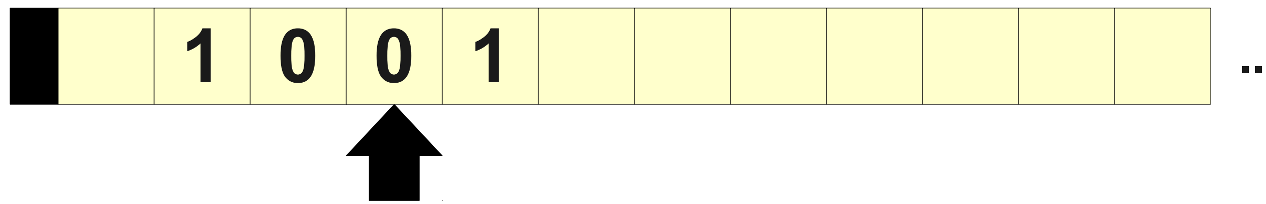


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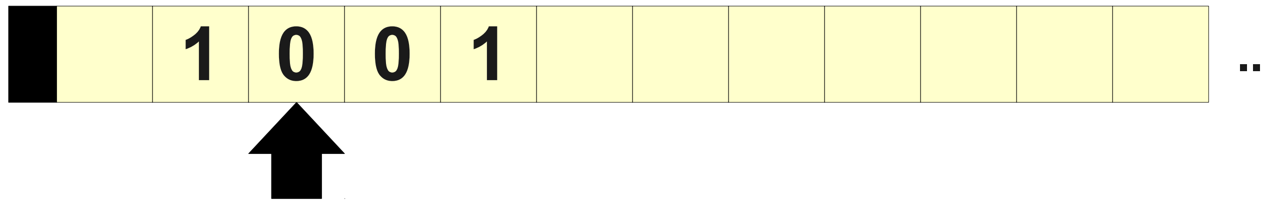


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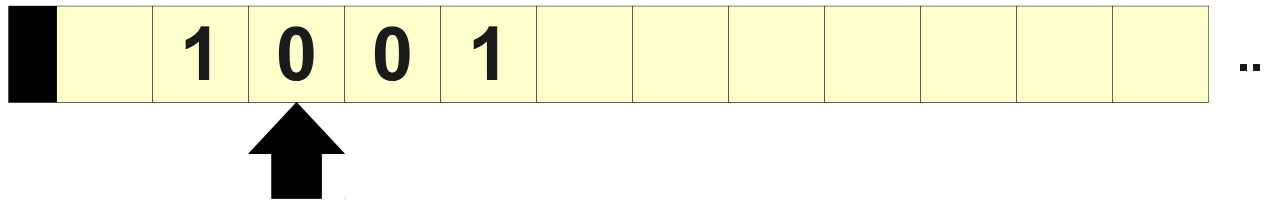


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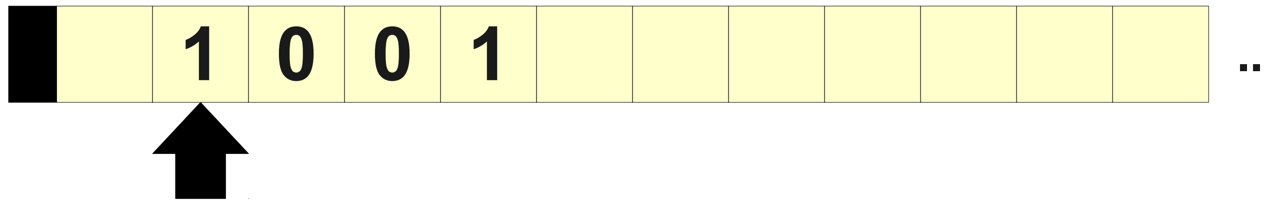


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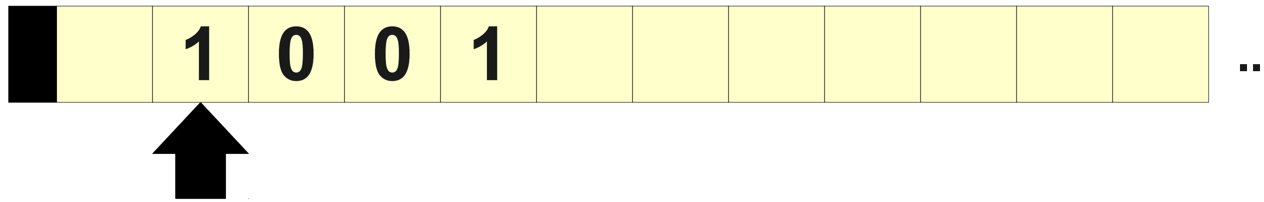
A **WB** Program for Even Palindromes



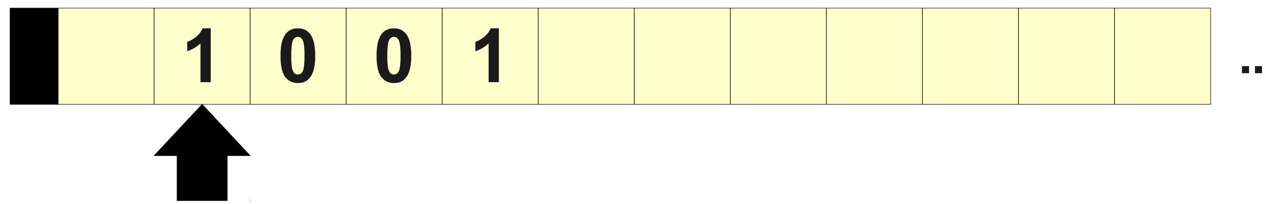

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.

// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
13: If reading 1, go to 11.
14: Move left.
15: If reading 1, go to Next.
16: Reject.

// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
20: If reading 1, go to 18
21: Move right
22: Go to Start.
```

A **WB** Program for Even Palindromes




```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
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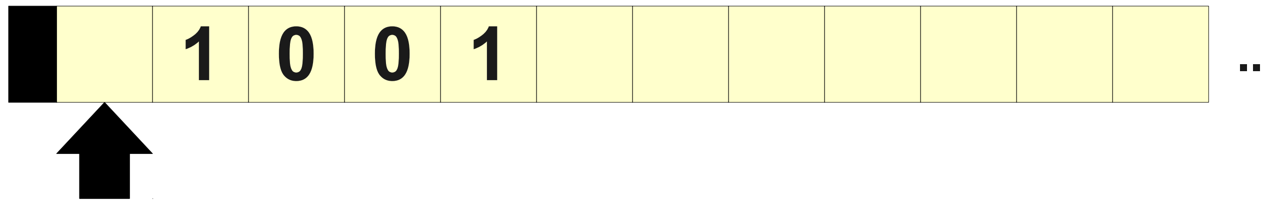
A **WB** Program for Even Palindromes




```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.
```

```
// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
13: If reading 1, go to 11.
14: Move left.
15: If reading 1, go to Next.
16: Reject.

// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
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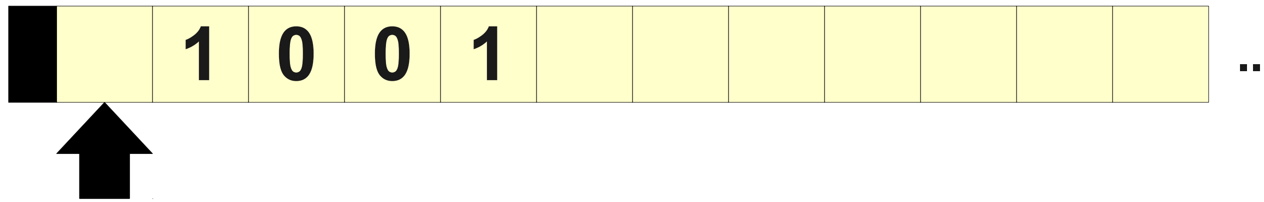
22: Go to Start.
```

A **WB** Program for Even Palindromes




```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.

// M1
10: Write B.
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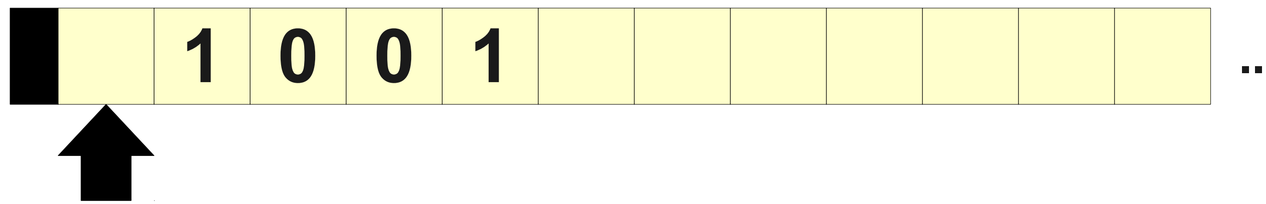
A **WB** Program for Even Palindromes




```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.

// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
13: If reading 1, go to 11.
14: Move left.
15: If reading 1, go to Next.
16: Reject.

// Next
17: Write B.
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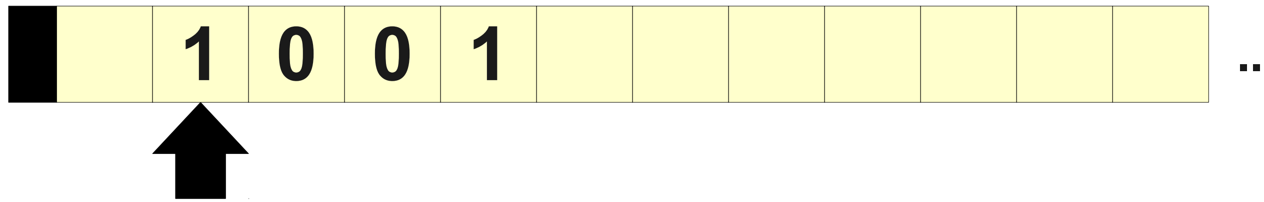
21: Move right
22: Go to Start.
```

A **WB** Program for Even Palindromes




```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.
```

```
// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
13: If reading 1, go to 11.
14: Move left.
15: If reading 1, go to Next.
16: Reject.

// Next
17: Write B.
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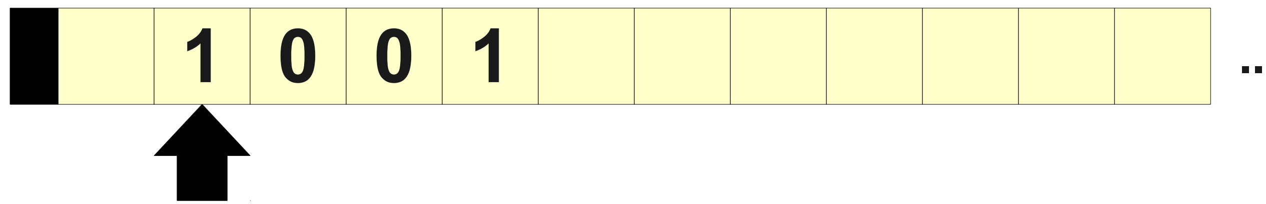
```

A **WB** Program for Even Palindromes




```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
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// M1
10: Write B.
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16: Reject.

// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
20: If reading 1, go to 18
21: Move right
22: Go to Start.
```

WB and Turing Machines

- *Recall:* A language L is **recursively enumerable** iff there is a TM for it.
- **Theorem:** A language L is recursively enumerable iff there is a **WB** program for it.
- Need to show the following:
  - Any TM can be converted into an equivalent **WB** program.
  - Any **WB** program can be converted into an equivalent TM.

From Turing Machines to **WB**

- Basic idea: Construct a small **WB** program for each state that simulates that state.
- Combine all programs together to get an overall **WB** program that simulates the Turing machine.

A State in a Turing Machine

- There are three kinds of states in a Turing machine:
  - Accepting states,
  - Rejecting states, and
  - “Working” states.
- We can easily build **WB** programs for the first two:

```
// q_acc
0: Accept
```

```
// q_rej
0: Reject
```

Working States

- At a given working state in a Turing machine, we will do exactly the following, in this order:
  - Read the current symbol.
  - Write back a new symbol based on this choice of symbol.
  - Transition to some destination state.
- Could we build a **WB** program for this?

Working States

	0			1			B		
q_0	B	R	q_1	0	L	q_0	B	R	q_{acc}

// q_0
0: If reading 0, go to $0q_0$.
1: If reading 1, go to $1q_0$.
2: If reading B, go to Bq_0.

// $0q_0$
3: Write B.
4: Move right.
5: Go to q_1

// $1q_0$
6: Write 0
7: Move left.
8: Go to q_0

// Bq_0
9: Write B
10: Move right.
11: Go to q_{acc}

A Complete Construction

	0			1			B		
q_0	0	R	q_1	1	R	q_{rej}	1	R	q_{acc}
q_1	0	R	q_{rej}	1	R	q_0	1	R	q_{acc}

// q_0

0: If reading 0, go to 3.
1: If reading 1, go to 6.
2: If reading B, go to 9.
3: Write 0.
4: Move right.
5: Go to q_1.
6: Write 1.
7: Move right.
8: Go to q_{rej}.
9: Write 1.
10: Move right.
11: Go to q_{acc}.

// q_{acc}

12: Accept.

// q_1

13: If reading 0, go to 16.
14: If reading 1, go to 19.
15: If reading B, go to 22.
16: Write 0.
17: Move right.
18: Go to q_{rej}.
19: Write 1.
20: Move right.
21: Go to q_0.
22: Write 1.
23: Move right.
24: Go to q_{acc}.

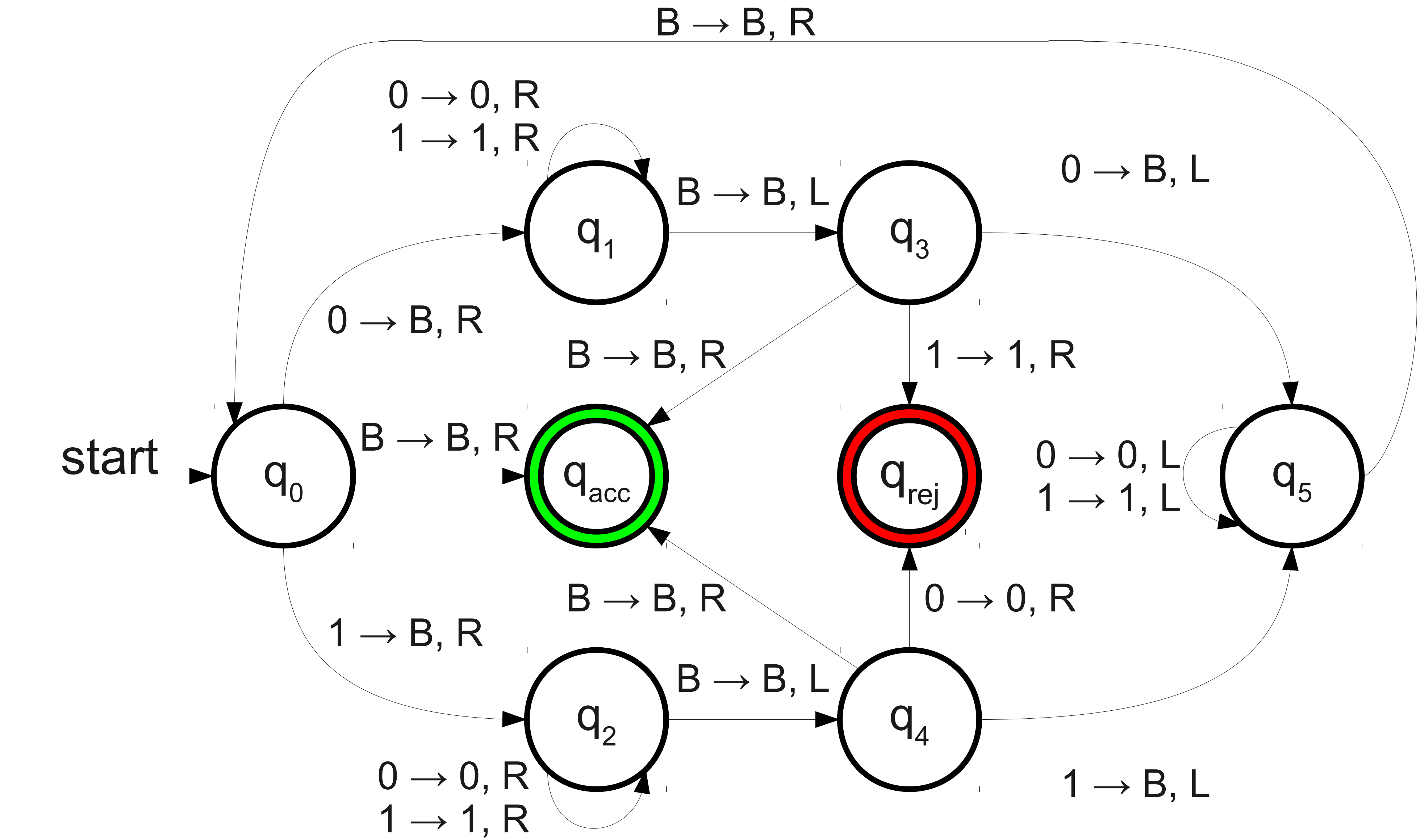
// q_{rej}

25: Reject.

From **WB** to Turing Machines

- We now need a way to convert a **WB** program into a Turing machine.
- Construction sketch:
  - Create a state in the TM for each line of the **WB** program.
  - Introduce extra "helper" states to implement some of the trickier instructions.
  - Connect the states by transitions that simulate the **WB** program.
- We will show how to translate each **WB** command into a collection of states plus transitions.

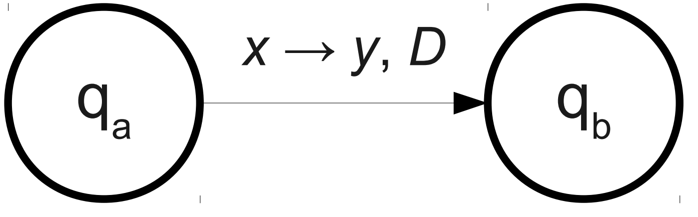
Refresher: Turing Machine Notation



Refresher: Turing Machine Notation

- The accept and reject states are denoted

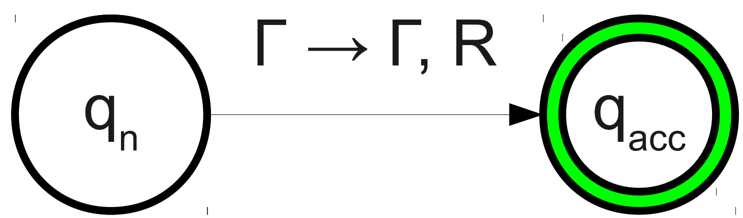
- A transition of the form



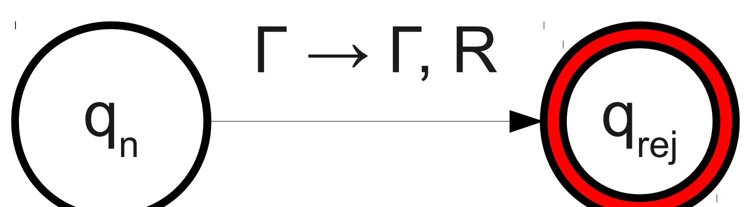
means "on seeing *x*, write *y* and move direction *D*."

`Accept` and `Reject`

- The **`Accept`** and **`Reject`** commands are the easiest to translate.
- To translate **`N: Accept`** into TM states, construct the following:

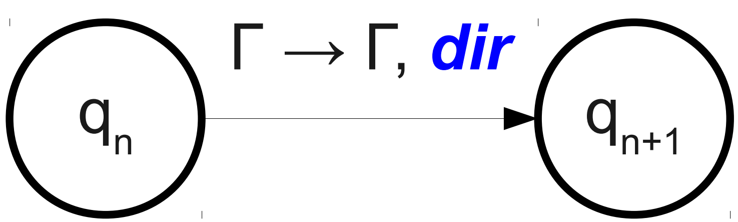


- To translate **`N: Reject`** into TM states, construct the following:



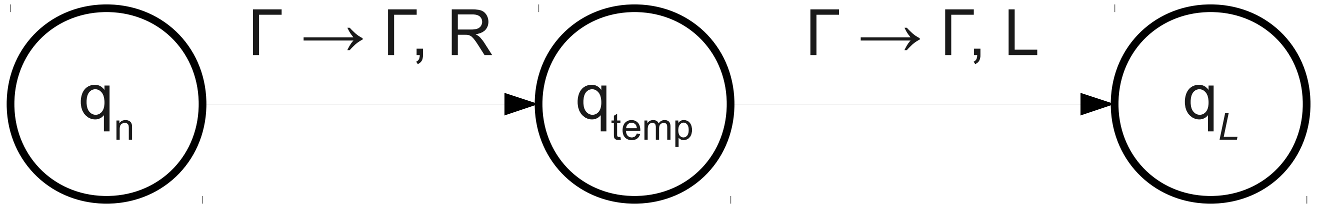
`Move left` and **`Move right`**

- We can translate **`N: Move left`** and **`N: Move right`** by having the TM do the following:
  - Write back the same symbol that was already on the tape (ensuring that we don't change the tape).
  - Move in the indicated direction.
  - Transition into the state representing line **`N`** + 1.



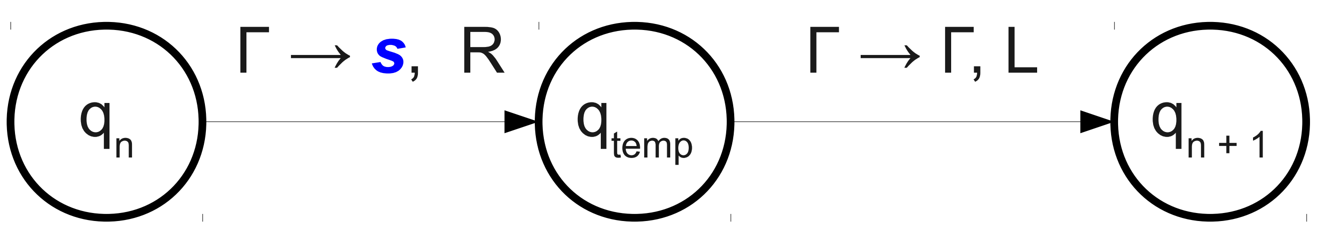
Go to *L*

- The line **`N: Go to M`** needs to change into the state for line **`M`** without moving the tape head.
- All TM transitions move the tape head; how might we address this?
- Move right and change into a new state that then moves back to the left.



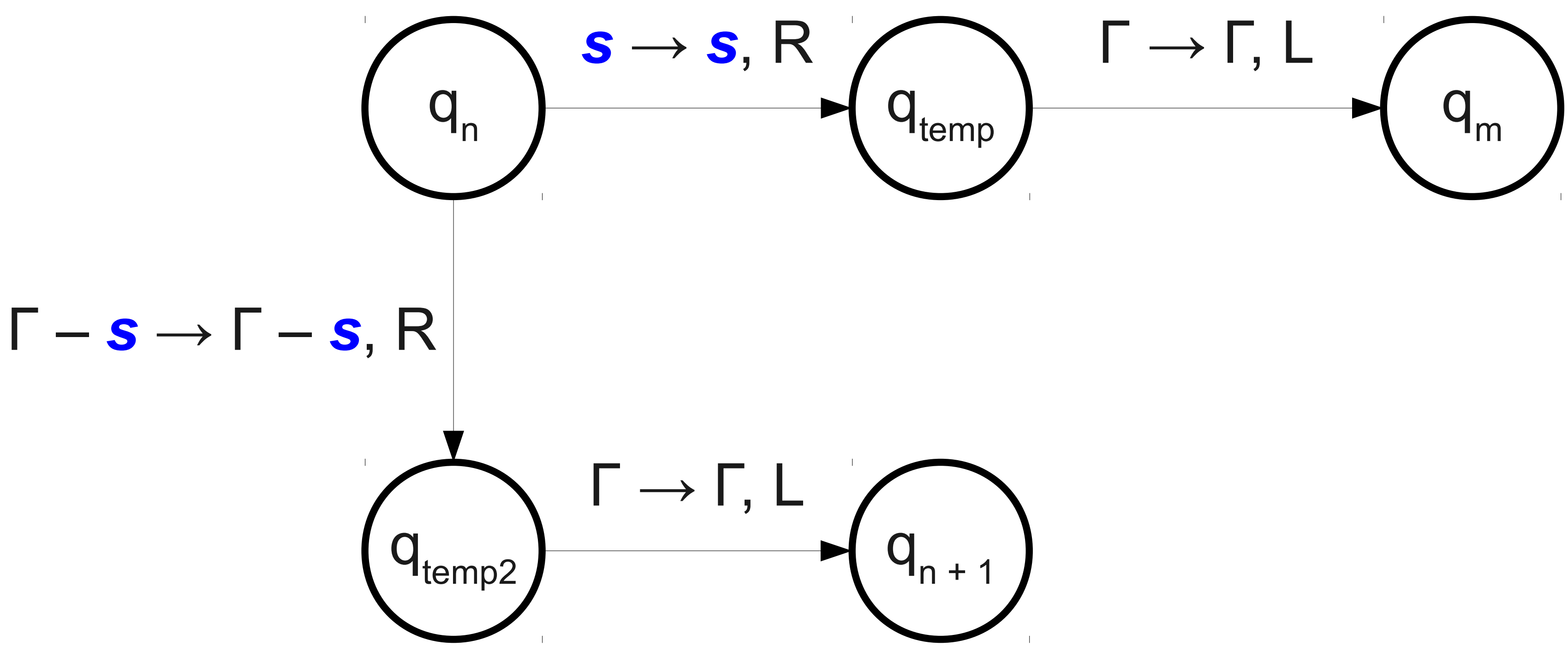
`Write` ***s***

- The line **`N: Write`** ***s*** needs to
  - Write the symbol ***s***,
  - Leave the tape head where it is, and
  - Move to line **`N`** + 1.
- We use a similar trick as before:

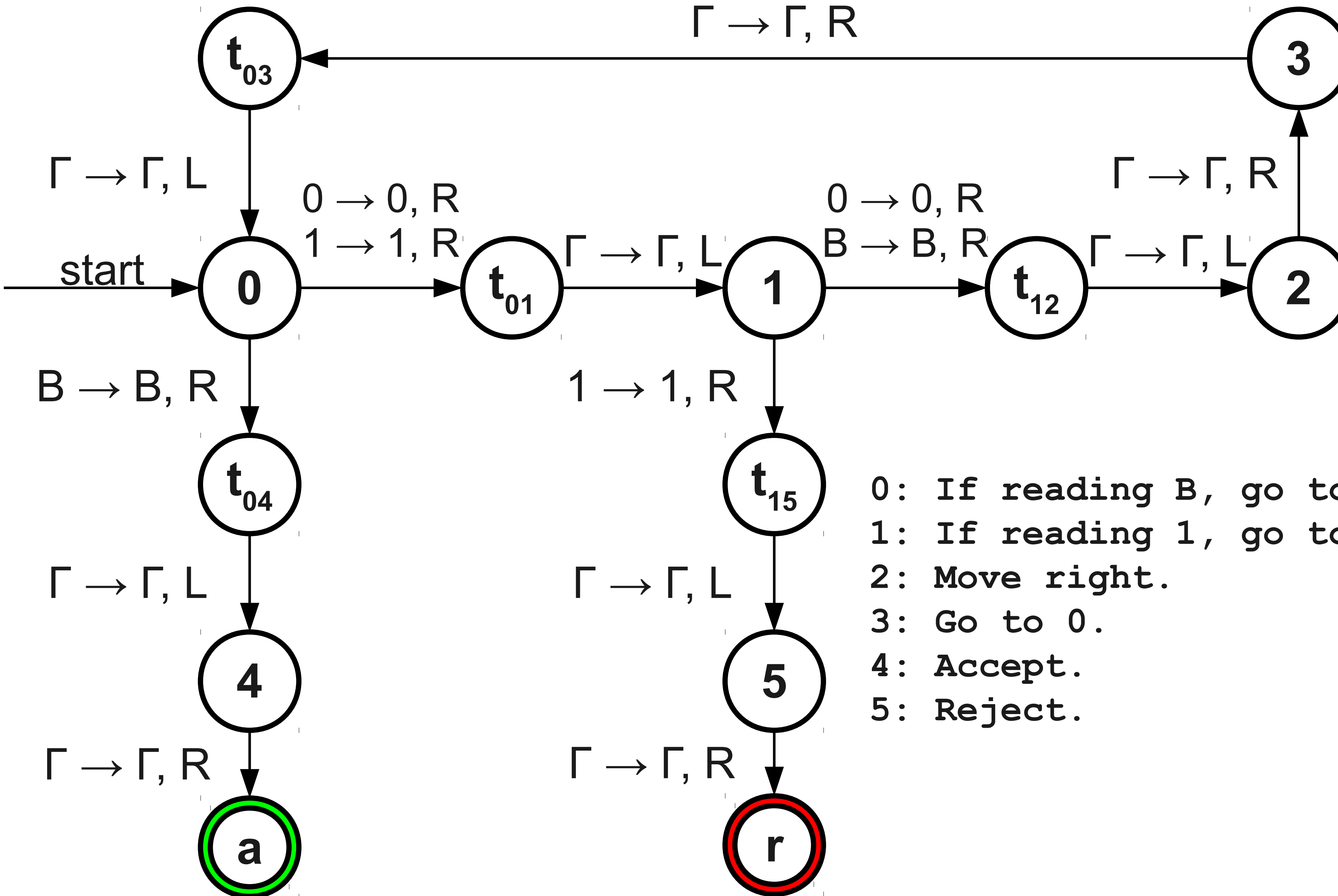


If reading *s*, go to M

- The line **`N: If reading s, go to M`** either
  - Executes a "go to **`M`**" step as before if reading **`s`**, or
  - Does nothing and transitions to state **`N`** + 1.



A Complete Conversion




```
0: If reading B, go to 4.
1: If reading 1, go to 5.
2: Move right.
3: Go to 0.
4: Accept.
5: Reject.
```

The Story So Far

- We have just built a simple programming language that is equivalent in power to a Turing machine.
- This language, however, makes for some very complicated programs.
- Let's add some new features to our programming language to make it a bit easier to work with.

Revisiting Even Palindromes

```
// M0
3: Write B.
4: Move right.
5: If reading 0, go to 4.
6: If reading 1, go to 4.
7: Move left.
8: If reading 0, go to Next.
9: Reject.
```

```
// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
20: If reading 1, go to 18
21: Move right.
22: Go to Start.
```

- Steps 4 – 6 essentially say “move right, then move right until you read a blank.”
- Steps 18 – 20 essentially say “move left, then move left until you read a blank.”
- Is it really necessary to write this out each time?

Introducing **WB2**

- The programming language **WB2** is the language **WB** with two new commands:
  - `Move left until {`$s_1, s_2, \ldots, s_n$`}.`
    - Moves the tape head left until we read one of $s_1, s_2, s_3, \ldots, s_n$.
  - `Move right until {`$s_1, s_2, \ldots, s_n$`}.`
    - Moves the tape head right until we read one of $s_1, s_2, s_3, \ldots, s_n$.
- Both commands are no-ops if we're already reading one of the specified symbols.
- We can write programs in **WB2** that are much easier to read than in **WB**.

A **WB** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.

// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
13: If reading 1, go to 11.
14: Move left.
15: If reading 1, go to Next.
16: Reject.

// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
20: If reading 1, go to 18
21: Move right
22: Go to Start.
```

A **WB2** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: Move right until {B}.
 6: Move left.
 7: If reading 0, go to Next.
 8: Reject.

// M1
 9: Write B.
10: Move right.
11: Move right until {B}.
12: Move left.
13: If reading 1, go to Next.
14: Reject.

// Next
15: Write B.
16: Move left.
17: Move left until {B}.
18: Move right.
19: Go to Start.
```

A **WB2** Program for *BALANCE*

- Let $\Sigma = \{$ `0`, `1` $\}$ and consider the language *BALANCE*:

  $$\{ w \in \Sigma^* \mid w \text{ has the same number of } \texttt{0}\text{s and } \texttt{1}\text{s.} \}$$

- Let's write a **WB2** program for *BALANCE*.

A **WB2** Program for *BALANCE*

```
// Start
 0: Move right until {0, 1, B}.
 1: If reading 0, go to Match0.
 2: If reading 1, go to Match1.
 3: Accept.

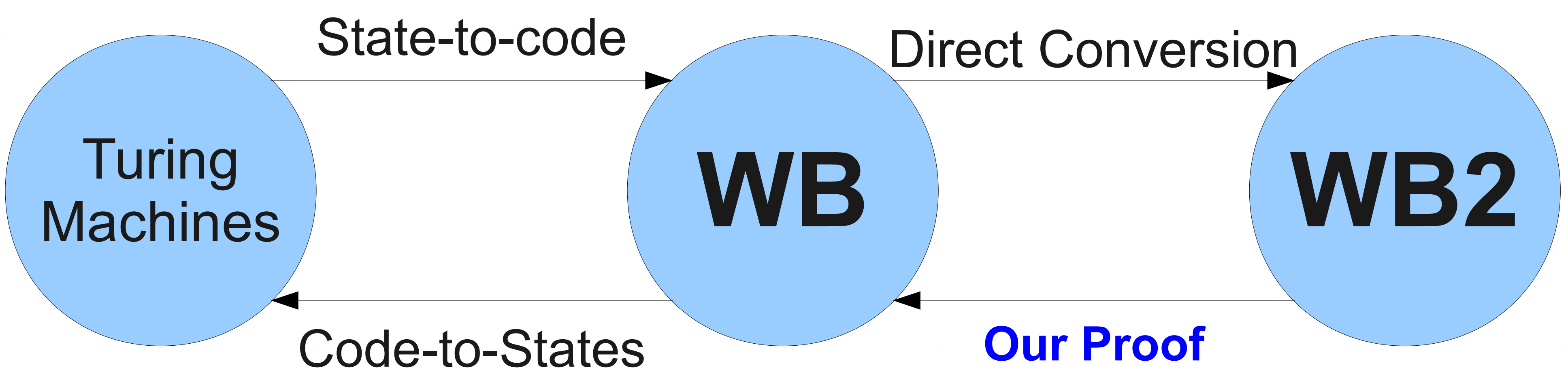
// Match0
 4: Write B.
 5: Move right.
 6: Move right until {1, B}.
 7: If reading 1, go to Found.
 8: Reject.

// Match1
 9: Write B.
10: Move right.
11: Move right until {0, B}.
12: If reading 0, go to Found.
13: Reject.

// Found
14: Write x.
15: Move left until {B}.
16: Move right.
17: Go to Start.
```

WB2 and Turing Machines

- **Theorem:** A language is recursively enumerable iff there is a **WB2** program for it.
- We could directly prove this again by showing equivalence with Turing machines.
- Instead, we'll connect it to **WB**:



From **WB2** to **WB**

- We will show how to turn any **WB2** program into an equivalent **WB** program.
- All old instructions are still valid.
- We need to show how to implement the new **`Move`** ... **`until`** commands using just **WB**.

Implementing `Move ... until`

- Replace `N: Move` *dir* `until {`s_1`, ..., `s_n`}` as follows:

```
N+0:      If reading s1, go to N+n+2.
N+1:      If reading s2, go to N+n+2.
N+2:      If reading s3, go to N+n+2.
…
N+(n–1):  If reading sn, go to N+n+2.
N+n:      Move dir.
N+n+1:    Go to N
```

- Renumber other lines as appropriate.

Why This Matters

- We are starting to move more and more away from the Turing machine with from we started.
- The structure of our approach is
  - Find some simple programming language that can be directly translated into a Turing machine (and vice-versa).
  - Add new features to the language, and show how to implement those new features using the old language.
  - Add new features to *that* language, and show how to implement those features using the previous language.
  - (etc.)
  - Conclude that the final language is equivalent to a Turing machine.

A Repeating Pattern

```
// Match0
 4: Write B.
 5: Move right.
 6: Move right until {1, B}.
 7: If reading 1, go to Found.
 8: Reject.
```

```
// Match1
 9: Write B.
10: Move right.
11: Move right until {0, B}.
12: If reading 0, go to Found.
13: Reject.
```

A Simple Memory

- Right now, our programming language **WB2** has no variables in it.
- To solve larger classes of problems, let's invent a new language **WB3** that has support for variables.
- We will severely limit the scope of our variables:
  - Only **finitely many** total variables throughout the program.
  - Each variable can only hold a single tape symbol.
  - Each variable initially holds the blank symbol.

Our New Commands

- We will define **WB3** as **WB2** with the following extra commands:
  - **`Load`** ***s*** **`into`** ***v*****.**
    - Sets the variable v equal to tape symbol s.
  - **`Load current into`** ***v*****.**
    - Sets the variable v equal to the currently-scanned tape symbol.
  - **`If`** v_1 **`=`** v_2**`, go to`** ***L*****.**
    - If v_1 and v_2 have the same value, go to instruction L.
    - These may be constants or variables.
- Additionally, any command that referenced a tape symbol (for example, **`write`**, **`if reading`**, **`move`** … **`until`**) can refer to variables in addition to constants.

A **WB2** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: Move right until {B}.
 6: Move left.
 7: If reading 0, go to Next.
 8: Reject.

// M1
 9: Write B.
10: Move right.
11: Move right until {B}.
12: Move left.
13: If reading 1, go to Next.
14: Reject.

// Next
15: Write B.
16: Move left.
17: Move left until {B}.
18: Move right.
19: Go to Start.
```

A **WB3** Program for Even Palindromes

```
// Start
 0: Read current into X.
 1: If X = B, go to Acc.
 2: Write B.
 3: Move right.
 4: Move right until {B}.
 5: Move left.
 6: If reading X, go to Match.
 7: Reject.

// Match
 8: Write B.
 9: Move left.
10: Move left until B.
11: Move right.
12: Go to Start.

// Acc:
13: Accept.
```

A **WB2** Program for *BALANCE*

```
// Start
 0: Move right until {0, 1, B}.
 1: If reading 0, go to Match0.
 2: If reading 1, go to Match1.
 3: Accept.

// Match0
 4: Write B.
 5: Move right.
 6: Move right until {1, B}.
 7: If reading 1, go to Found.
 8: Reject.

// Match1
 9: Write B.
10: Move right.
11: Move right until {0, B}.
12: If reading 0, go to Found.
13: Reject.

// Found
14: Write x.
15: Move left until {B}.
16: Move right.
17: Go to Start.
```

A **WB3** Program for *BALANCE*

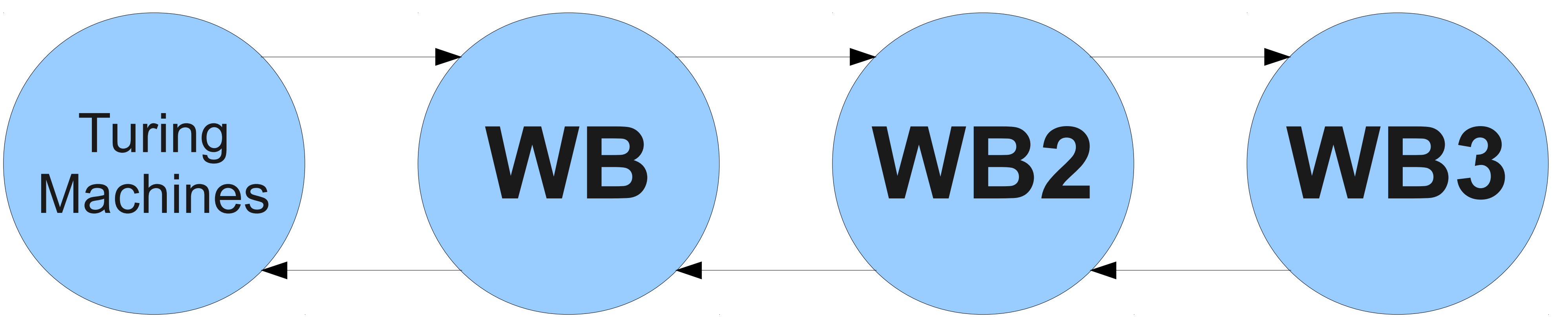
```
// Start
 0: Move right until {0, 1, B}.
 1: If reading B, go to Acc.
 2: If reading 0, go to 5.
 3: Load 0 into Y.
 4: Go to Scan.
 5: Load 1 into Y.
 6: Go to Scan.
```

```
// Scan
 8: Write B.
 9: Move right.
10: Move right until {Y, B}
11: If reading Y, go to 13.
12: Reject.
13: Write x.
14: Move left until B.
15: Move right.
16: Go to Start.

// Acc:
17: Accept.
```

Equivalence of **WB2** and **WB3**

- **Theorem:** A language is recursively enumerable iff there is a **WB3** program for it.
- Adding in these sorts of variables adds *no power* to our model of computation!
- To prove the theorem, we will show
  - Any WB2 program can be converted to a WB3 program, and
  - Any WB3 program can be converted to a WB2 program.



The Proof: An Intuition

- Our programs allow only finitely many variables holding only one of finitely many different values (tape symbols).
- We could just **replicate the program** for each possible assignment to the variables, then hardcode in the behavior in each of these cases.
- Could make the program staggeringly huge, but it will still be finite!

The Transformation, Part I

- Let V_1, V_2, …, V_n be the variables referenced in the program.
  - We can just look at the source code to determine this.
- Make $|\Gamma|^n$ copies of the initial program, one for each possible assignment of tape symbols to the variables V_i.
- Order the copies arbitrarily, but make the version where all variables hold B come first.

The Transformation, Part II

- We now have a whole bunch of copies of **WB3** programs.
- We need to convert them into legal **WB2** programs.
- This works in two steps:
  - Removing variables from older **WB2** commands like **`Write`**, **`If reading`** …, and **`Move`** … **`while`**.
    - For example: "**`Write`** ***`X`***," where ***`X`*** is a variable.
  - Rewriting all new **WB3** commands that reference variables to use only **WB2** commands.
    - For example: "**`Load current into`** ***`X`***."

Eliminating Variables from **WB2**

- Removing variables from purely **WB2** statements is easy because we've copied the program so many times.
- For each copy, replace all variables in **WB2** statements with the value that the variable has in that copy.

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

Y = B

```
3: Load 0 into Y.
4: Write Y.
5: Accept
```

Y = 0

```
6: Load 0 into Y.
7: Write Y.
8: Accept
```

Y = 1

Eliminating Variables from **WB2**

- Removing variables from purely **WB2** statements is easy because we've copied the program so many times.
- For each copy, replace all variables in **WB2** statements with the value that the variable has in that copy.

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

```
0: Load 0 into Y.
1: Write B.
2: Accept
```

Y = B

```
3: Load 0 into Y.
4: Write 0.
5: Accept
```

Y = 0

```
6: Load 0 into Y.
7: Write 1.
8: Accept
```

Y = 1

Eliminating Variables from **WB3**

- We can eliminate commands that manipulate variables by replacing them with **`Go to`**s.
- There are three commands to eliminate:
  - **`Load`** ***s*** **`into`** ***v***.
  - **`Load current into`** ***v***.
  - **`If`** v_1 **`=`** v_2**`, go to`** ***L***.

`If` v_1 `=` v_2`, go to` L

- We can eliminate this statement by just hardcoding the jump in place.
- If in the current copy of the program v_1 and v_2 have the same values, replace with

  `Go to` L

  where L is the corresponding version of L in this copy.
- Otherwise, replace with

  `Go to` N

  where N is the number of the next line in the program.

Load ***s*** into ***v***

- To simulate the effect of loading ***s*** into ***v***, we can jump out of the current copy of the program into the copy where ***v*** has value ***s***.

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

Y = B

```
3: Load 0 into Y.
4: Write Y.
5: Accept
```

Y = 0

```
6: Load 0 into Y.
7: Write Y.
8: Accept
```

Y = 1

Load *s* into *v*

- To simulate the effect of loading ***s*** into ***v***, we can jump out of the current copy of the program into the copy where ***v*** has value ***s***.

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

```
0: Load 0 into Y.
1: Write B.
2: Accept
```

```
3: Load 0 into Y.
4: Write 0.
5: Accept
```

```
6: Load 0 into Y.
7: Write 1.
8: Accept
```

Y = B

Y = 0

Y = 1

Load ***s*** into ***v***

- To simulate the effect of loading ***s*** into ***v***, we can jump out of the current copy of the program into the copy where ***v*** has value ***s***.

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

```
0: Go to 4.
1: Write B.
2: Accept
```

Y = B

```
3: Go to 4.
4: Write 0.
5: Accept
```

Y = 0

```
6: Go to 4.
7: Write 1.
8: Accept
```

Y = 1

Load current into *v*

- We can simulate this instruction using a similar trick to before.
- Replace this instruction as follows:

If reading s_1, go to $LoadS_1$.

If reading s_2, go to $LoadS_2$.

…

If reading s_n, go to $LoadS_n$.

// $LoadS_1$:

Load s_1 into *v*.

Go to Done.

…

// $LoadS_n$

Load s_n into *v*.

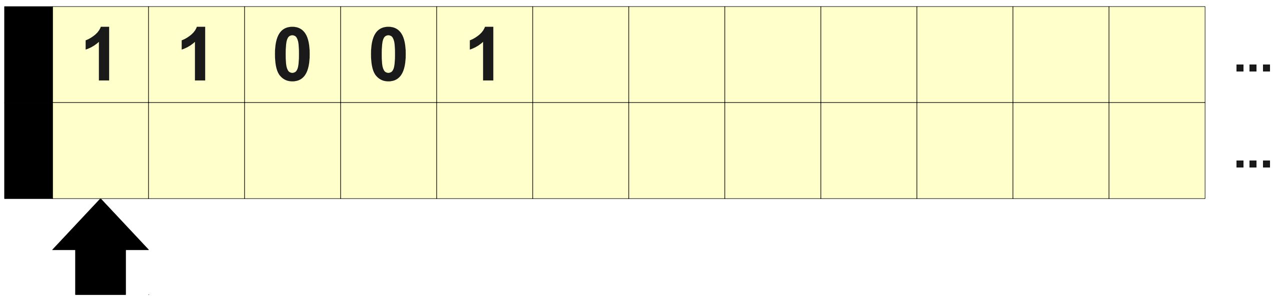
Go to Done.

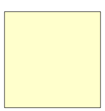
// Done:

Souping up our Tape

- Up to this point, we've been improving our **WB** programming language by adding in new ways of scanning over the tape.
- What if we made changes to the tape itself?

A Multitrack Tape





X

```
// Start
 0: Read track 1 into X.
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 4: Go to 0
 5: /* … */
```

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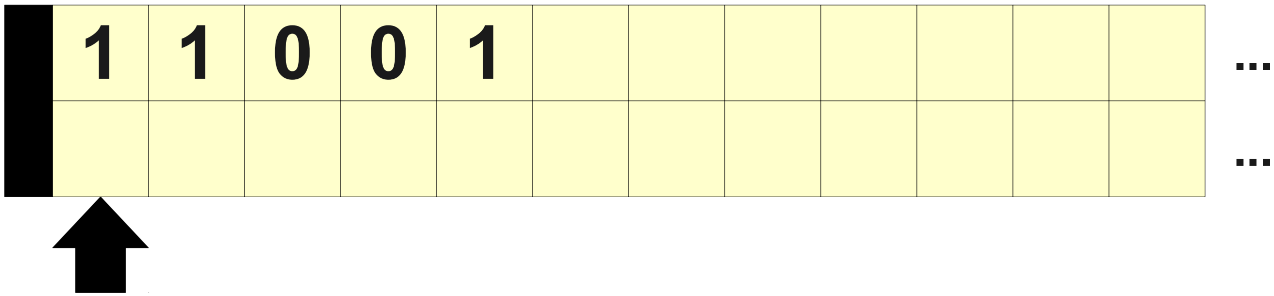

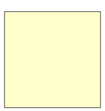

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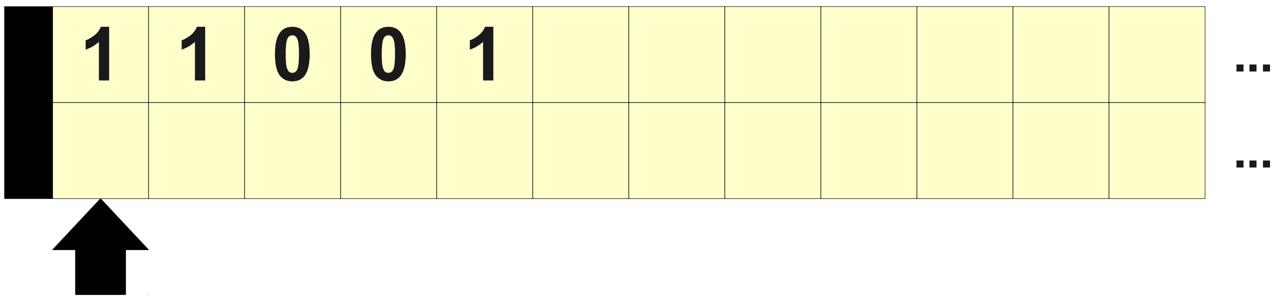

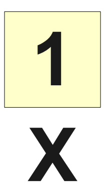


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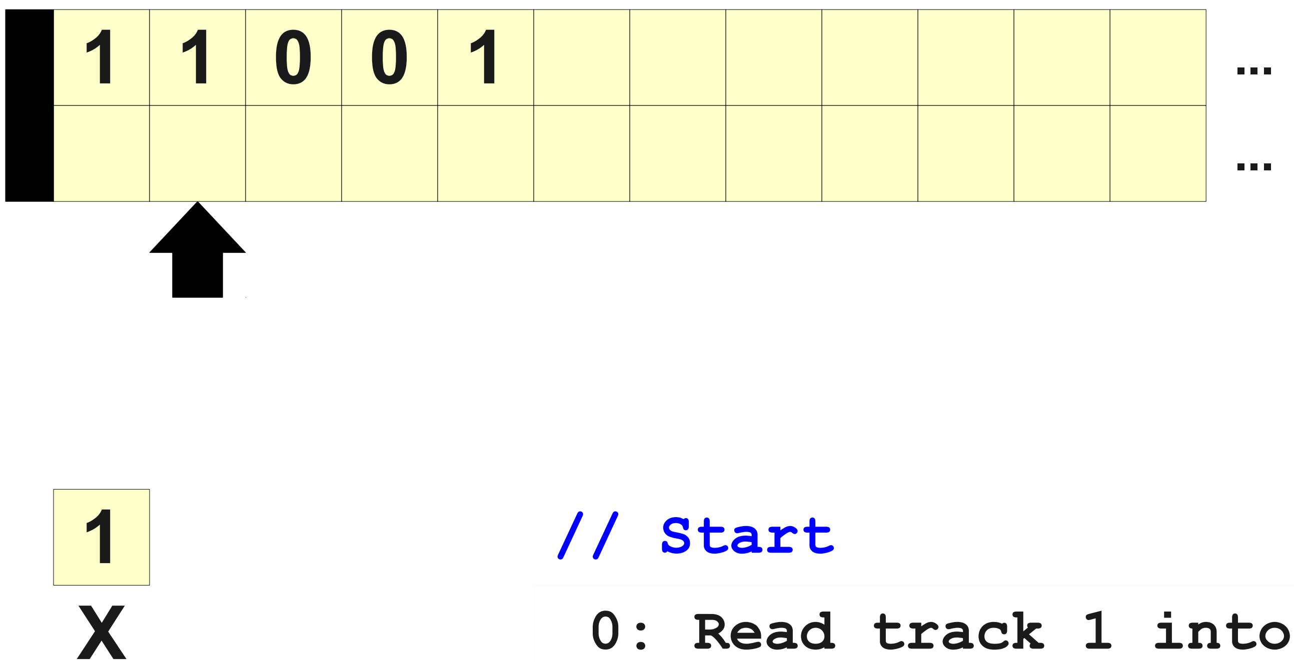


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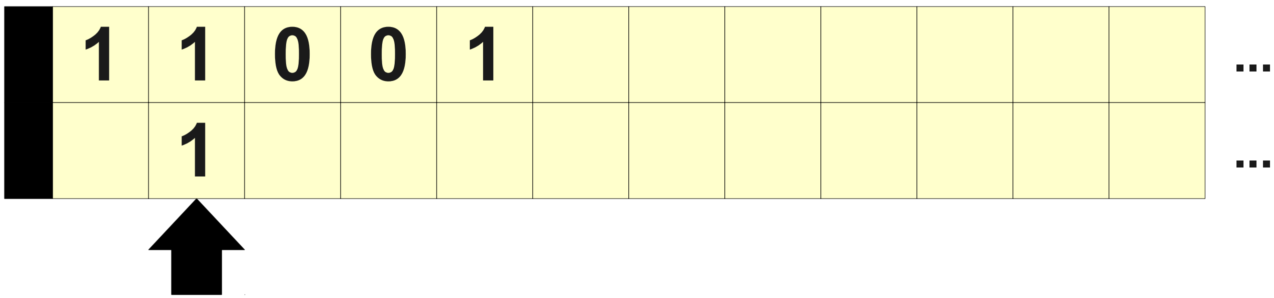

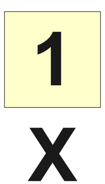


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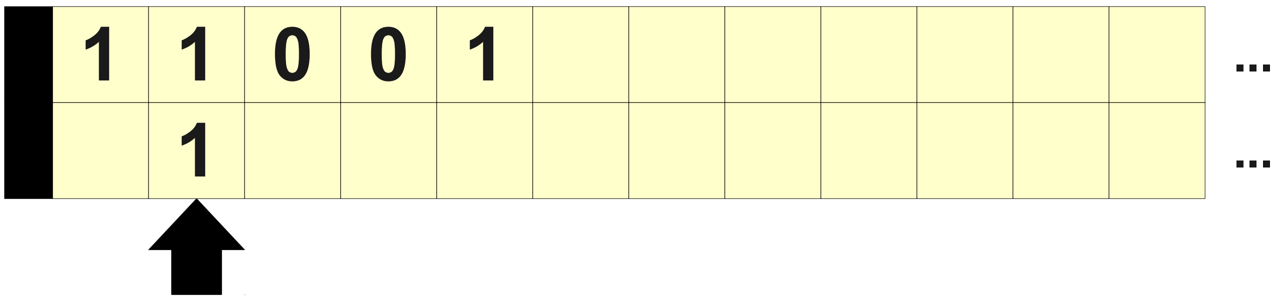

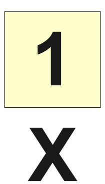


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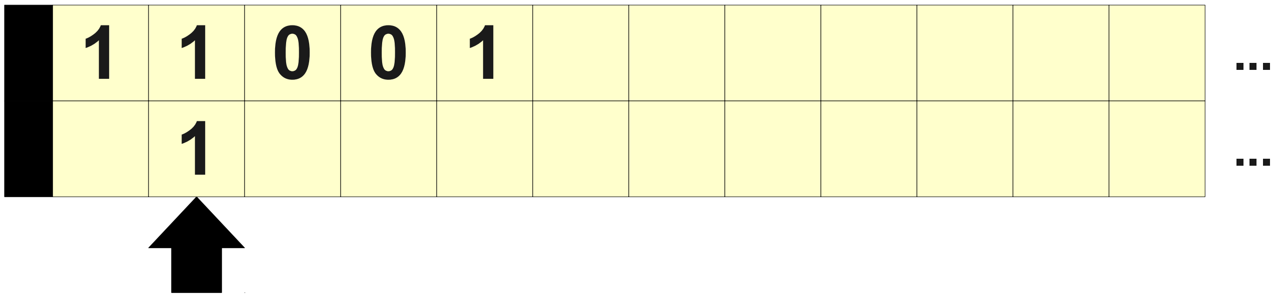

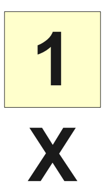


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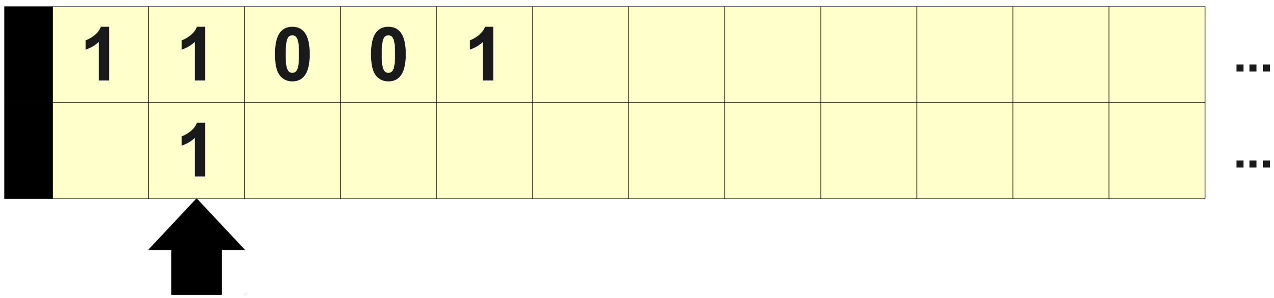

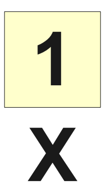


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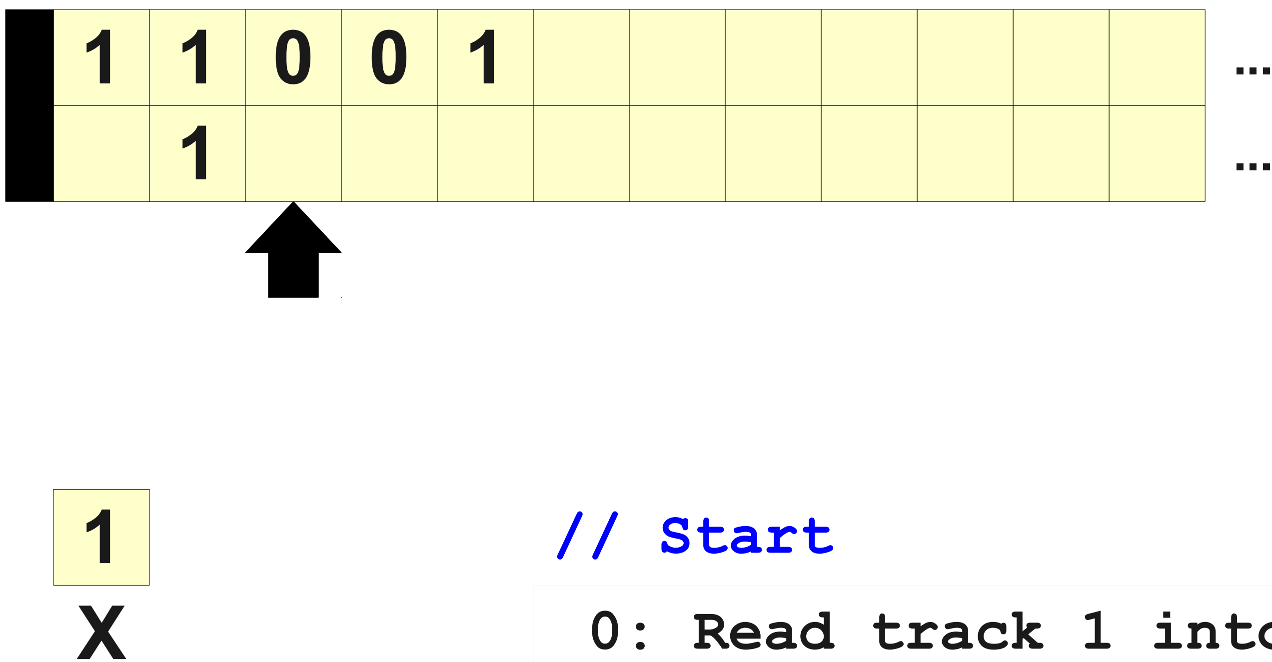


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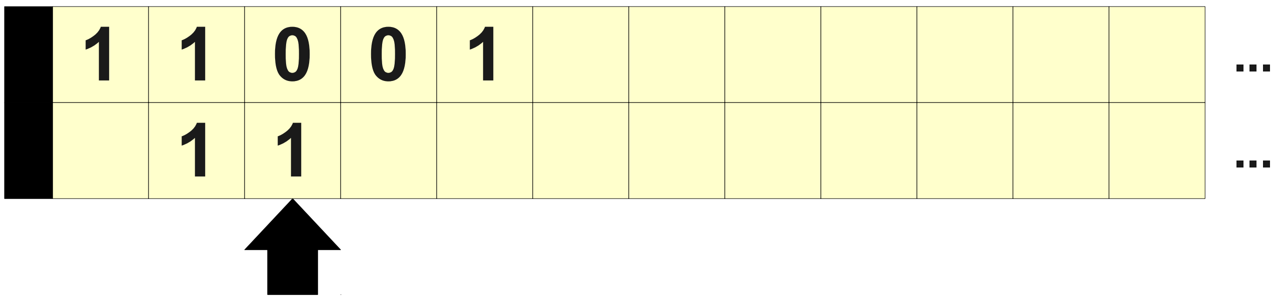

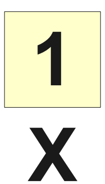


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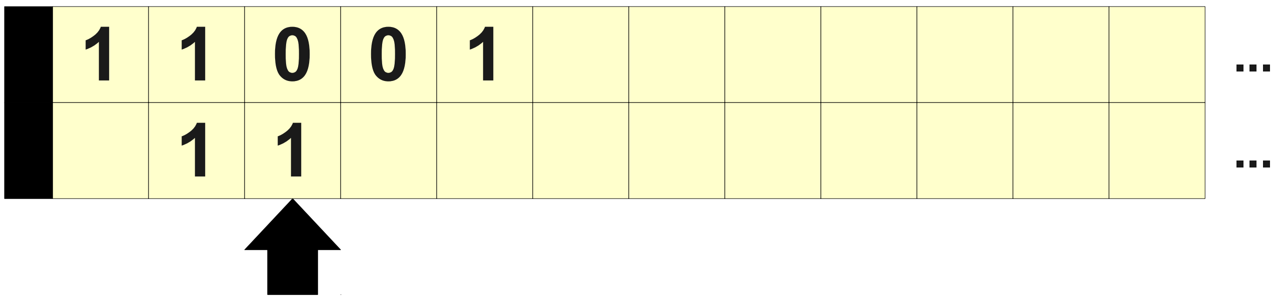

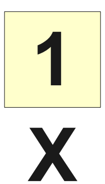


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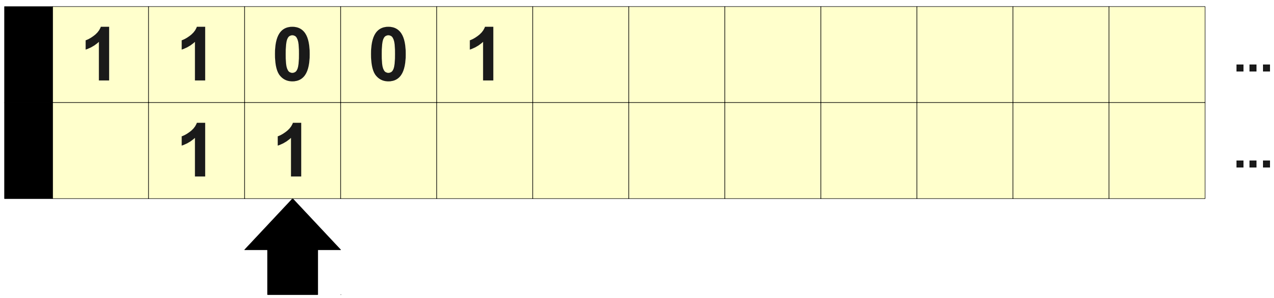

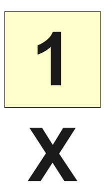


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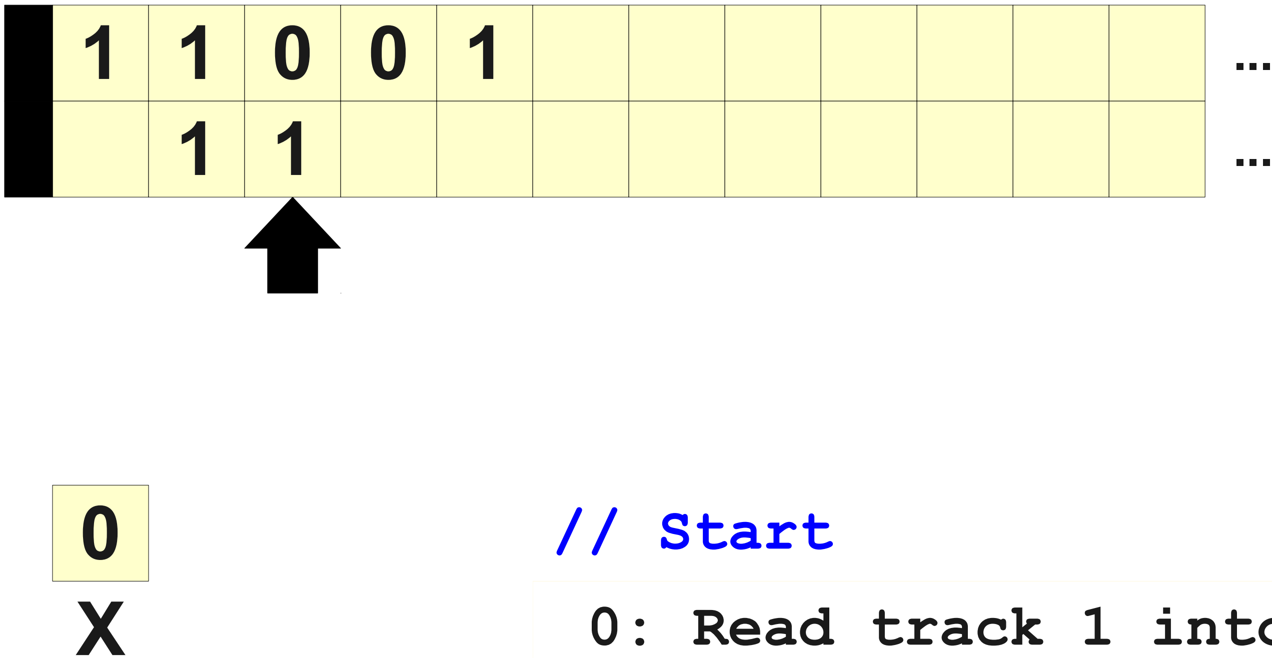


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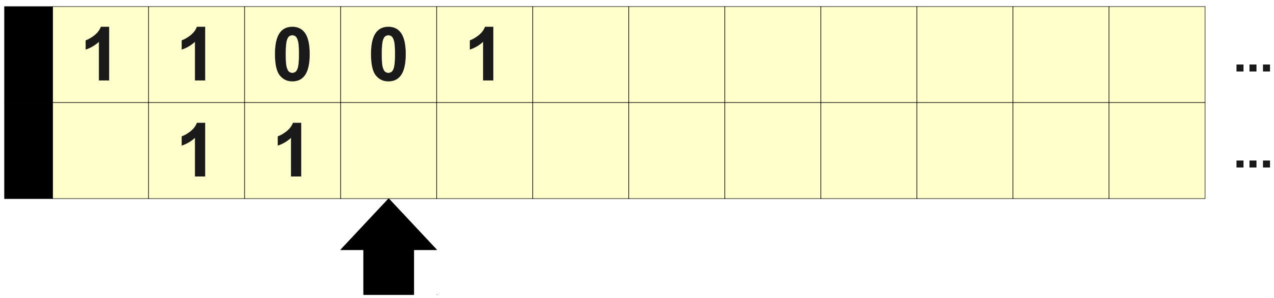

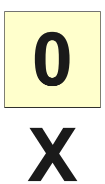


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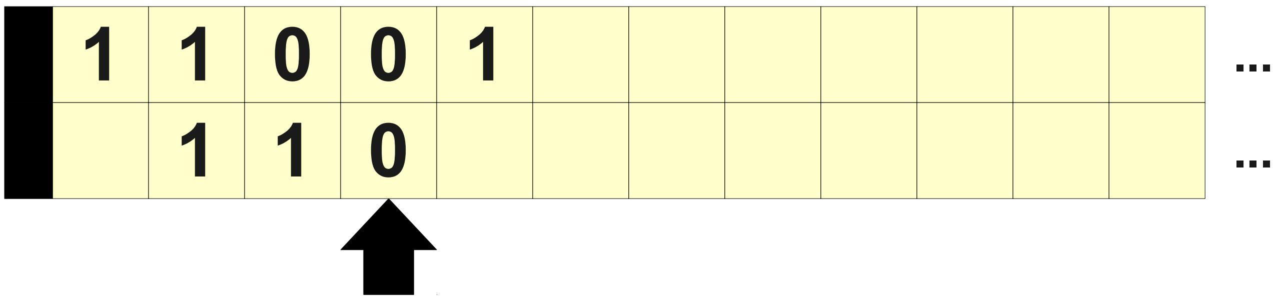

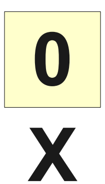


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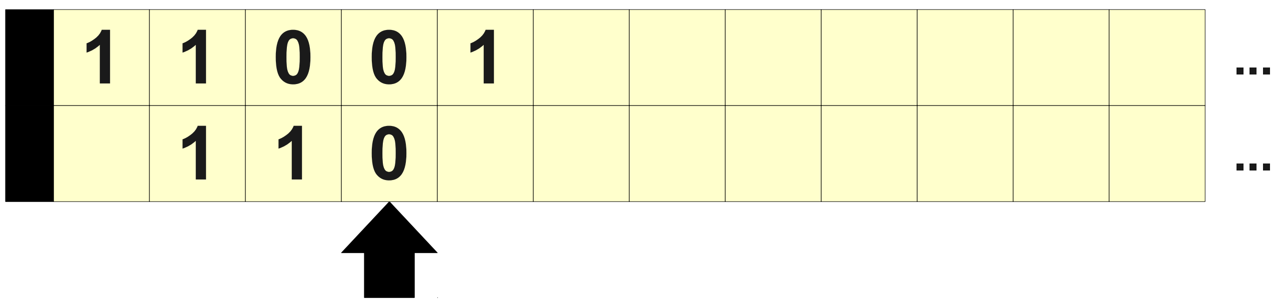

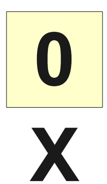


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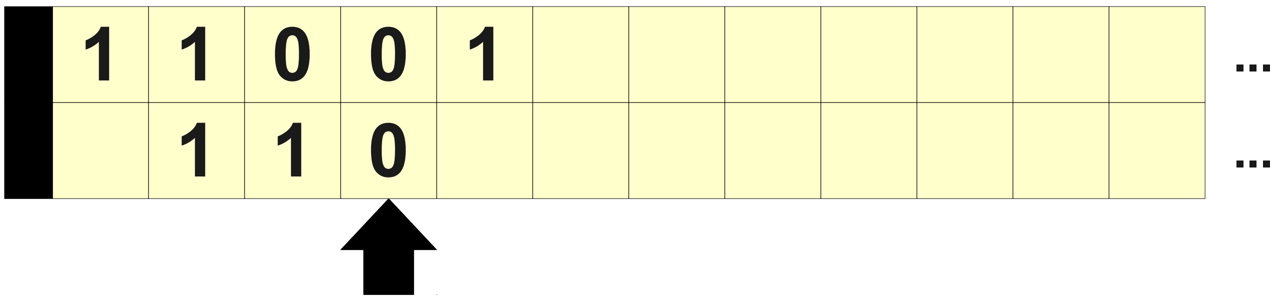

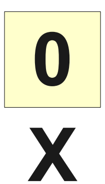


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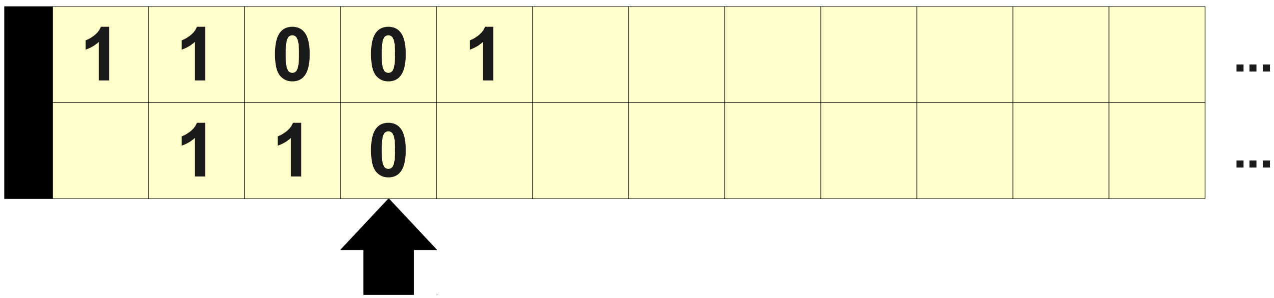

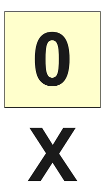


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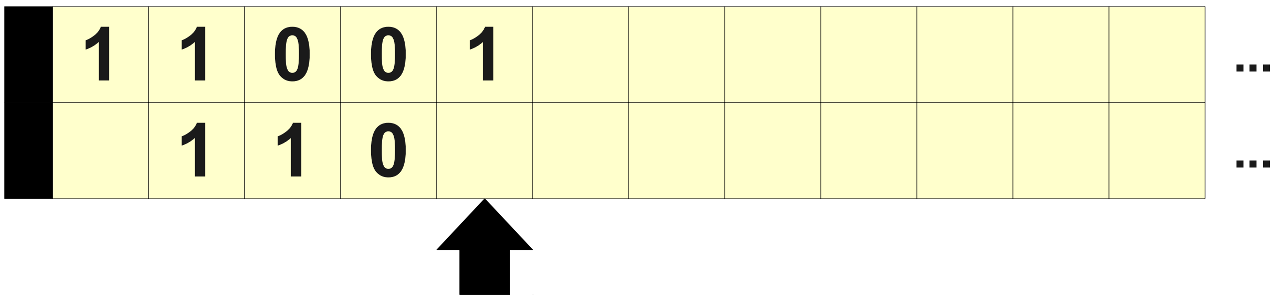

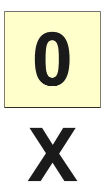


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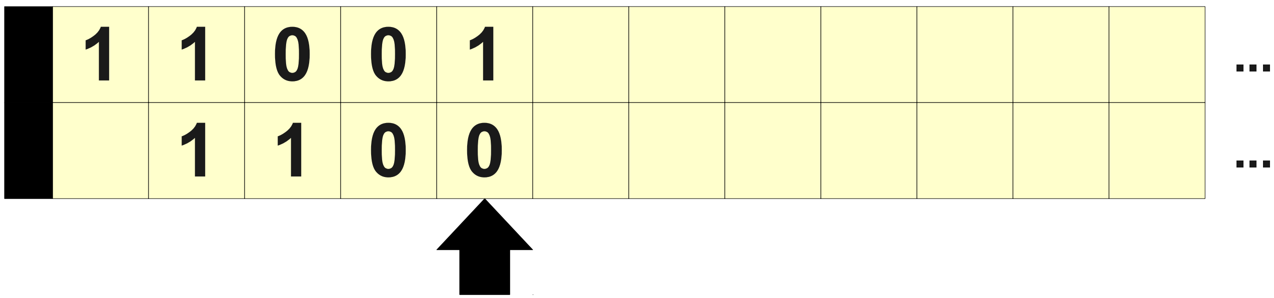

0

X

```
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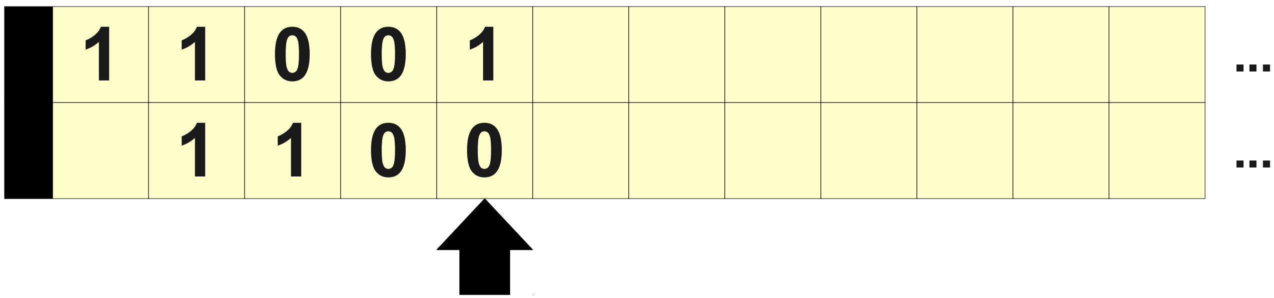

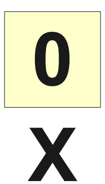


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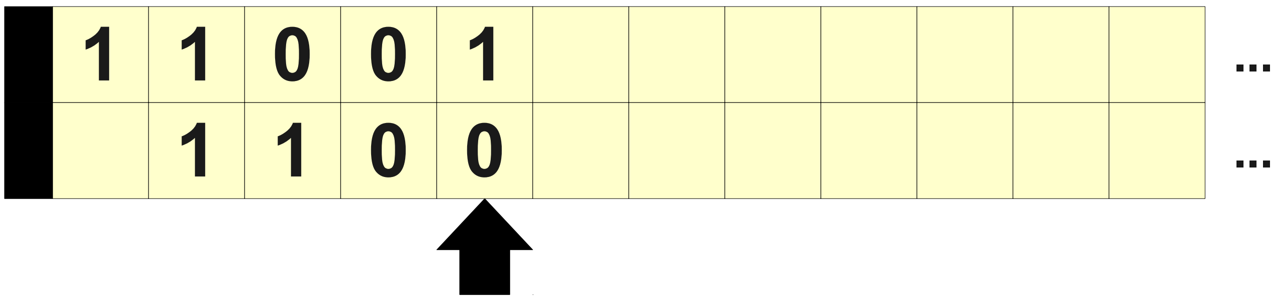

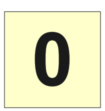

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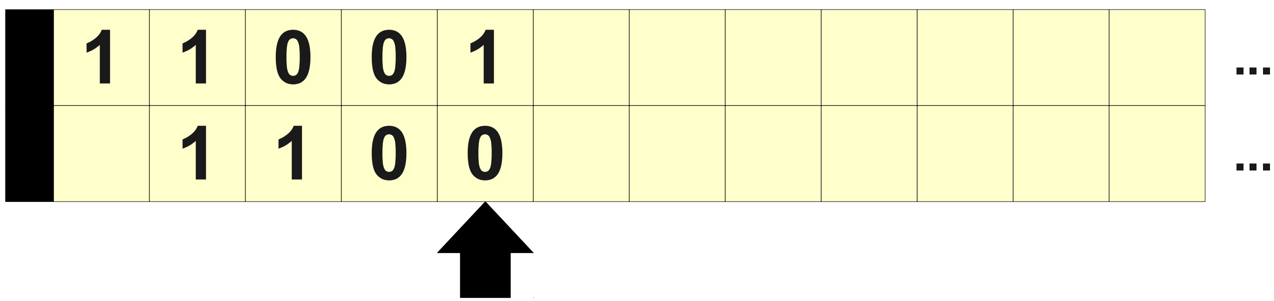

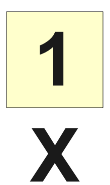


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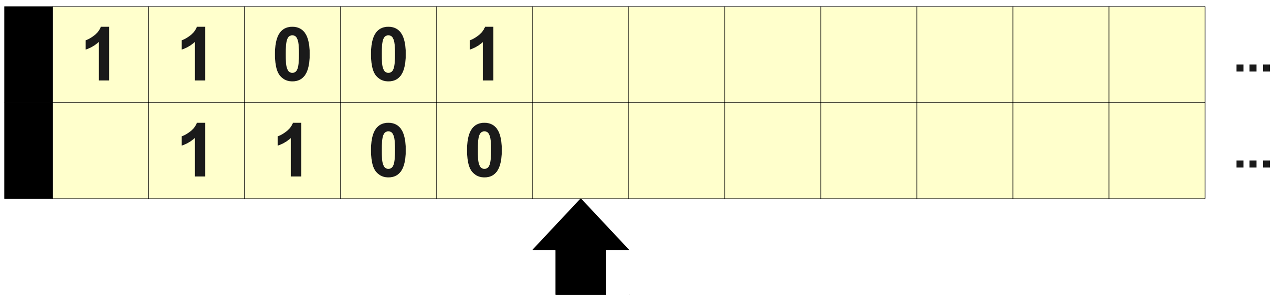

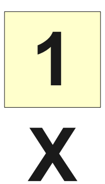


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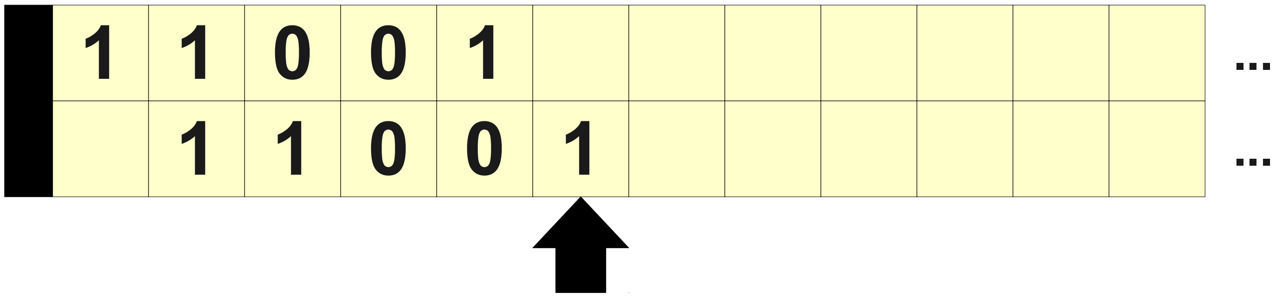

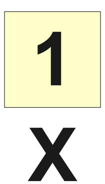


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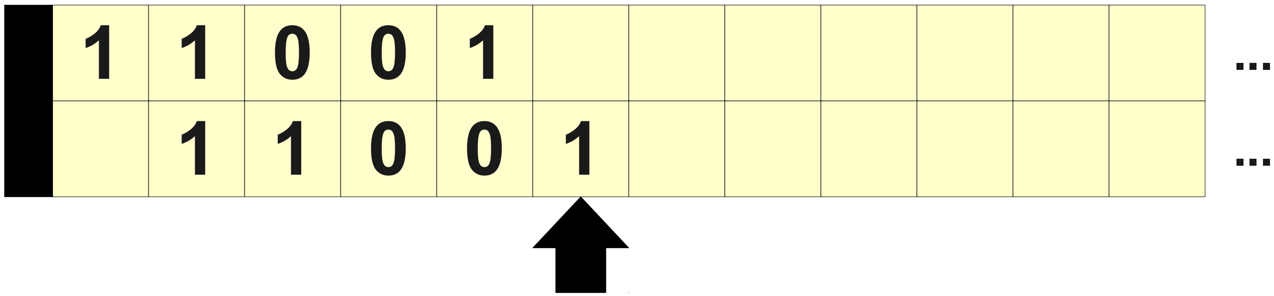
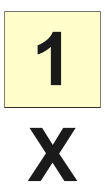
2: Write X into track 2
3: If reading B on track 1, go to 5.
4: Go to 0
5: /* … */
```

A Multitrack Tape





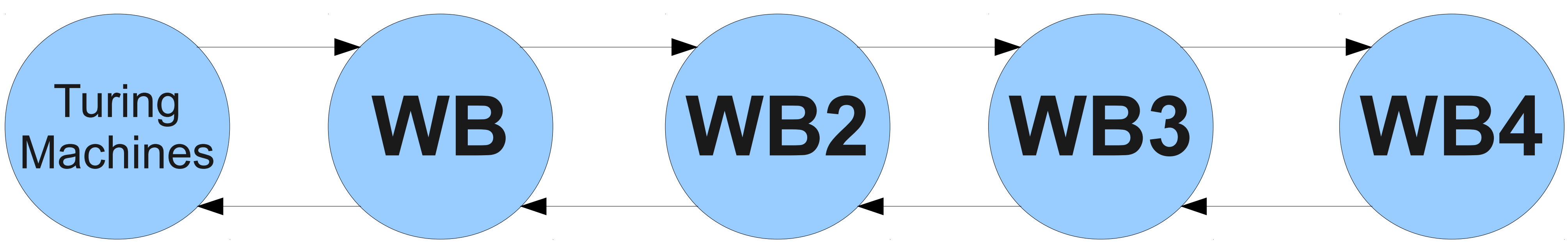

```
// Start
0: Read track 1 into X.
1: Move right.
2: Write X into track 2
3: If reading B on track 1, go to 5.
4: Go to 0
5: /* … */
```

Introducing **WB4**

- Let's define **WB4** to be **WB3** with the introduction of finitely many **tracks** on the tape.
- The tape head still moves as a unit to the left or right, but we can now issue read and write commands to any cell in the current track.
- All previous commands updated to specify which track is to be read or written.

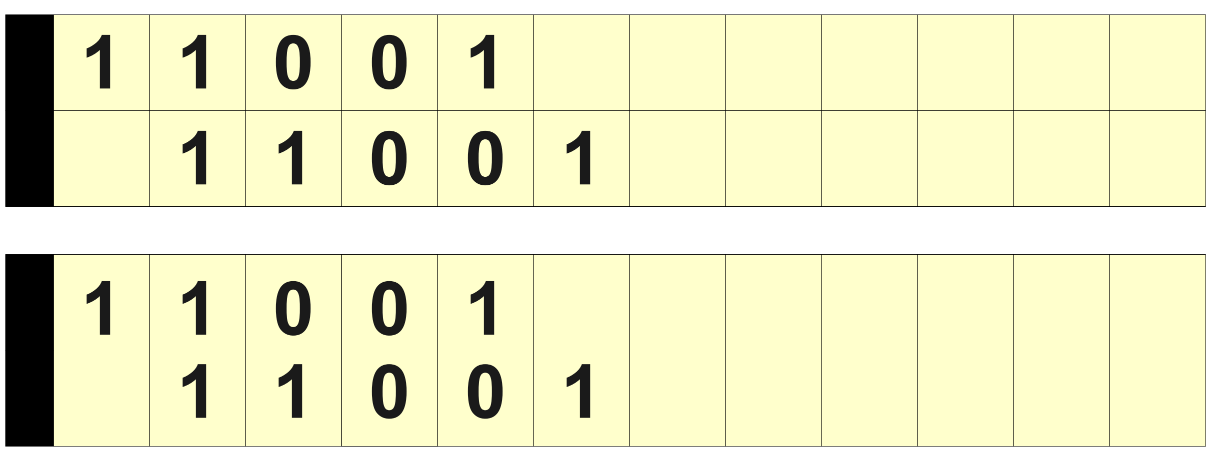
A Surprising Theorem

- **Theorem:** A language is recursively enumerable iff there is a **WB4** program for it.
- This is not obvious... it seems like adding in more tracks should increase the power of our programming language!
- As with before, will prove that all **WB4** programs are equivalent to **WB3** programs.



The Intuition

- Treat a single tape as a “fat tape” where each tape symbol encodes the contents of the cells of all four tracks.
- Each read or write to a specific location replaces the entire tape cell with a new symbol representing the change.

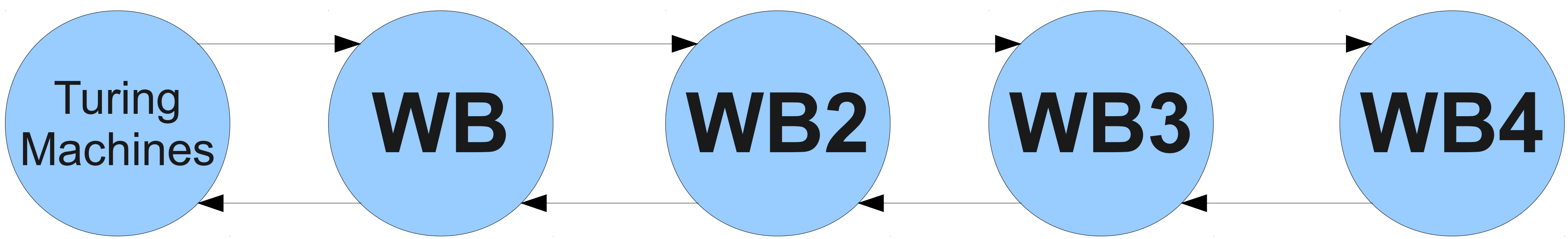


A Sketch of the Construction

- Replace each instruction that reads or writes a track with a huge cascading “if” that checks for every possible tape symbol and reacts accordingly.
- Can make the program enormously bigger, but it still ends up finite.
- I'm not even going to attempt to fit something like that onto these slides.

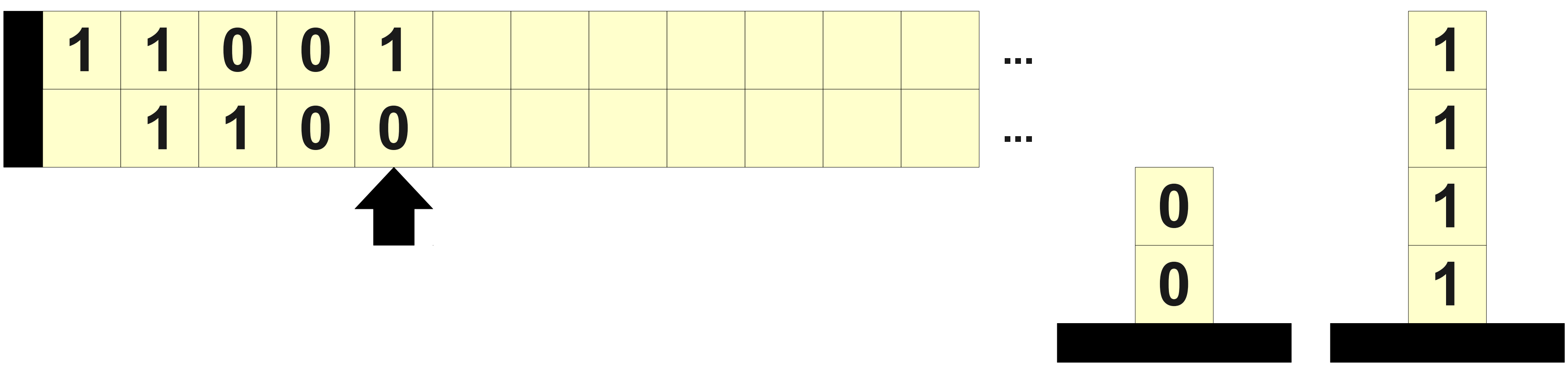
Where We Are Now

- Starting with **WB**, we have added
  - Loops to search for a value. (**WB2**)
  - Variables with finite storage. (**WB3**)
  - Multiple tracks. (**WB4**)
- Yet we still accept exactly the same set of languages.
- Every **WB*n*** program can be converted back to a TM.



Making Things Crazier

- What do you get when you combine a PDA and a **WB4** program?
- A program with an infinite tape, plus multiple stacks!



Introducing **WB5**

- The programming language **WB5** is the programming language **WB4** with the addition of a finite number of stacks.
- We add three extra commands:
  - `Push` ***s*** `onto stack` ***v***`.`
    - Pushes the symbol ***s*** onto the stack named ***v***.
  - `If stack` ***v*** `is empty, go to` ***L***`.`
    - If stack ***v*** is empty, go to instruction ***L***.
  - `Pop stack` ***v*** `into` ***w***`.`
    - If stack ***v*** is nonempty, pops ***v*** and puts the top into ***w***.

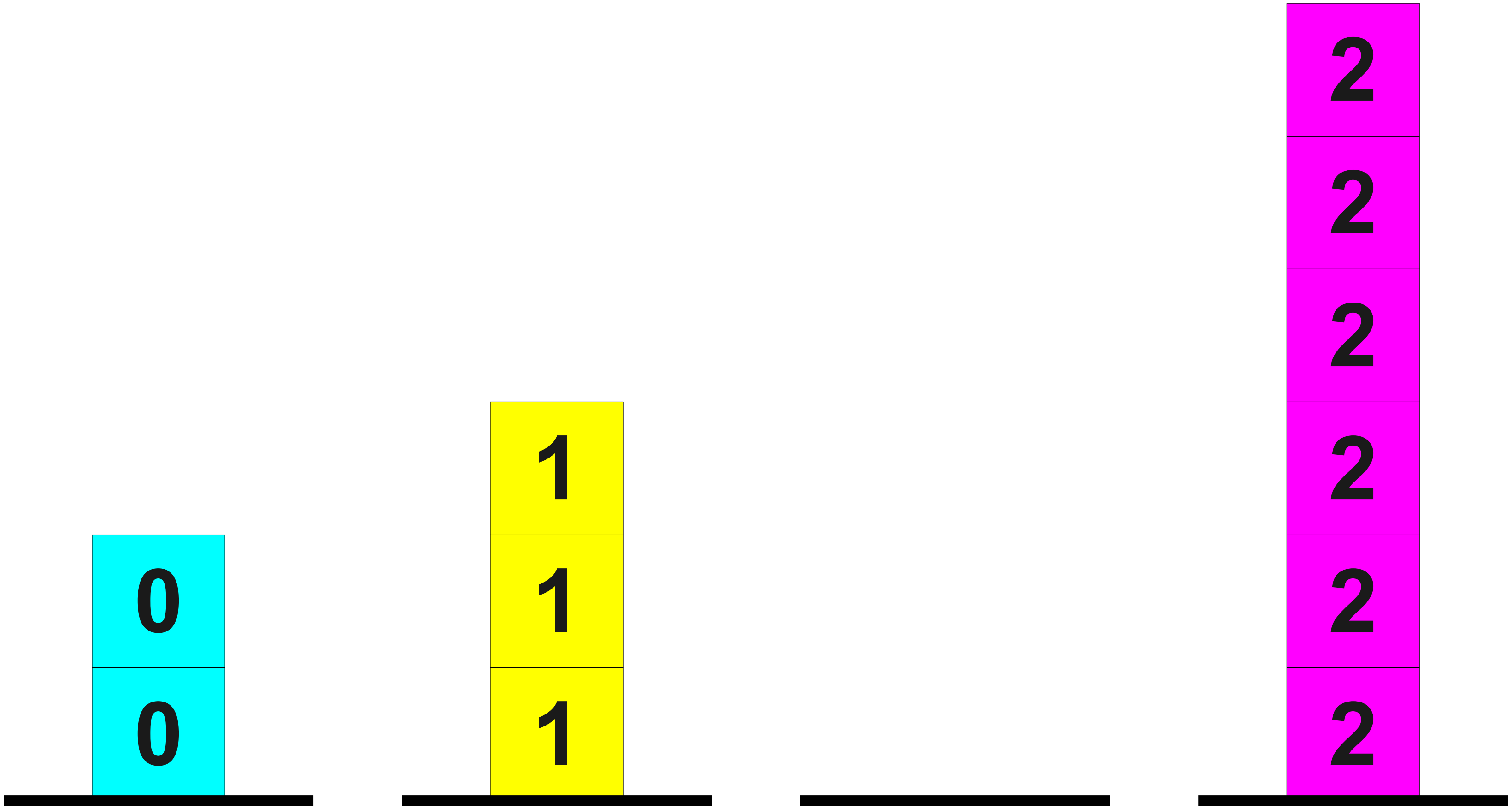
The Multiplication Language

- Let Σ = { `0`, `1`, `2` } and consider the language *01MULT* defined as

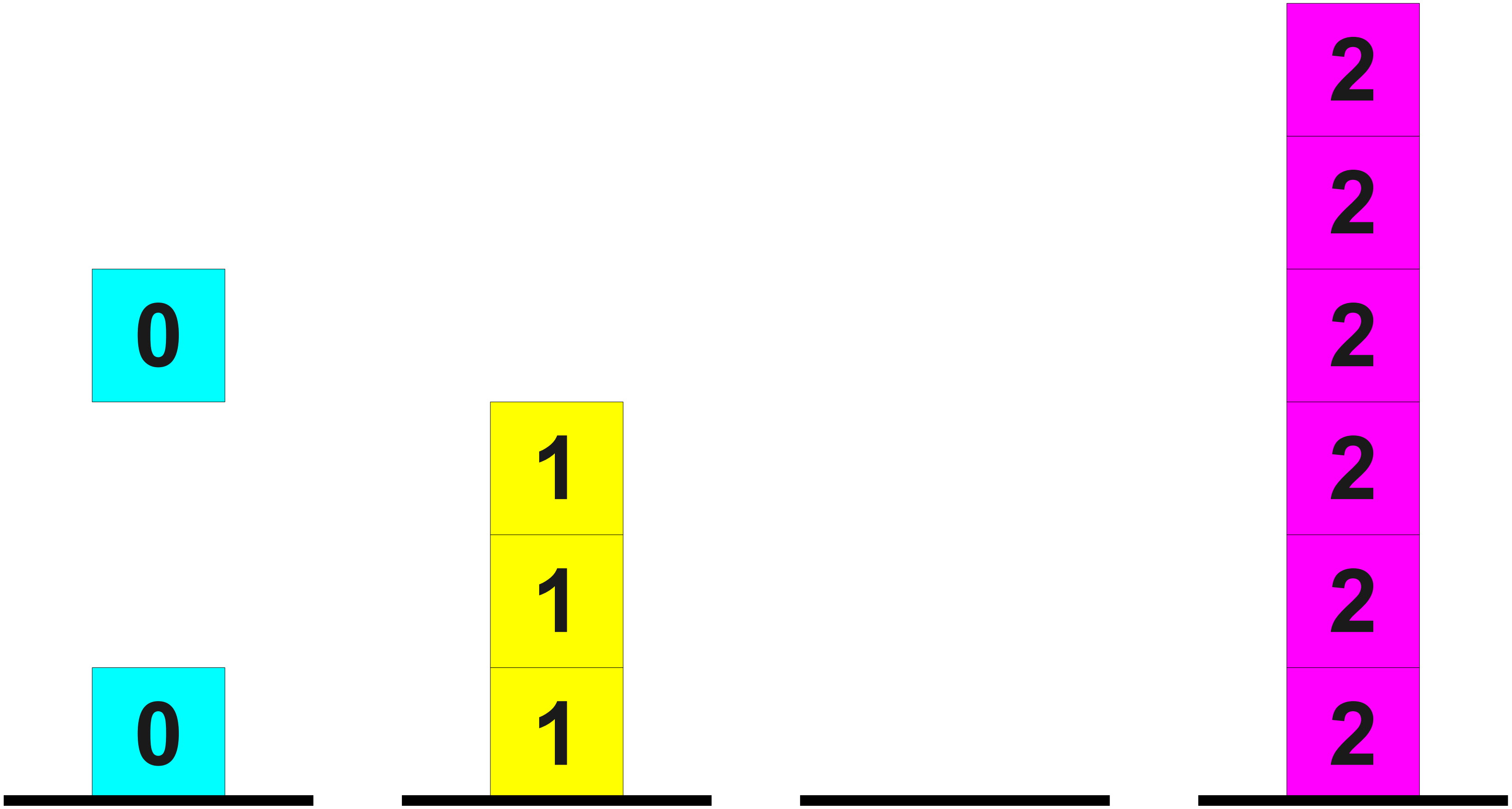
  { *w* ∈ Σ* | the number of `2`'s in *w* is the product of the number of `1`'s and the number of `0`'s. }

- For example:
  - `00112222` ∈ *01MULT*
  - `22001122122` ∈ *01MULT*
- This language is neither context-free nor regular.
- How could we write a **WB5** program for it?

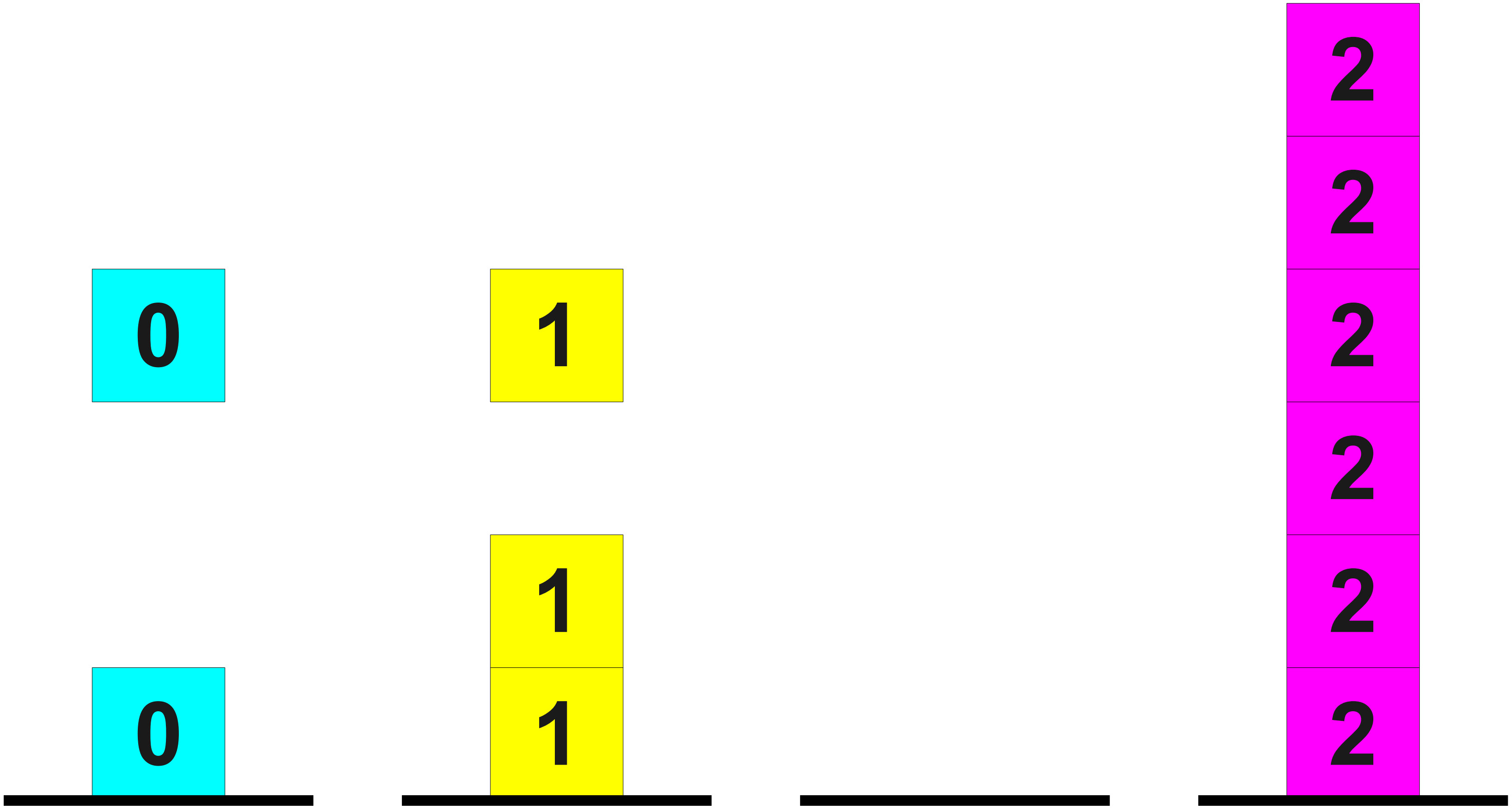
One Approach



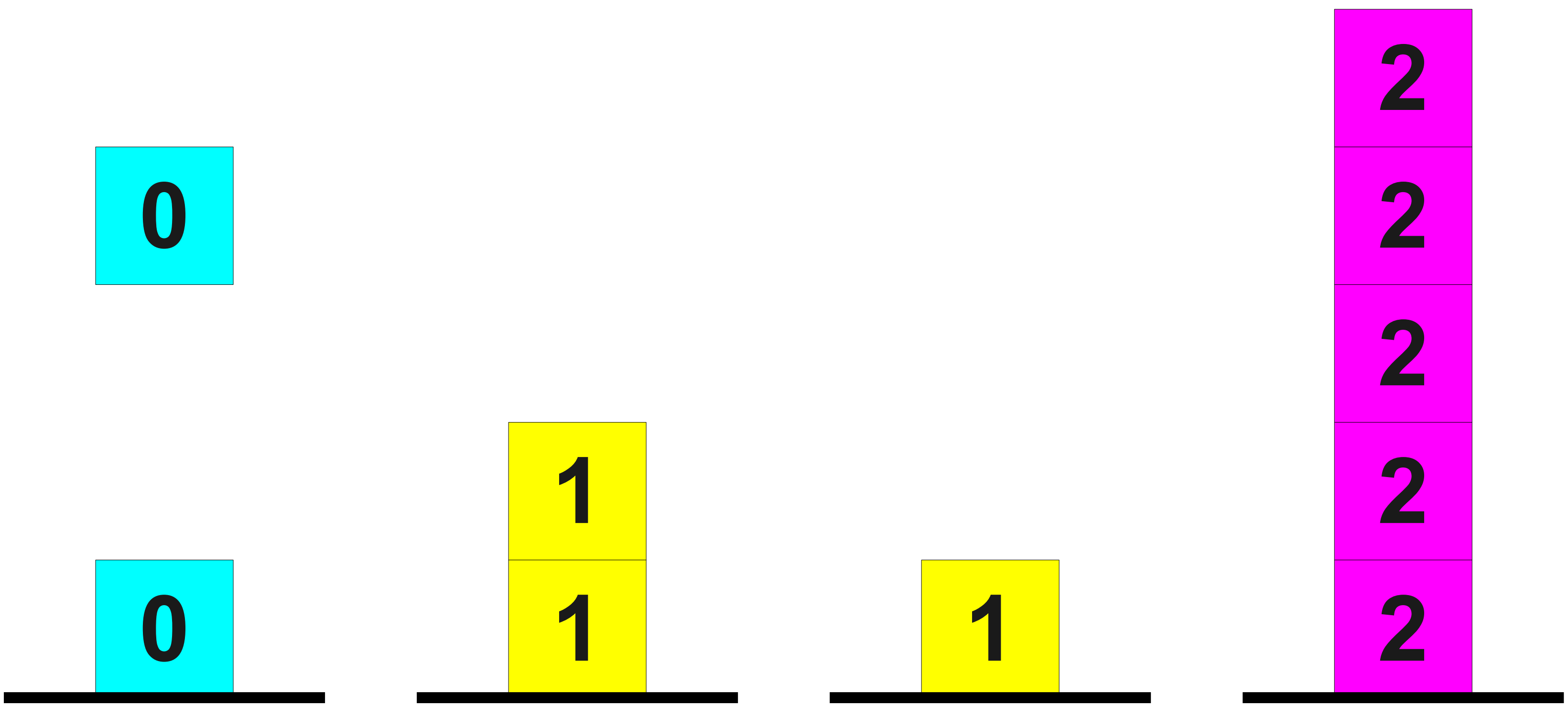
One Approach



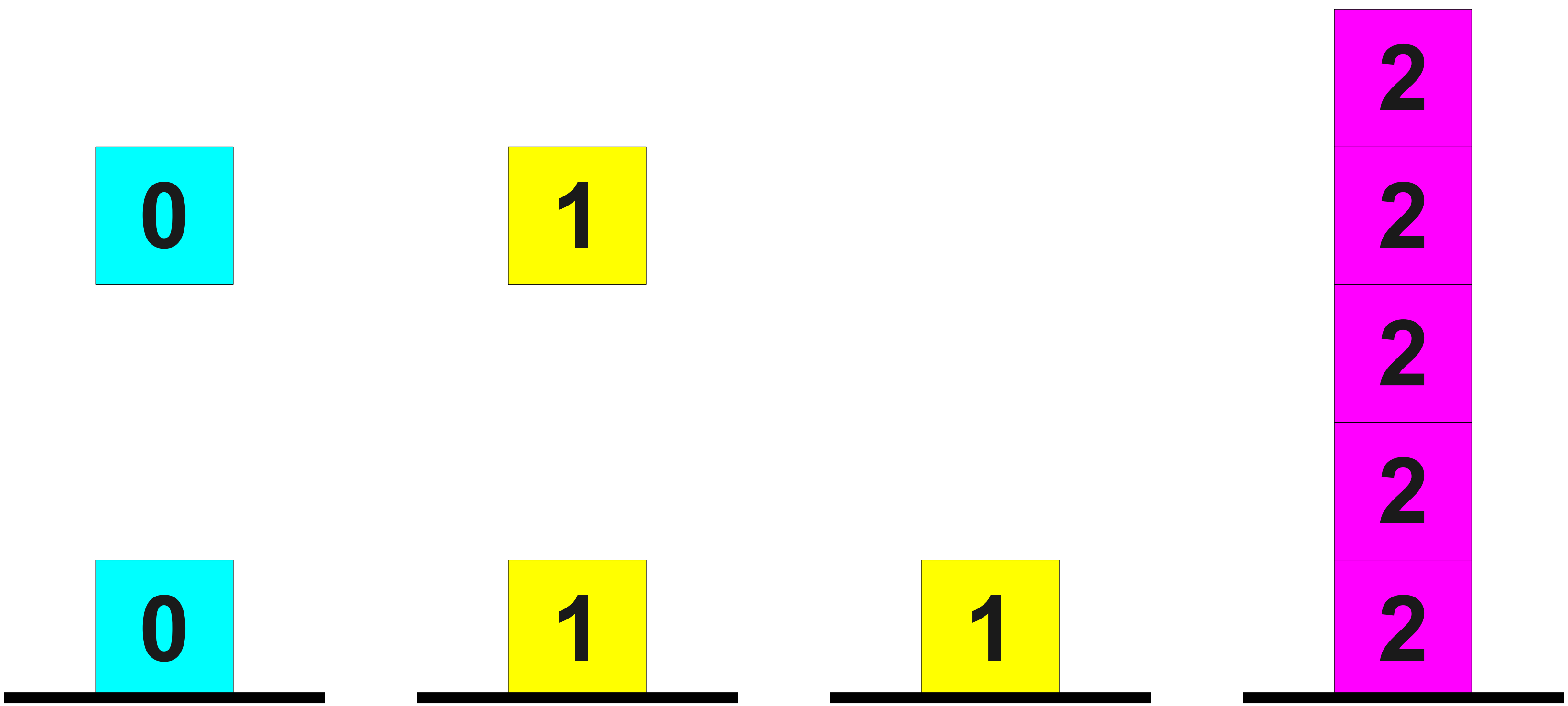
One Approach



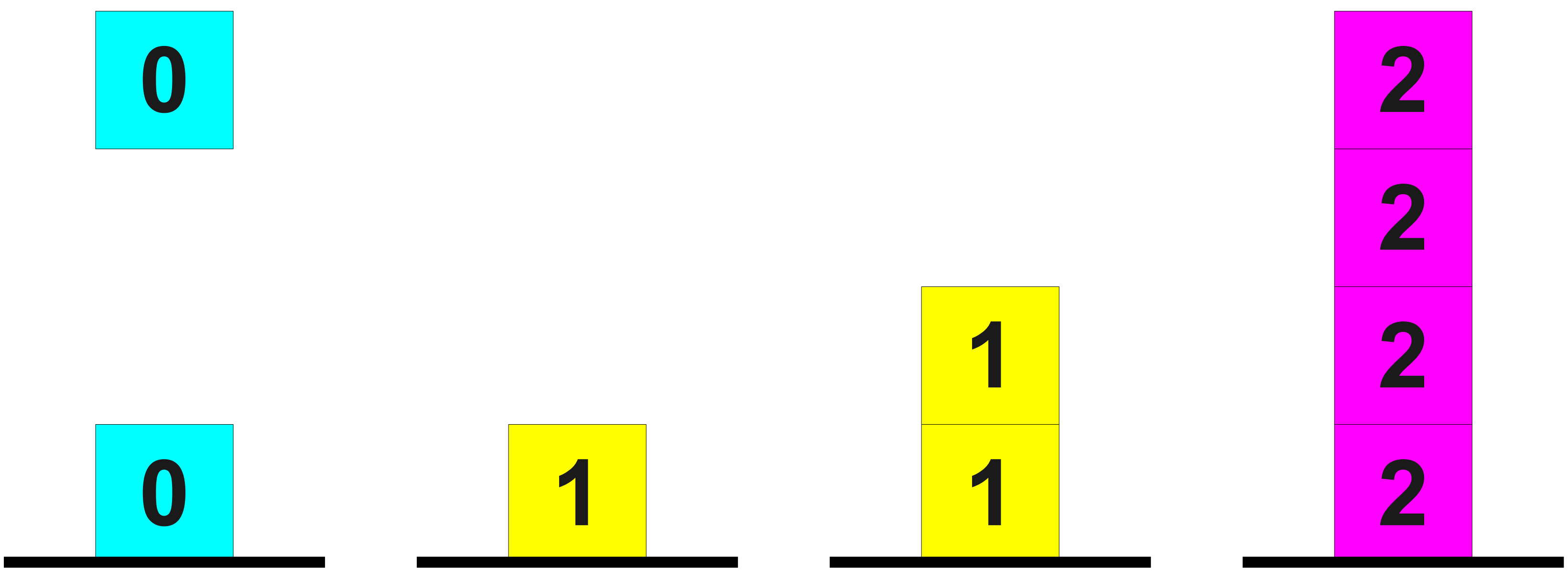
One Approach



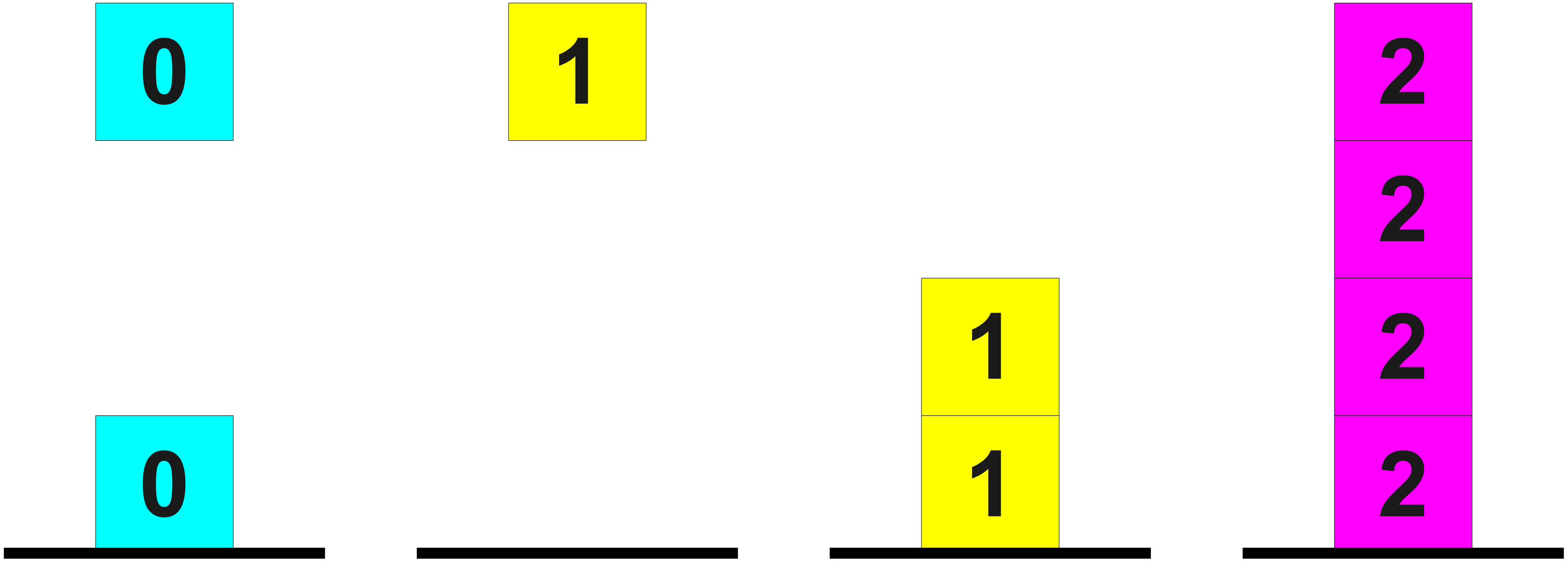
One Approach



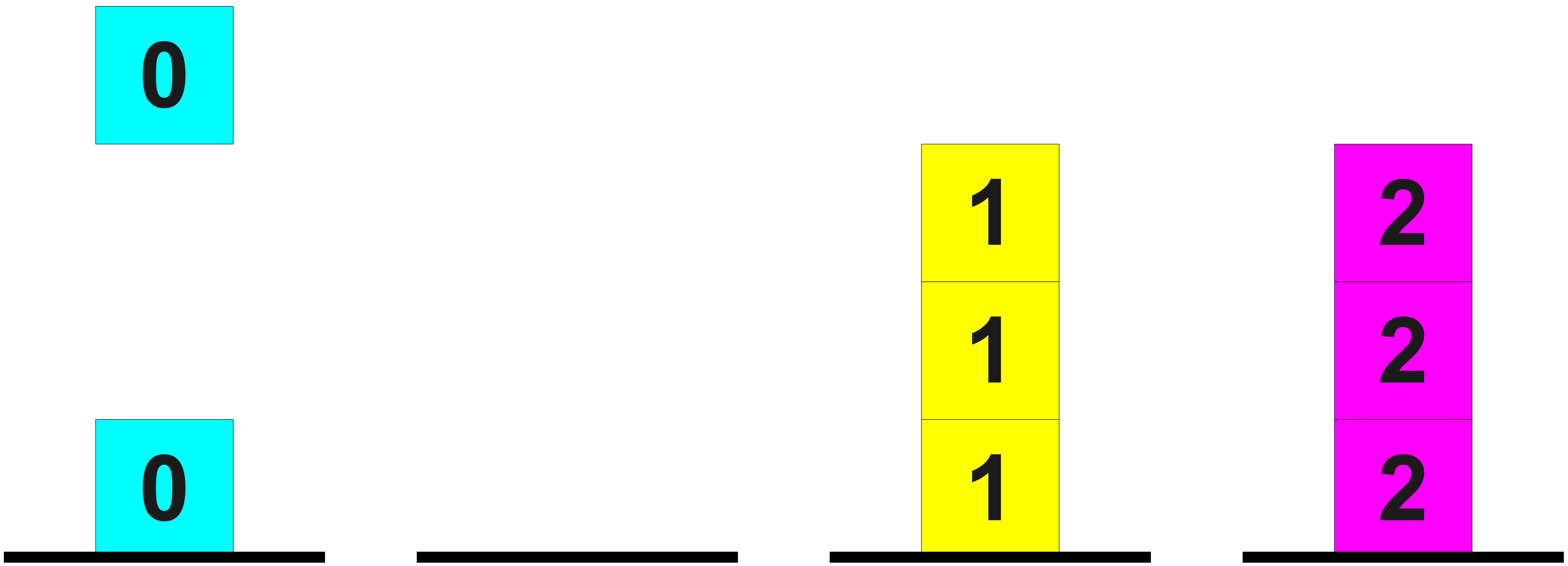
One Approach



One Approach

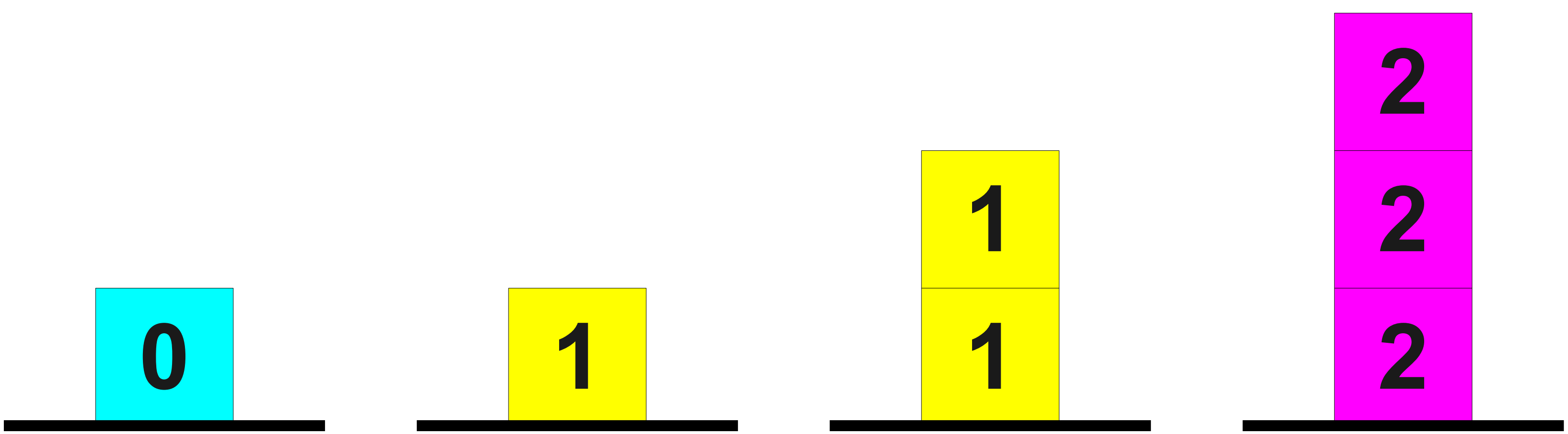


One Approach

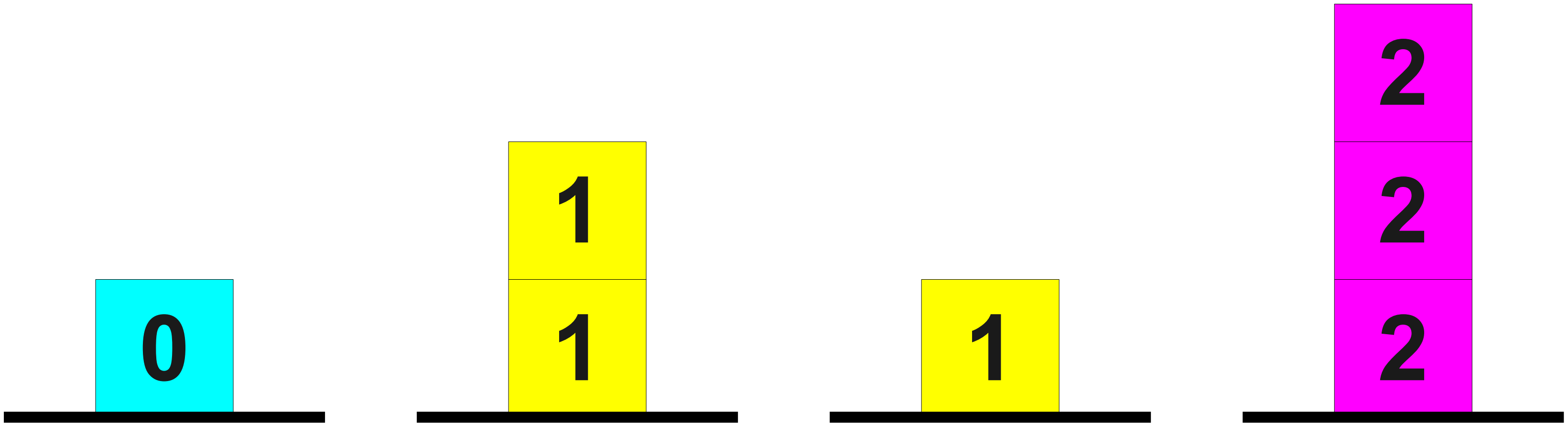


One Approach

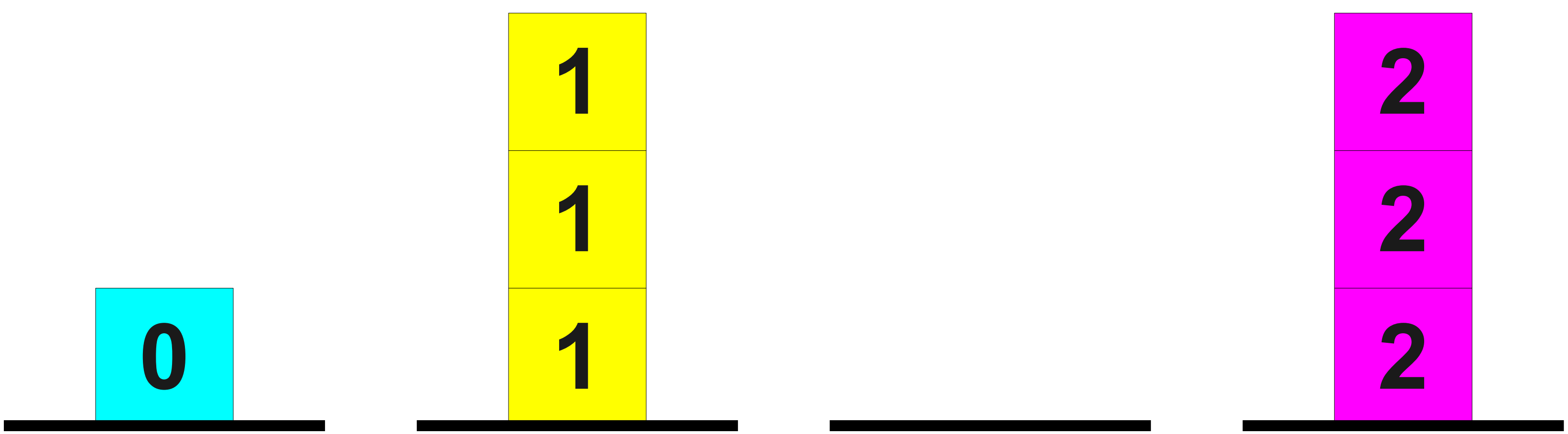
One Approach



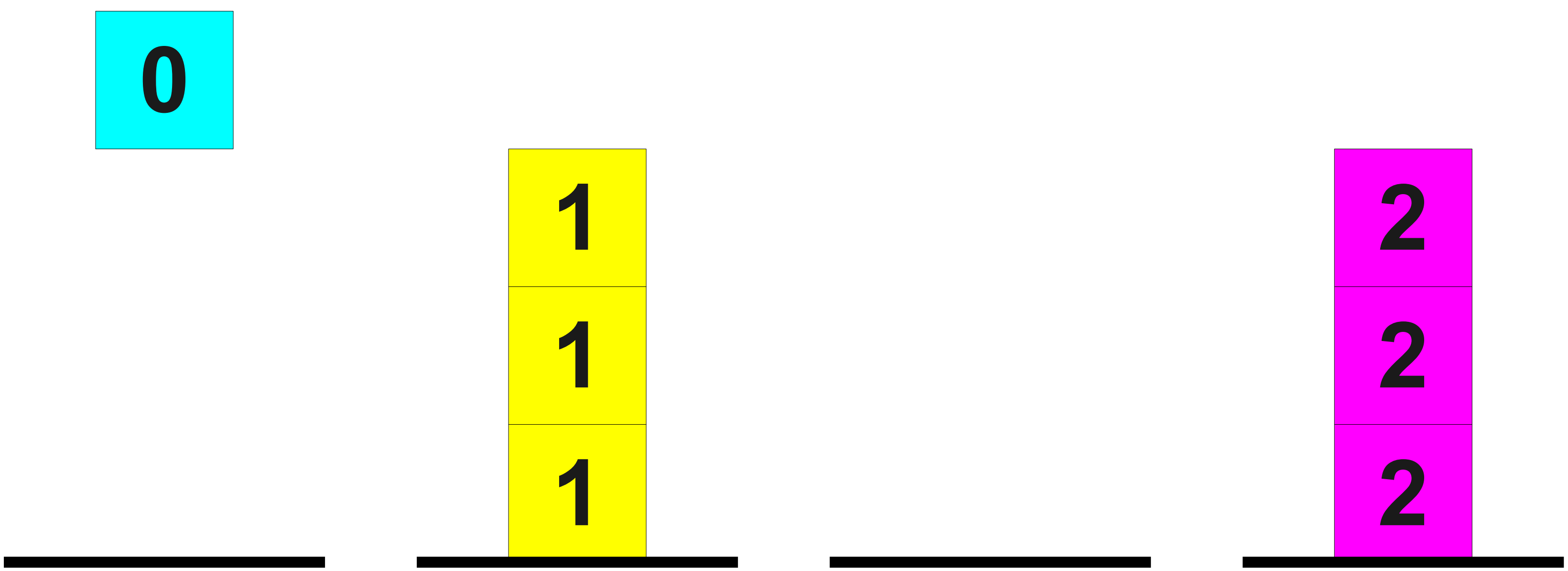
One Approach



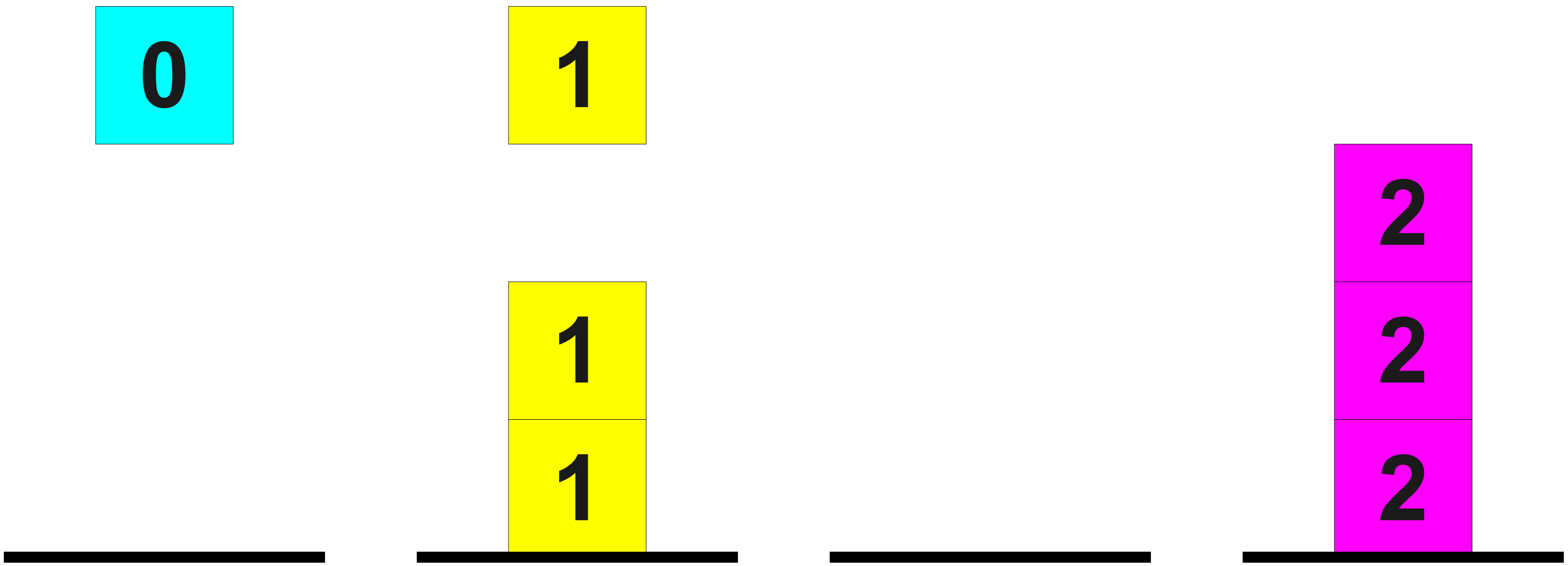
One Approach



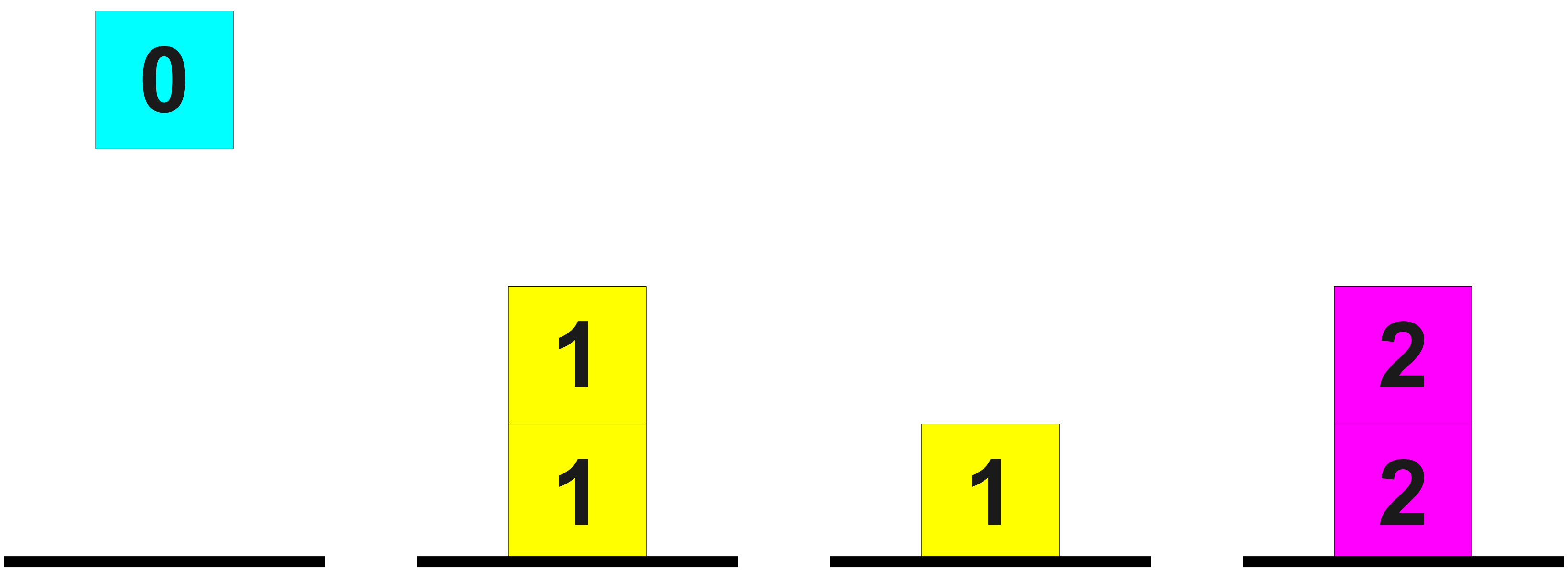
One Approach



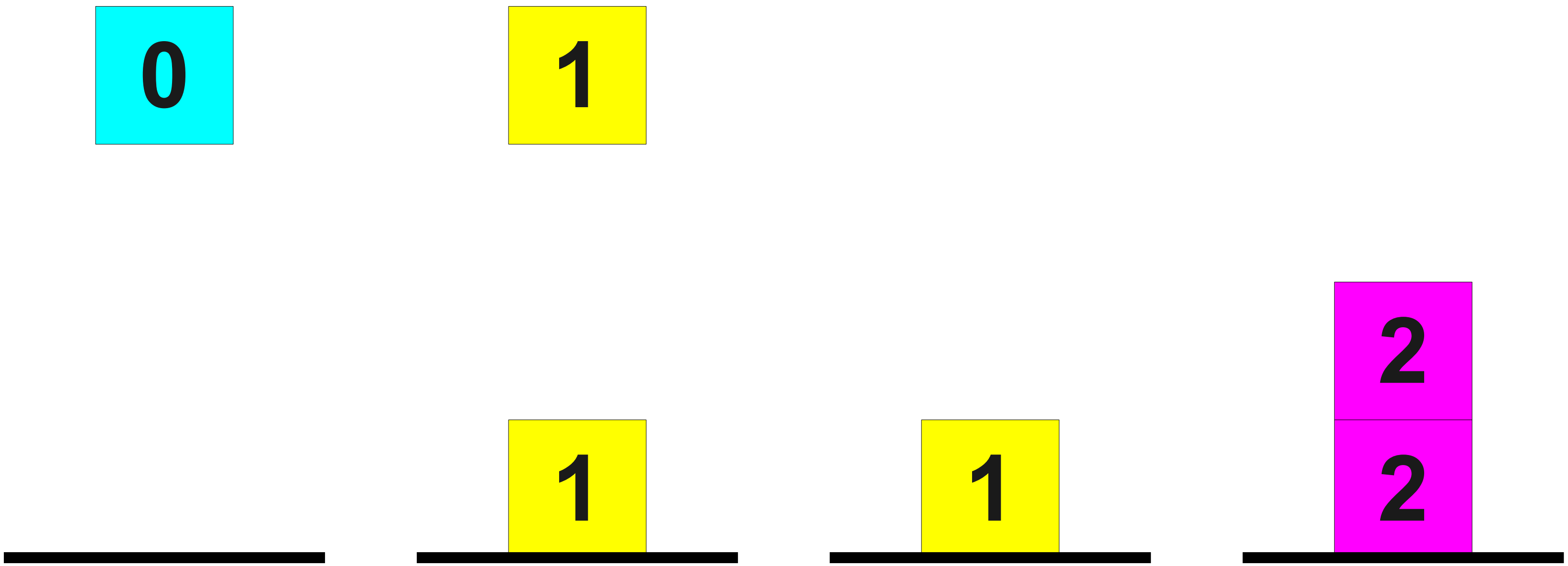
One Approach



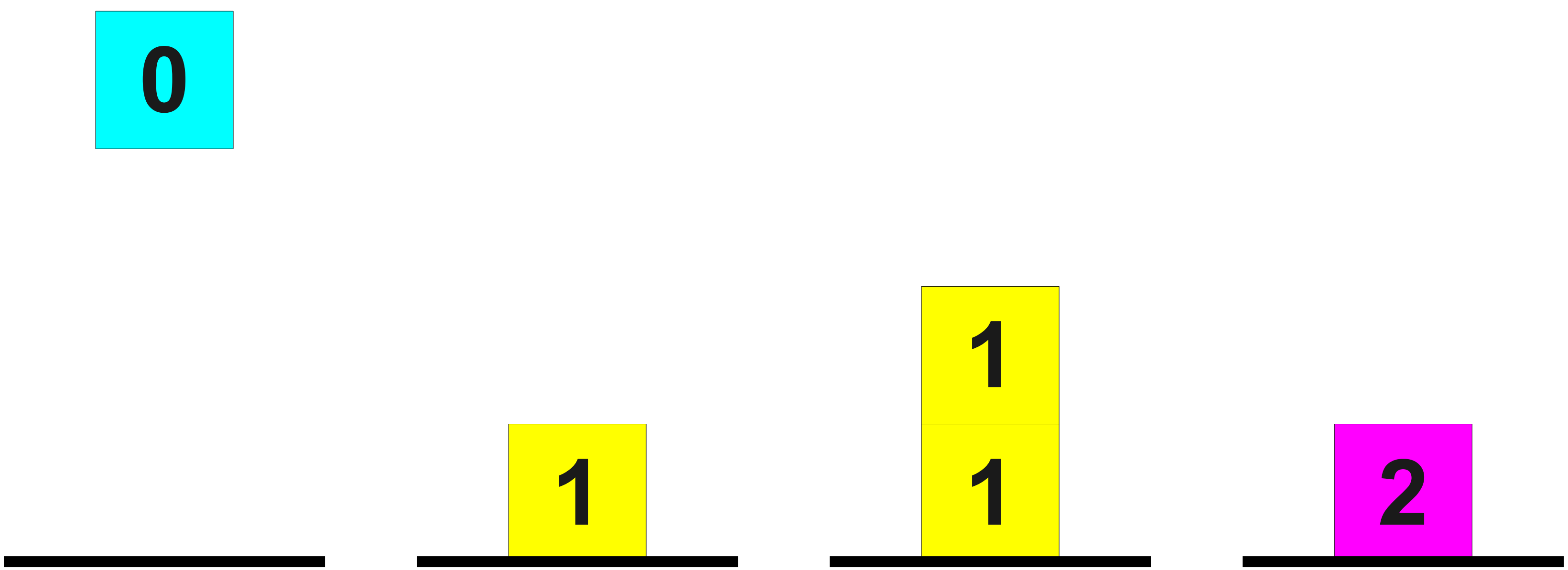
One Approach



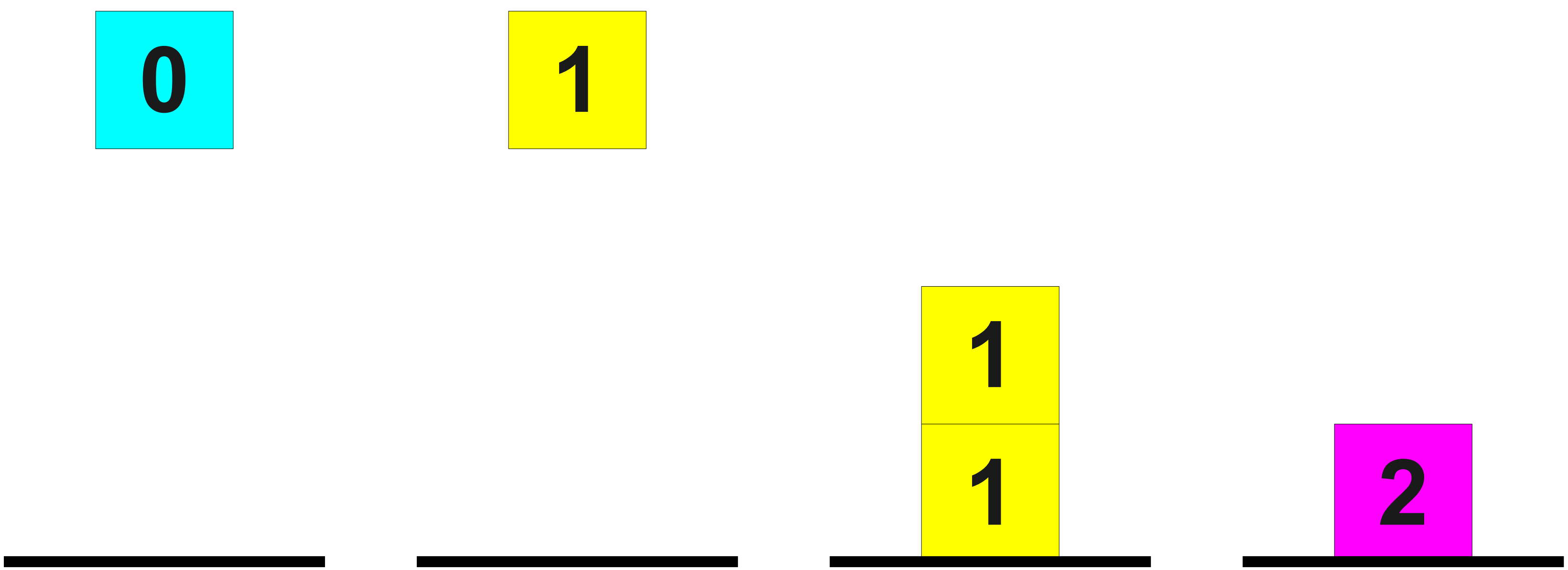
One Approach



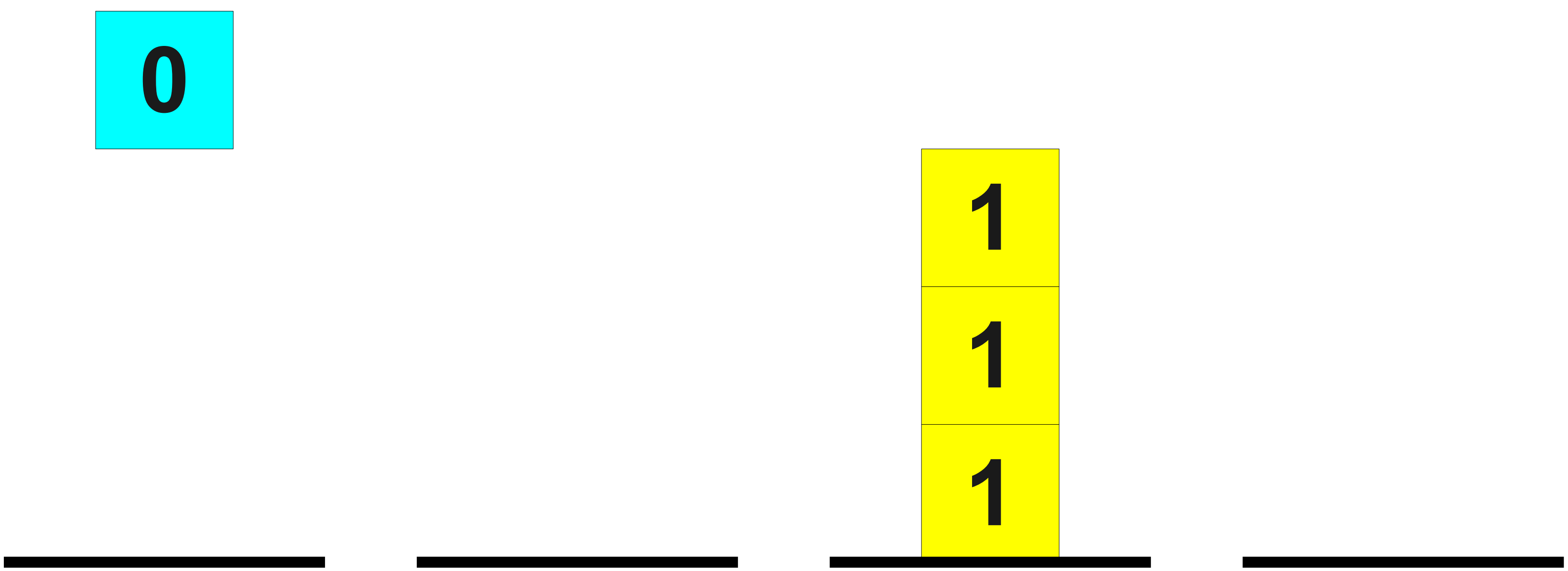
One Approach



One Approach

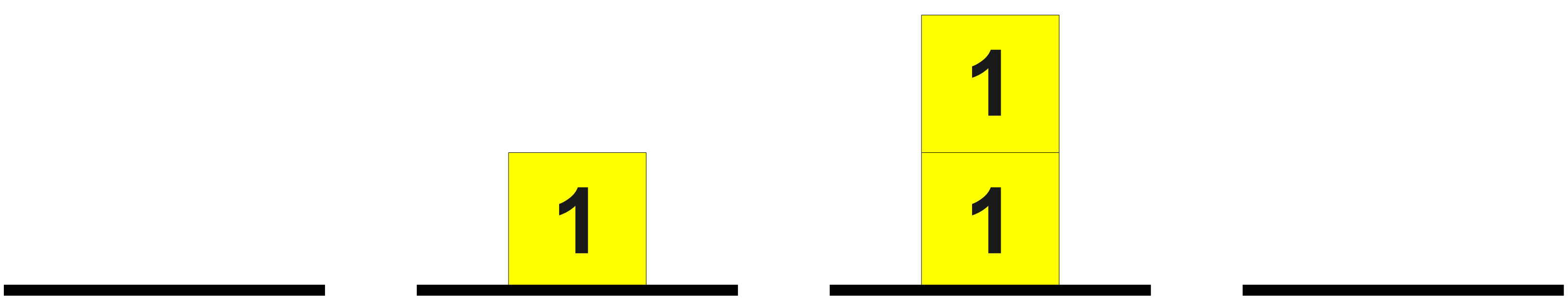


One Approach

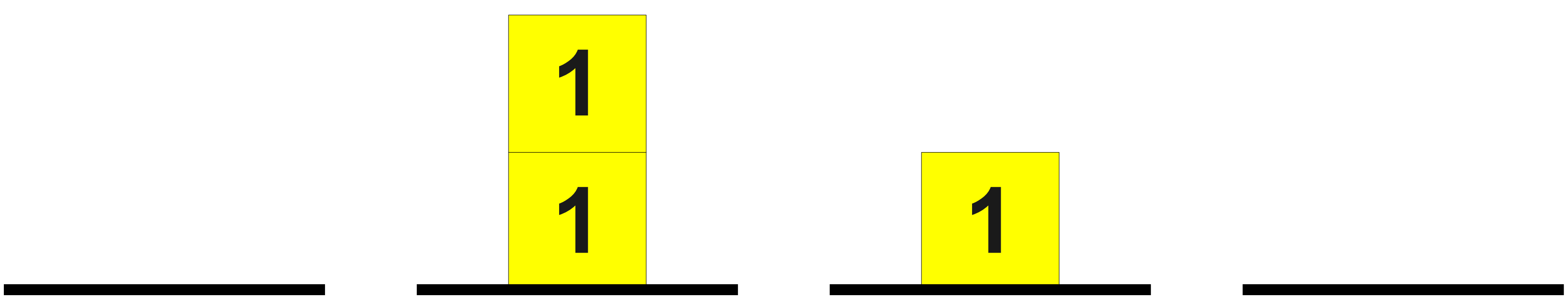


One Approach

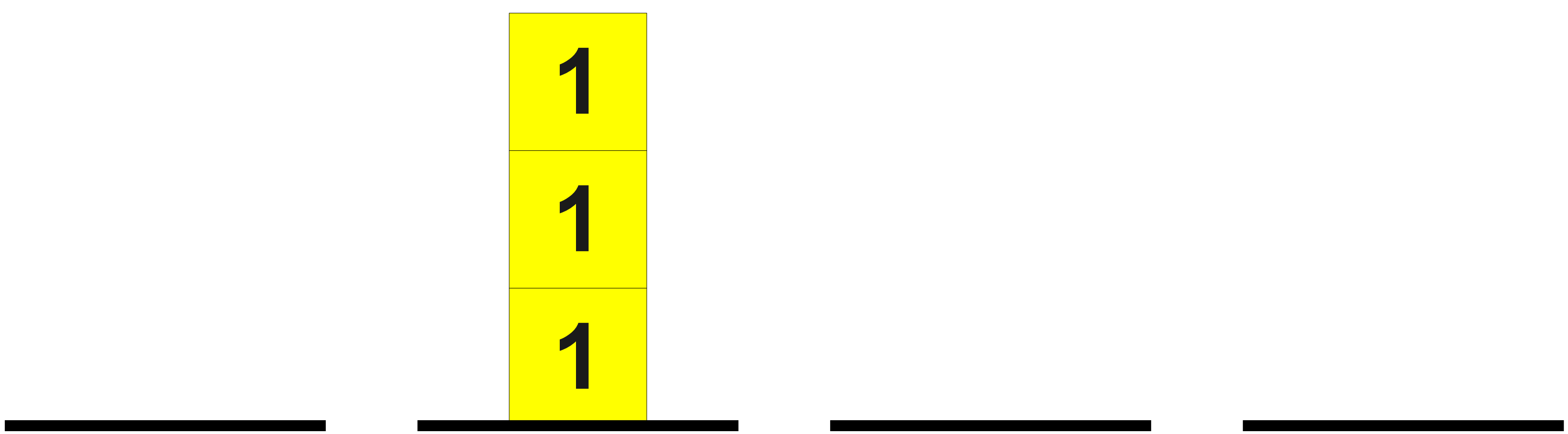
One Approach



One Approach



One Approach



WB5 Program for *01MULTI*

```
// Start
 0: If reading 0, go to Load0.
 1: If reading 1, go to Load1.
 2: If reading 2, go to Load2.
 3: Go to Check.

// Load0
 4: Push 0 onto Stack 0.
 5: Move right.
 6: Go to Start.

// Load1
 7: Push 1 onto Stack 1.
 8: Move right.
 9: Go to Start.

// Load2
10: Push 2 onto Stack 2.
11: Move right.
12: Go to Start.
```

WB5 Program for *01MULTI*

```
// Check:
13: If Stack 0 is empty, go to Ver.
14: Pop Stack 0.
15: If Stack 1 is empty, go to Fix.
16: Pop Stack 1.
17: Push 1 onto Stack 1T.
18: If Stack 2 is empty, go to Rej.
19: Pop Stack 2.
20: Go to 15.

// Rej:
21: Reject.

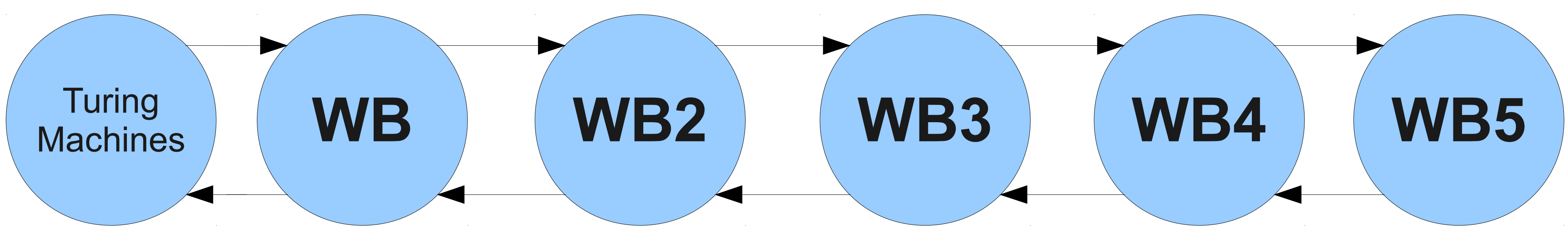
// Fix:
22: If St 1T is empty, go to Check.
23: Pop Stack 1T.
24: Push 1 onto Stack 1.
25: Go to Fix.

// Ver:
26: If Stack 2 is empty, go to Acc.
27: Reject.

// Acc:
28: Accept.
```

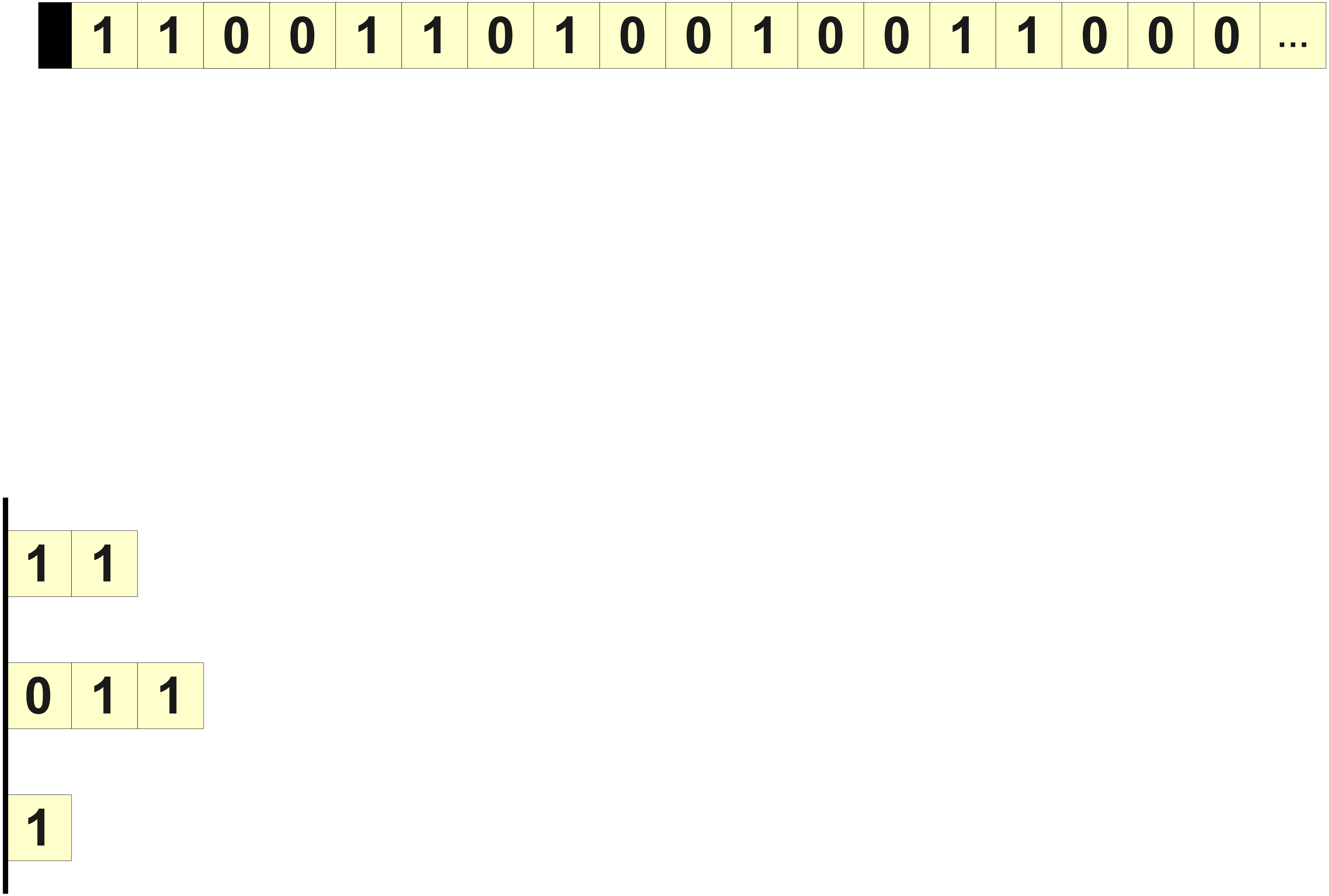
A Pretty Ridiculous Theorem

- **Theorem:** A language is recursively enumerable iff there is a **WB5** program for it.
- So adding in finitely many infinite stacks doesn't give us any more expressive power!
- As with before, will prove that all **WB5** programs are equivalent to **WB4** programs.



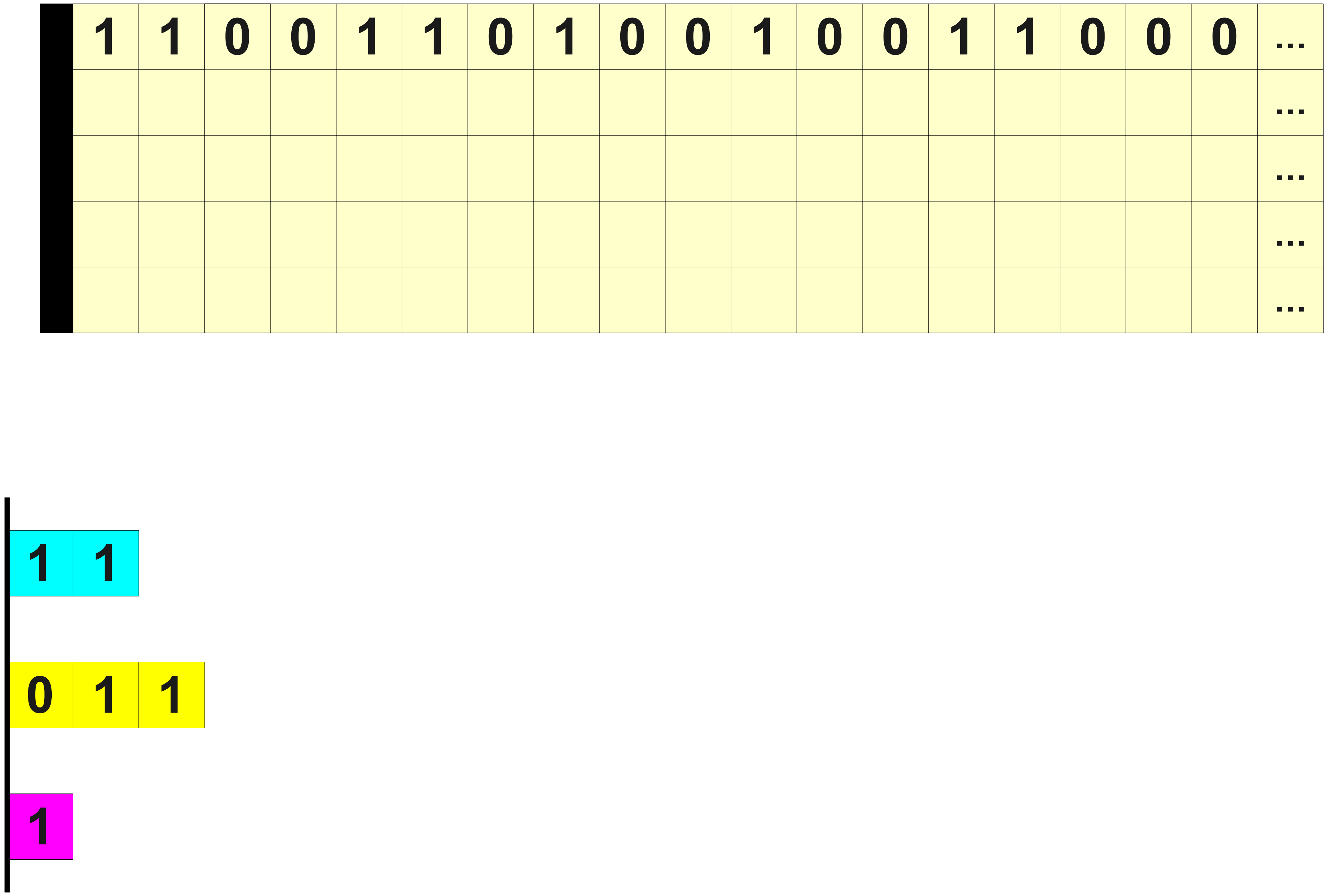
From Stacks to Tracks

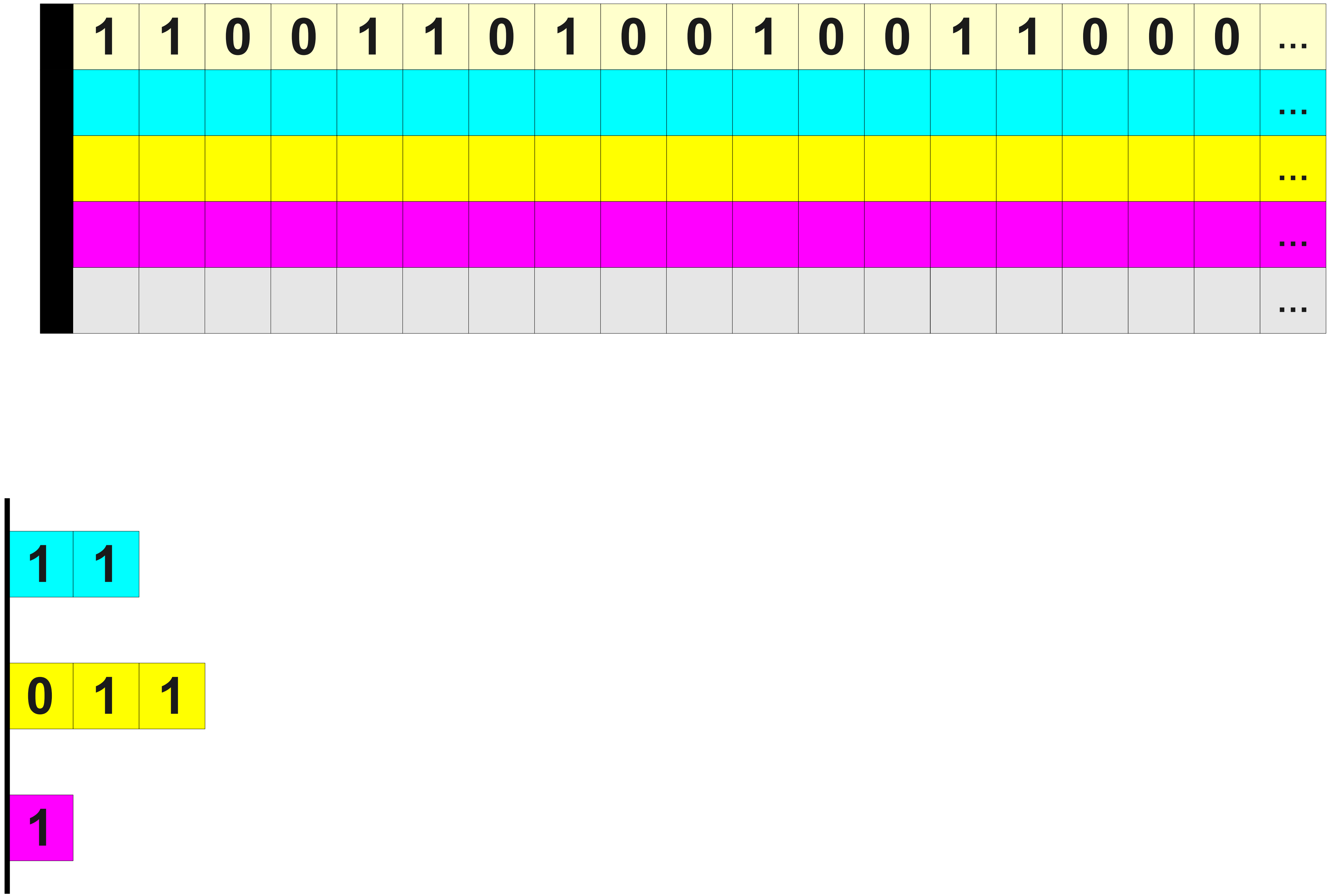
- The key idea behind the construction for converting **WB5** programs into **WB4** programs is to represent each stack with its own track.
- If there are n stacks in the program, we will add $n + 1$ tracks:
  - One track for each of the n stacks, and
  - One track for bookkeeping.
- If the **WB5** program was using any tracks, we'll keep them as well and add these new ones in separately.


1 1 0 0 1 1 0 1 0 0 1 0 0 1 1 0 0 0 ...
1 1
0 1 1
1

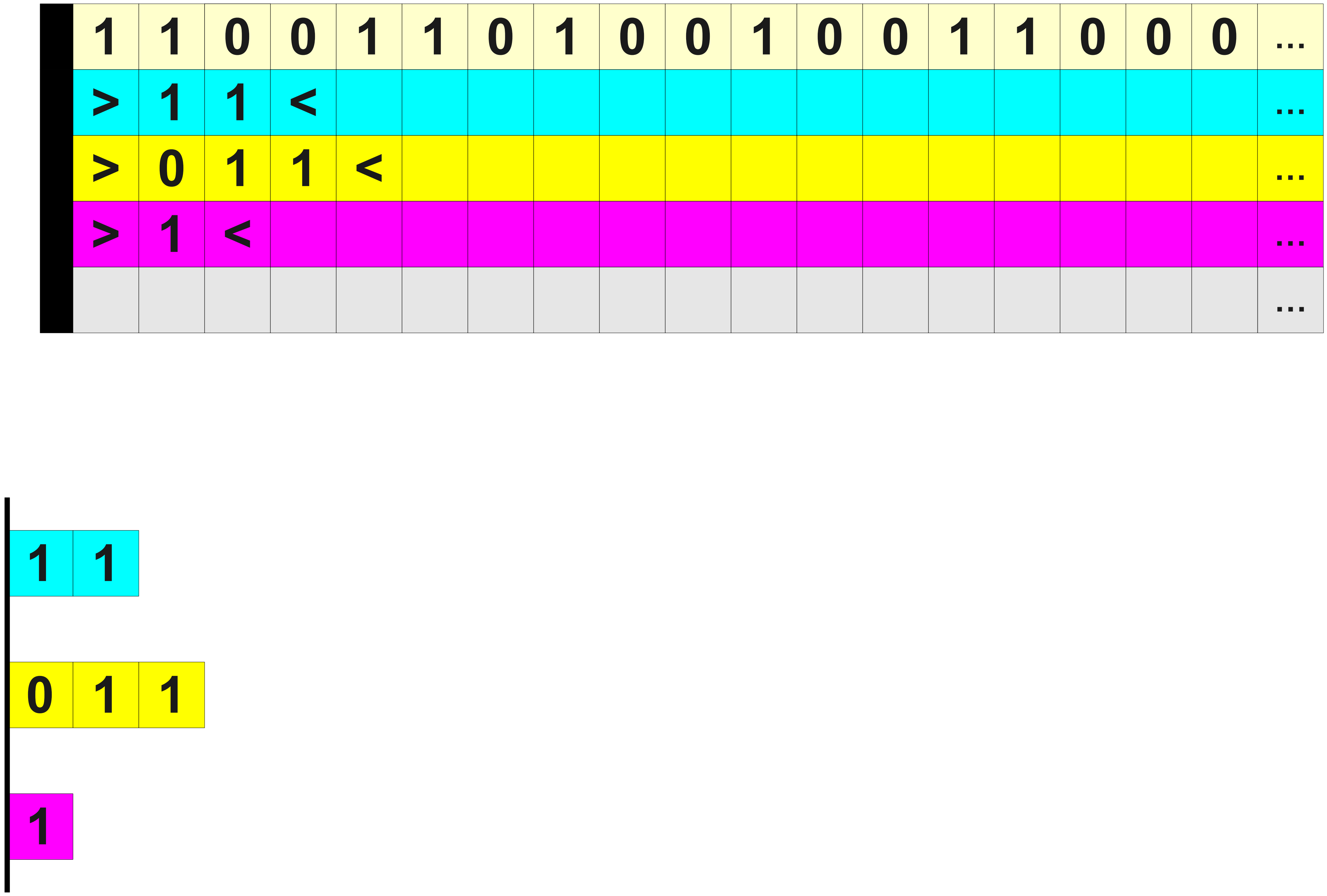
1	1	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0	0	...
																		...
																		...
																		...
																		...

1	1	
0	1	1
1		

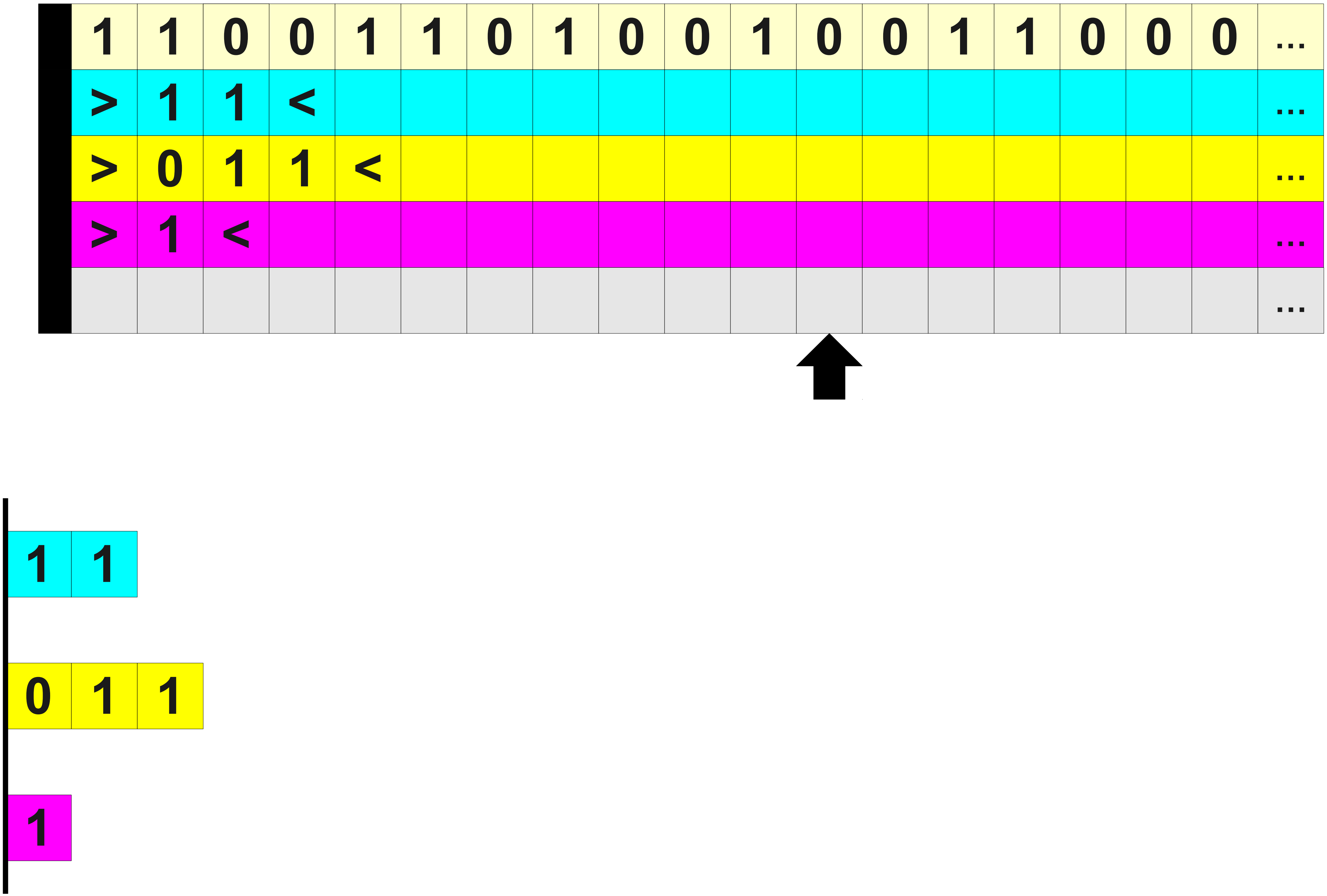

1 1 0 0 1 1 0 1 0 0 1 0 0 1 1 0 0 0 ...
...
...
...
...
1 1
0 1 1
1

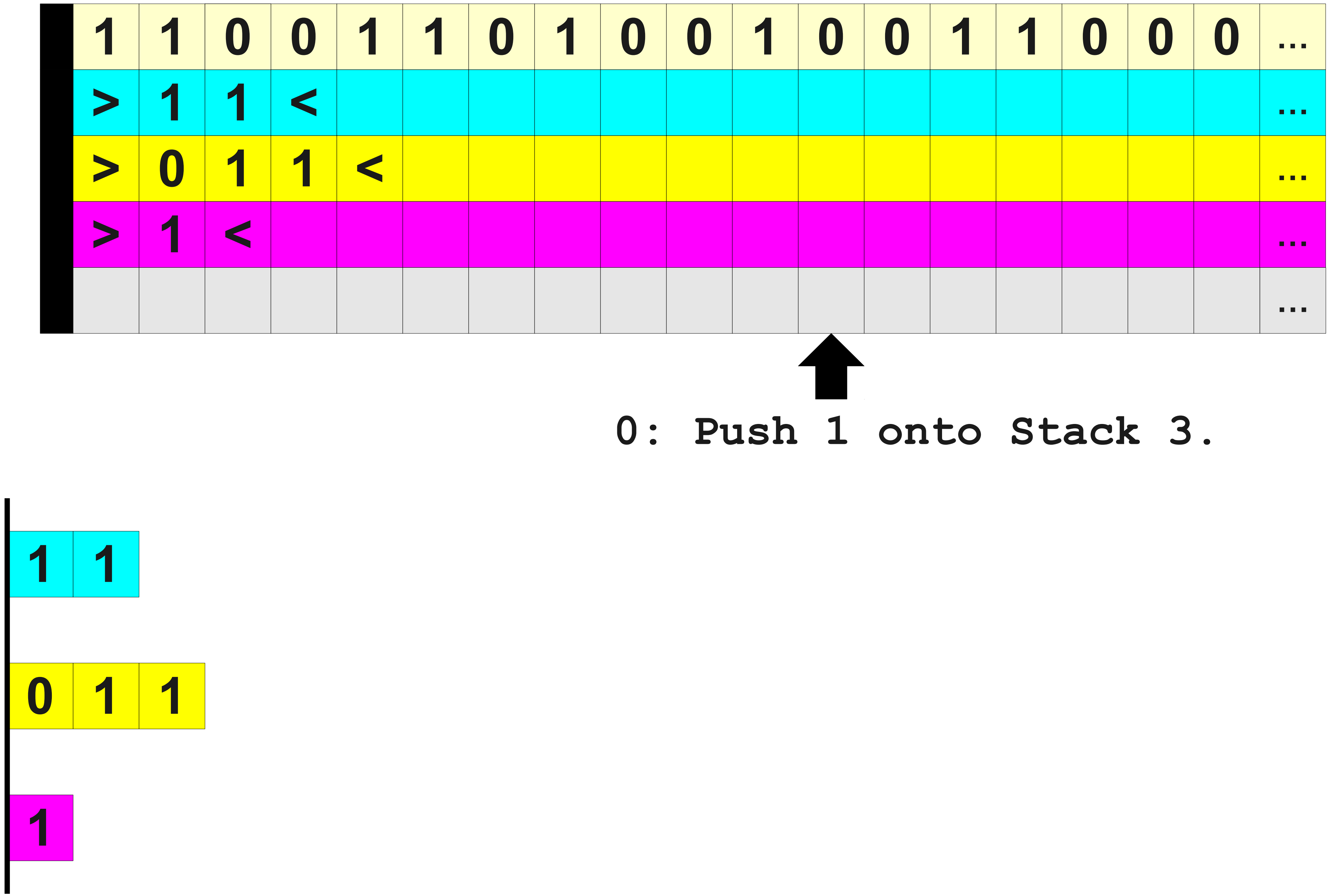


1 1 0 0 1 1 0 1 0 0 1 0 0 1 1 0 0 0 ...
1 1
0 1 1
1



1 1 0 0 1 1 0 1 0 0 1 0 0 1 1 0 0 0 ...
> 1 1 < ...
> 0 1 1 < ...
> 1 < ...
...
1 1
0 1 1
1


1 1 0 0 1 1 0 1 0 0 1 0 0 1 1 0 0 0 ...
> 1 1 < ...
> 0 1 1 < ...
> 1 < ...
...
1 1
0 1 1
1



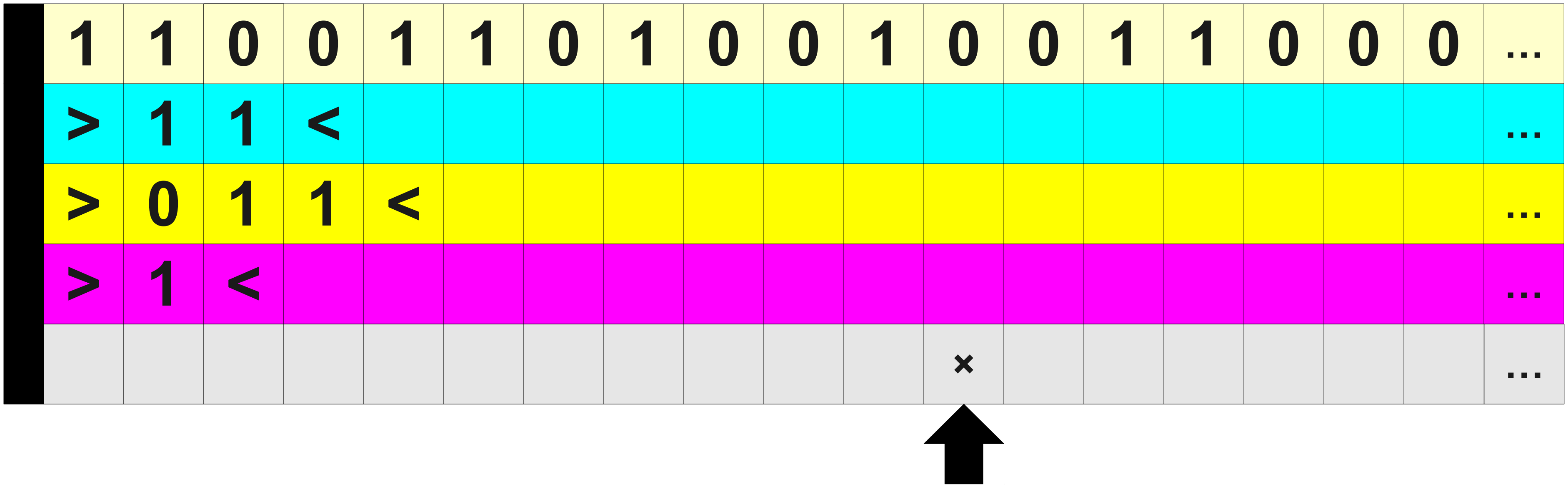
1 1 0 0 1 1 0 1 0 0 1 0 0 1 1 0 0 0 ...
> 1 1 < ...
> 0 1 1 < ...
> 1 < ...
...
0: Push 1 onto Stack 3.
1 1
0 1 1
1

1	1	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0	0	…
>	1	1	<															…
>	0	1	1	<														…
>	1	<																…
																		…

0: Push 1 onto Stack 3.

0: Write × on track 5.

1	1	
0	1	1
1		



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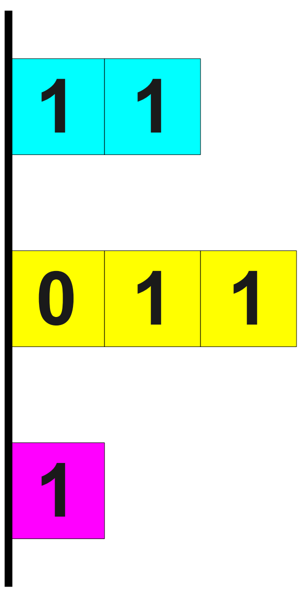
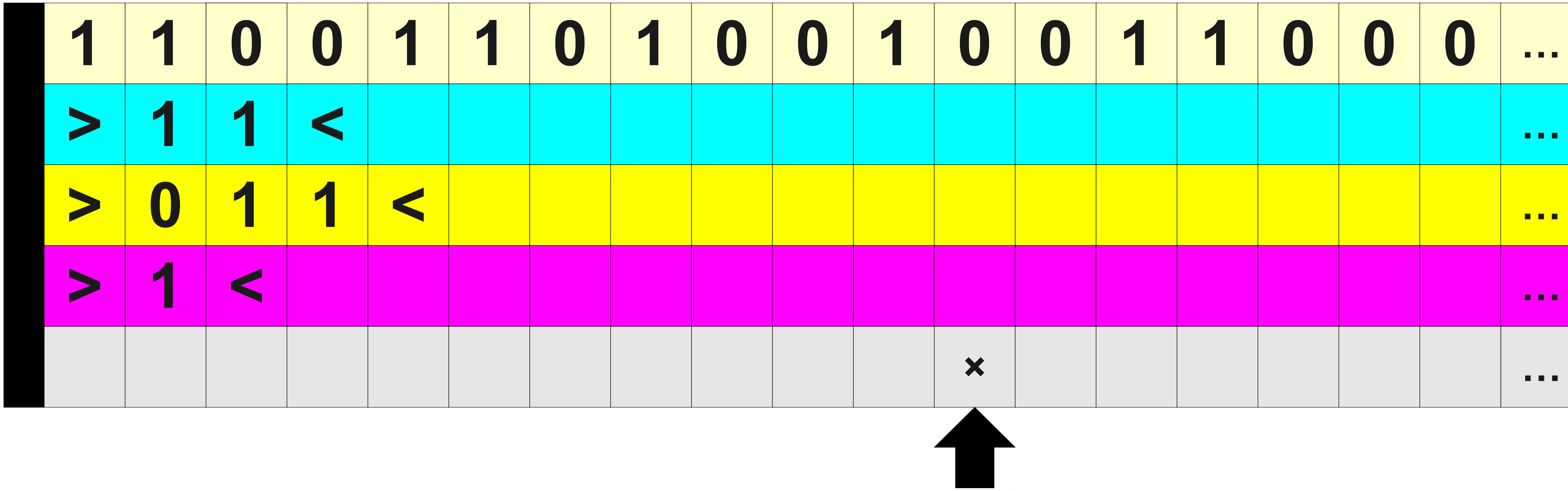
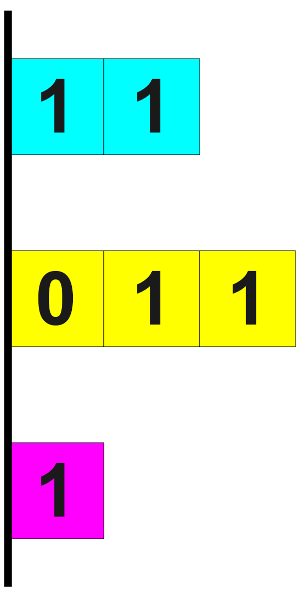


0: Write × on track 5.

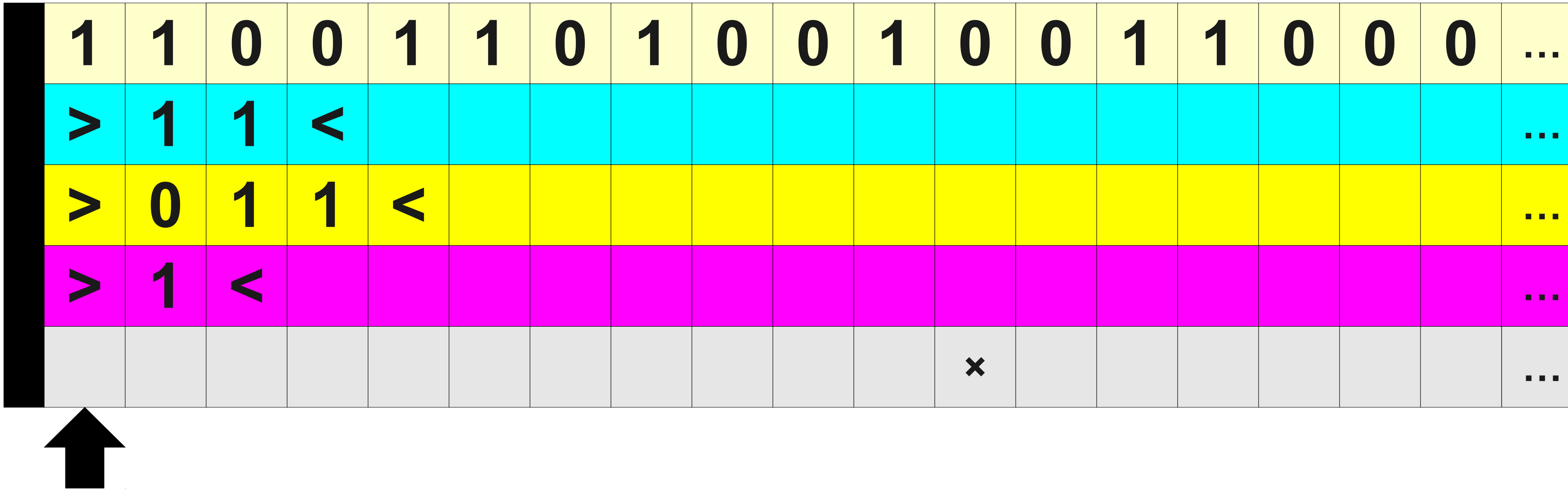


0: Push 1 onto Stack 3.

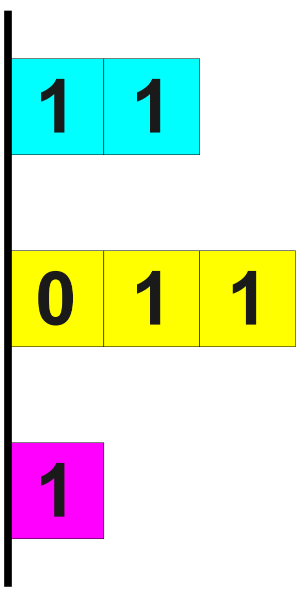


0: Write × on track 5.

1: Move left until {>} on track 4.

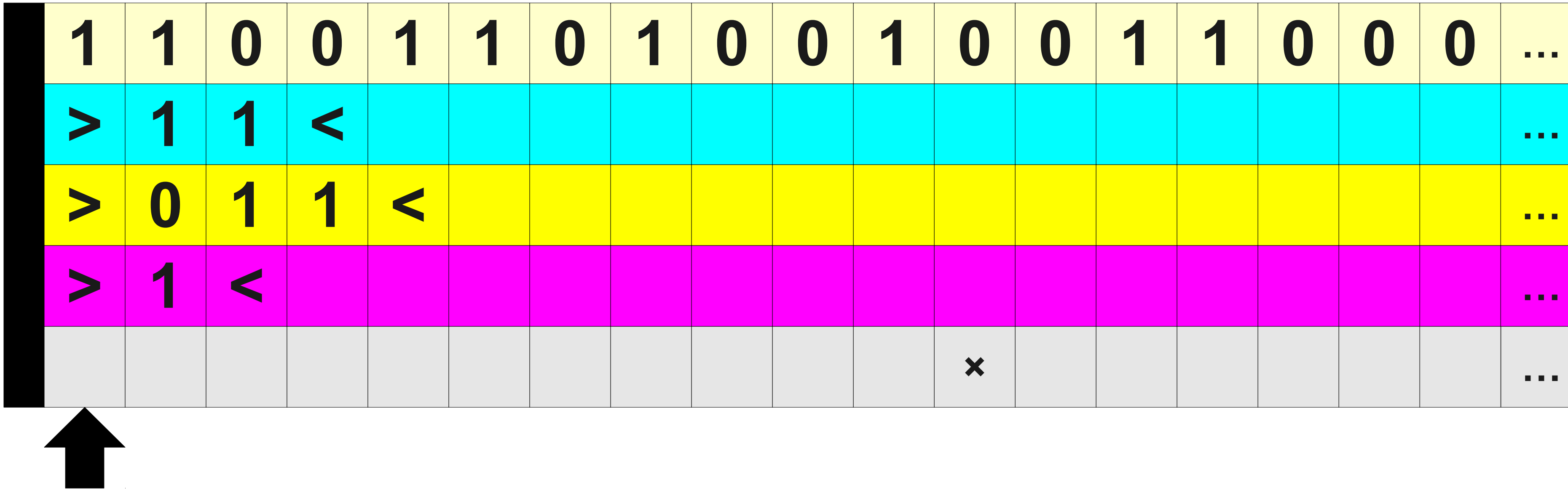


0: Push 1 onto Stack 3.

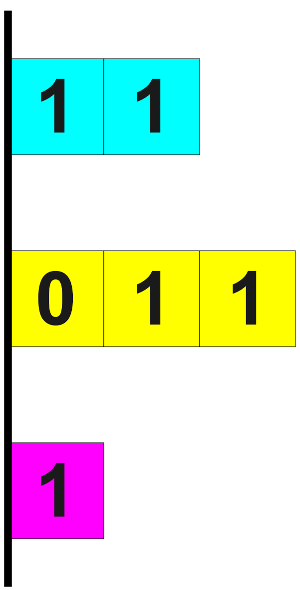


0: Write × on track 5.

1: Move left until {>} on track 4.



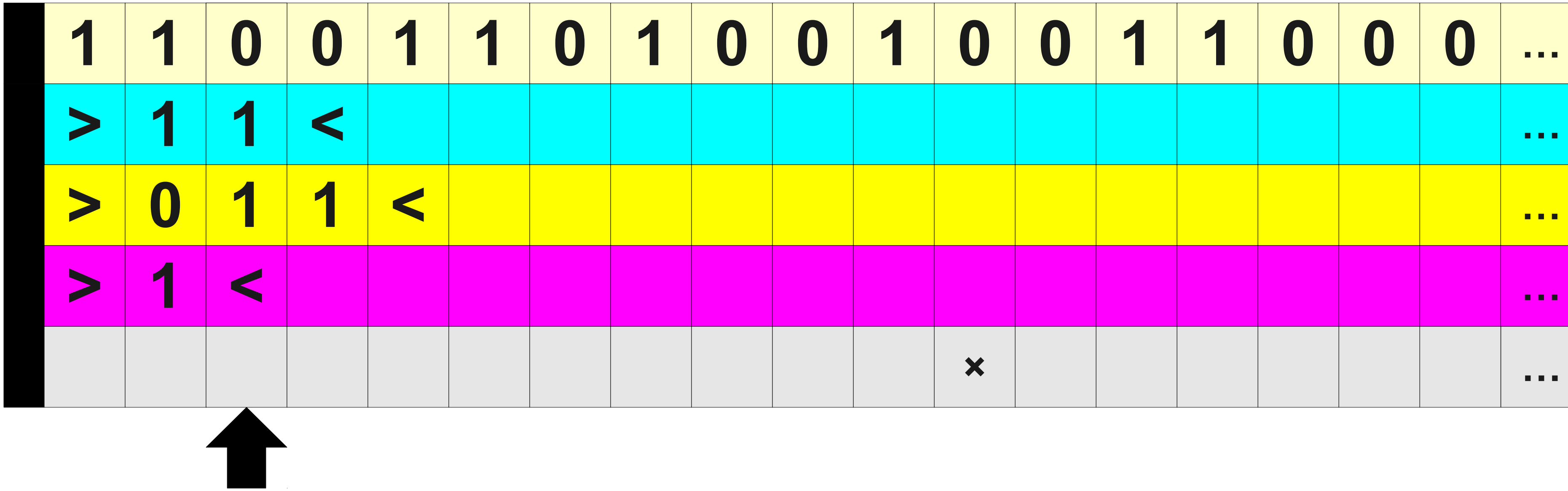
0: Push 1 onto Stack 3.



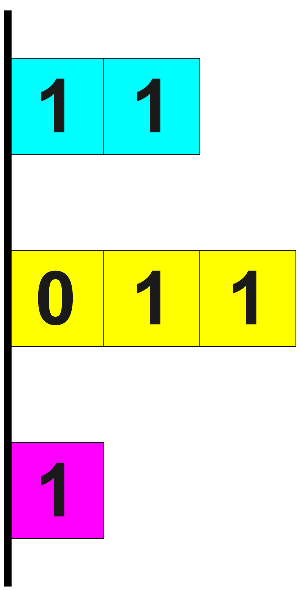
0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.



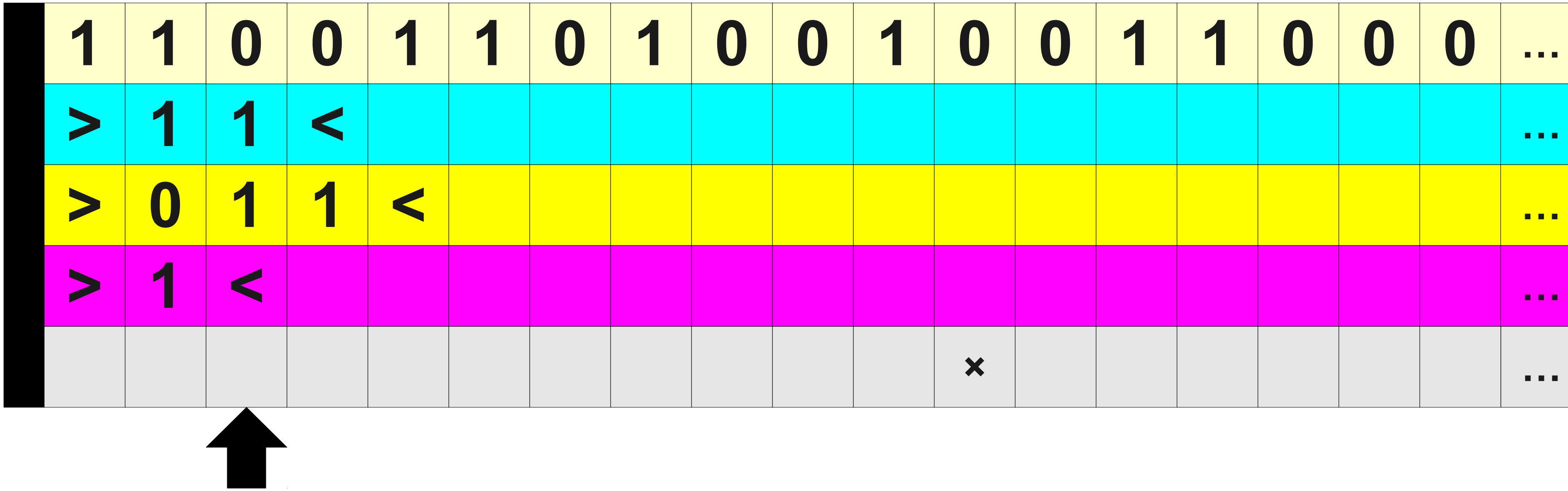
0: Push 1 onto Stack 3.



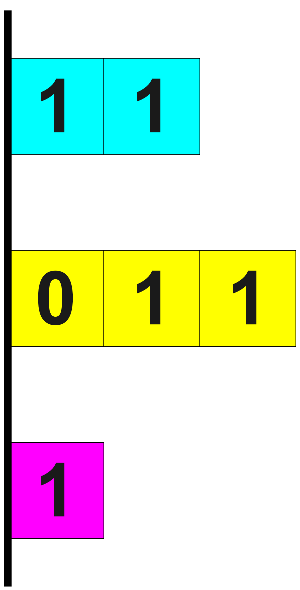
0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.



0: Push 1 onto Stack 3.

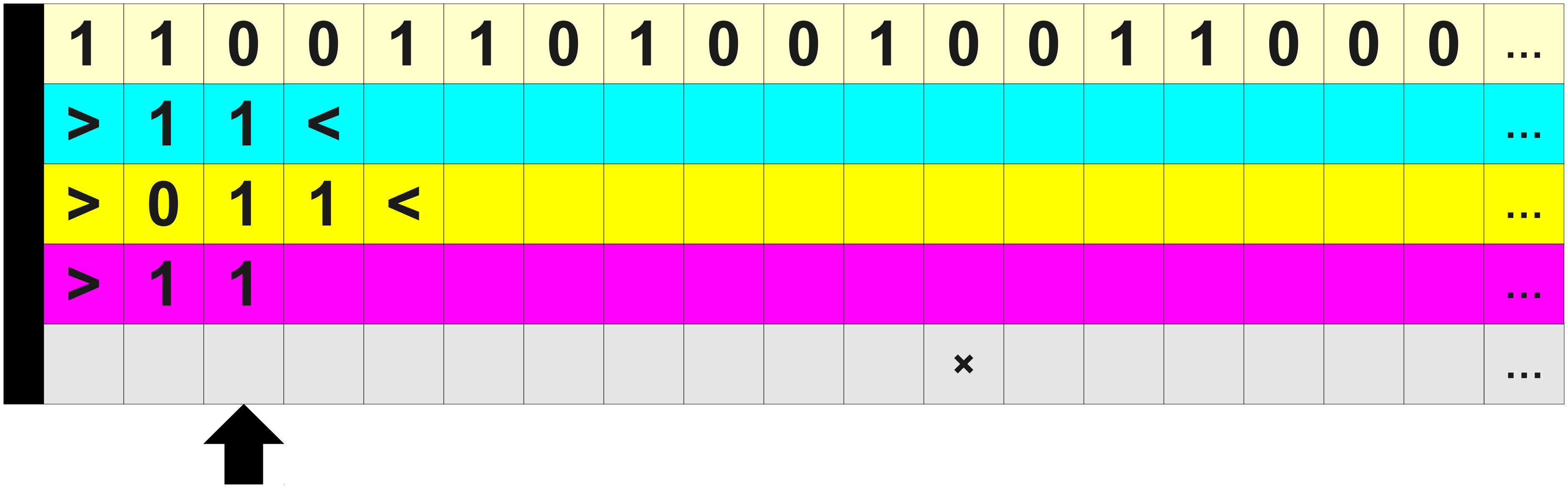


0: Write × on track 5.

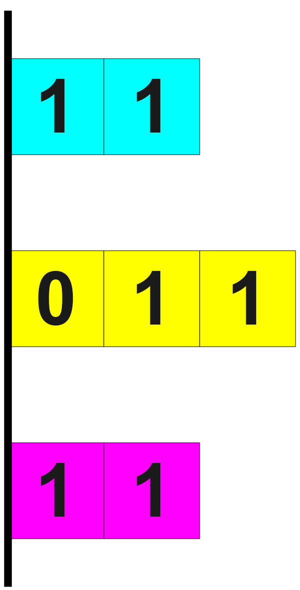
1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

3: Write 1 on track 4.



0: Push 1 onto Stack 3.

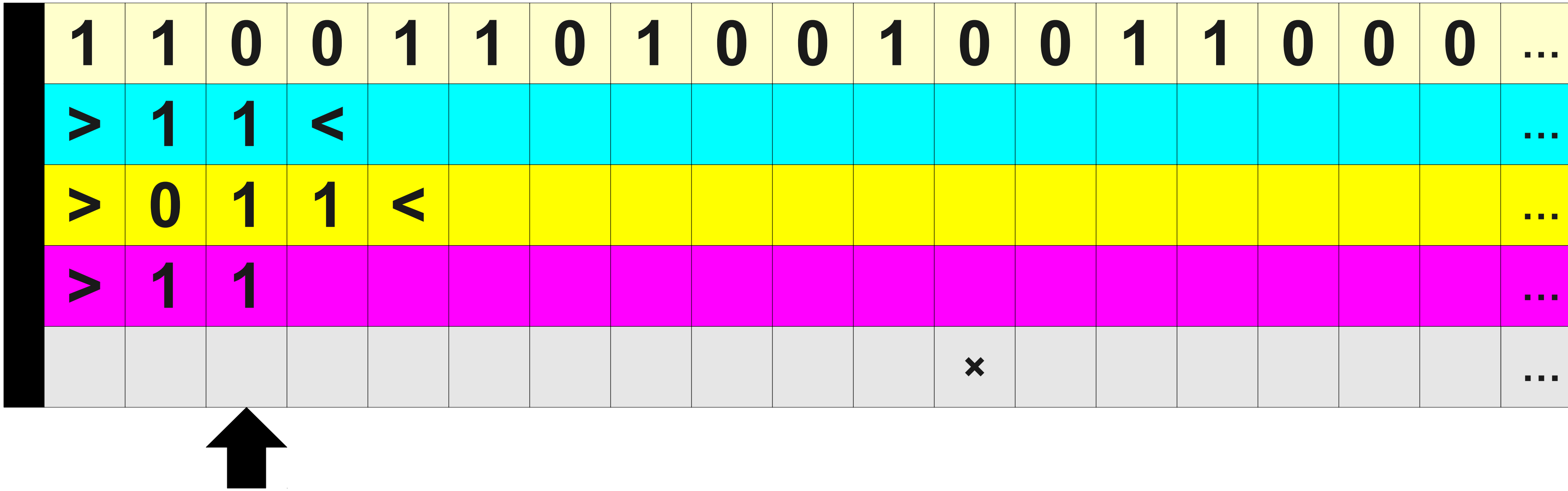


0: Write × on track 5.

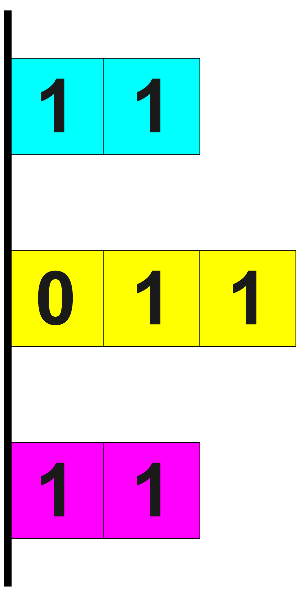
1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

3: Write 1 on track 4.



0: Push 1 onto Stack 3.



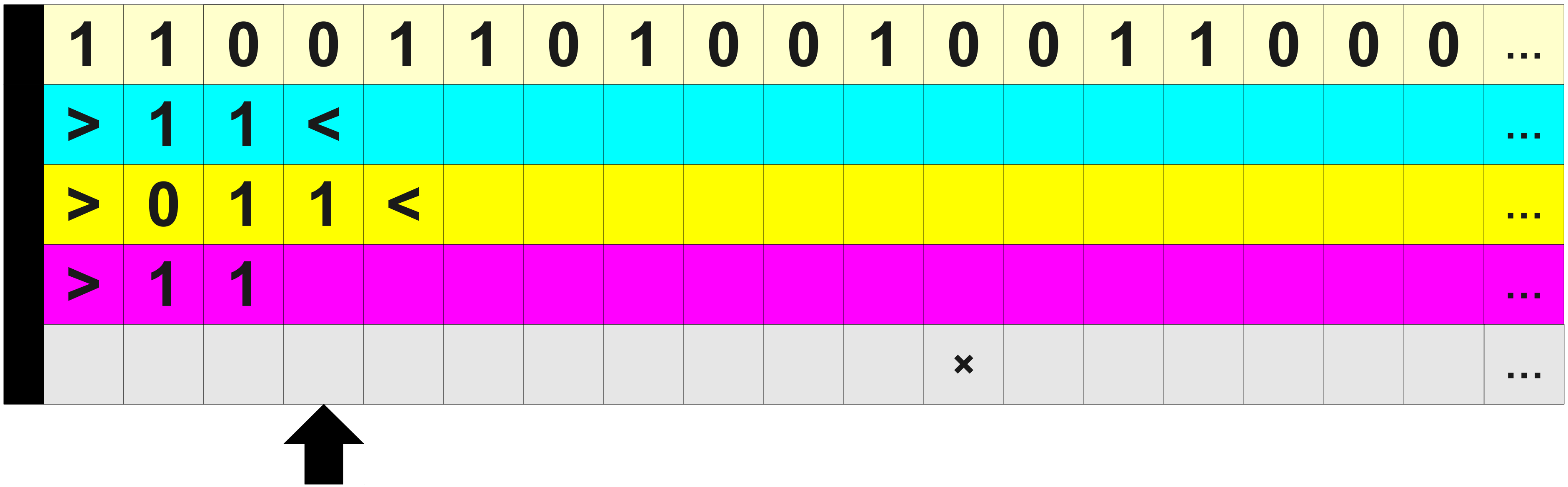
0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

3: Write 1 on track 4.

4: Move right.



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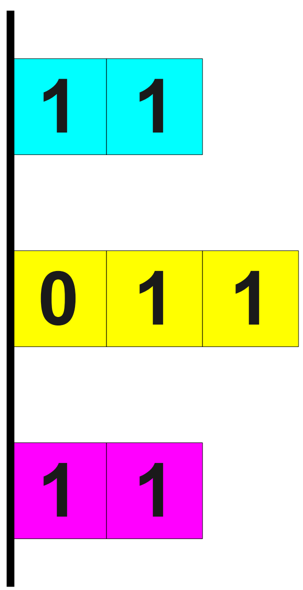


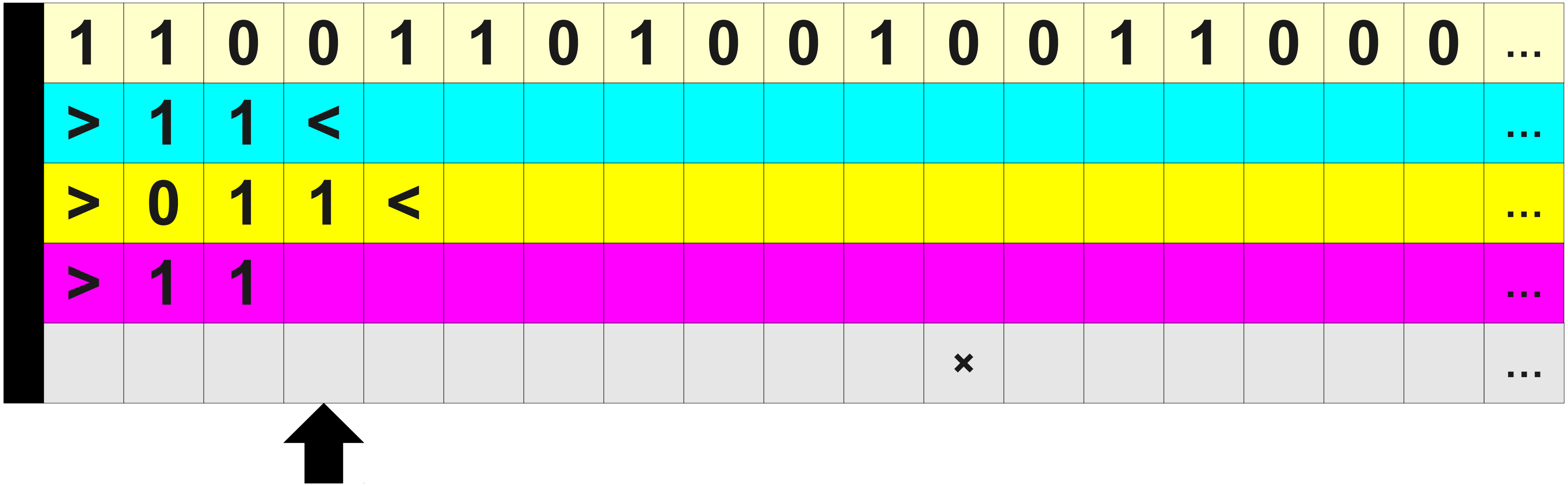
0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

3: Write 1 on track 4.

4: Move right.



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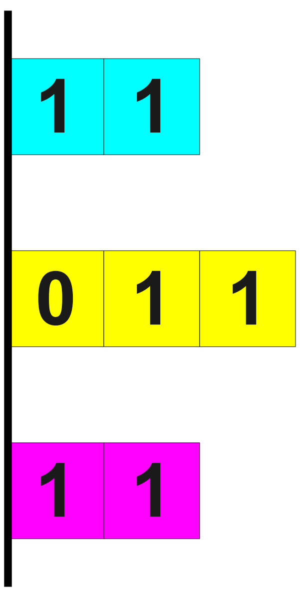


0: Write × on track 5.

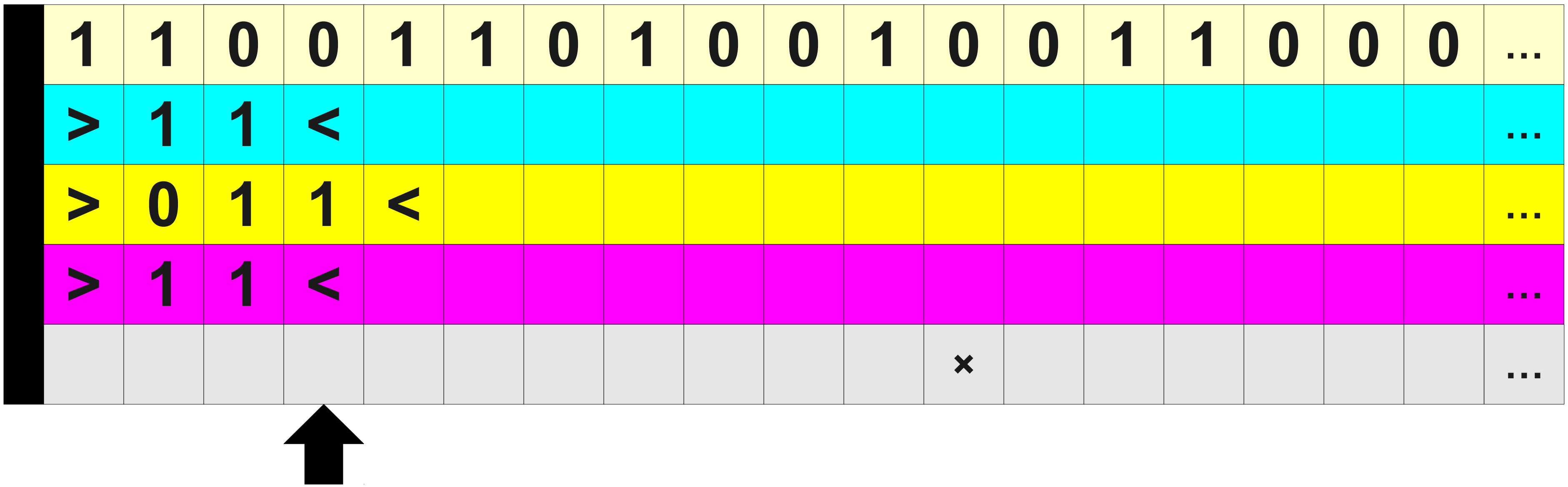
1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

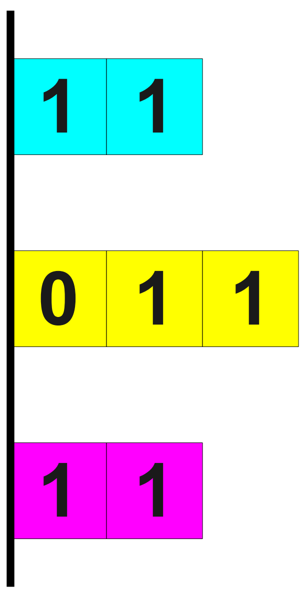
3: Write 1 on track 4.

4: Move right.

5: Write < on track 4



0: Push 1 onto Stack 3.



0: Write × on track 5.

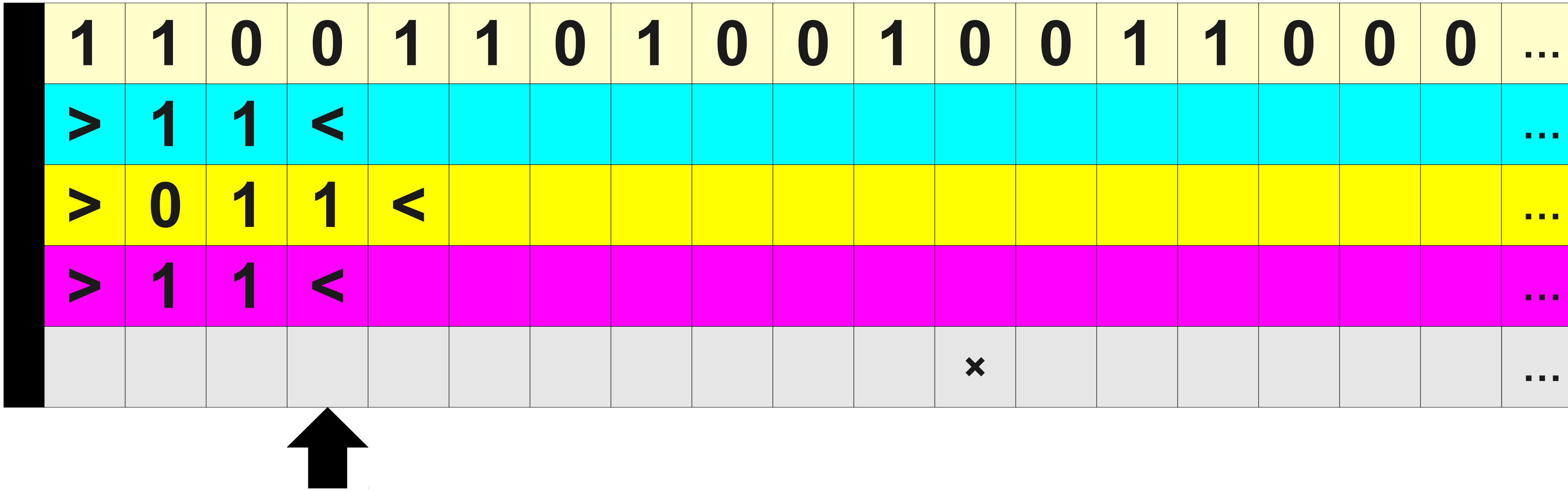
1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

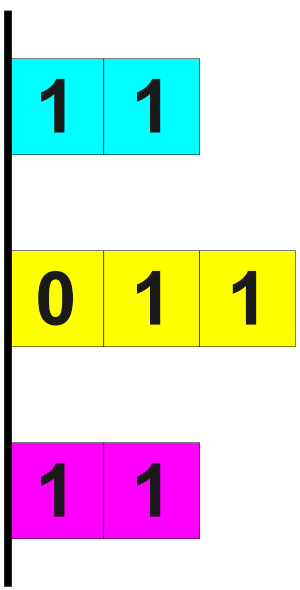
3: Write 1 on track 4.

4: Move right.

5: Write < on track 4



0: Push 1 onto Stack 3.



0: Write × on track 5.

1: Move left until {>} on track 4.

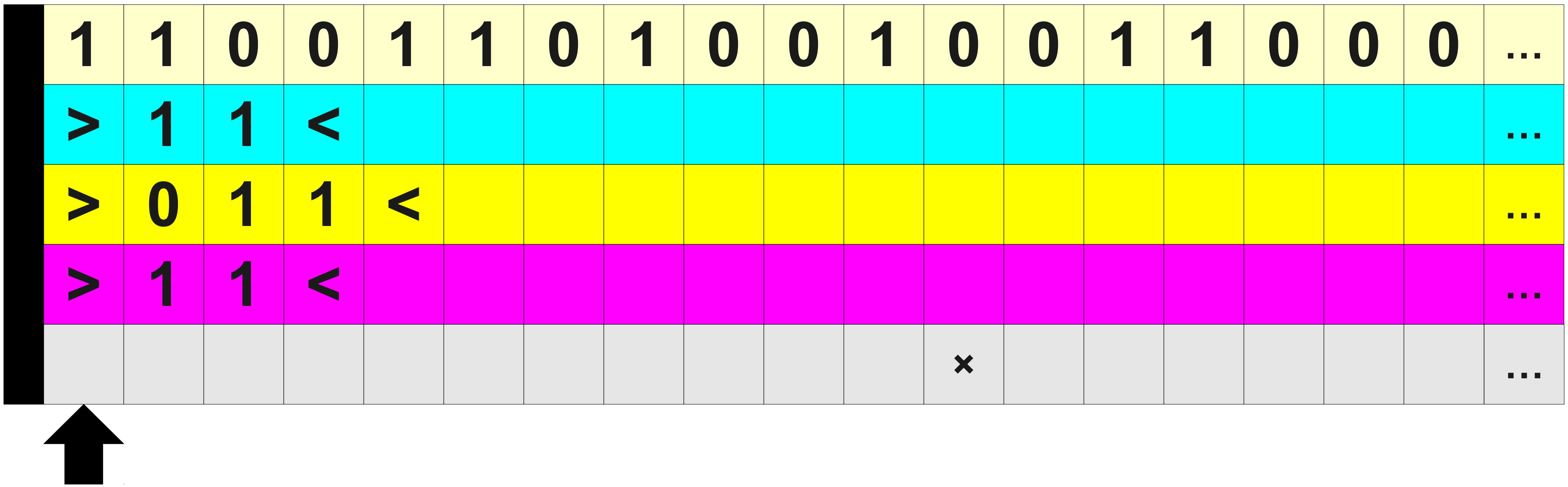
2: Move right until {<} on track 4.

3: Write 1 on track 4.

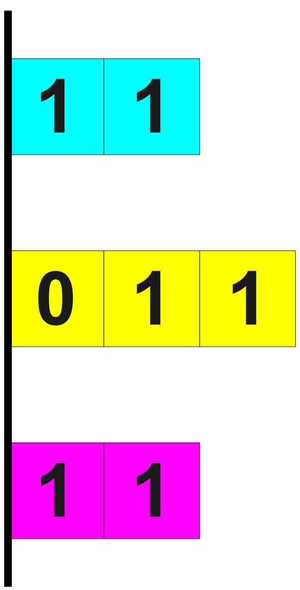
4: Move right.

5: Write < on track 4

6: Move left until {>} on track 4.



0: Push 1 onto Stack 3.



0: Write × on track 5.

1: Move left until {>} on track 4.

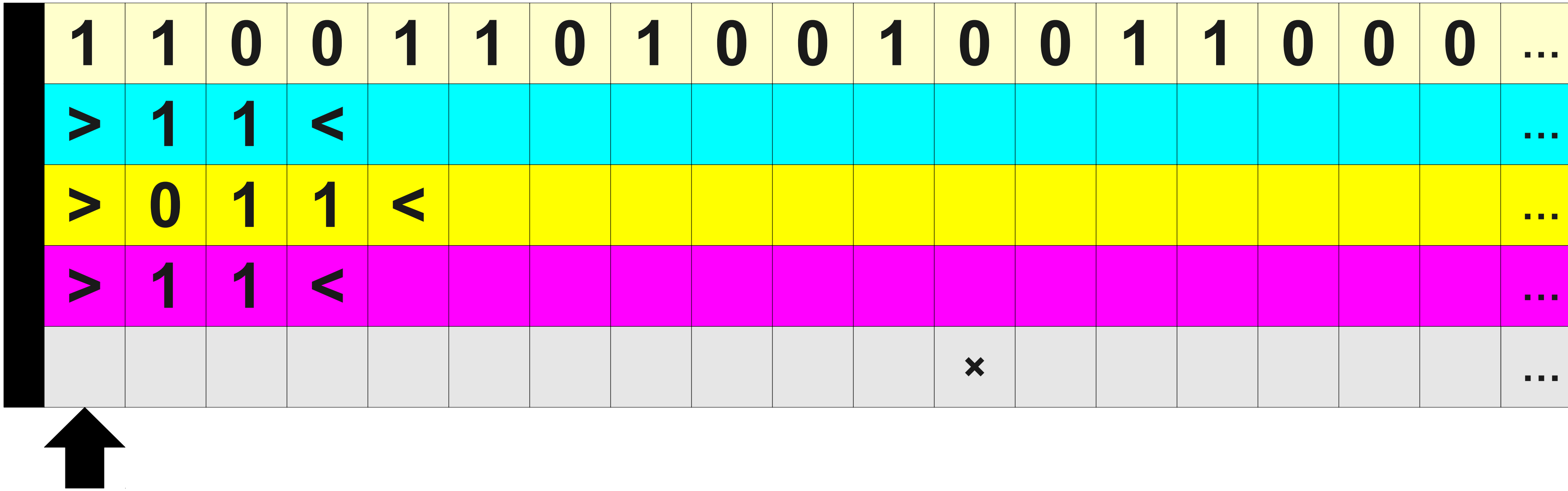
2: Move right until {<} on track 4.

3: Write 1 on track 4.

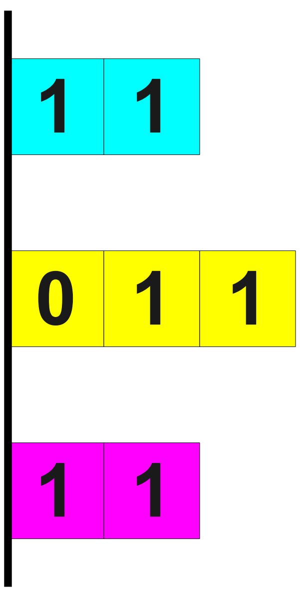
4: Move right.

5: Write < on track 4

6: Move left until {>} on track 4.



0: Push 1 onto Stack 3.



0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

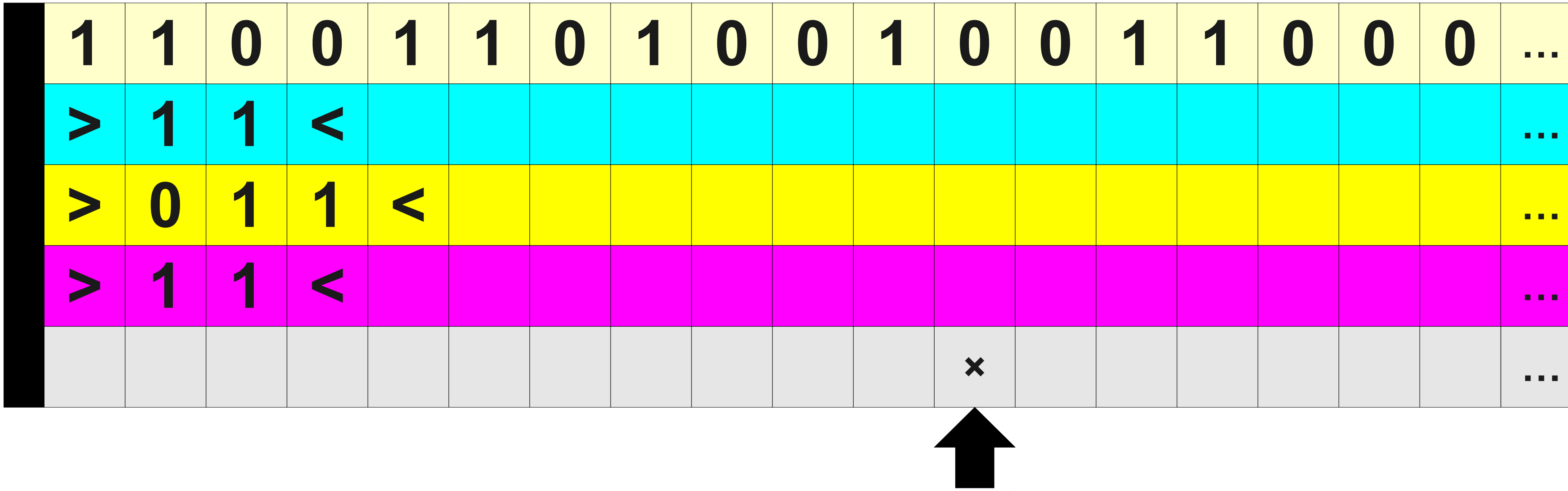
3: Write 1 on track 4.

4: Move right.

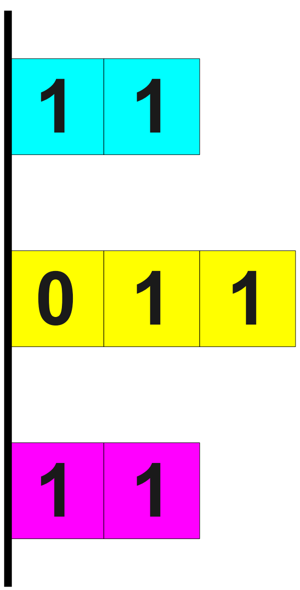
5: Write < on track 4

6: Move left until {>} on track 4.

7: Move right until {×} on track 5.



0: Push 1 onto Stack 3.



0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

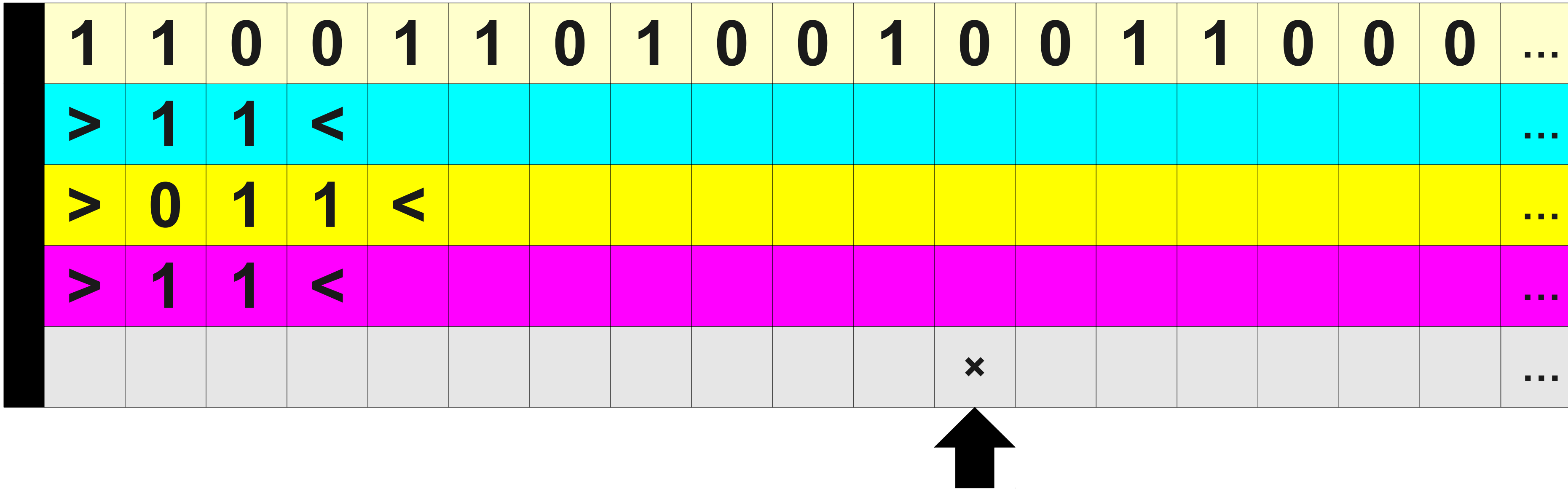
3: Write 1 on track 4.

4: Move right.

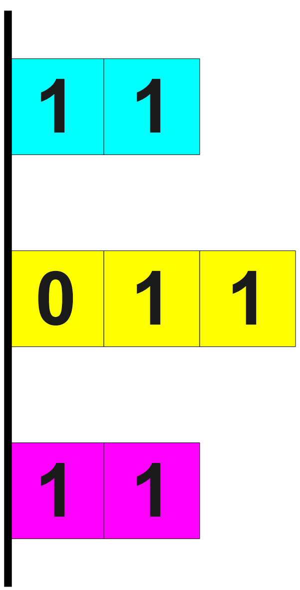
5: Write < on track 4

6: Move left until {>} on track 4.

7: Move right until {×} on track 5.



0: Push 1 onto Stack 3.



0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

3: Write 1 on track 4.

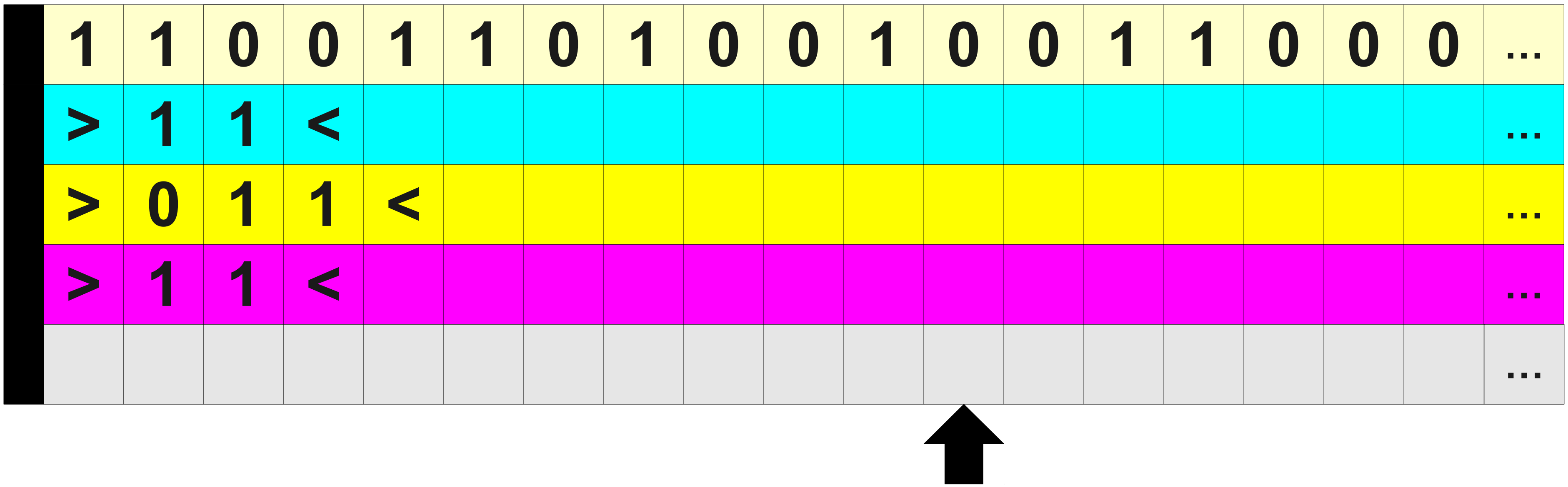
4: Move right.

5: Write < on track 4

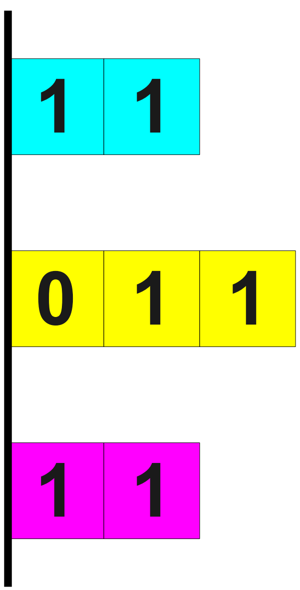
6: Move left until {>} on track 4.

7: Move right until {×} on track 5.

8: Write B on track 5.



0: Push 1 onto Stack 3.



0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

3: Write 1 on track 4.

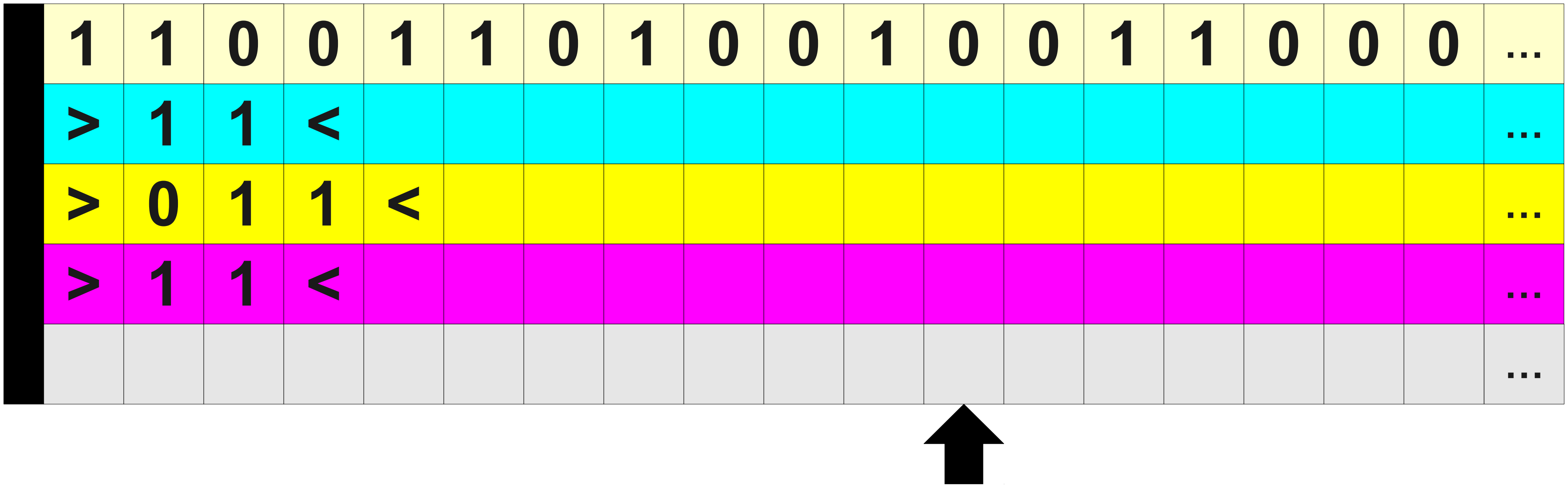
4: Move right.

5: Write < on track 4

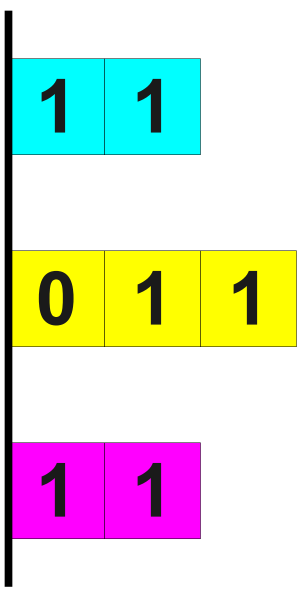
6: Move left until {>} on track 4.

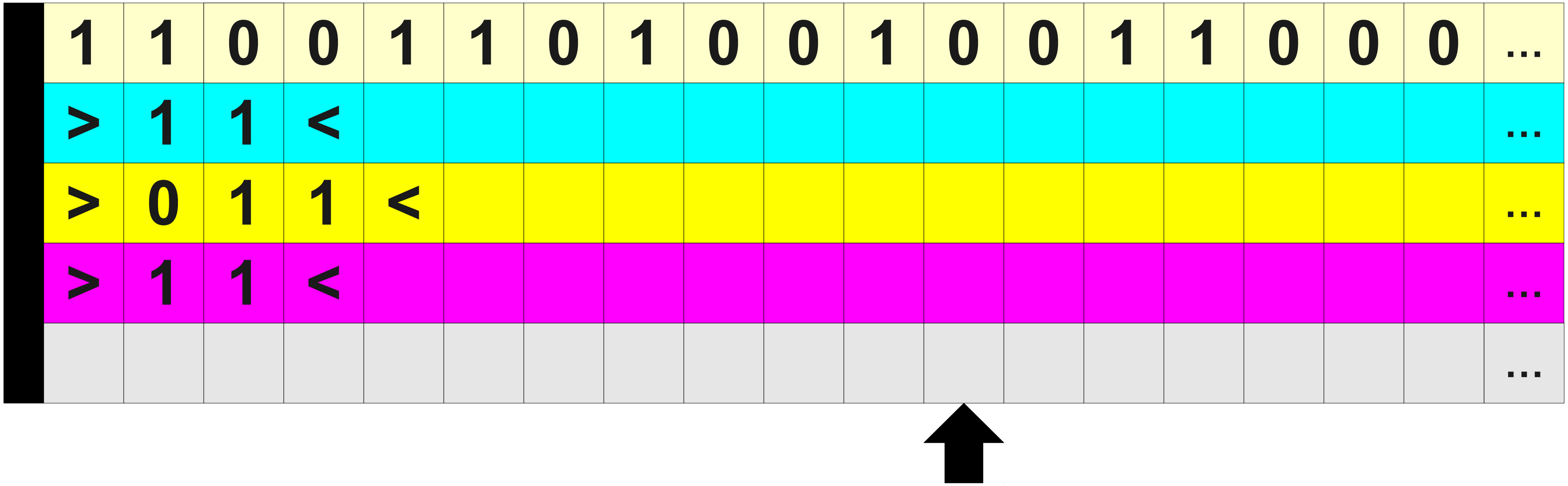
7: Move right until {×} on track 5.

8: Write B on track 5.

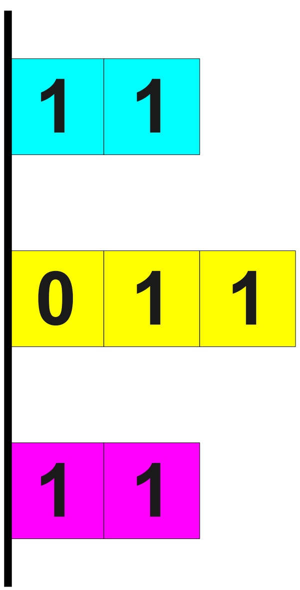


`1: If Stack 1 is empty, go to L`

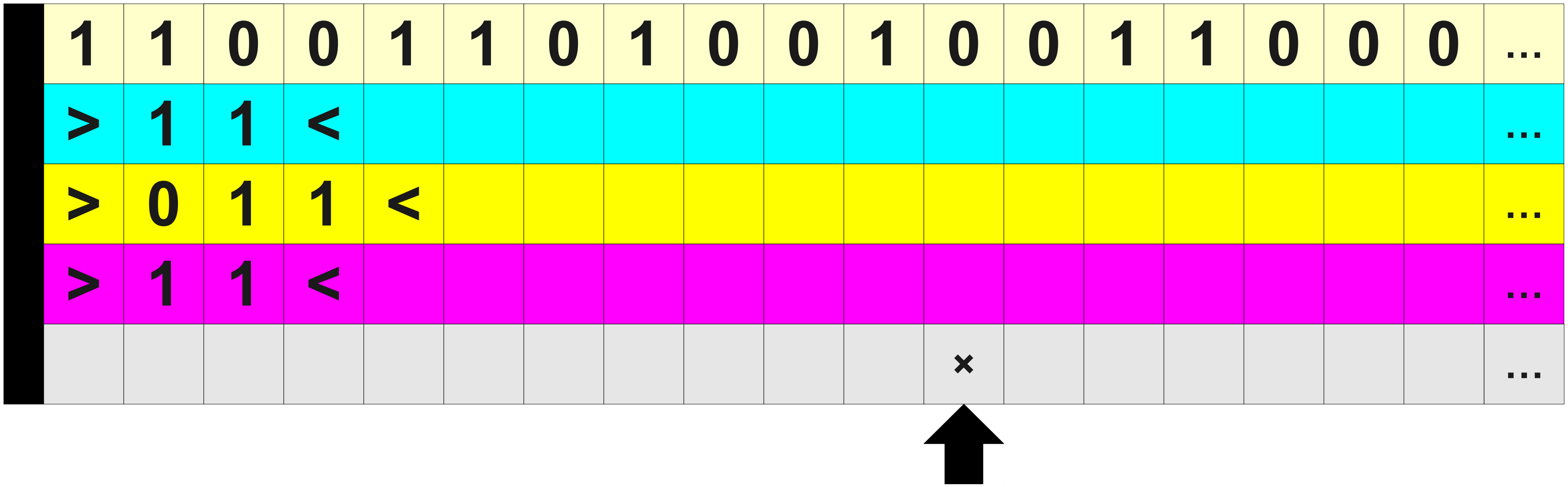




1: If Stack 1 is empty, go to L

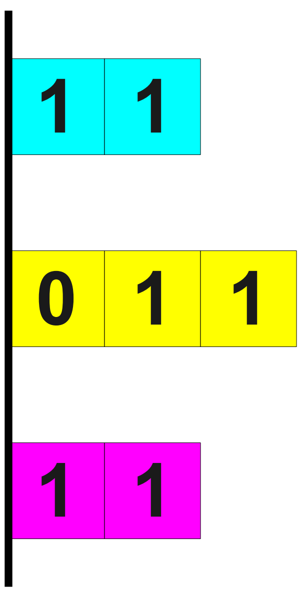


0: Write × on track 5.



1: If Stack 1 is empty, go to L

0: Write × on track 5.



1	1	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0	0	...
>	1	1	<															...
>	0	1	1	<														...
>	1	1	<															...
											×							...

1: If Stack 1 is empty, go to L

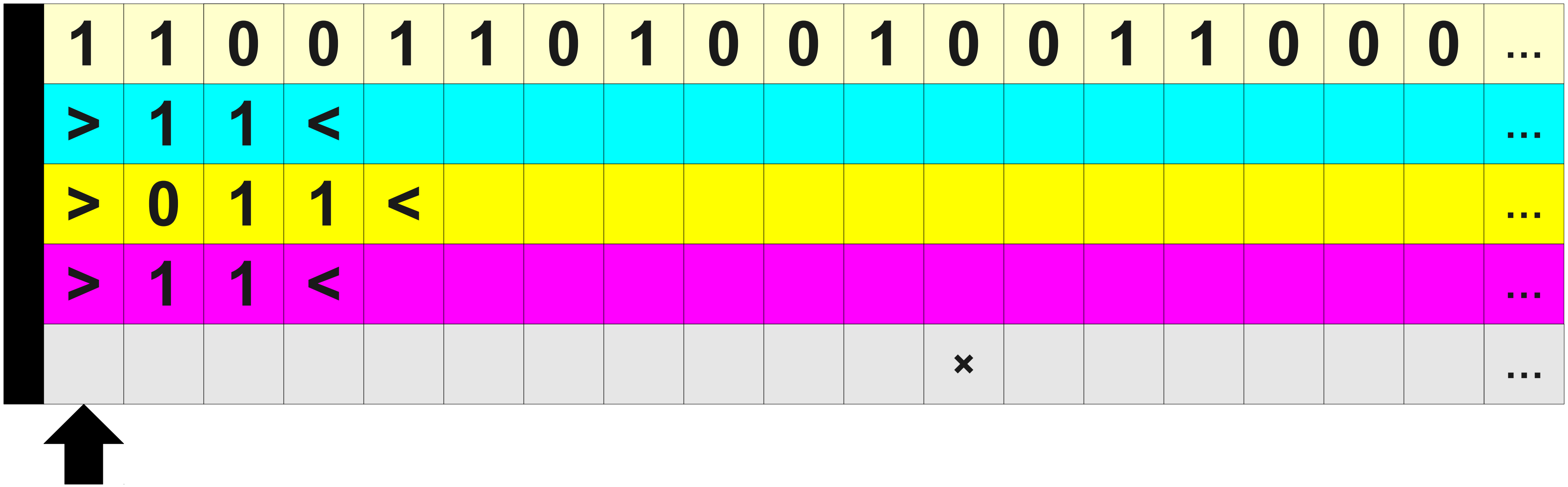
0: Write × on track 5.

1: Move left until {>} on track 2.

1	1

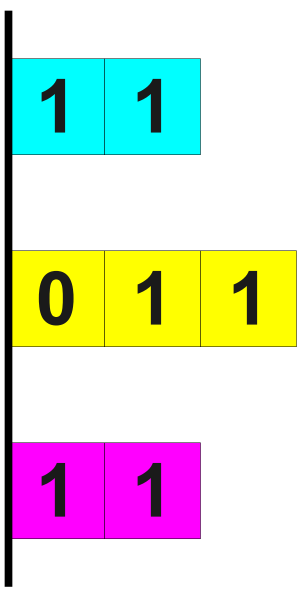
0	1	1

1	1




```
1: If Stack 1 is empty, go to L

  0: Write × on track 5.
  1: Move left until {>} on track 2.
```



1	1	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0	0	...
>	1	1	<															...
>	0	1	1	<														...
>	1	1	<															...
											×							...

1: If Stack 1 is empty, go to L

1	1

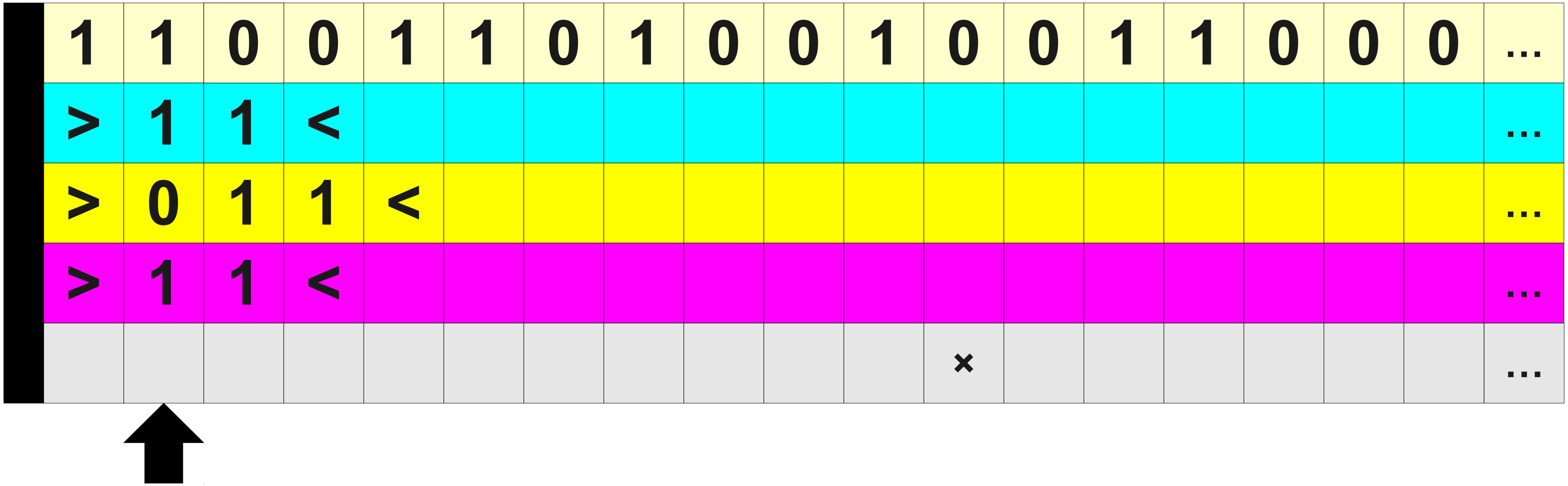
0	1	1

1	1

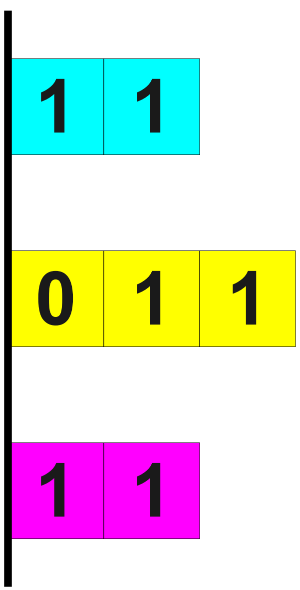
0: Write × on track 5.

1: Move left until {>} on track 2.

2: Move right.



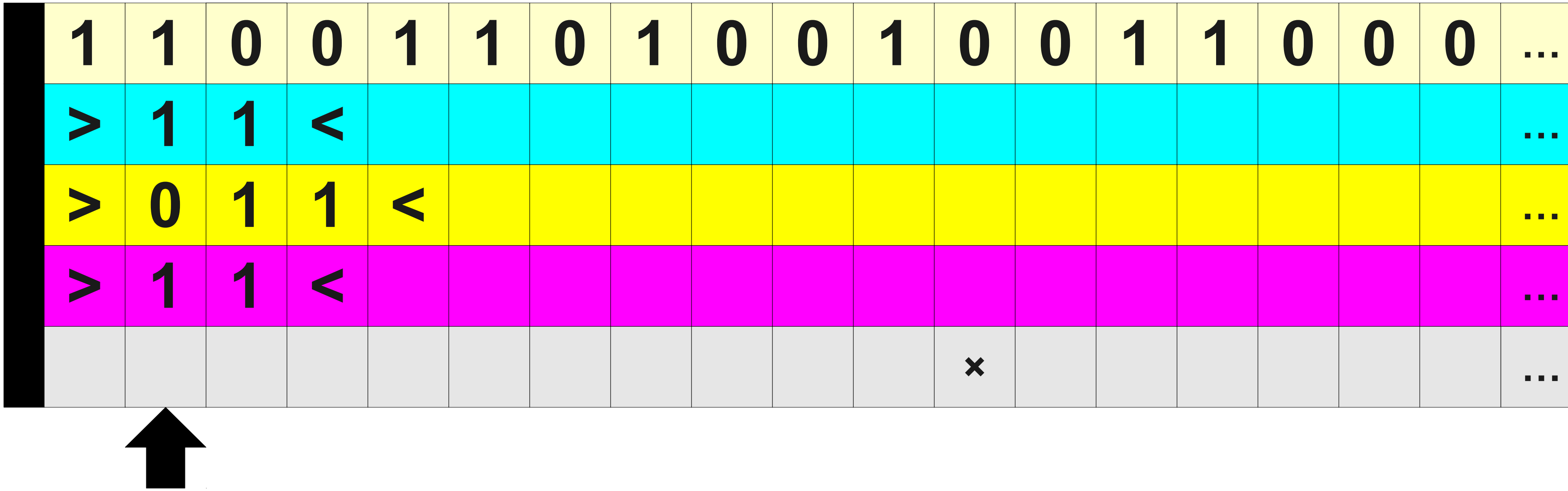
1: If Stack 1 is empty, go to L



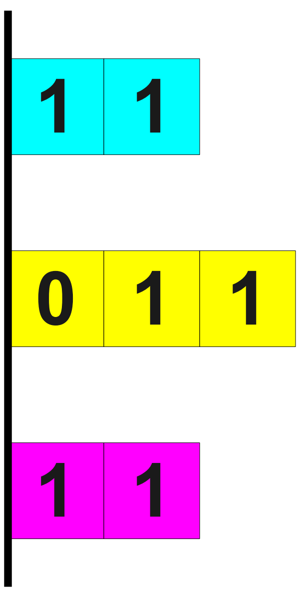
0: Write × on track 5.

1: Move left until {>} on track 2.

2: Move right.



1: If Stack 1 is empty, go to L

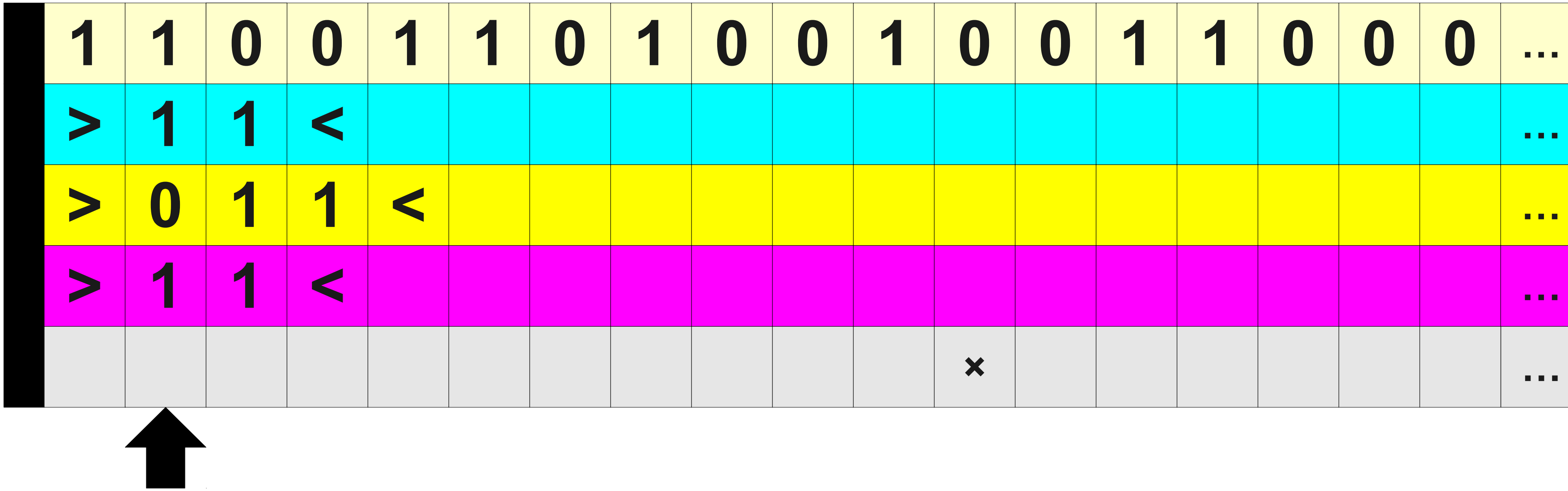


0: Write × on track 5.

1: Move left until {>} on track 2.

2: Move right.

3: Load current on track 2 into V



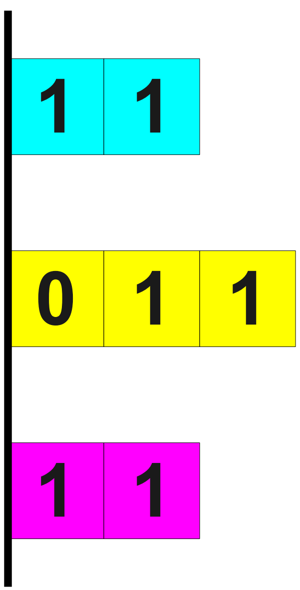
1: If Stack 1 is empty, go to L

0: Write × on track 5.

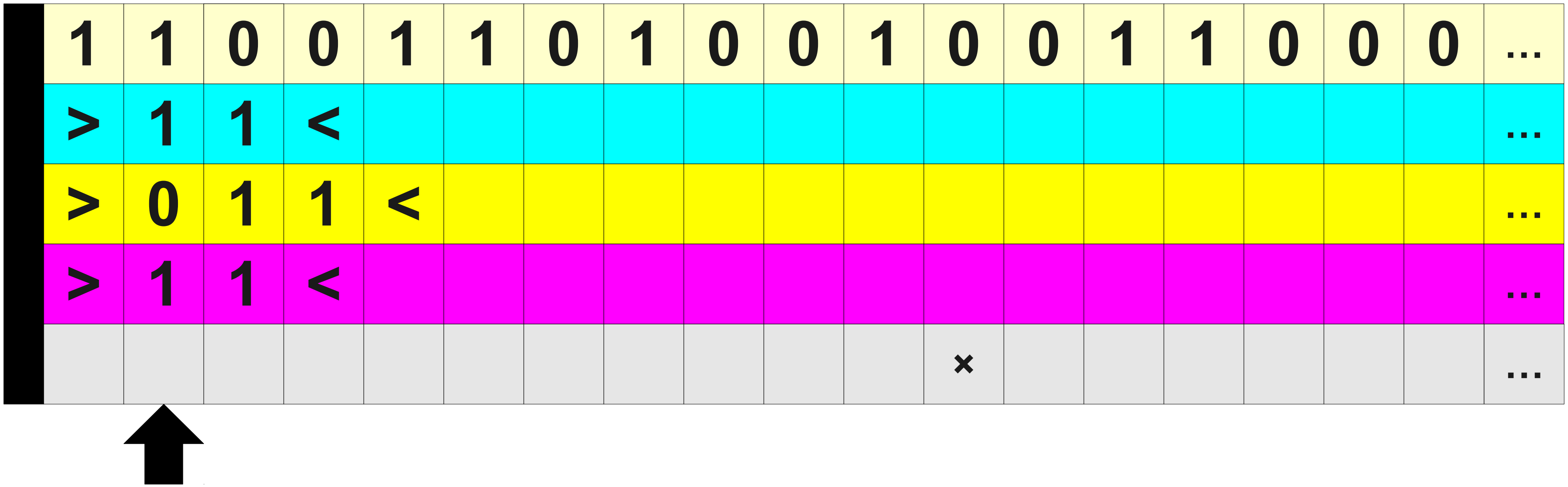
1: Move left until {>} on track 2.

2: Move right.

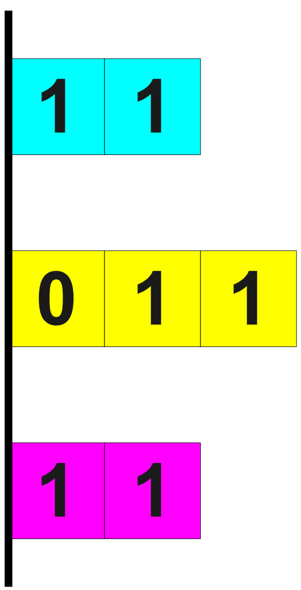
3: Load current on track 2 into V



V= 1



1: If Stack 1 is empty, go to L



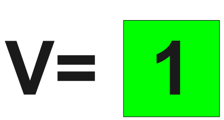
0: Write × on track 5.

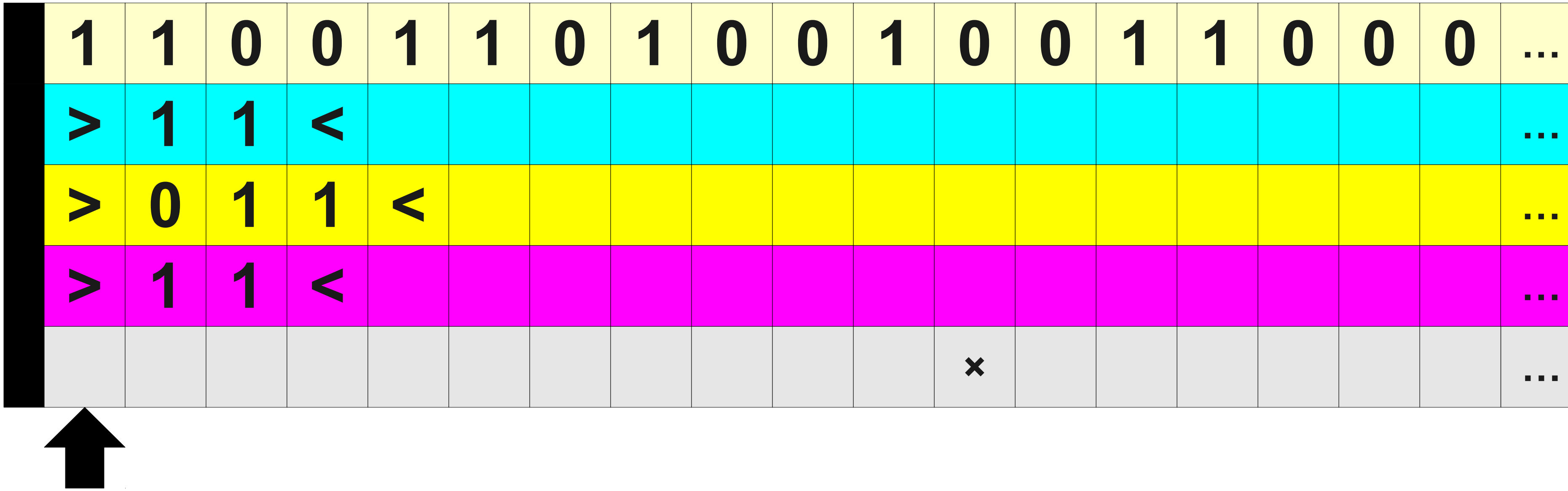
1: Move left until {>} on track 2.

2: Move right.

3: Load current on track 2 into V

4: Move left.





1: If Stack 1 is empty, go to L

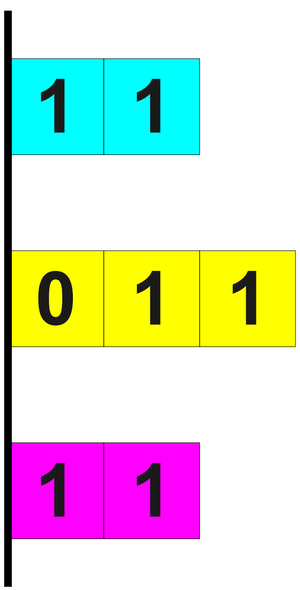
0: Write × on track 5.

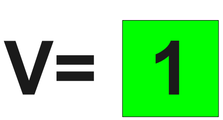
1: Move left until {>} on track 2.

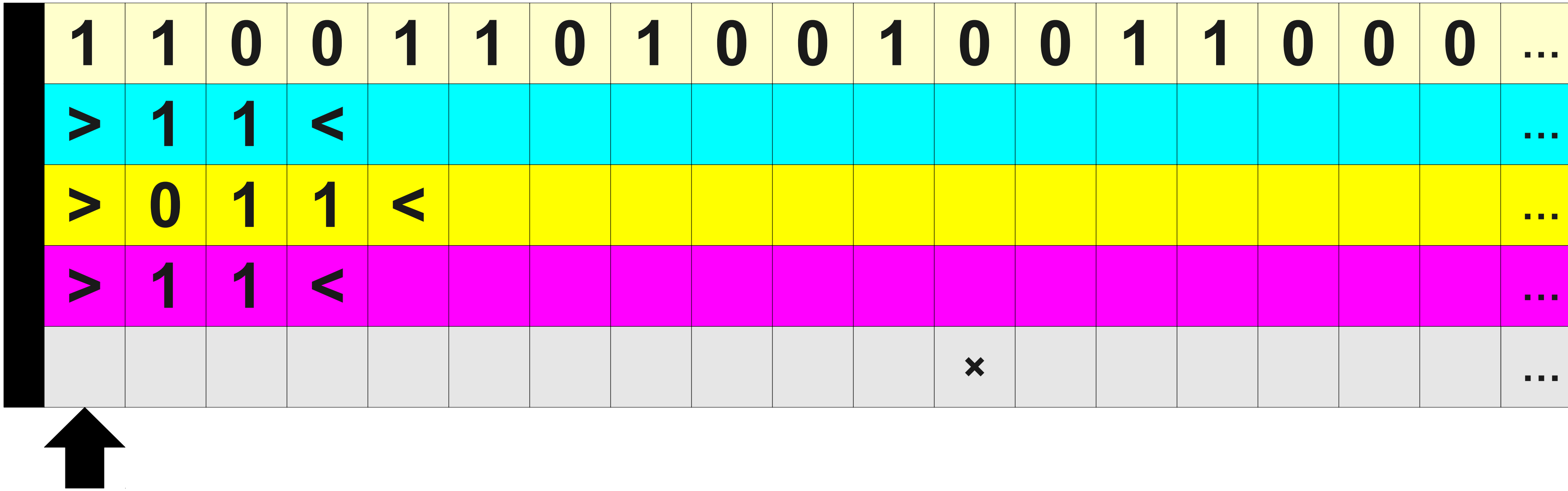
2: Move right.

3: Load current on track 2 into V

4: Move left.







1: If Stack 1 is empty, go to L

0: Write × on track 5.

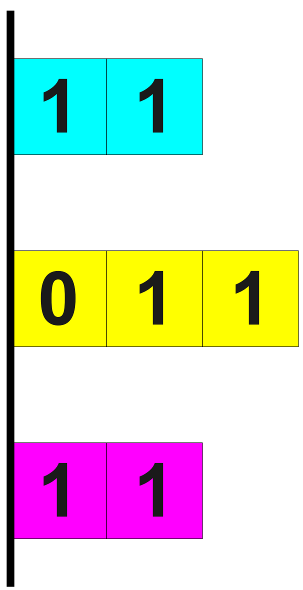
1: Move left until {>} on track 2.

2: Move right.

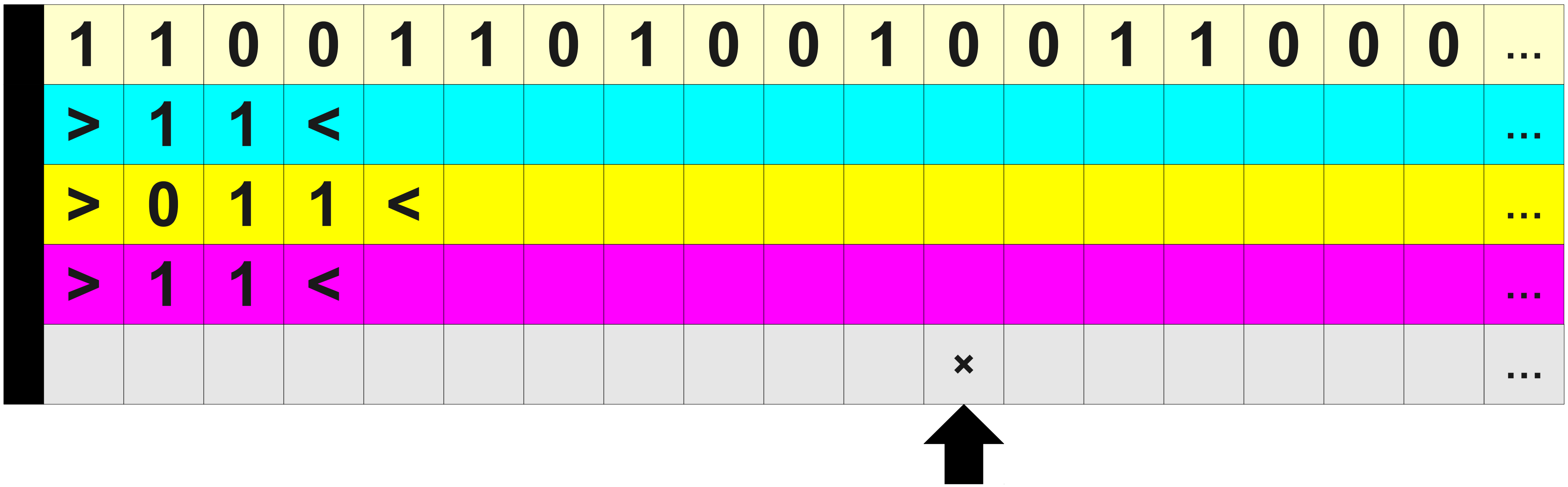
3: Load current on track 2 into V

4: Move left.

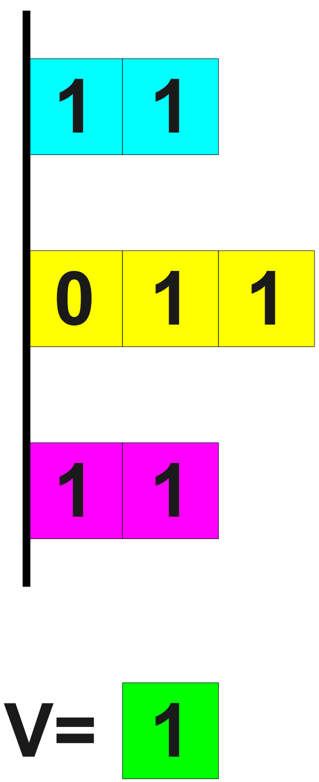
5: Move right until {×} on track 5.



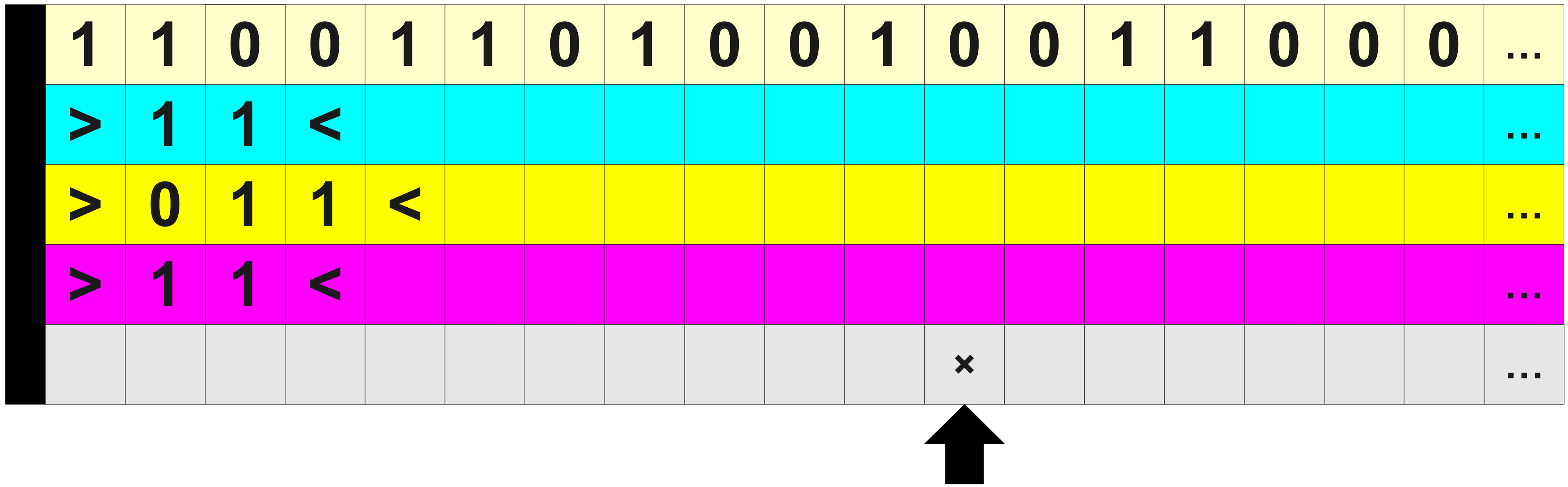
V= 1



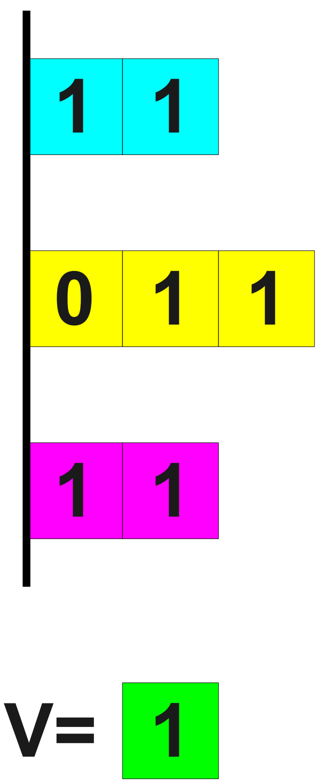

```
1: If Stack 1 is empty, go to L
```




```
0: Write × on track 5.
1: Move left until {>} on track 2.
2: Move right.
3: Load current on track 2 into V
4: Move left.
5: Move right until {×} on track 5.
```



1: If Stack 1 is empty, go to L



0: Write × on track 5.

1: Move left until {>} on track 2.

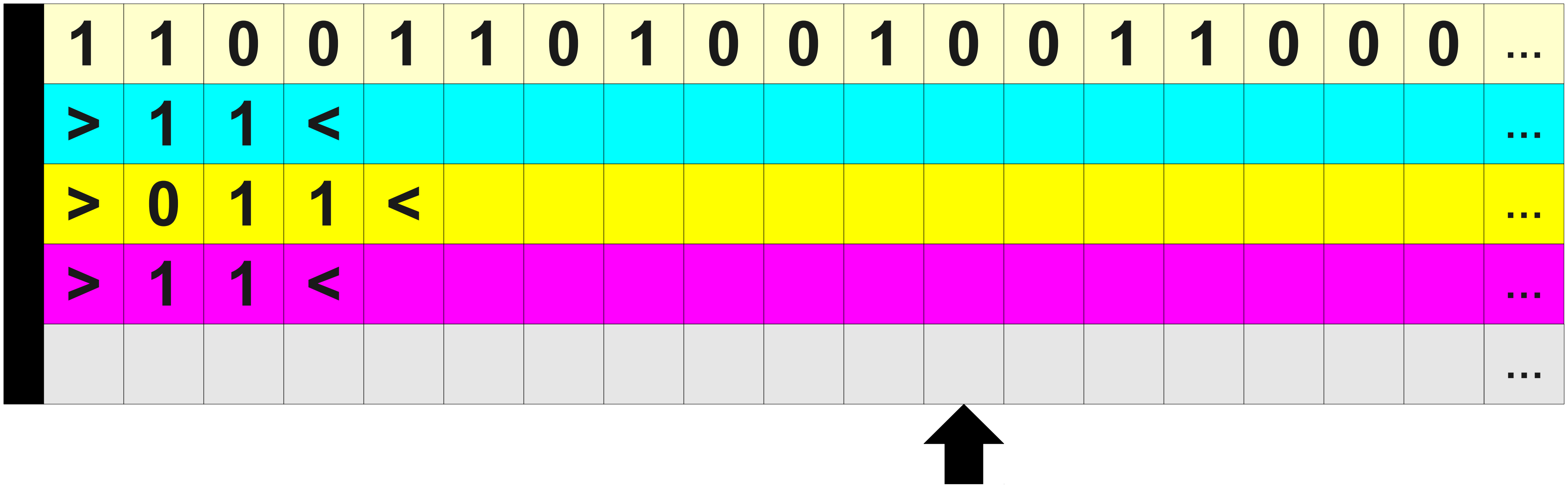
2: Move right.

3: Load current on track 2 into V

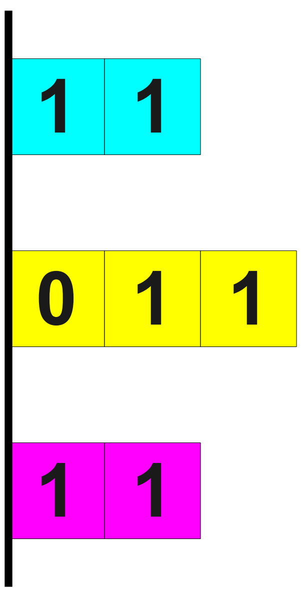
4: Move left.

5: Move right until {×} on track 5.

6: Write B on track 5.



1: If Stack 1 is empty, go to L



0: Write × on track 5.

1: Move left until {>} on track 2.

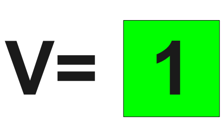
2: Move right.

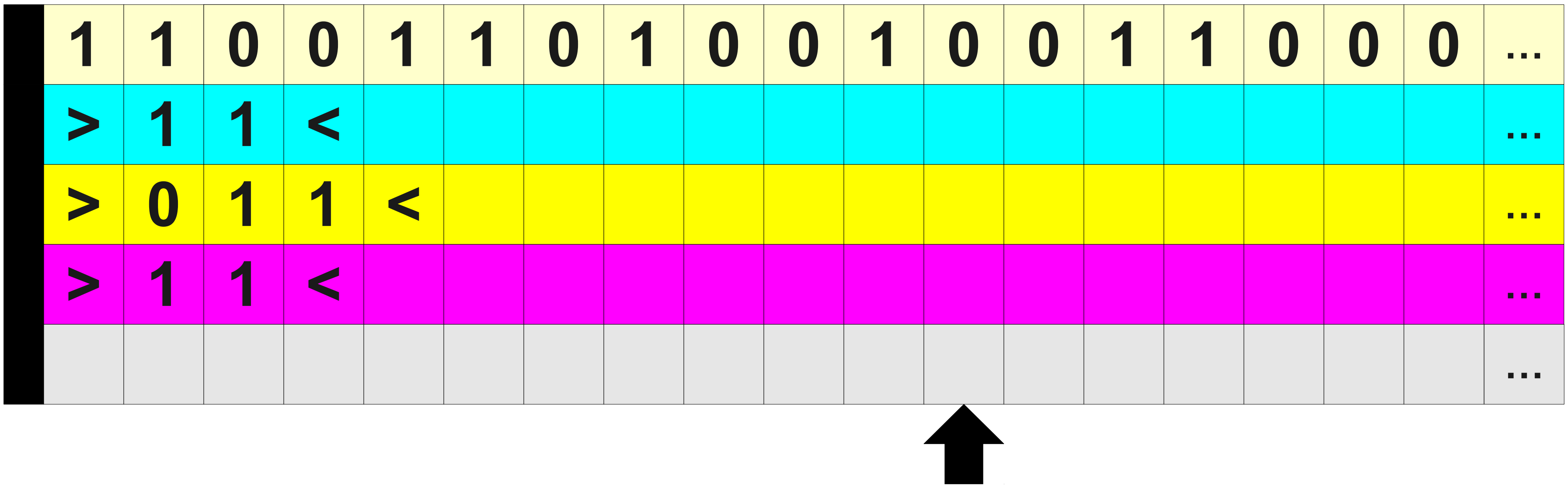
3: Load current on track 2 into V

4: Move left.

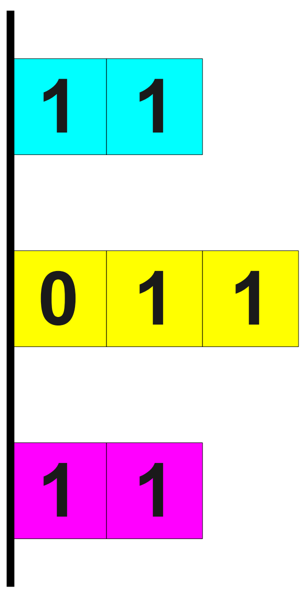
5: Move right until {×} on track 5.

6: Write B on track 5.





1: If Stack 1 is empty, go to L



0: Write × on track 5.

1: Move left until {>} on track 2.

2: Move right.

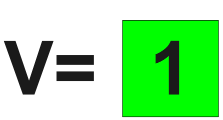
3: Load current on track 2 into V

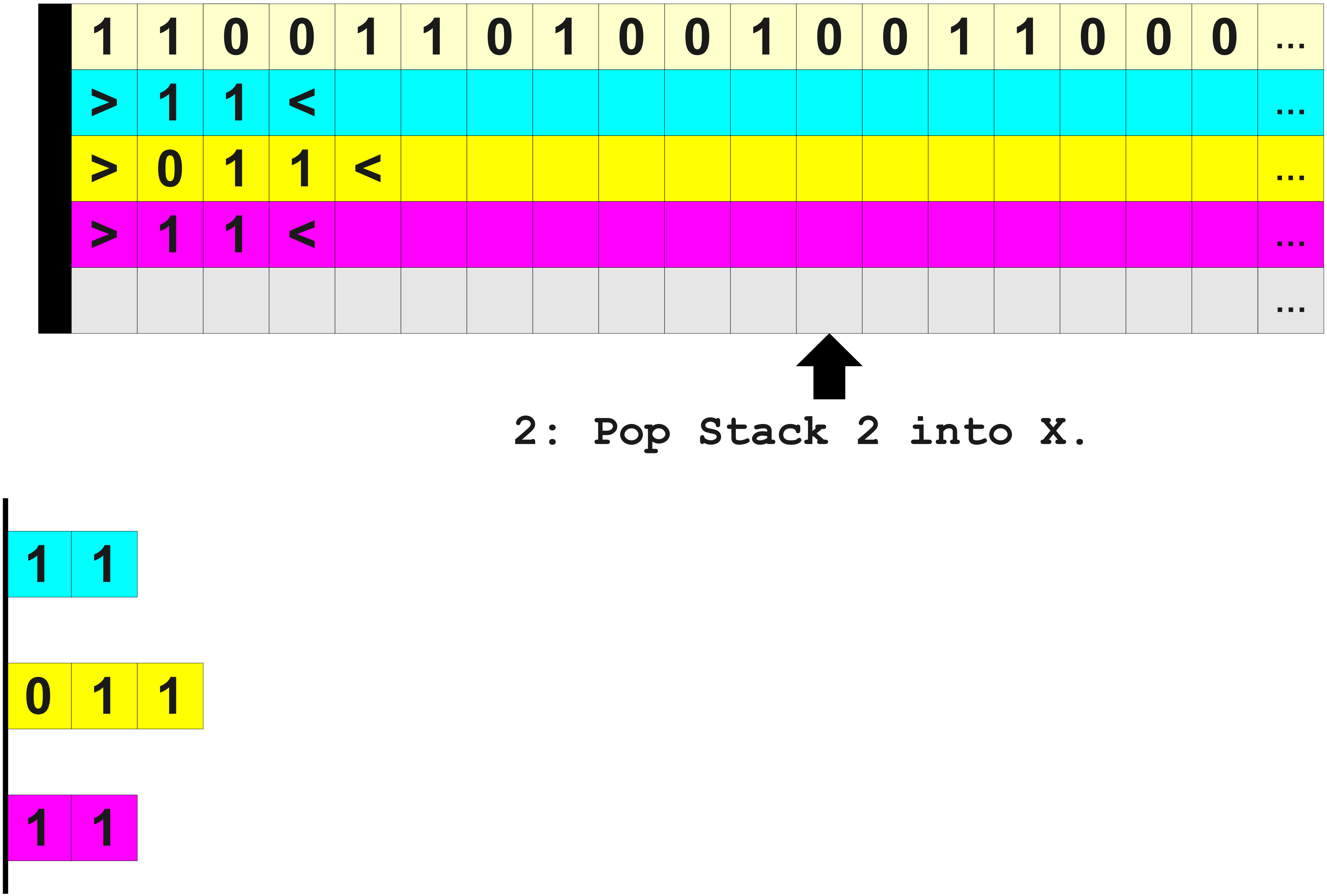
4: Move left.

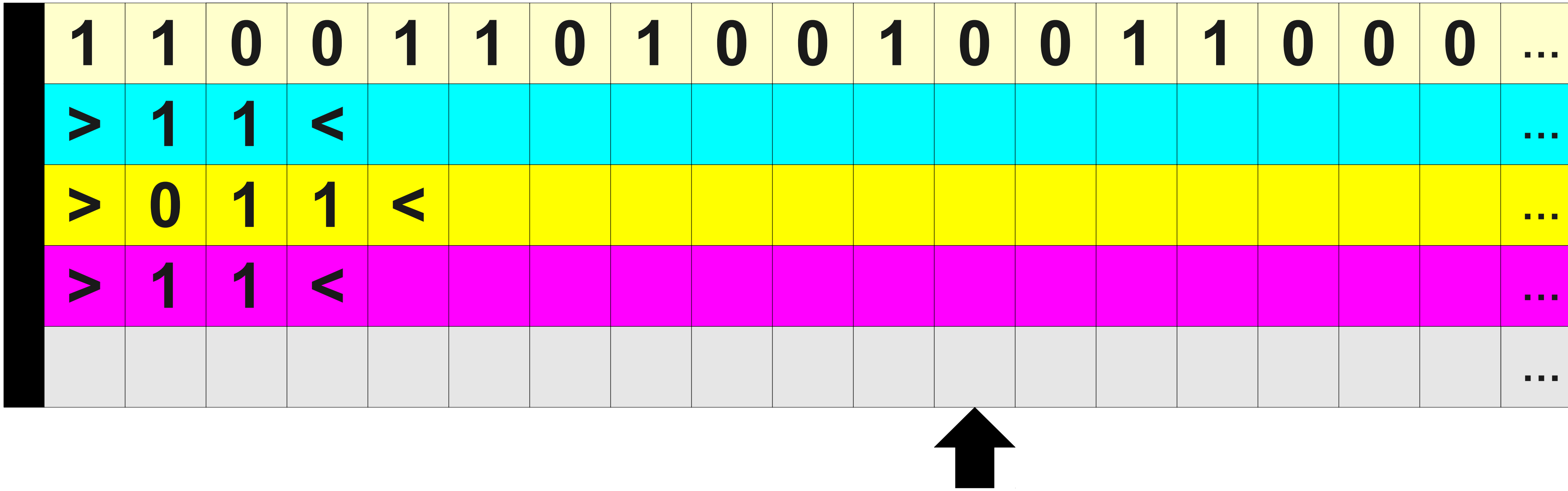
5: Move right until {×} on track 5.

6: Write B on track 5.

7: If V = <, go to L.

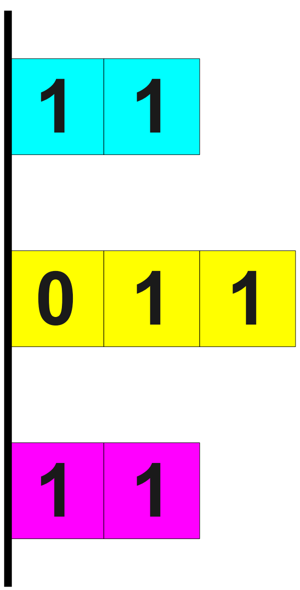


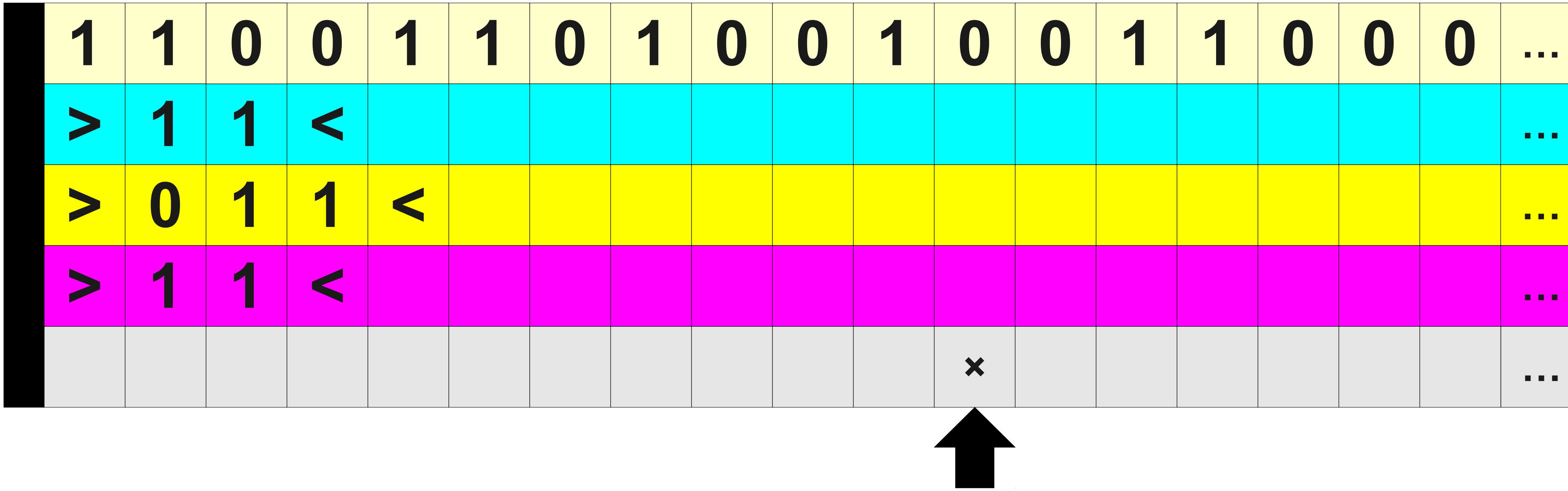

1 1 0 0 1 1 0 1 0 0 1 0 0 1 1 0 0 0 ...
> 1 1 < ...
> 0 1 1 < ...
> 1 1 < ...
...
2: Pop Stack 2 into X.
1 1
0 1 1
1 1



2: Pop Stack 2 into X.

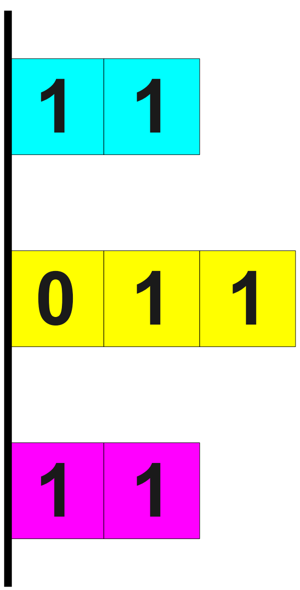
0: Write × on track 5.

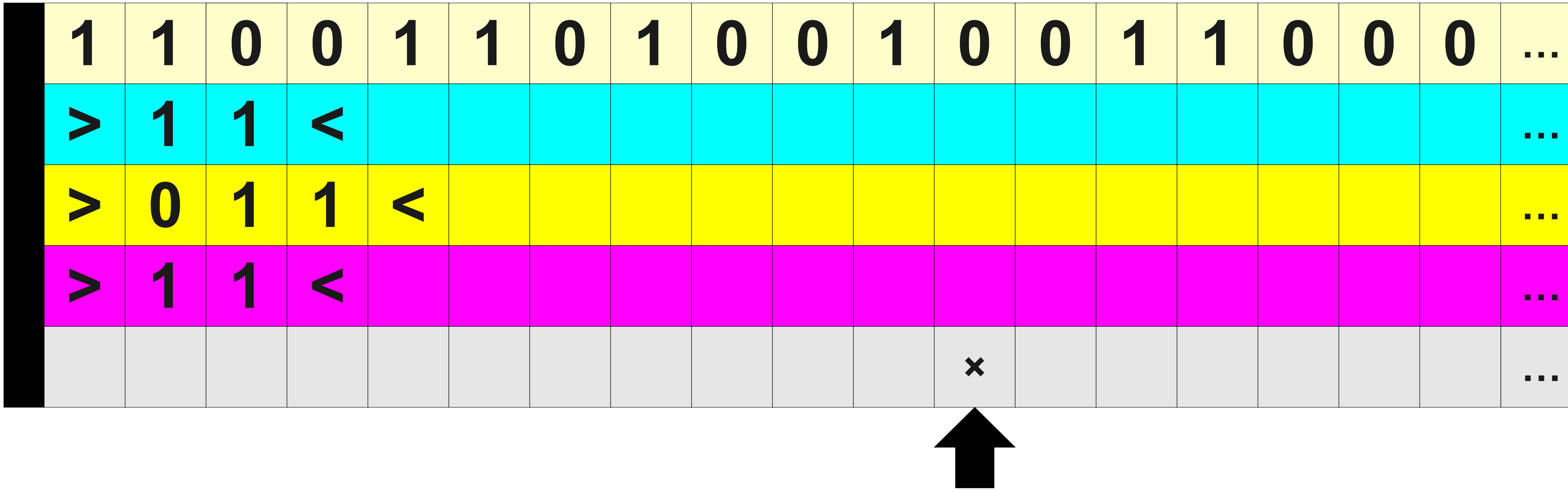




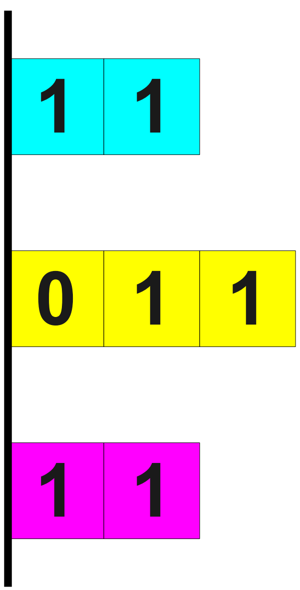
2: Pop Stack 2 into X.

0: Write × on track 5.



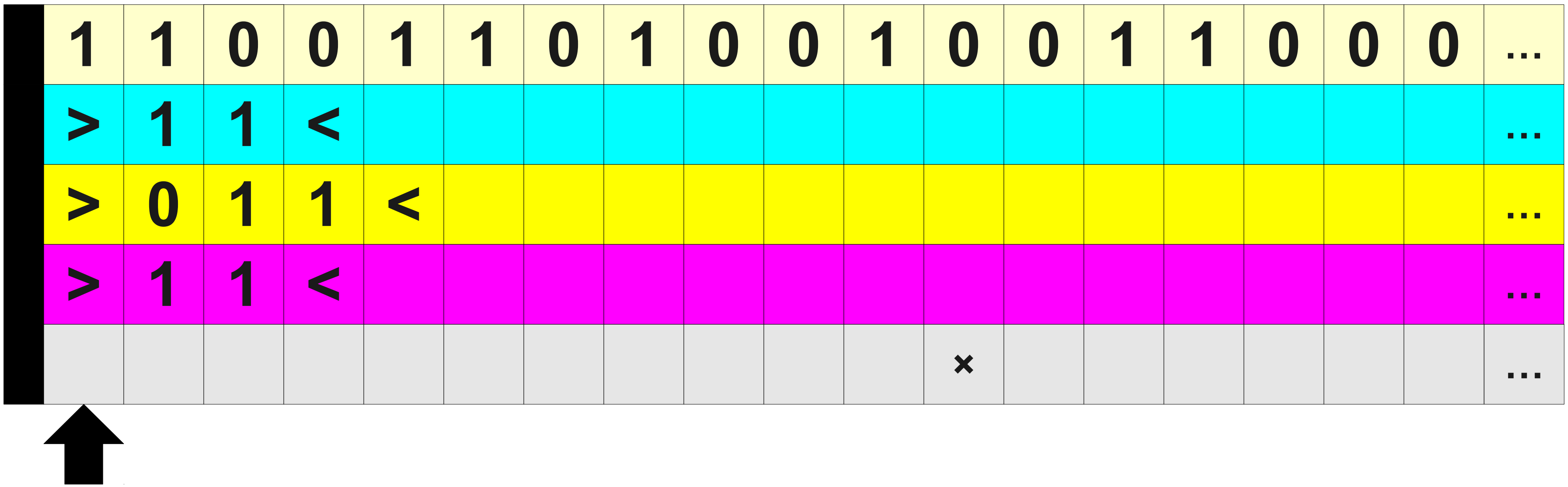


2: Pop Stack 2 into X.

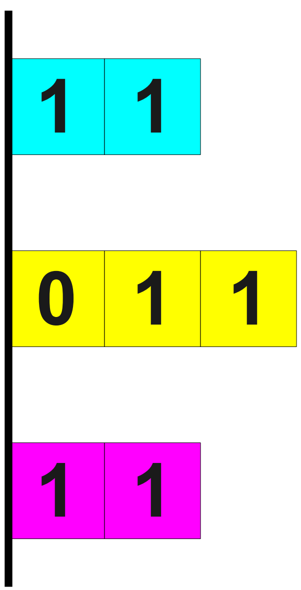


0: Write × on track 5.

1: Move left until {>} on track 3.

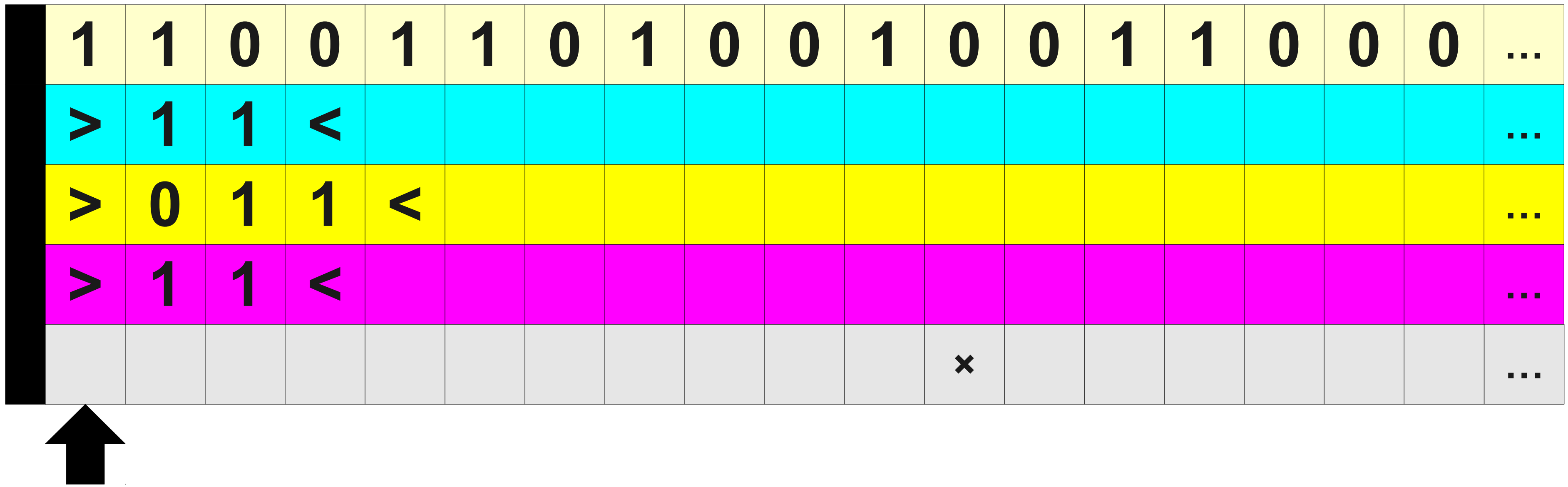


2: Pop Stack 2 into X.

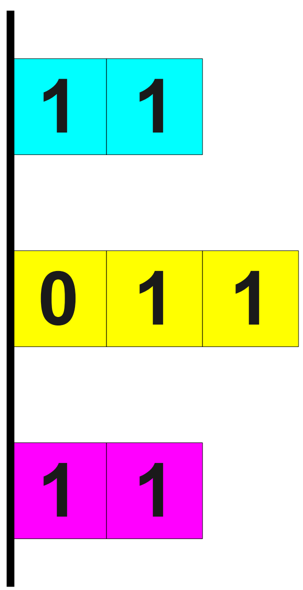


0: Write × on track 5.

1: Move left until {>} on track 3.



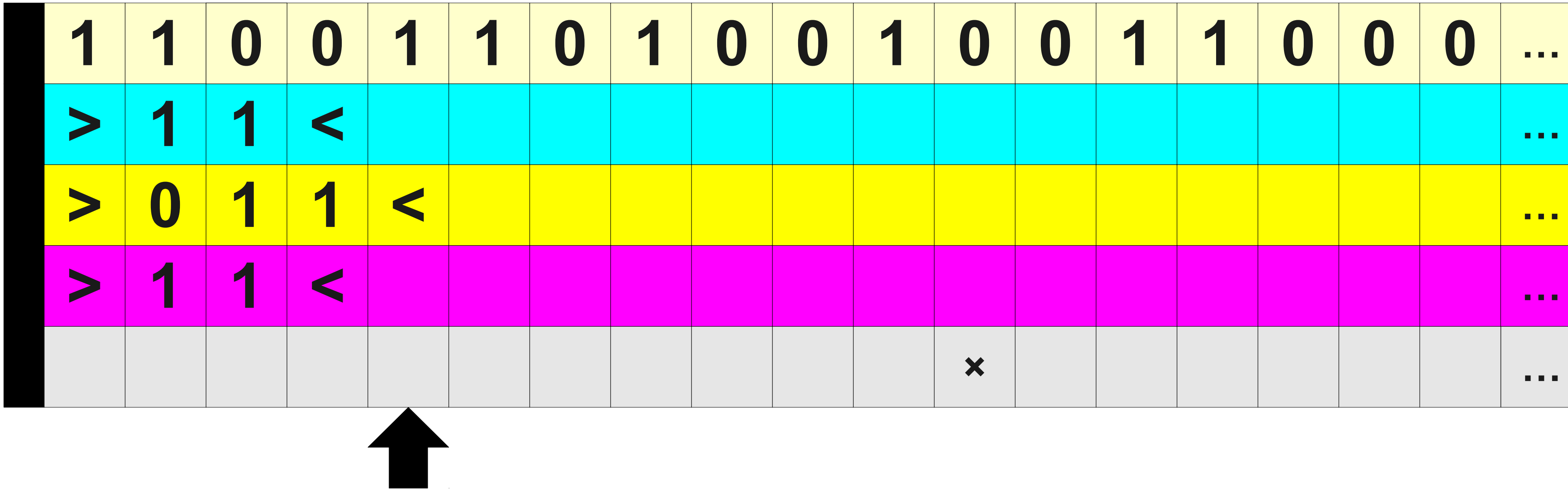
2: Pop Stack 2 into X.



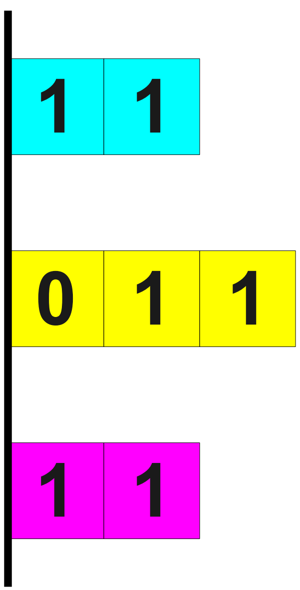
0: Write × on track 5.

1: Move left until {>} on track 3.

2: Move right until {<} on track 3.



2: Pop Stack 2 into X.



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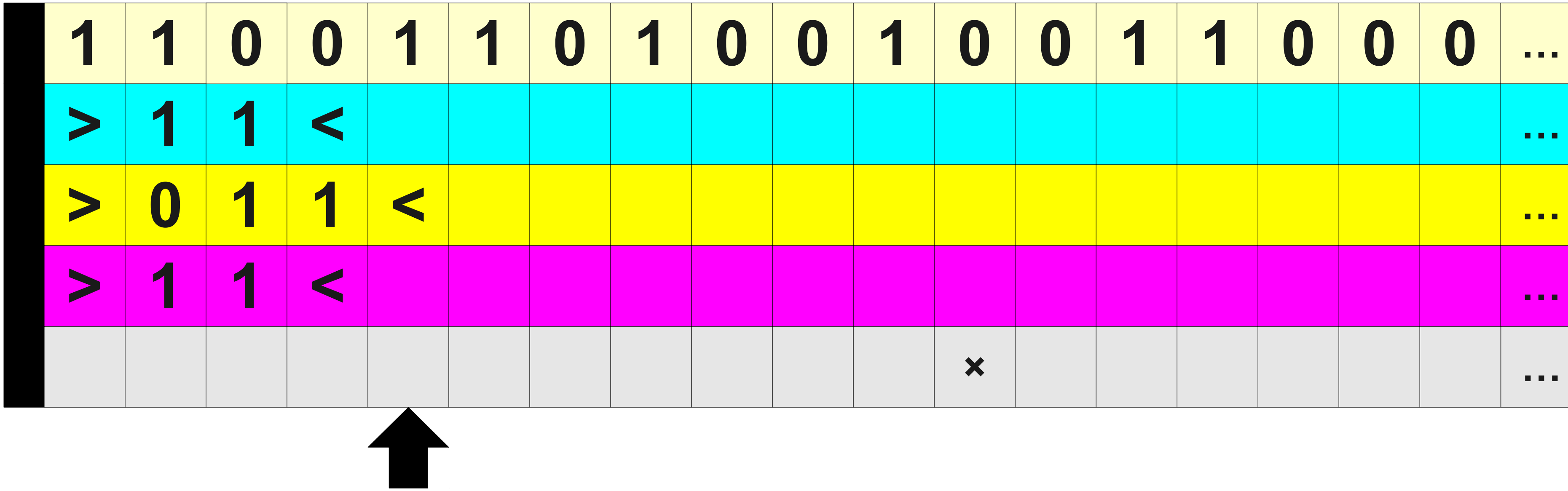

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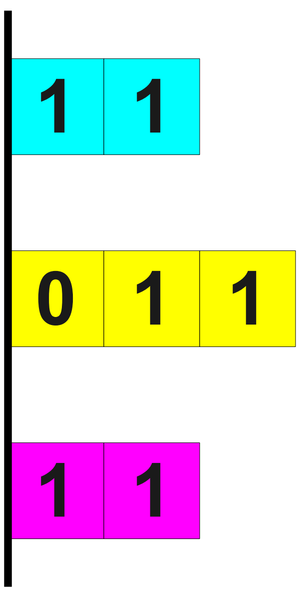

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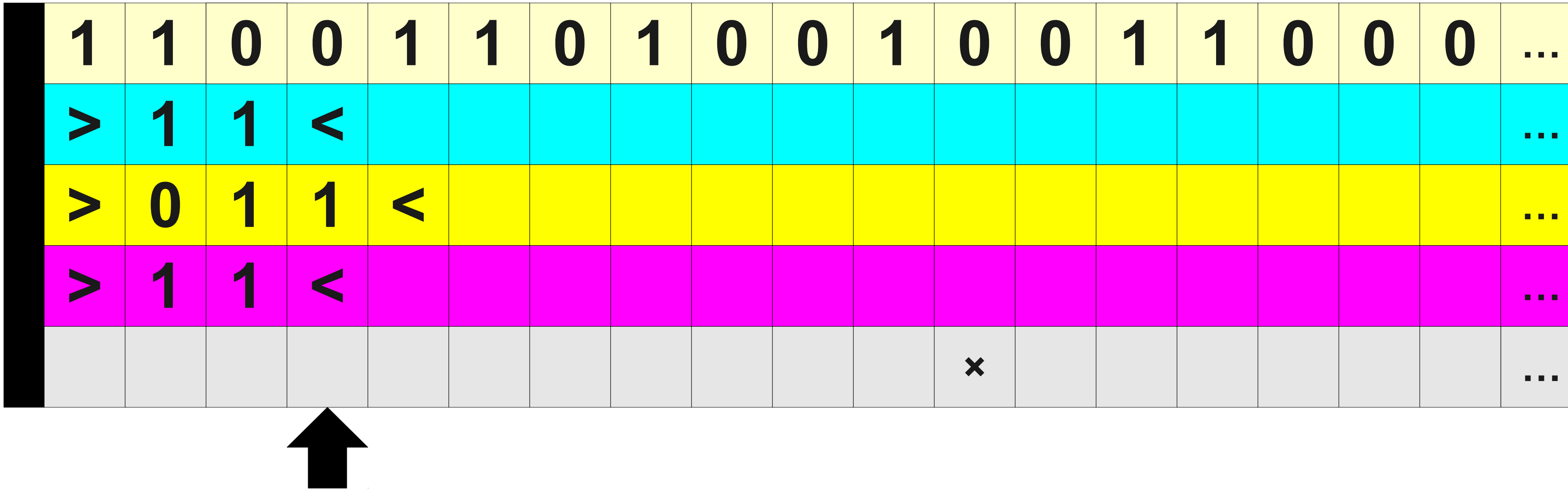

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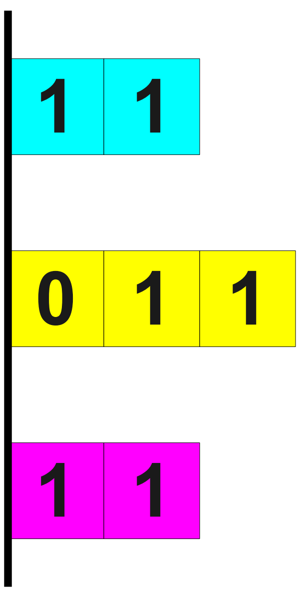

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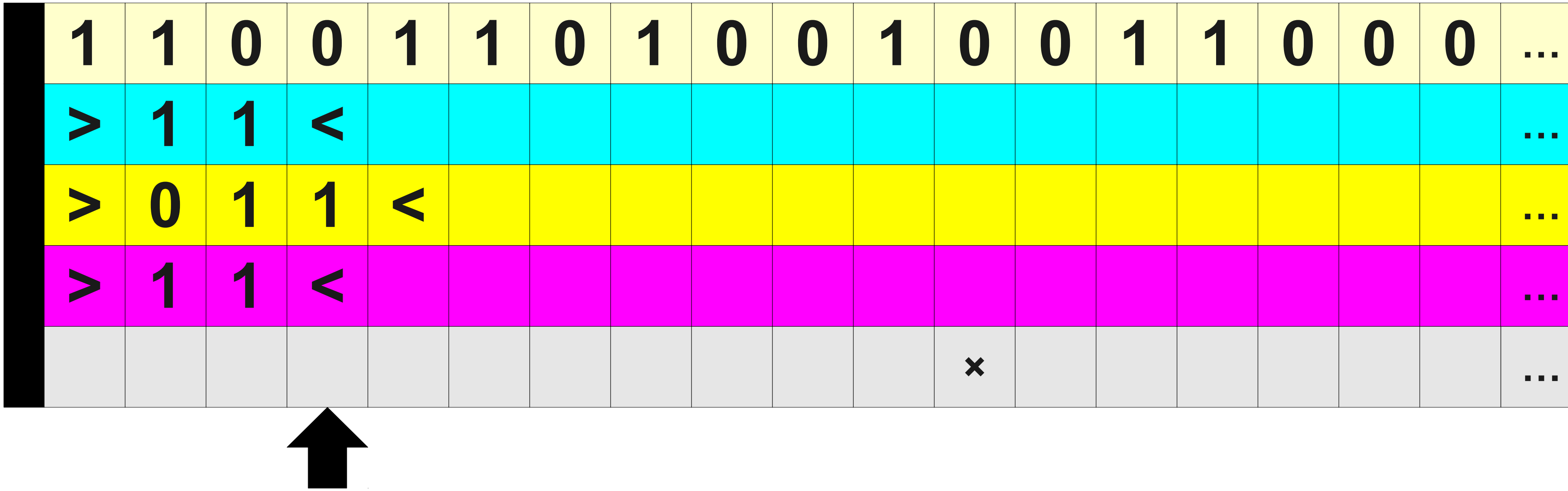

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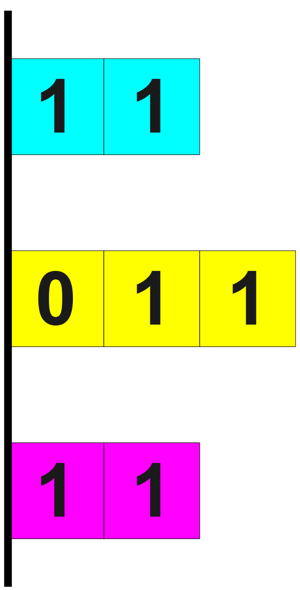

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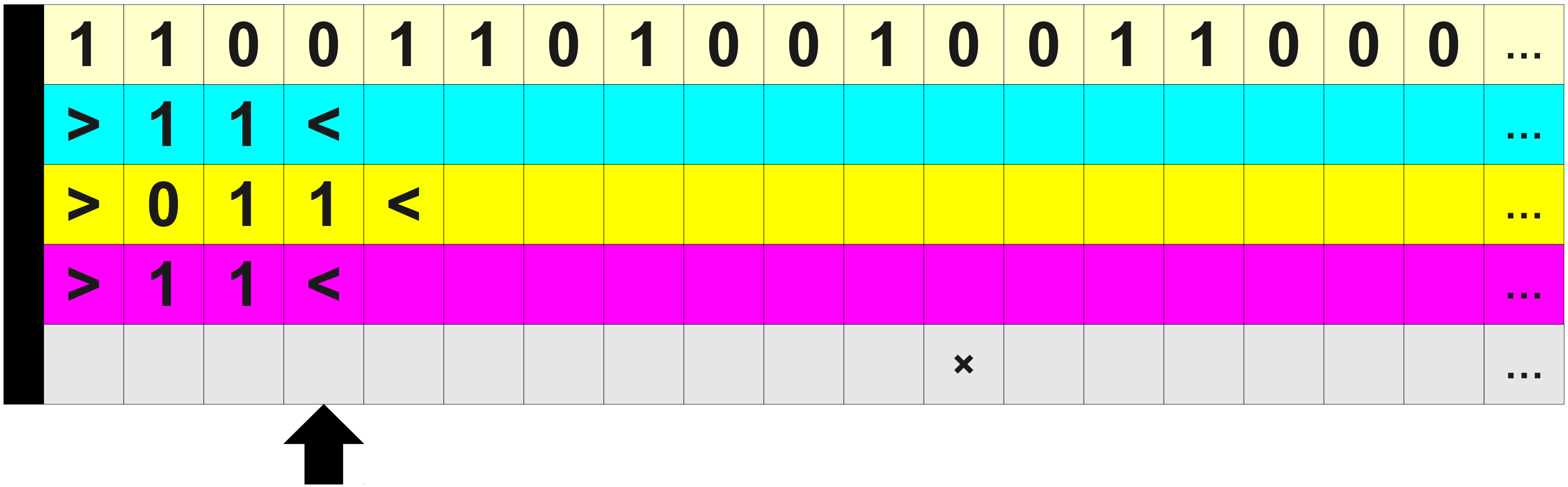
3: Move left.
4: Load current on track 3 into X.



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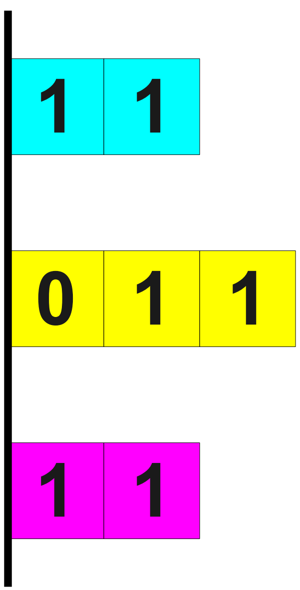

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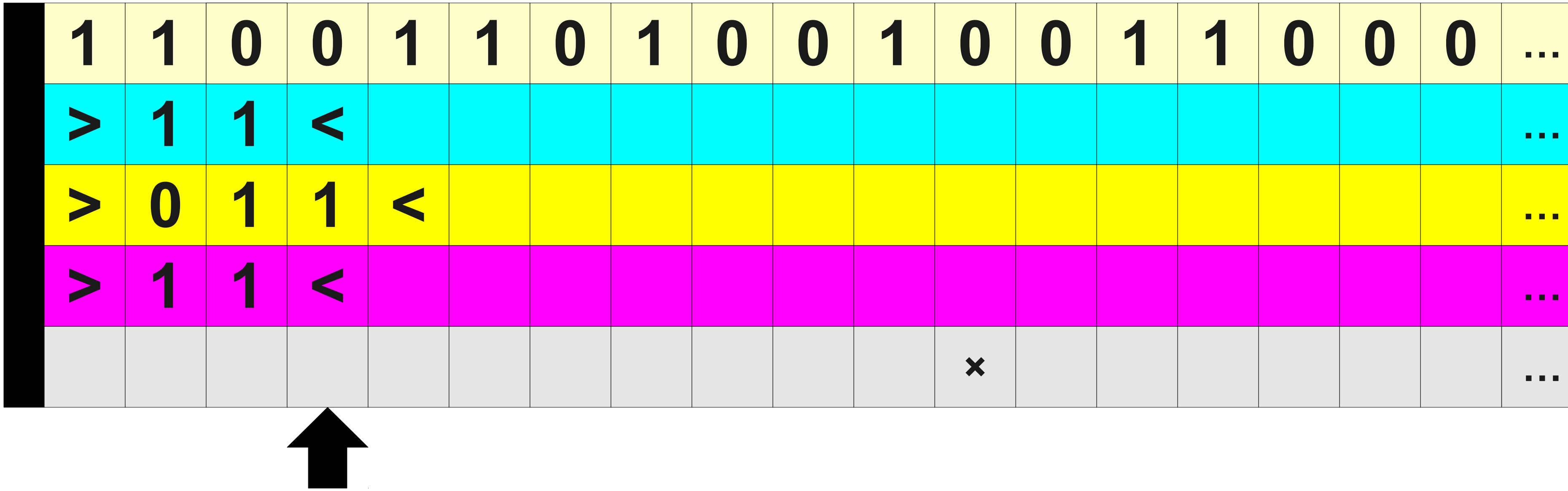
4: Load current on track 3 into X.

X= 1



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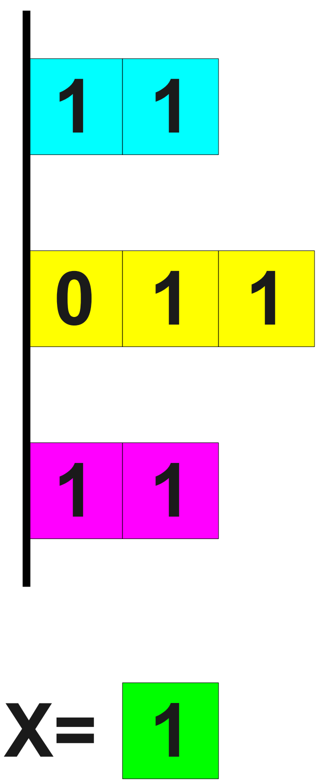


0: Write × on track 5.

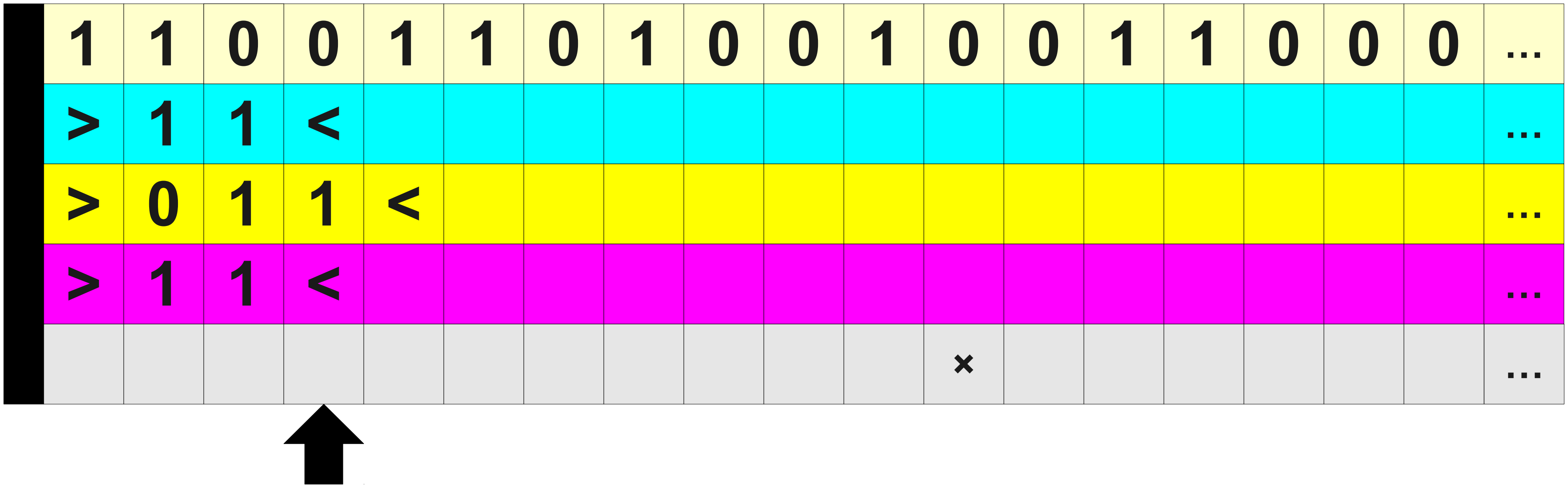
1: Move left until {>} on track 3.

2: Move right until {<} on track 3.

3: Move left.

4: Load current on track 3 into X.

5: If X = >, go to 7.



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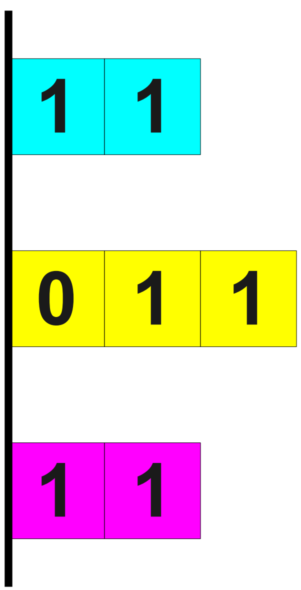


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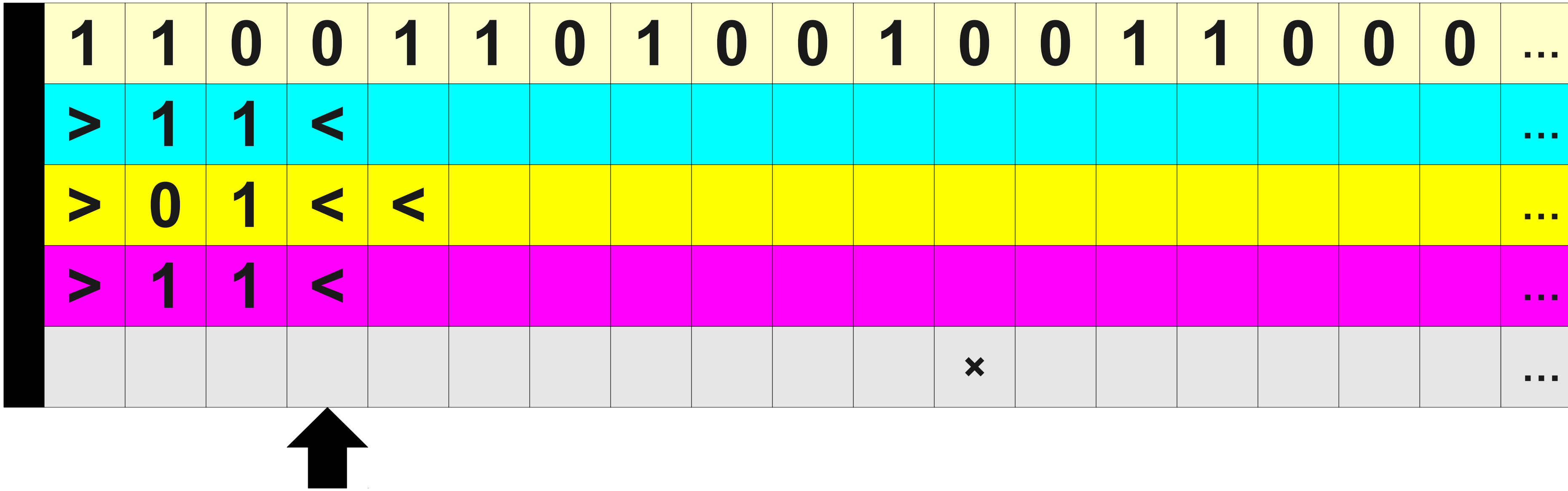
6: Write < on track 3
```

X= 1



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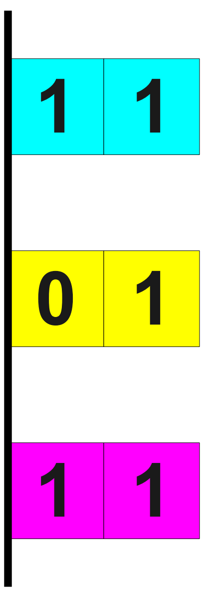


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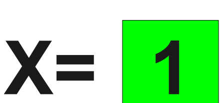
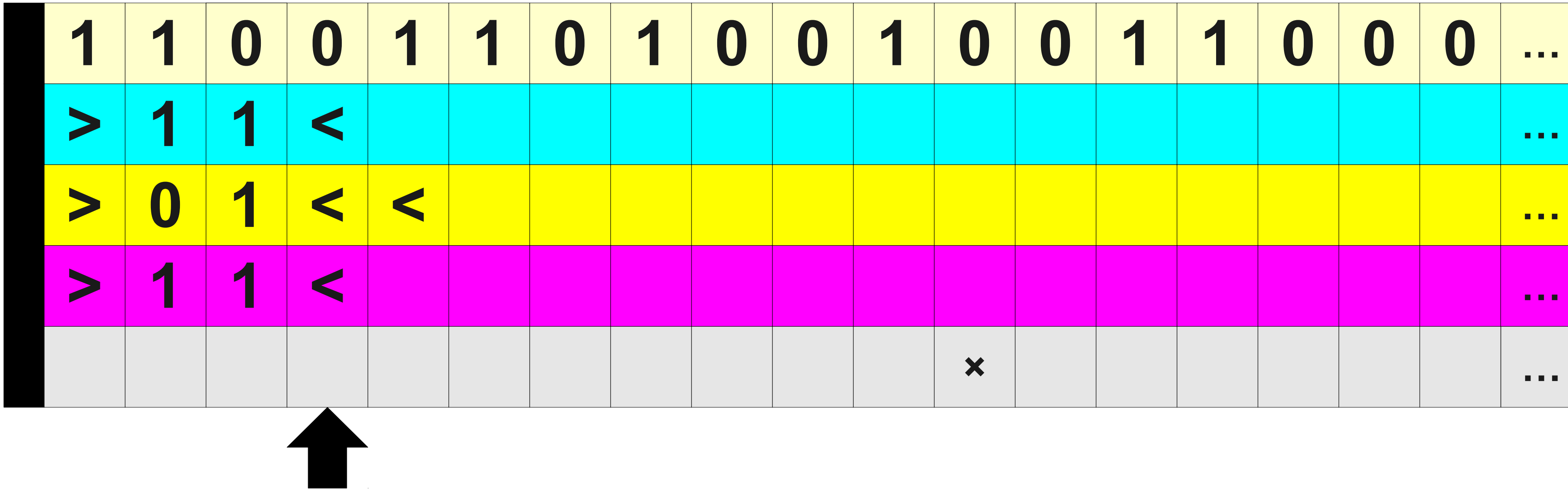
```





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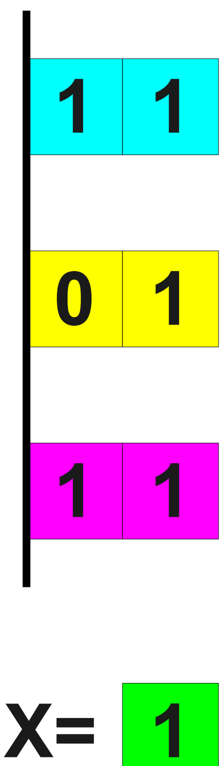


0: Write × on track 5.
1: Move left until {>} on track 3.
2: Move right until {<} on track 3.
3: Move left.
4: Load current on track 3 into X.
5: If X = >, go to 7.
6: Write < on track 3
7: Move left until {>} on track 3.

1	1	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0	0	...
>	1	1	<															...
>	0	1	<	<														...
>	1	1	<															...
											×							...

2: Pop Stack 2 into X.

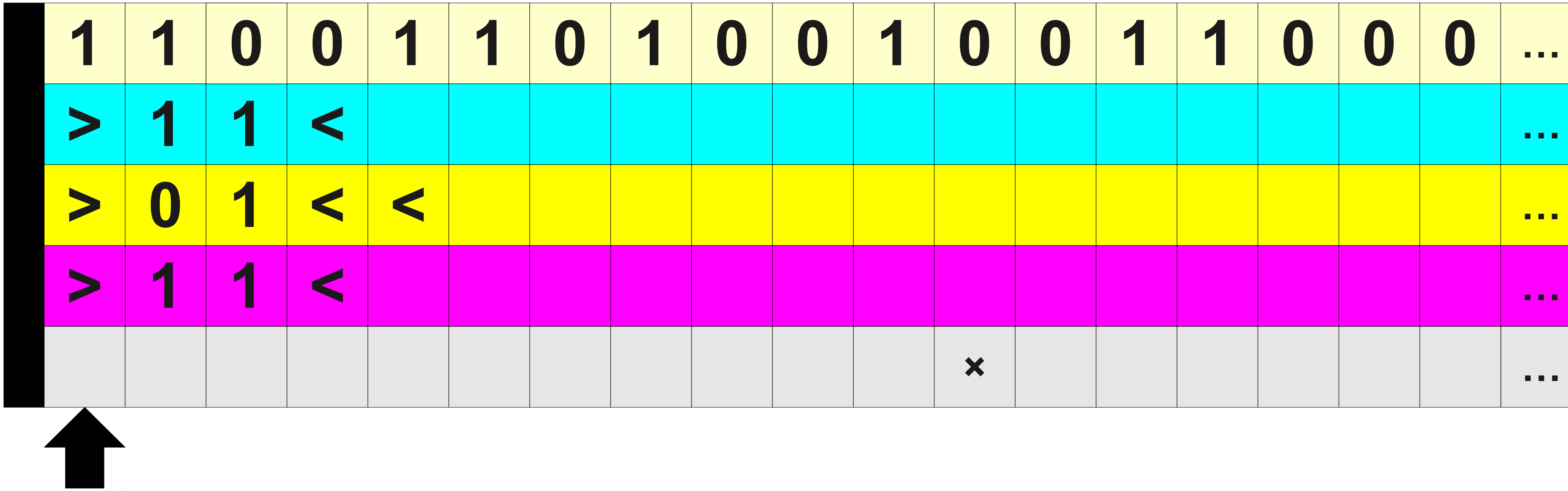
1	1

0	1

1	1

X= 1

0: Write × on track 5.
1: Move left until {>} on track 3.
2: Move right until {<} on track 3.
3: Move left.
4: Load current on track 3 into X.
5: If X = >, go to 7.
6: Write < on track 3
7: Move left until {>} on track 3.



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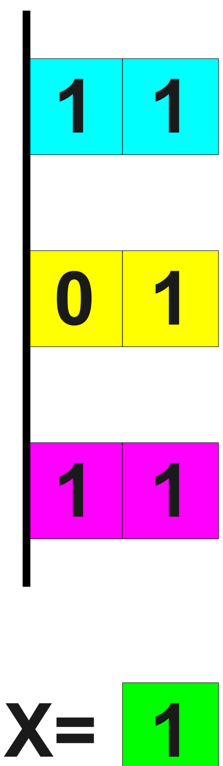
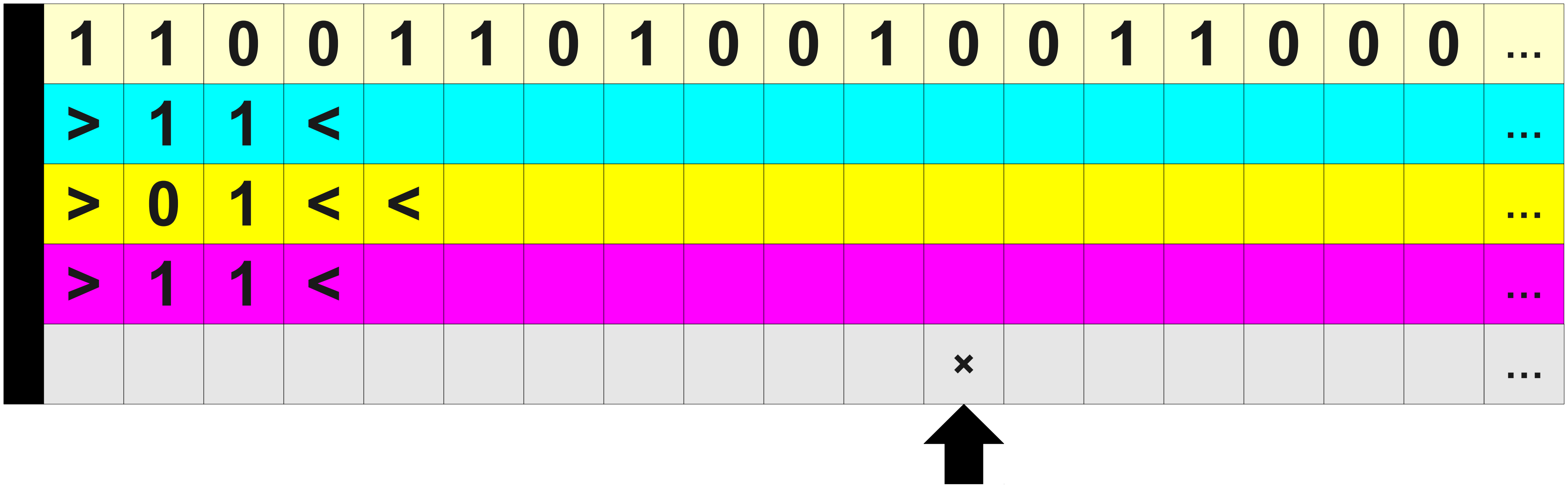



```
0: Write × on track 5.
1: Move left until {>} on track 3.
2: Move right until {<} on track 3.
3: Move left.
4: Load current on track 3 into X.
5: If X = >, go to 7.
6: Write < on track 3
7: Move left until {>} on track 3.
8: Move right until {×} on track 5.
```



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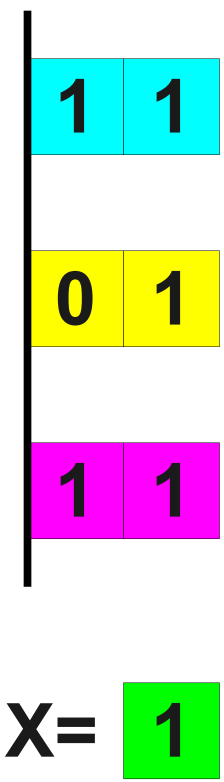

0: Write × on track 5.
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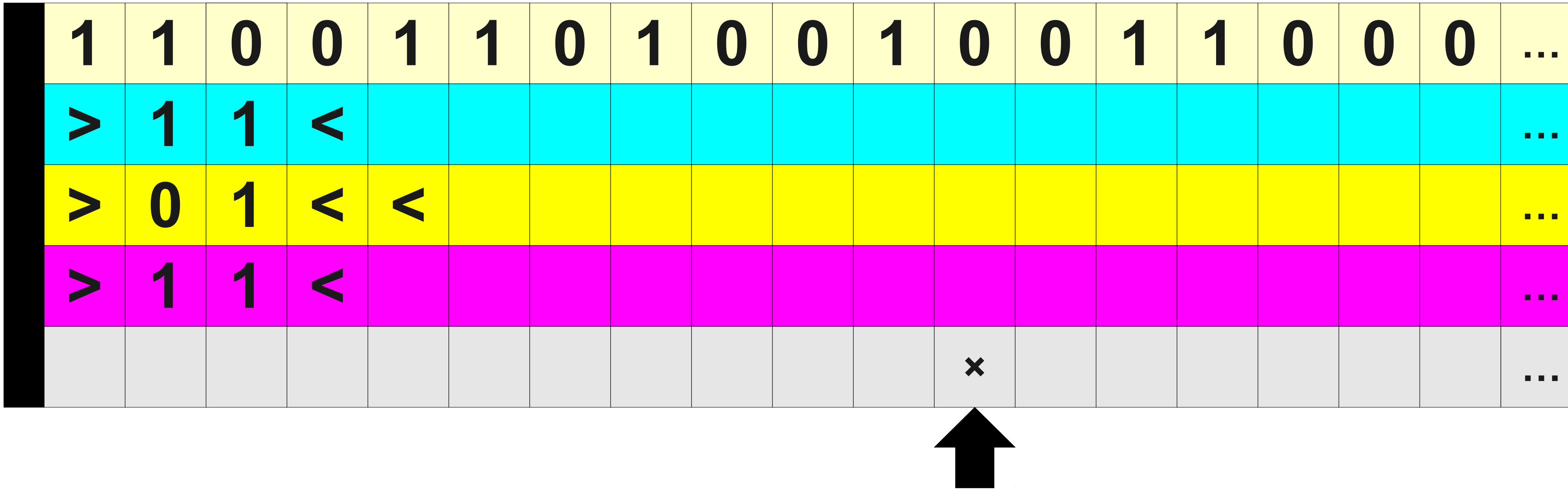
3: Move left.
4: Load current on track 3 into X.
5: If X = >, go to 7.
6: Write < on track 3
7: Move left until {>} on track 3.
8: Move right until {×} on track 5.



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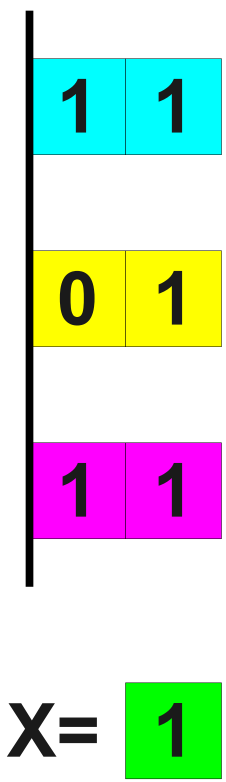


```
0: Write × on track 5.
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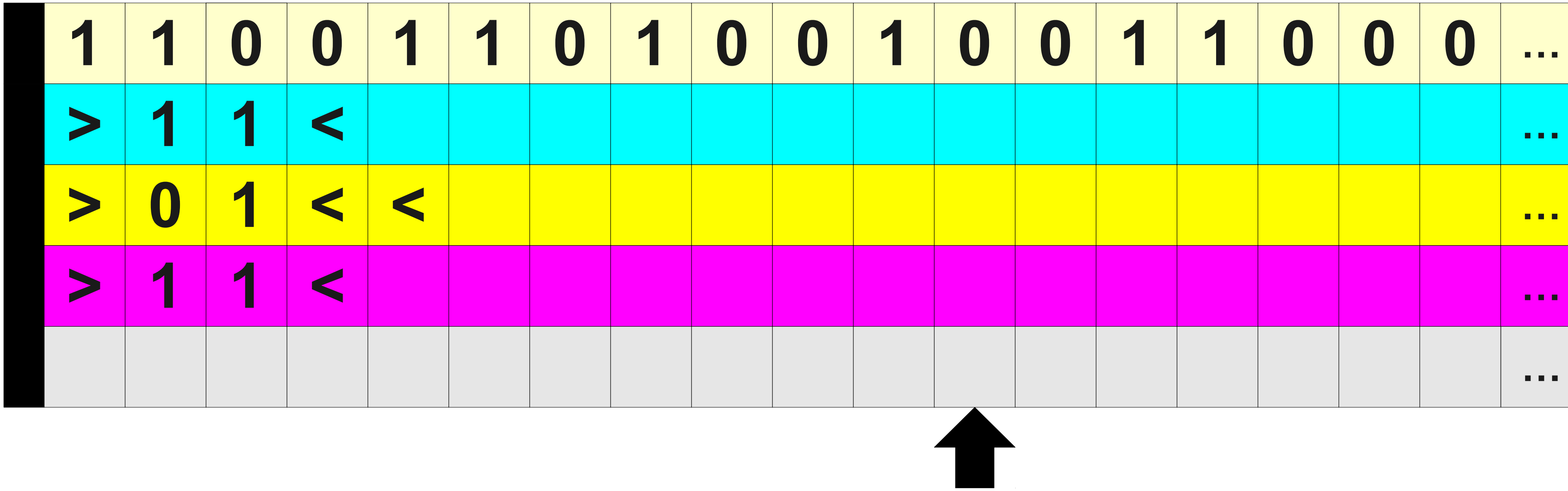
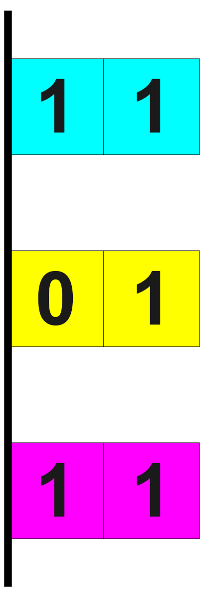
2: Move right until {<} on track 3.
3: Move left.
4: Load current on track 3 into X.
5: If X = >, go to 7.
6: Write < on track 3
7: Move left until {>} on track 3.
8: Move right until {×} on track 5.
9: Write B on track 5.
```




```
2: Pop Stack 2 into X.
```



X= 1

```
0: Write × on track 5.
1: Move left until {>} on track 3.
2: Move right until {<} on track 3.
3: Move left.
4: Load current on track 3 into X.
5: If X = >, go to 7.
6: Write < on track 3
7: Move left until {>} on track 3.
8: Move right until {×} on track 5.
9: Write B on track 5.
```

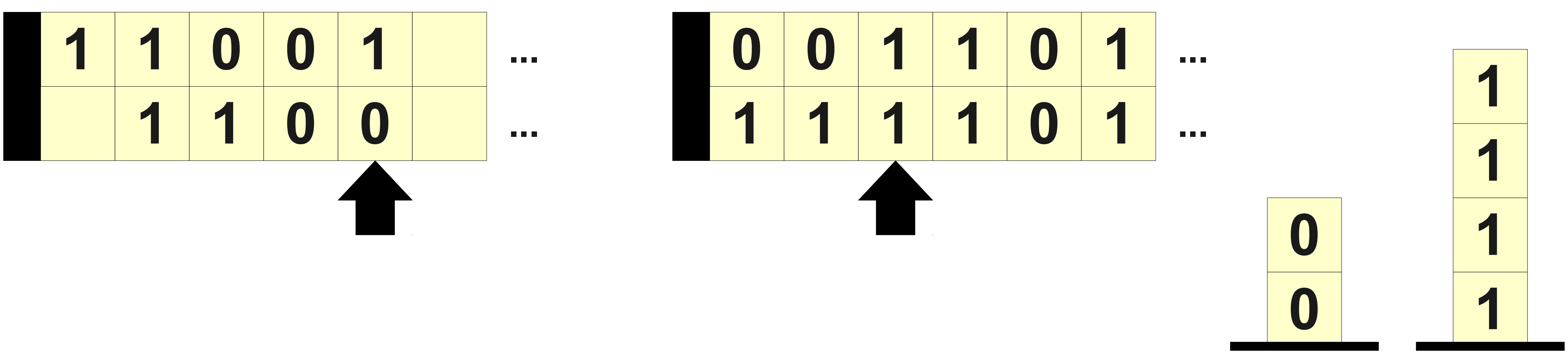
Completing the Construction

- We've seen how to convert the new **WB5** stack commands into **WB4** code.
- For this to work, the extra tracks must be set up correctly.
- Add preamble code to the generated **WB4** program to do this:

```
Write > to track 2.
...
Write > to track n.
Move right.
Write < to track 2.
...
Write < to track n.
Move left.
```

But Why Stop There?

- Adding finitely many stacks to **WB** doesn't increase its expressive power.
- What if we added finitely many **tapes** to **WB**?
- We now have a programming language controlling
  - Multiple tracks per tape,
  - Finitely many stacks, and
  - Finitely many tapes.



Introducing **WB6**

- The programming language **WB6** is **WB5** with the addition of multiple tapes.
- All tape commands have been updated to specify which tape they apply to.
- If tape unspecified, it's assumed that it's tape 1.

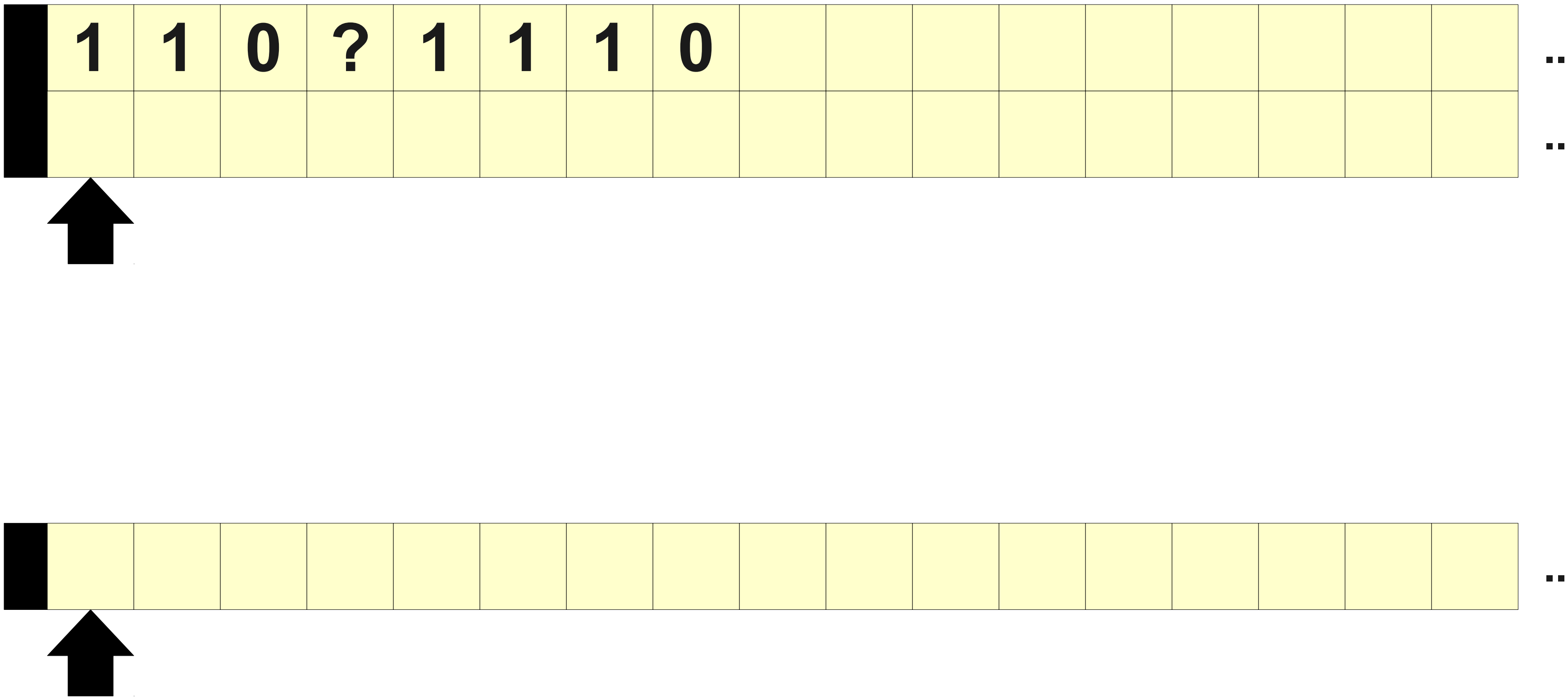
A **WB6** Program for *SEARCH*

- Recall from Problem Sets 5 and 6 that the language *SEARCH* over $\Sigma = \{\texttt{0}, \texttt{1}, \texttt{?}\}$ is the language

$$\{\ p\texttt{?}t \mid p, t \in \{\ \texttt{0}, \texttt{1}\ \}^* \text{ and } p \text{ is a substring of } t\ \}$$

- How would we write a **WB6** program for *SEARCH*?
- (For simplicity, we'll assume that the input is properly formatted).

A **WB6** Program for Search



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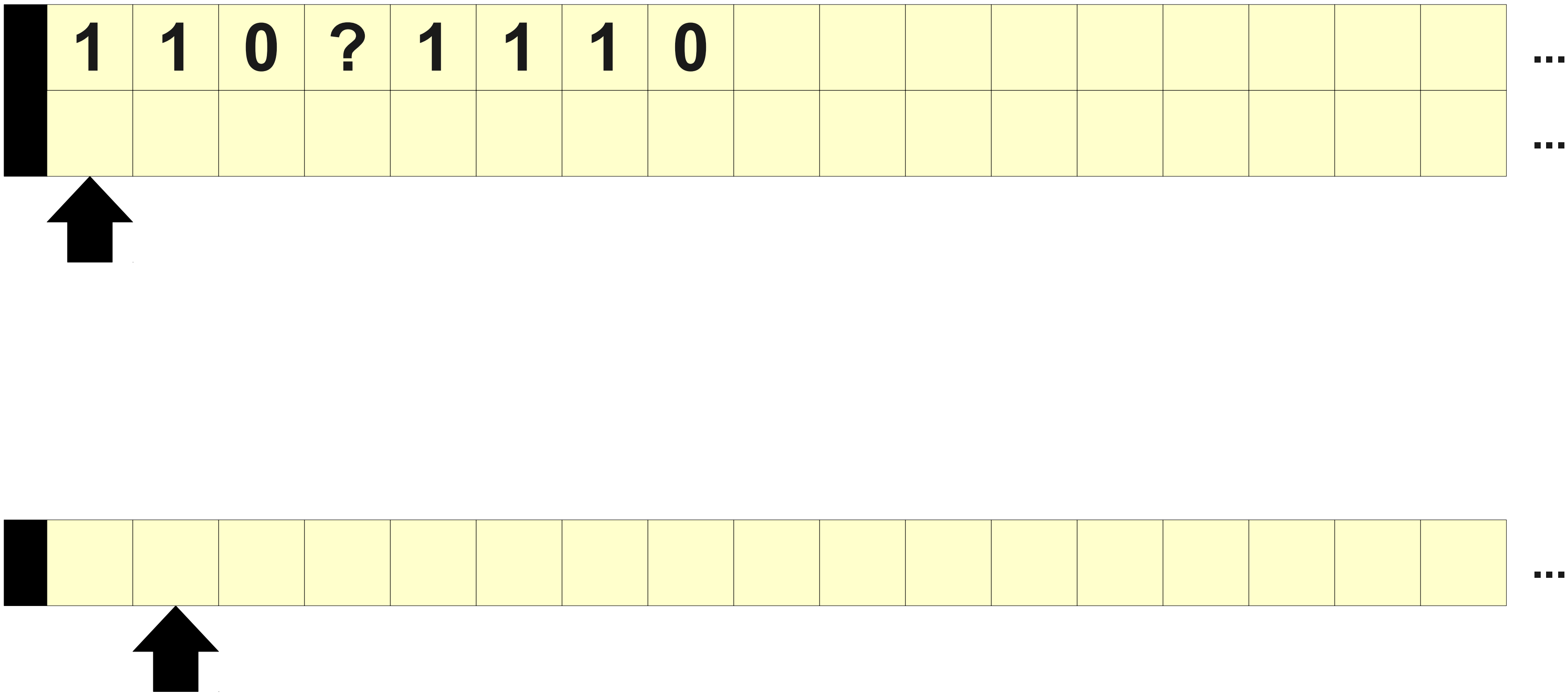

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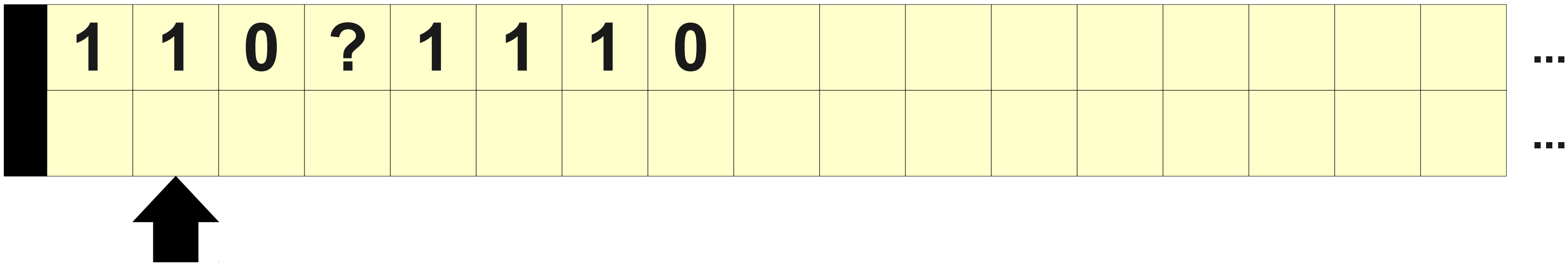

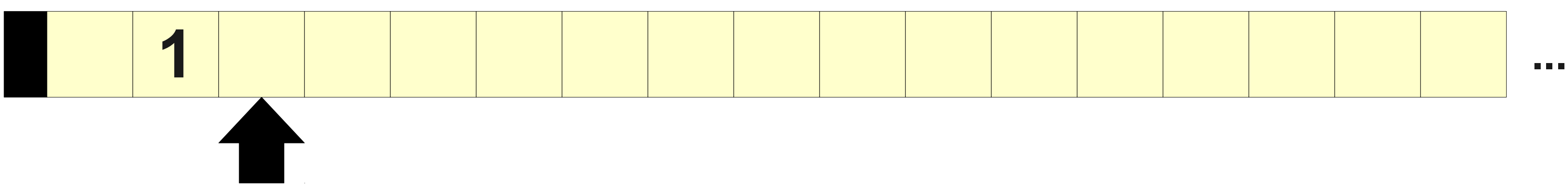

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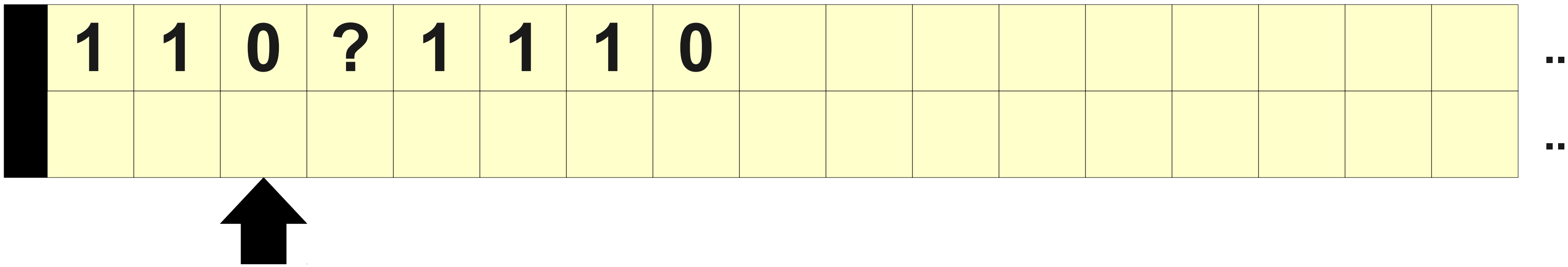

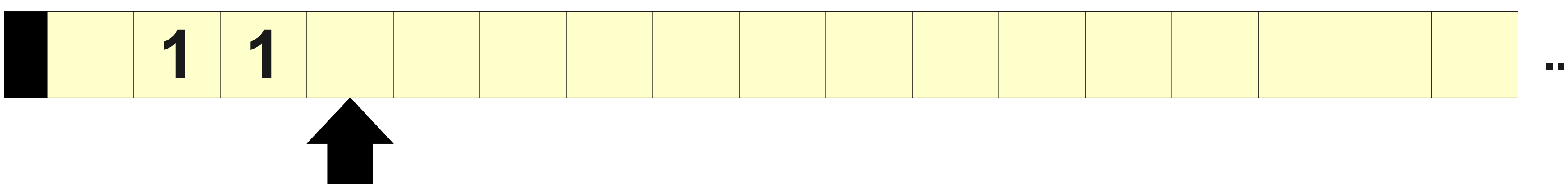

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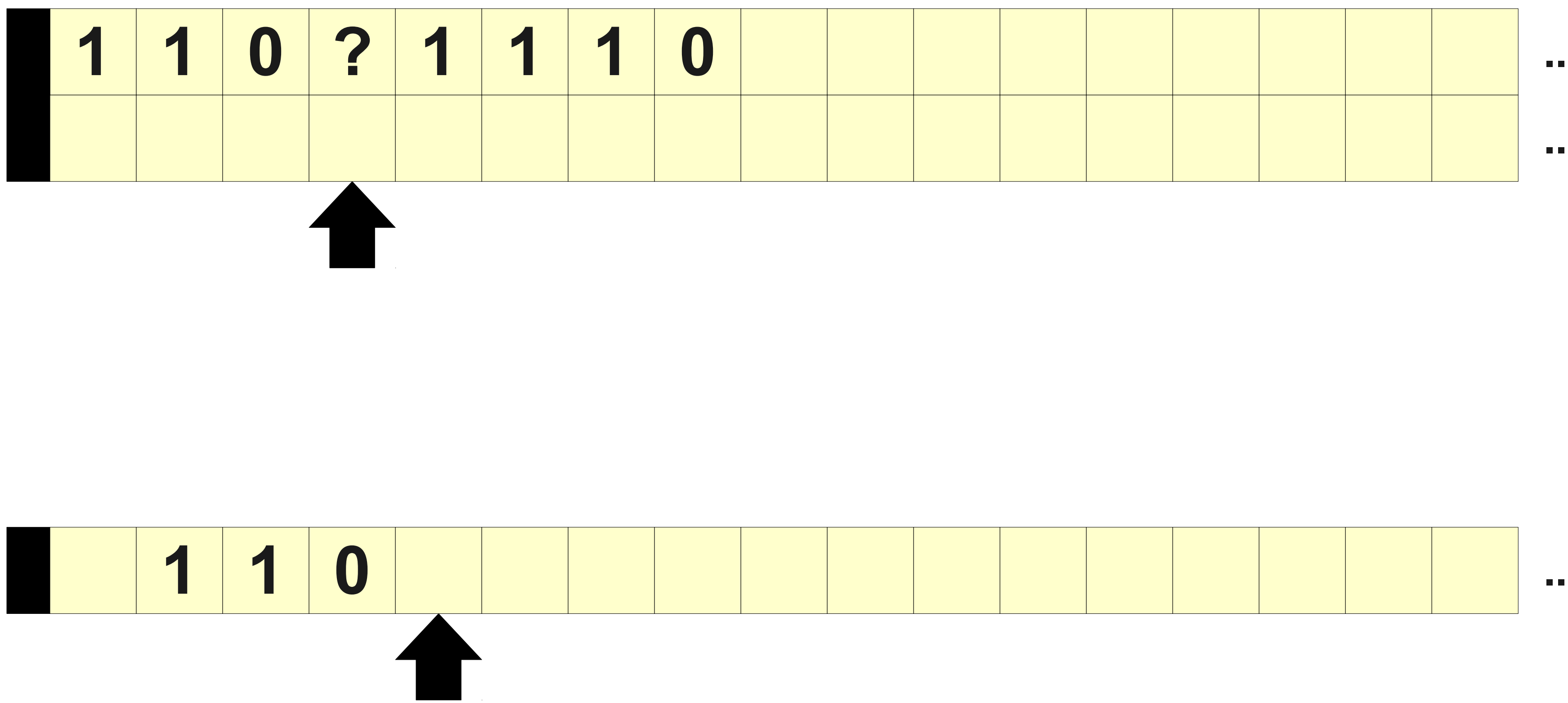

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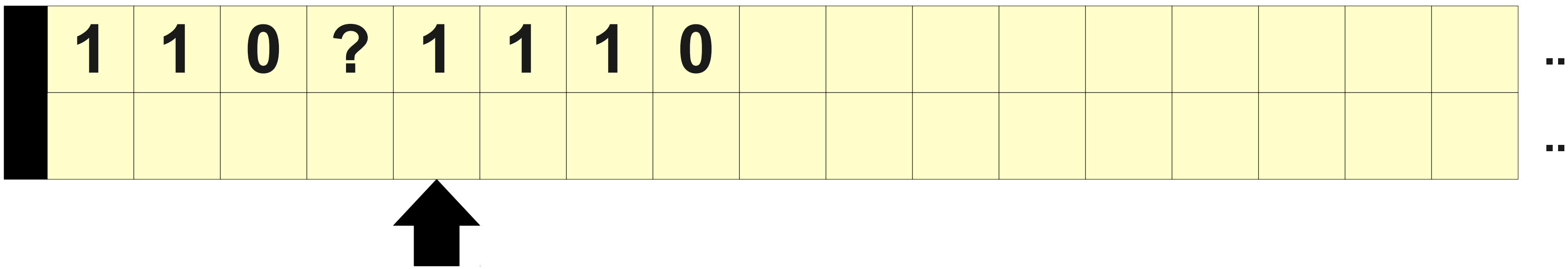

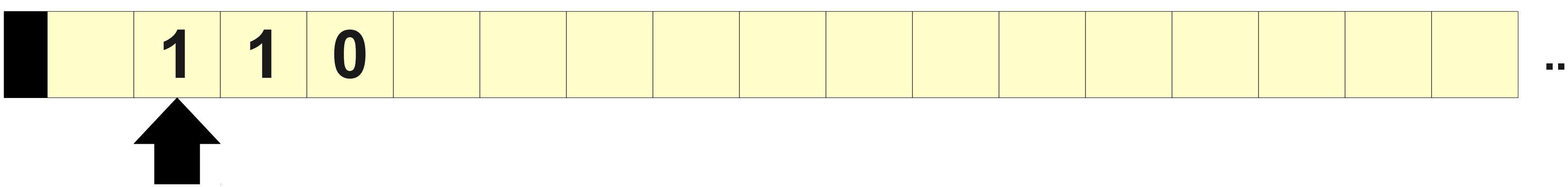

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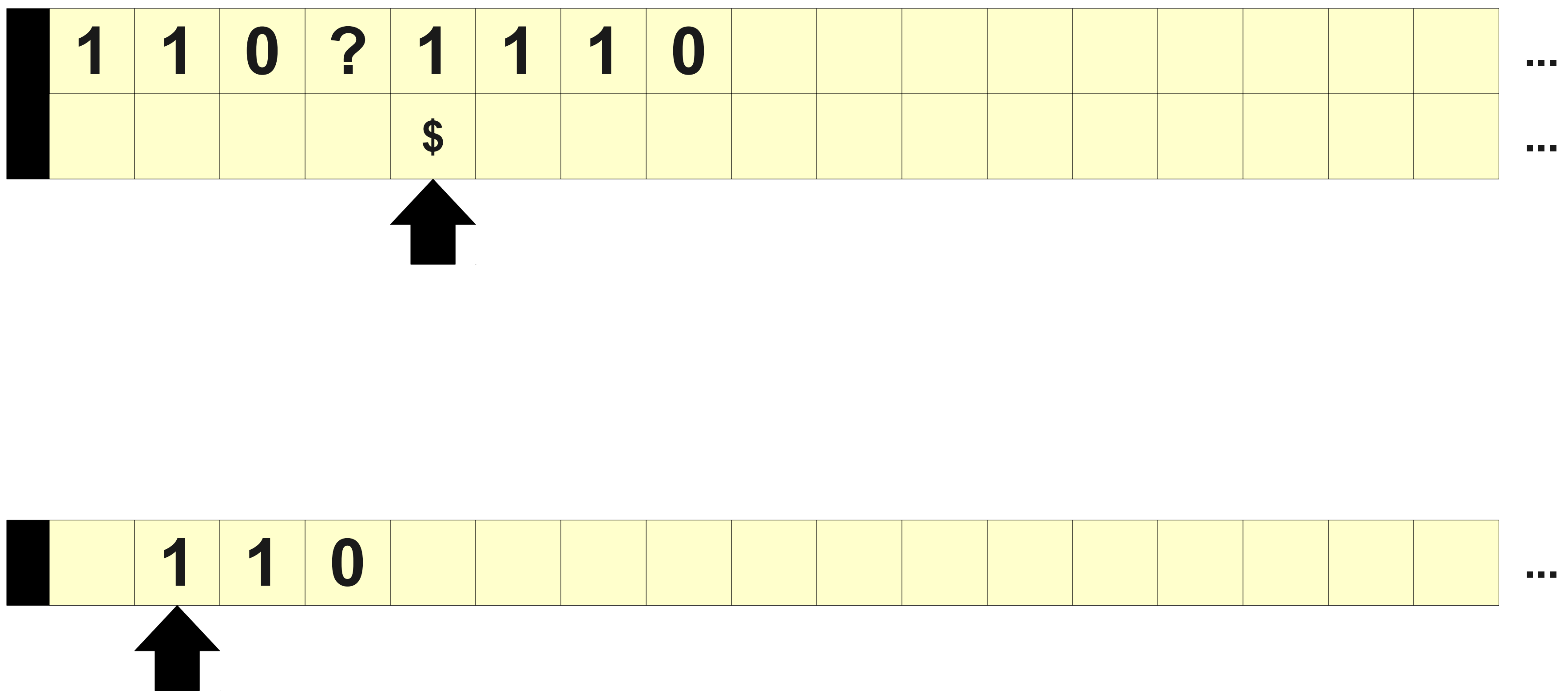

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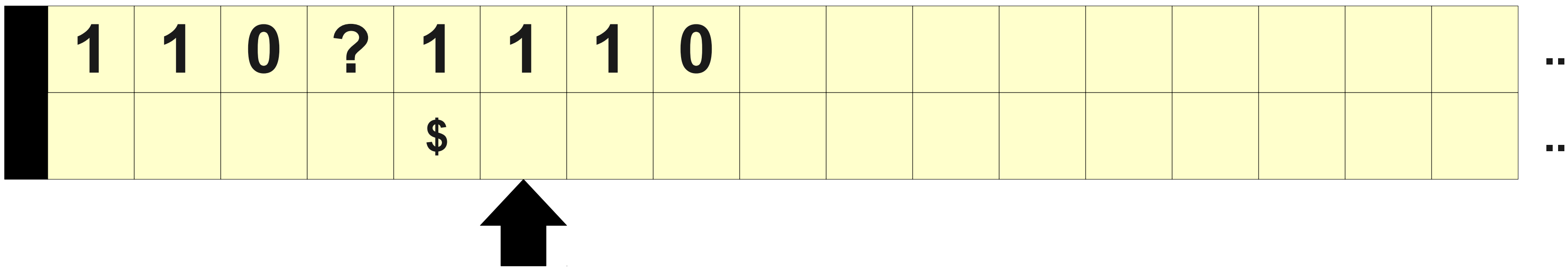

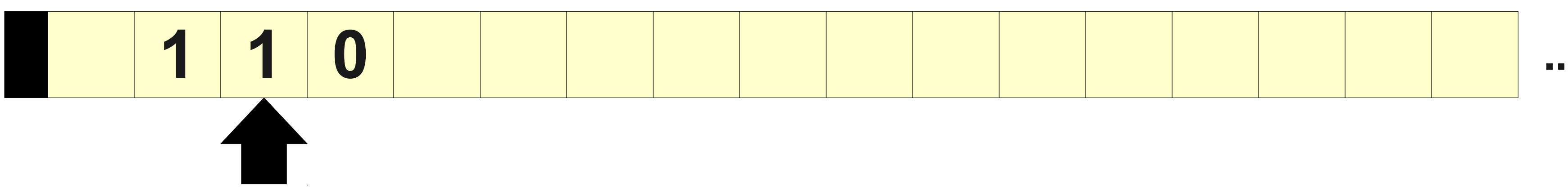

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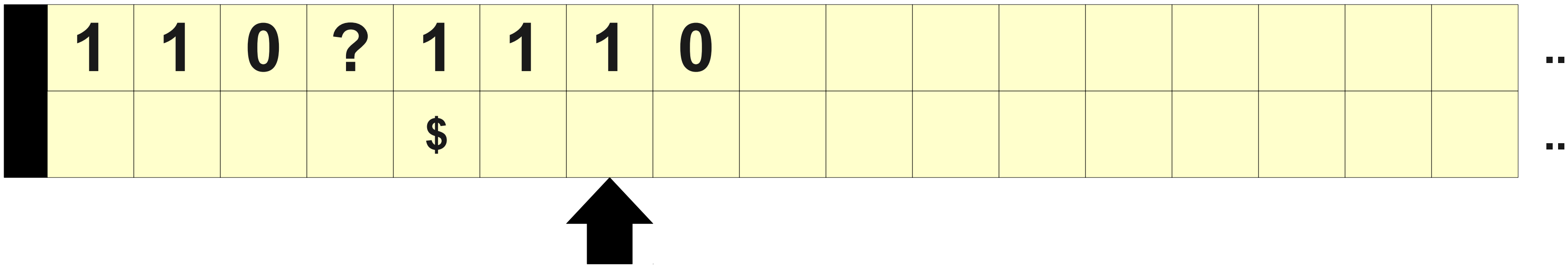

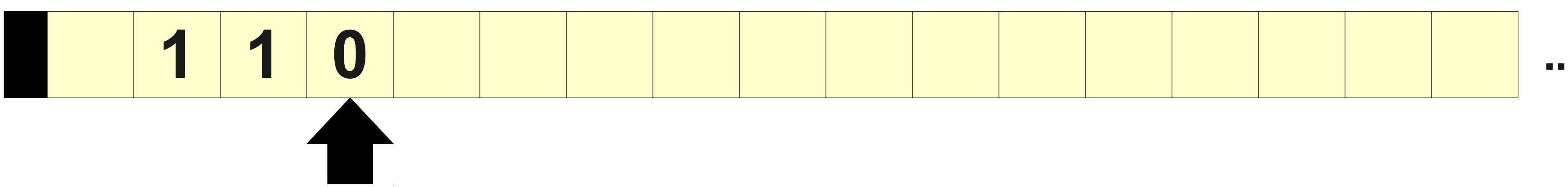

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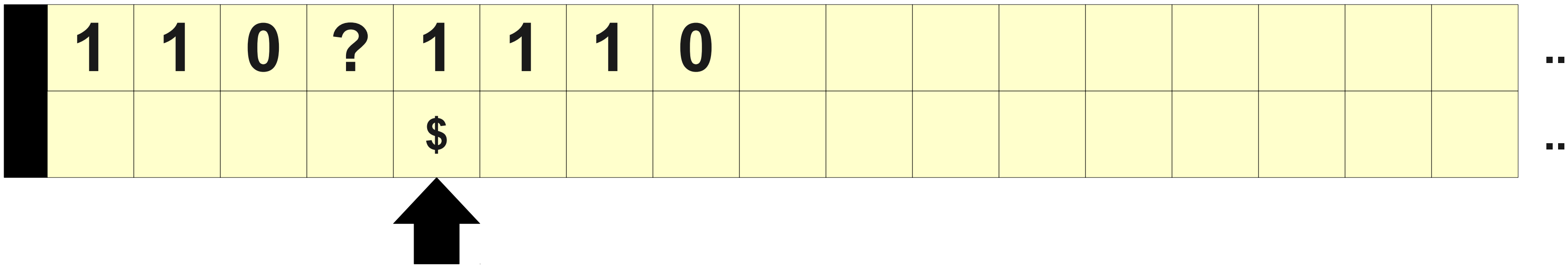

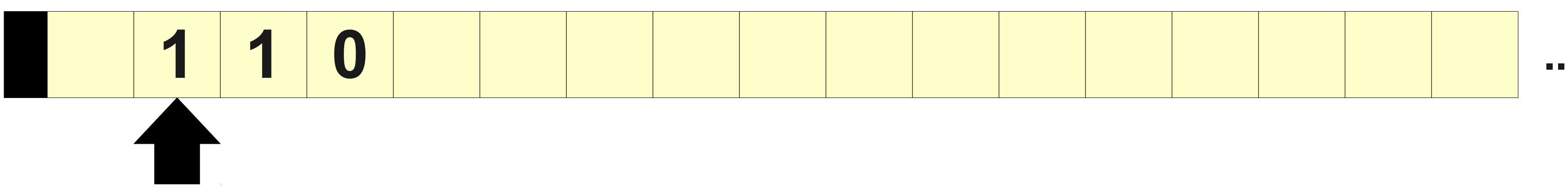

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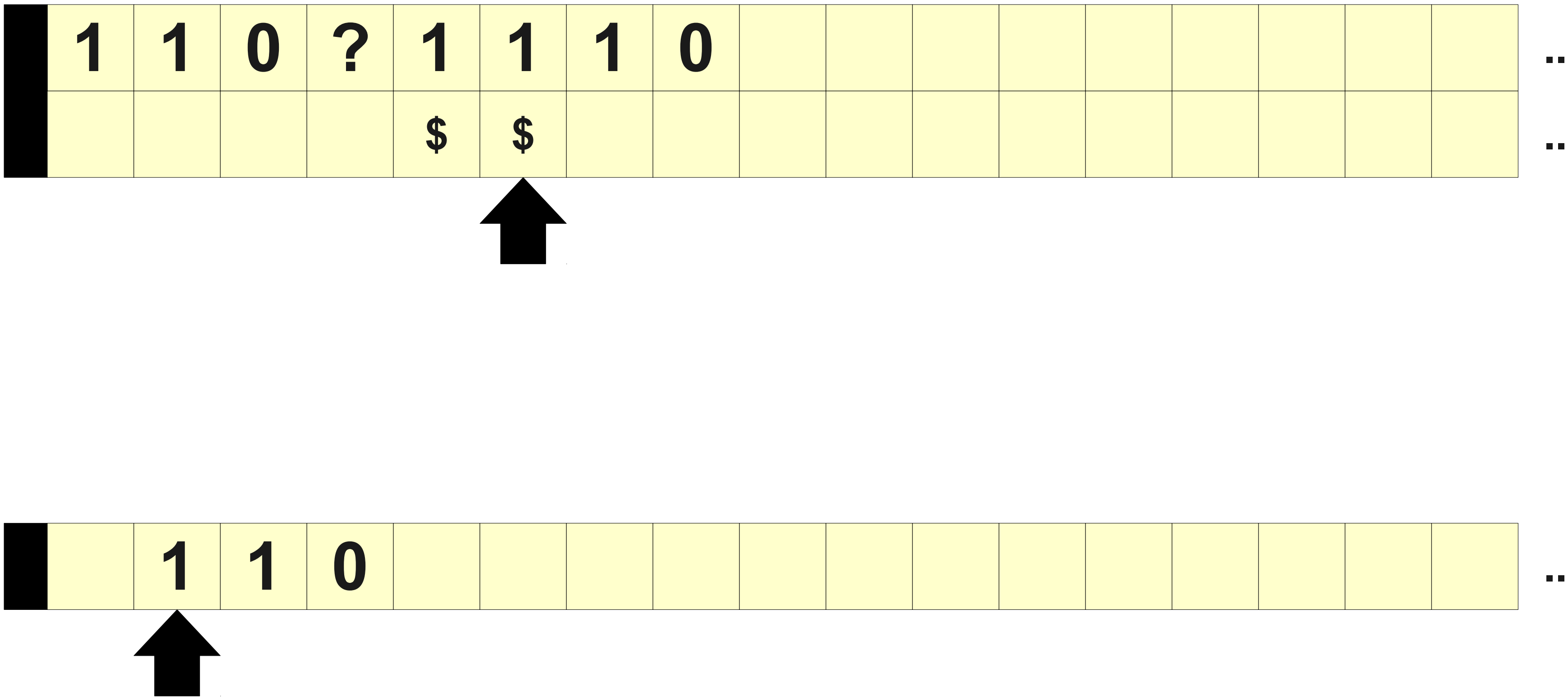

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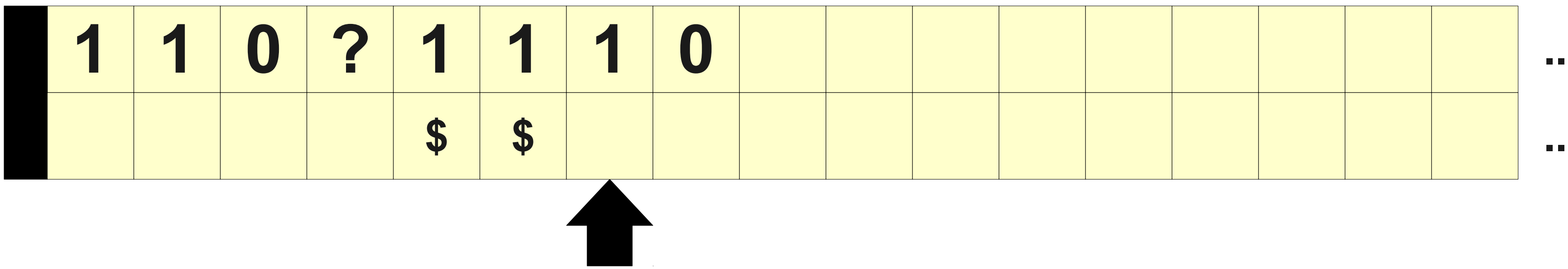

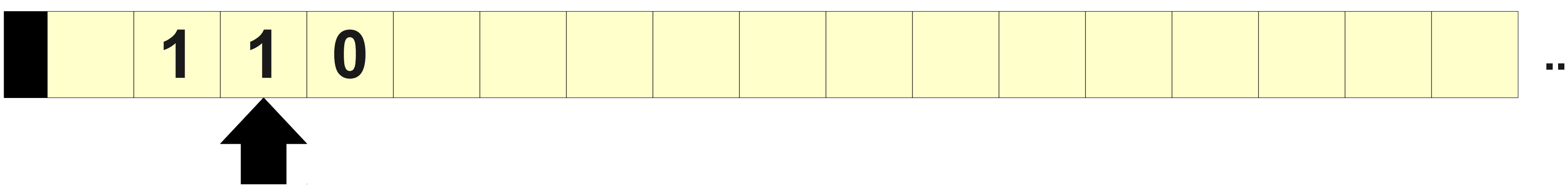

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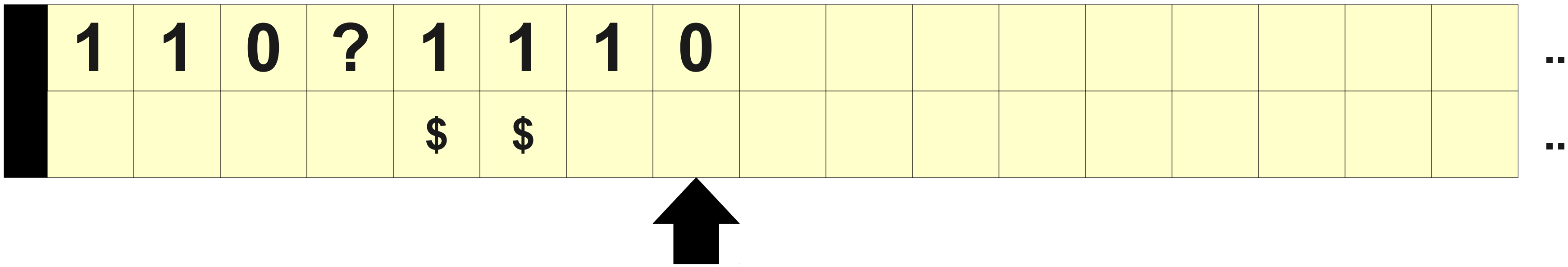

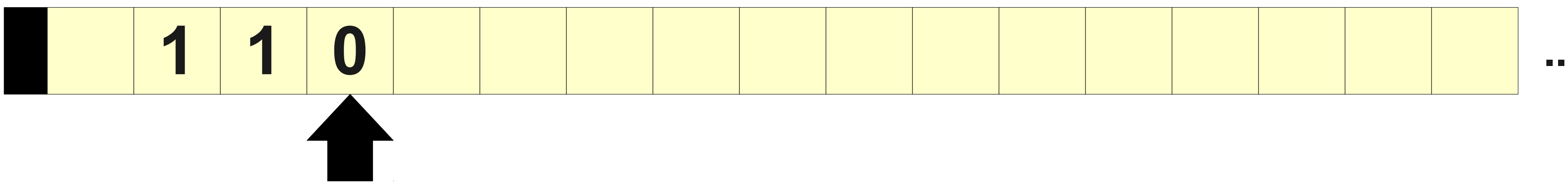

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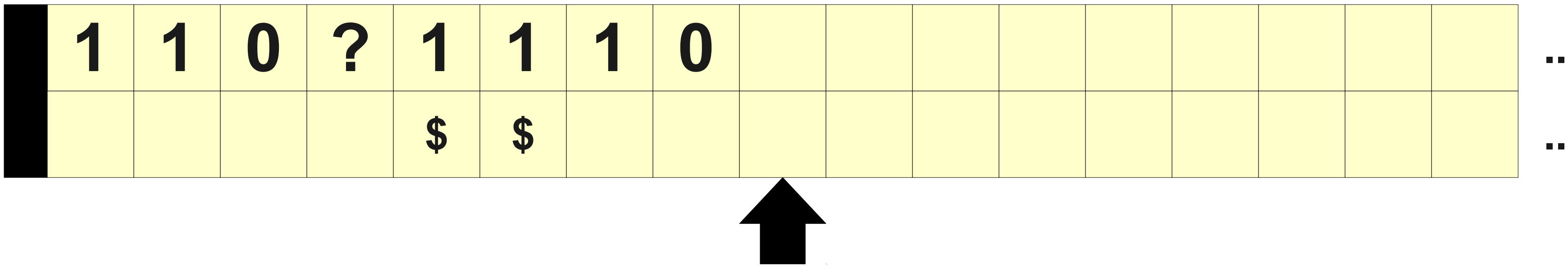
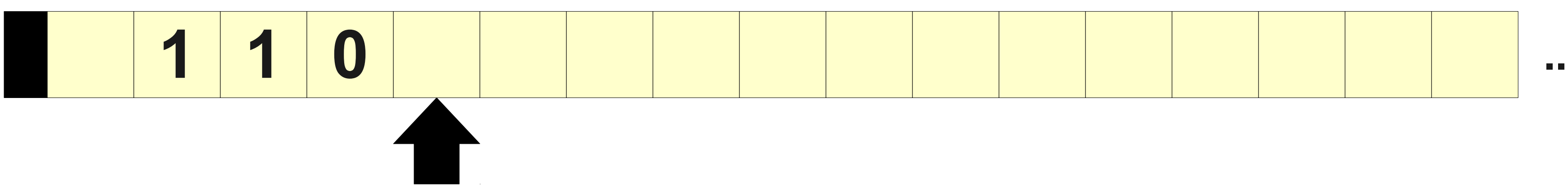




A **WB6** Program for *SEARCH*

```
// Start
 0: Move tape 2 right.
 1: If reading ? on tape 1.1, go to Match.
 2: Load curr on tape 1.1 into X.
 3: Write X to tape 2.
 4: Move tape 1 right.
 5: Move tape 2 right.
 6: Go to 1.
```

A **WB6** Program for *SEARCH*

```
// Match
 7: Move tape 2 left until {B}
 8: Move tape 2 right.
 9: Move tape 1 right.
10: Write $ to tape 1, track 2.
11: If B on tape 2, go to Acc.
12: If B on tape 1, go to Rej.
13: Load tape 1, track 1 into X.
14: Load tape 2 into Y.
15: If X = Y, go to 17.
16: Go to Mismatch.
17: Move tape 1 right.
18: Move tape 2 right.
19: Go to 11.
```

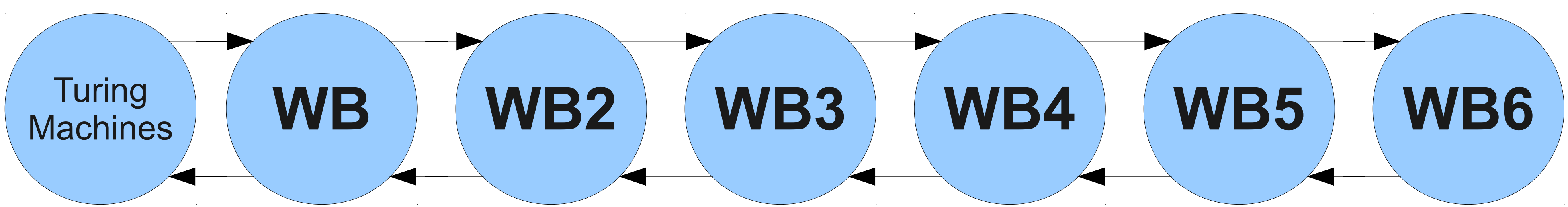
```
// Mismatch
20: Move tape 1.2 left until {$}
21: Go to Match.

// Acc
22: Accept.

// Rej
23: Reject.
```

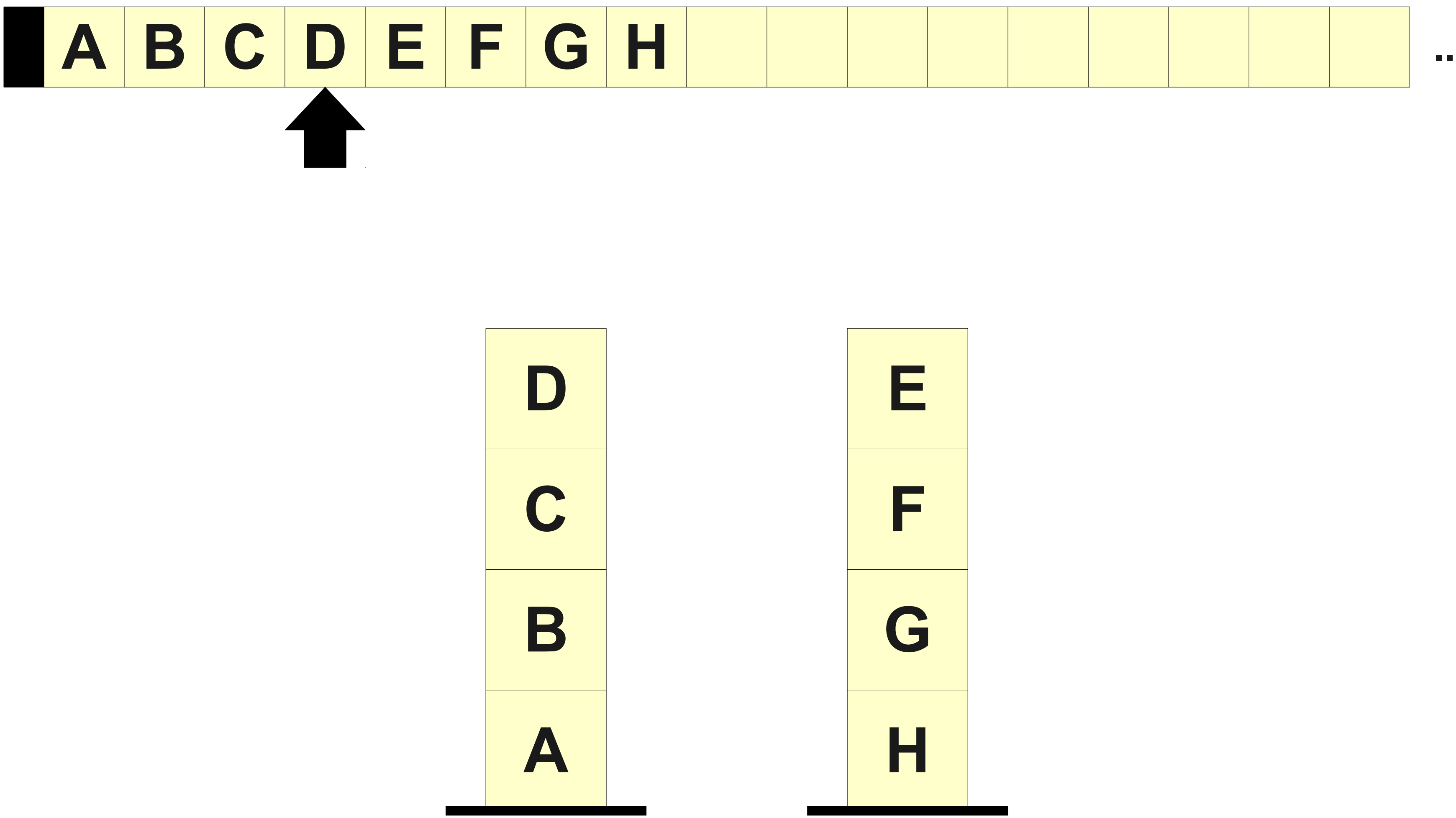
Oh, Come On Already...

- **Theorem:** A language is recursively enumerable iff there is a **WB6** program for it.
- We can really supercharge these languages without increasing our power!
- As with before, the construction will convert **WB6** programs into **WB5** programs.



The Key Idea

- Represent an infinite tape with two stacks.



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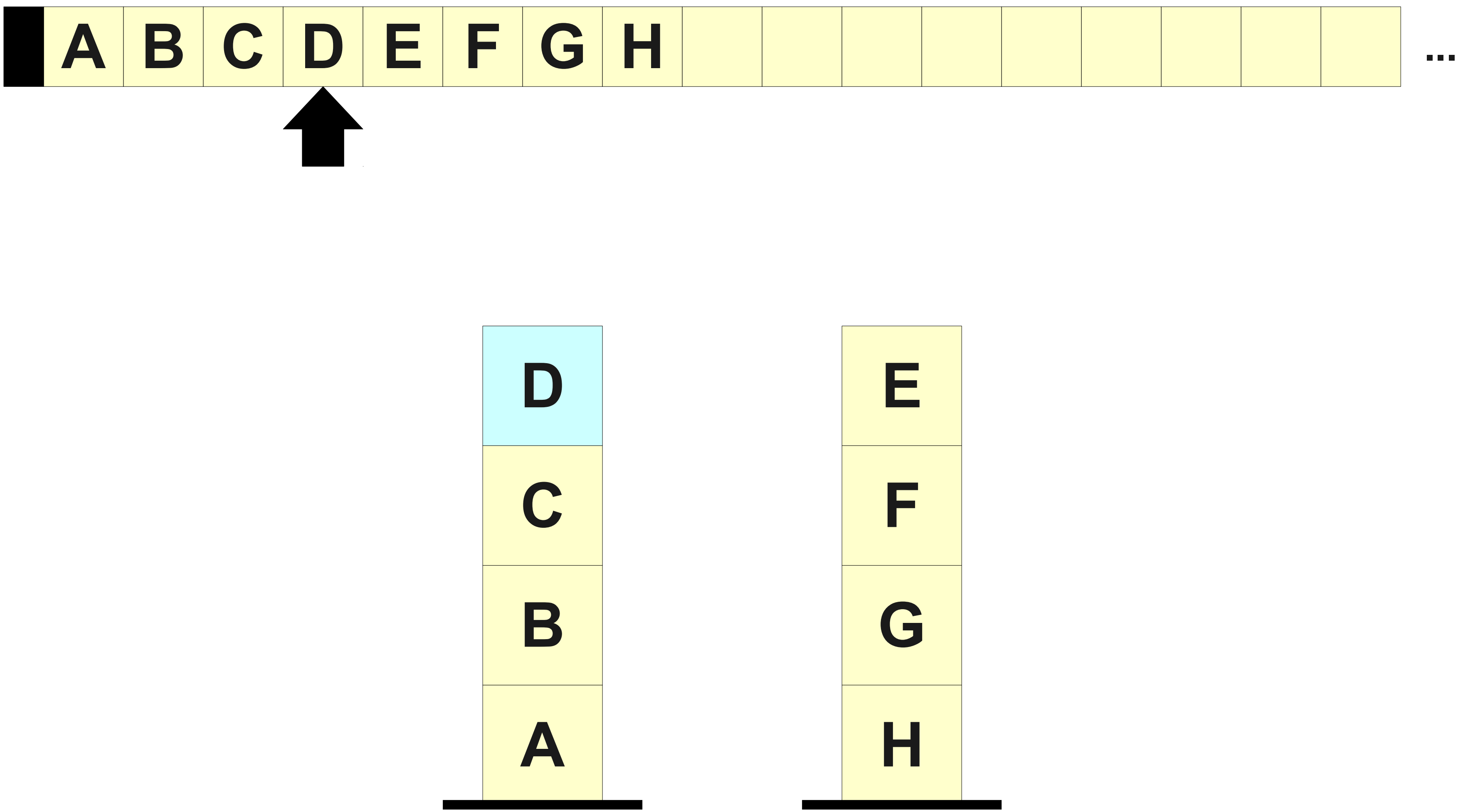

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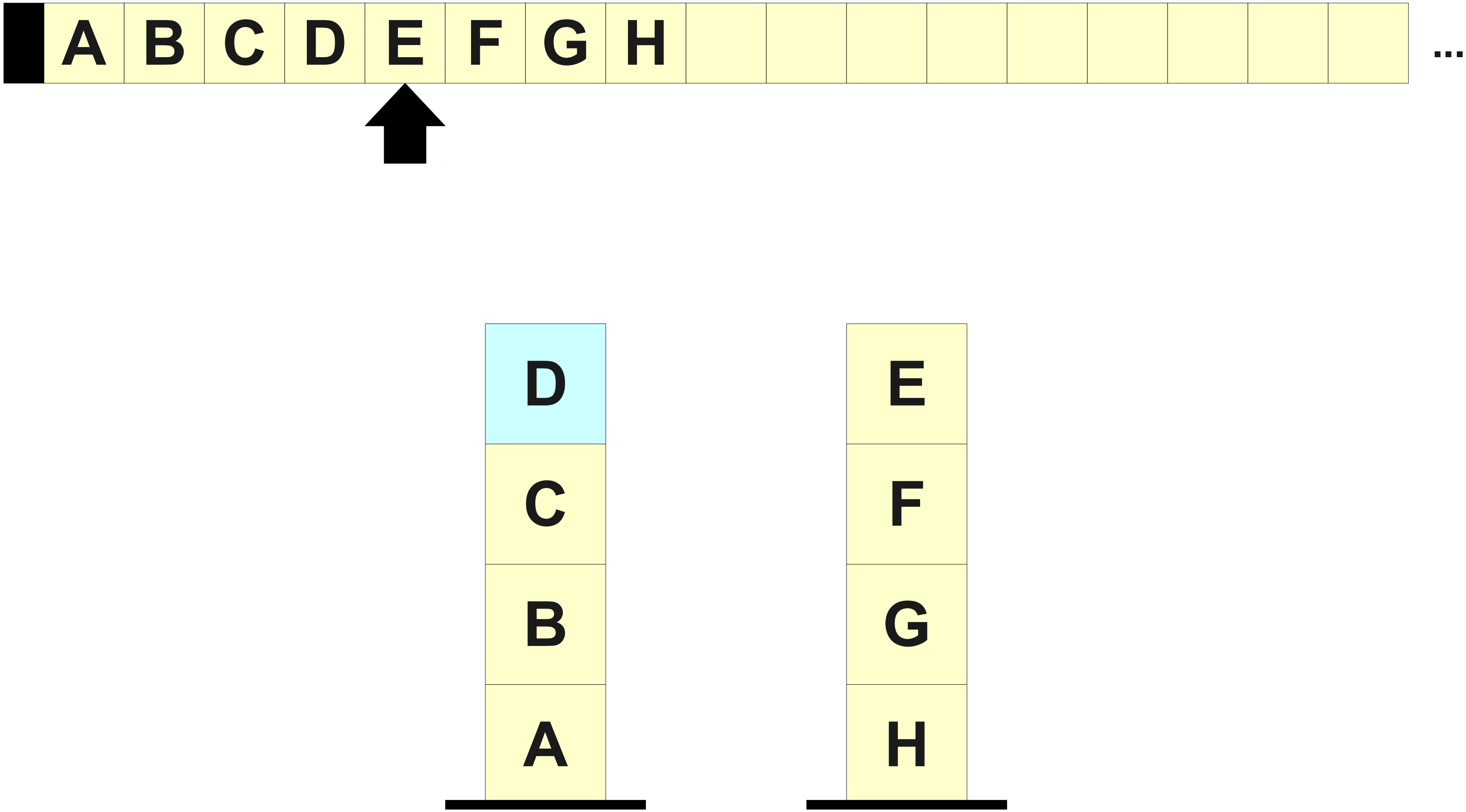
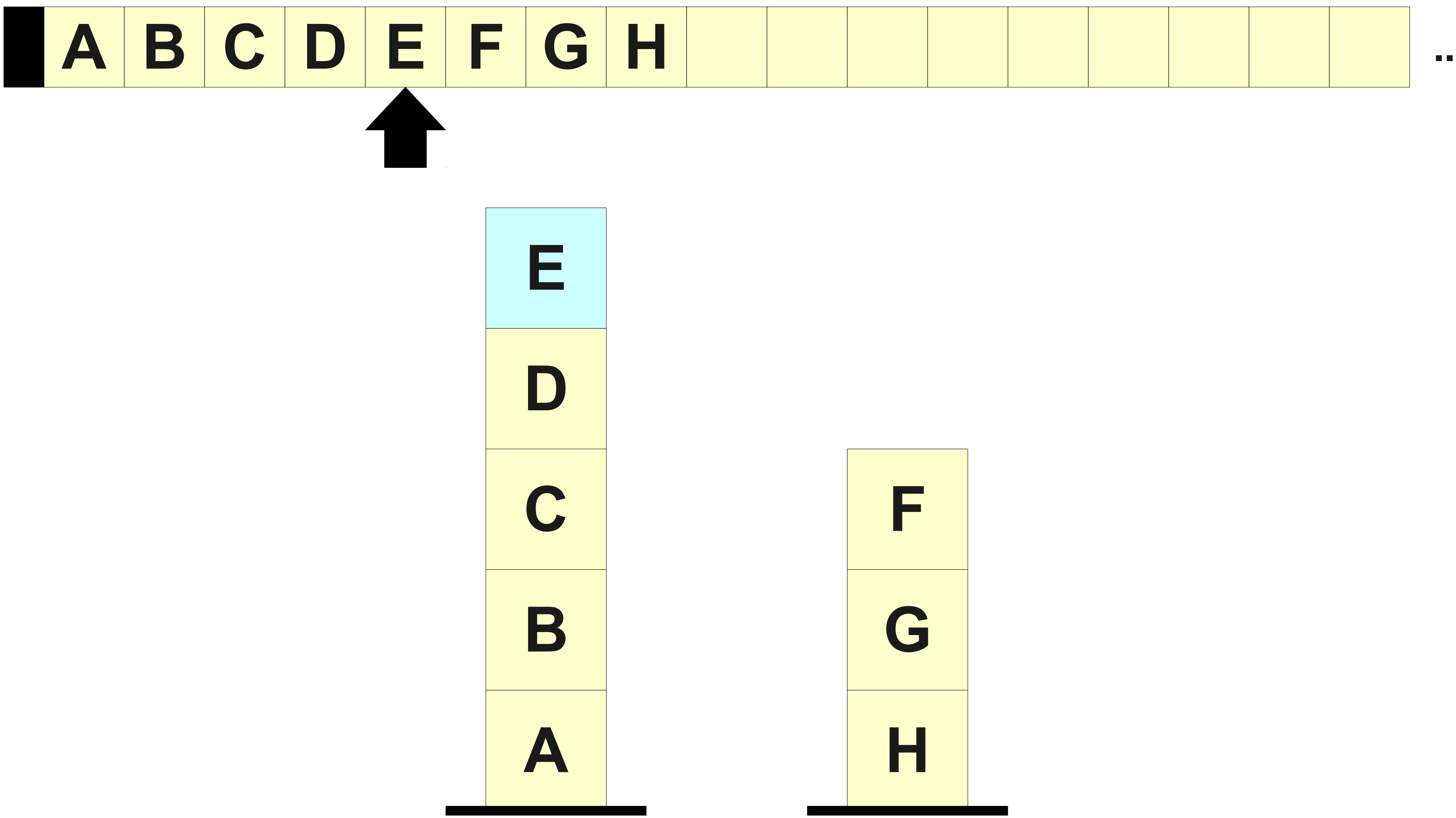


The Key Idea

- Represent an infinite tape with two stacks.



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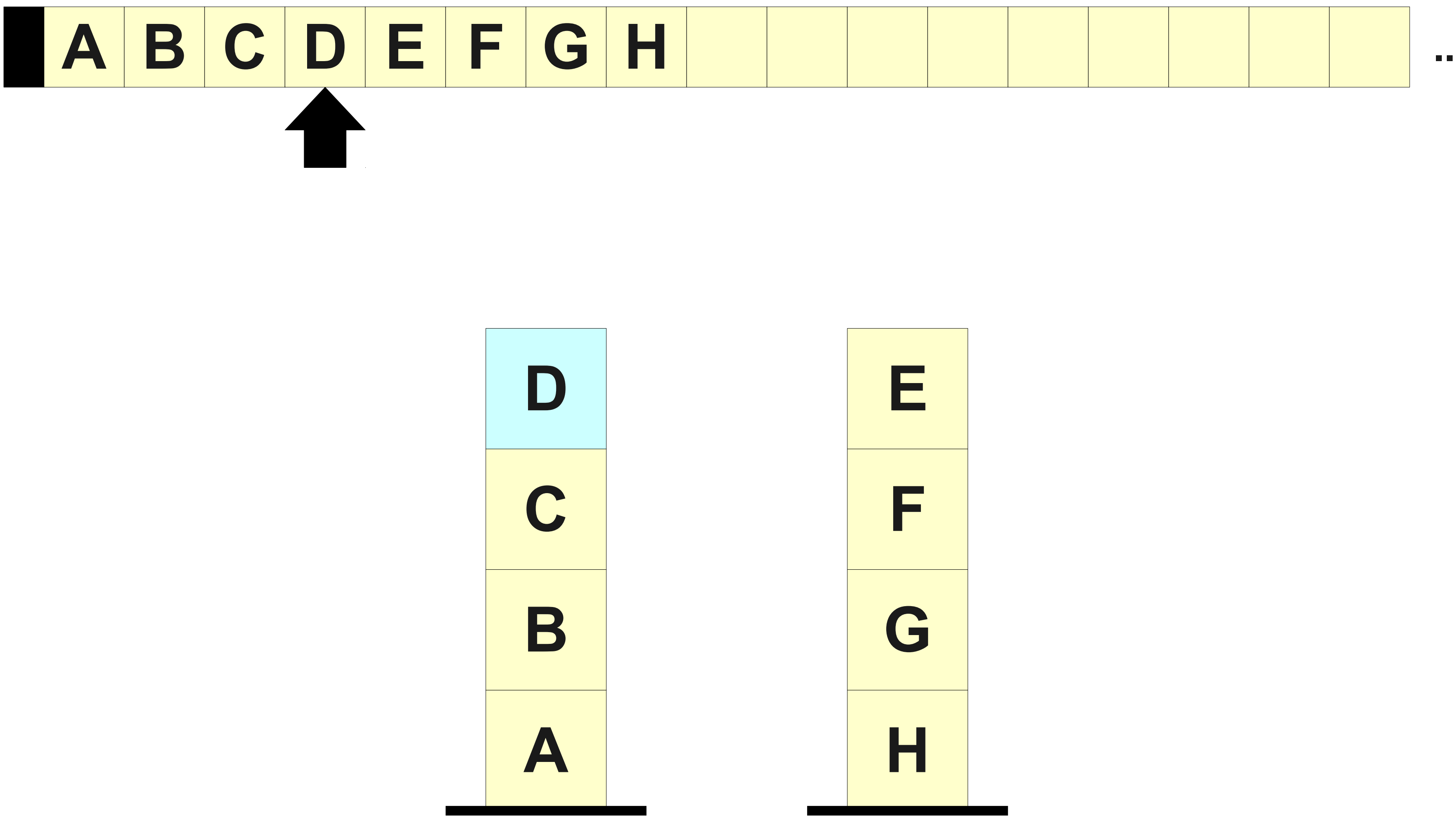

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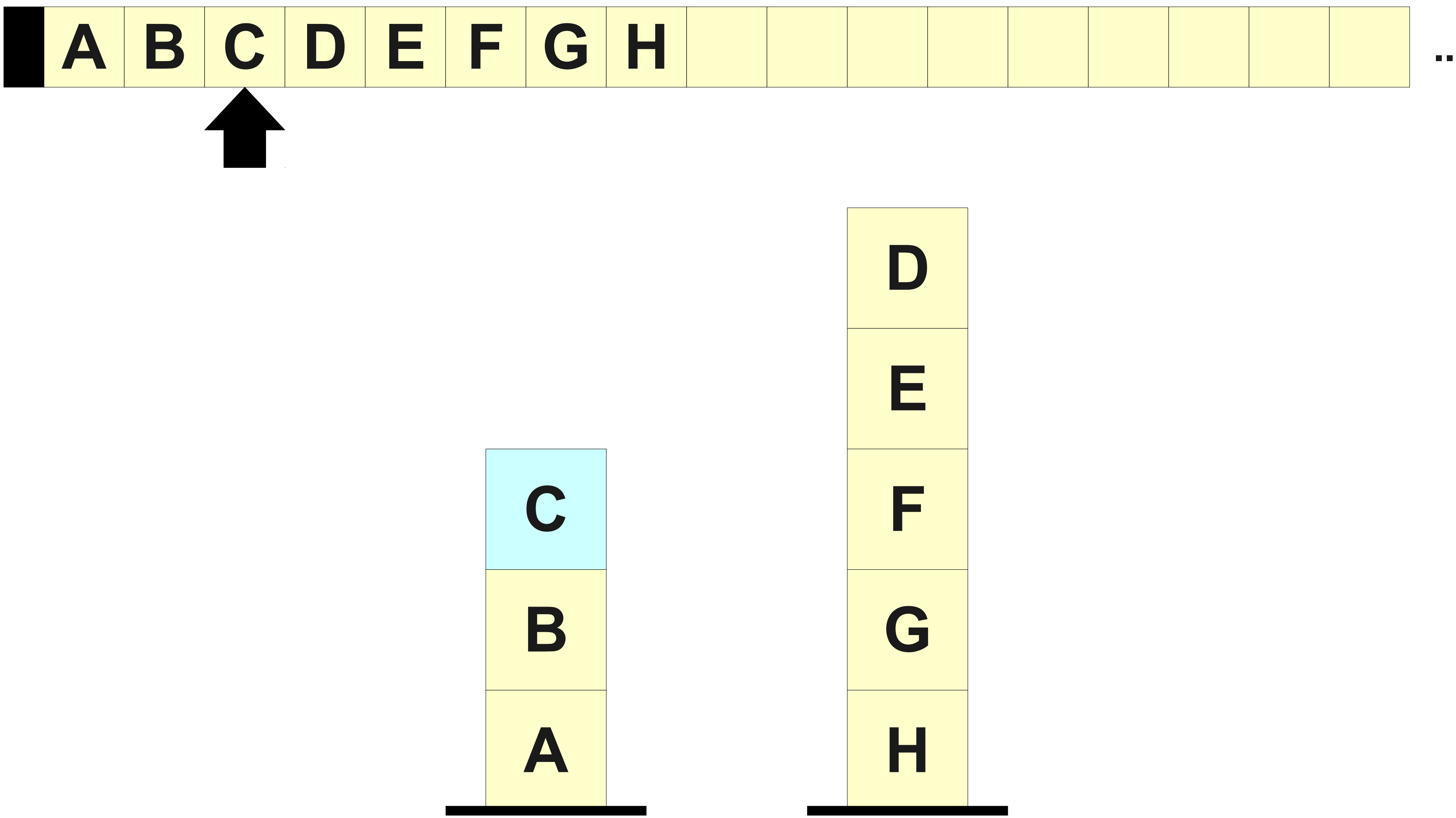

A Sketch of the Construction

- At the start of the program, copy the contents of the initial tape into a pair of stacks that will henceforth represent the first tape.
- Convert all motion operations into stack manipulation operations to push and pop values from the appropriate stacks.
- Use variables to hold temporary values (for example, when moving the top of one stack to another).

```
0: Move tape 1 right.
```

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0: If stack 1R is empty, go to 2.
1: Go to 3.
2: Push B onto stack 1R.
3: Pop stack 1R into X.
4: Push X onto stack 1L.
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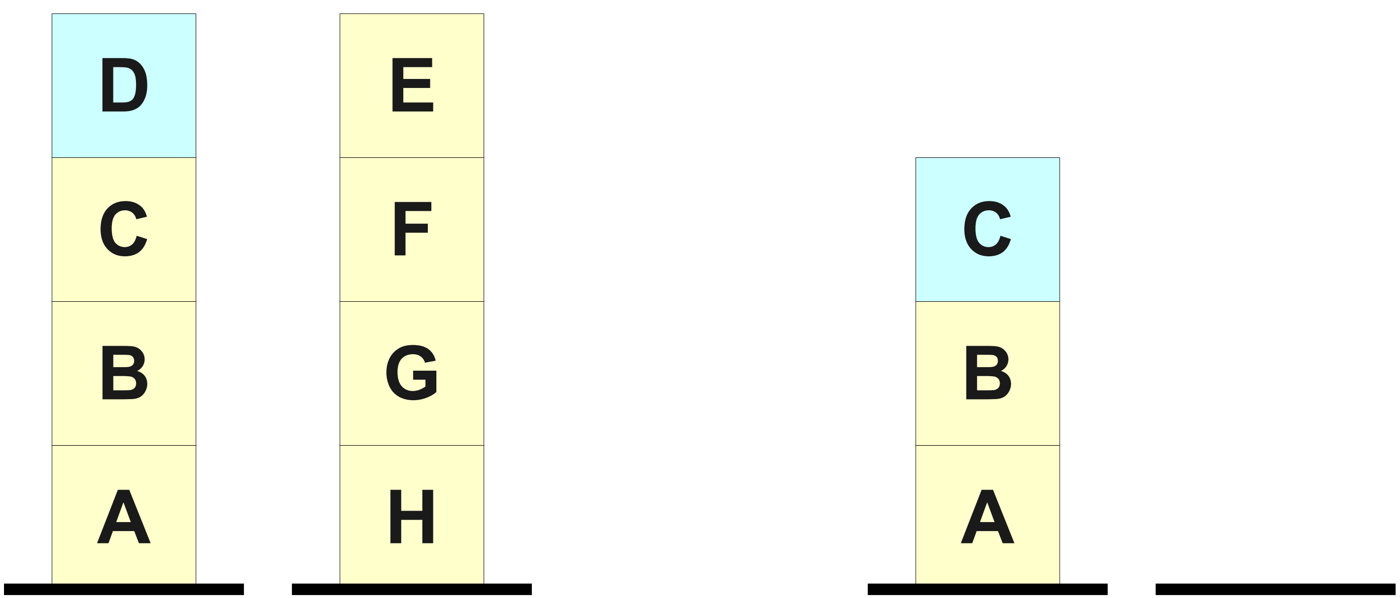

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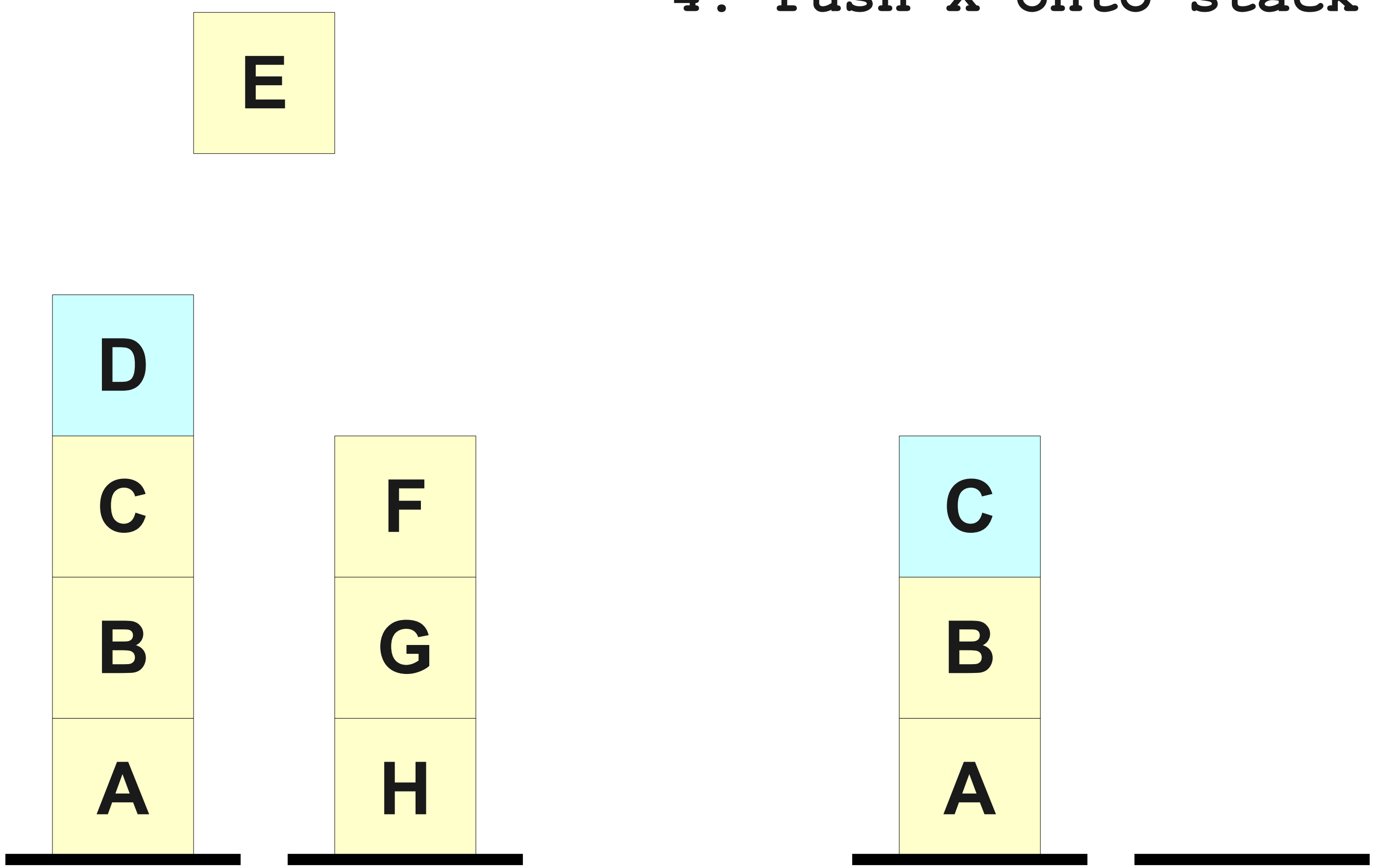

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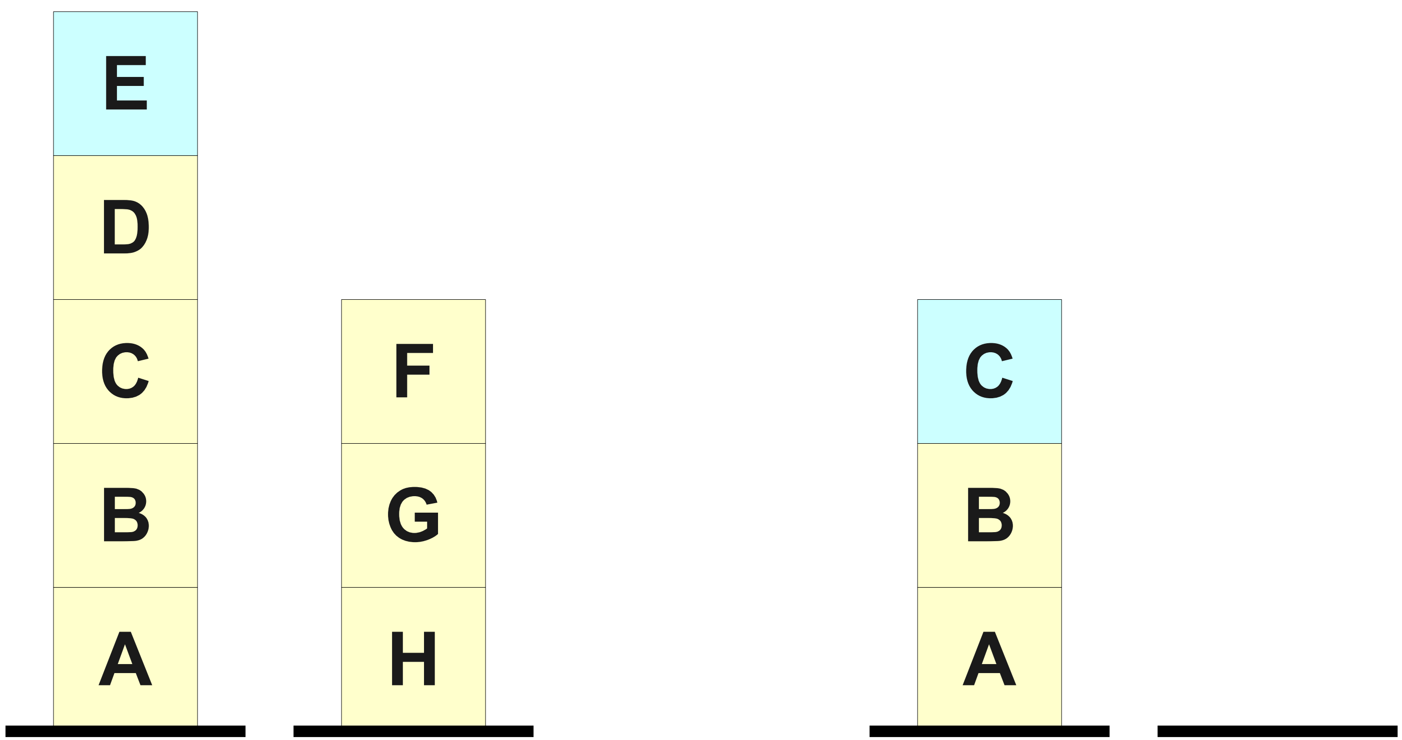

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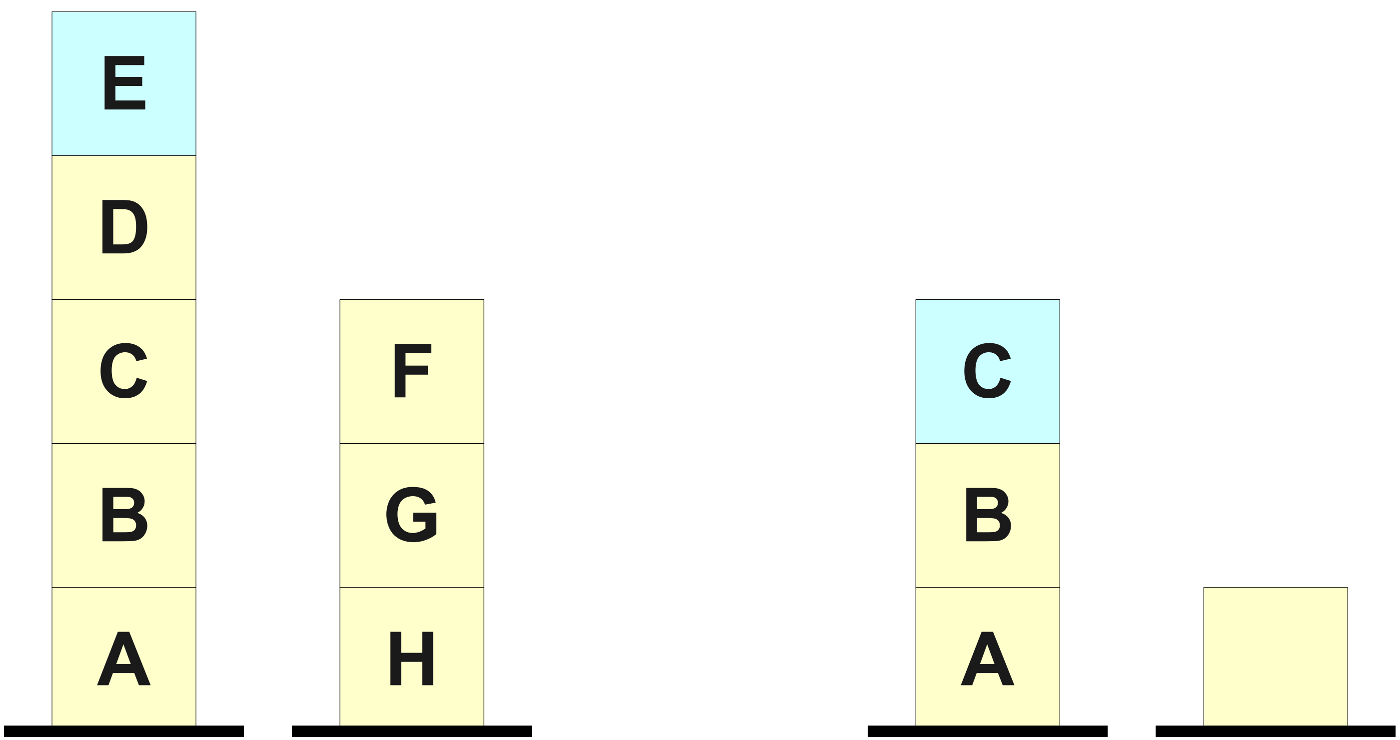

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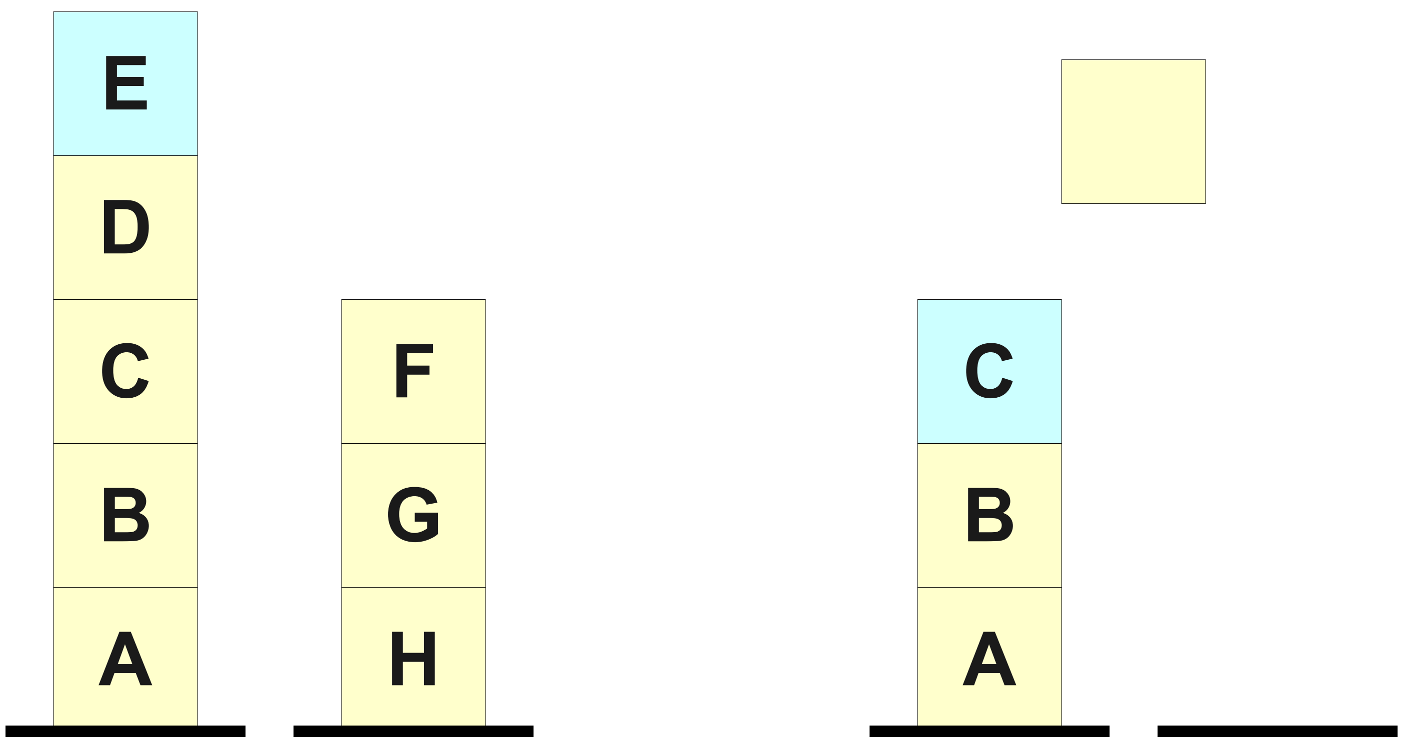

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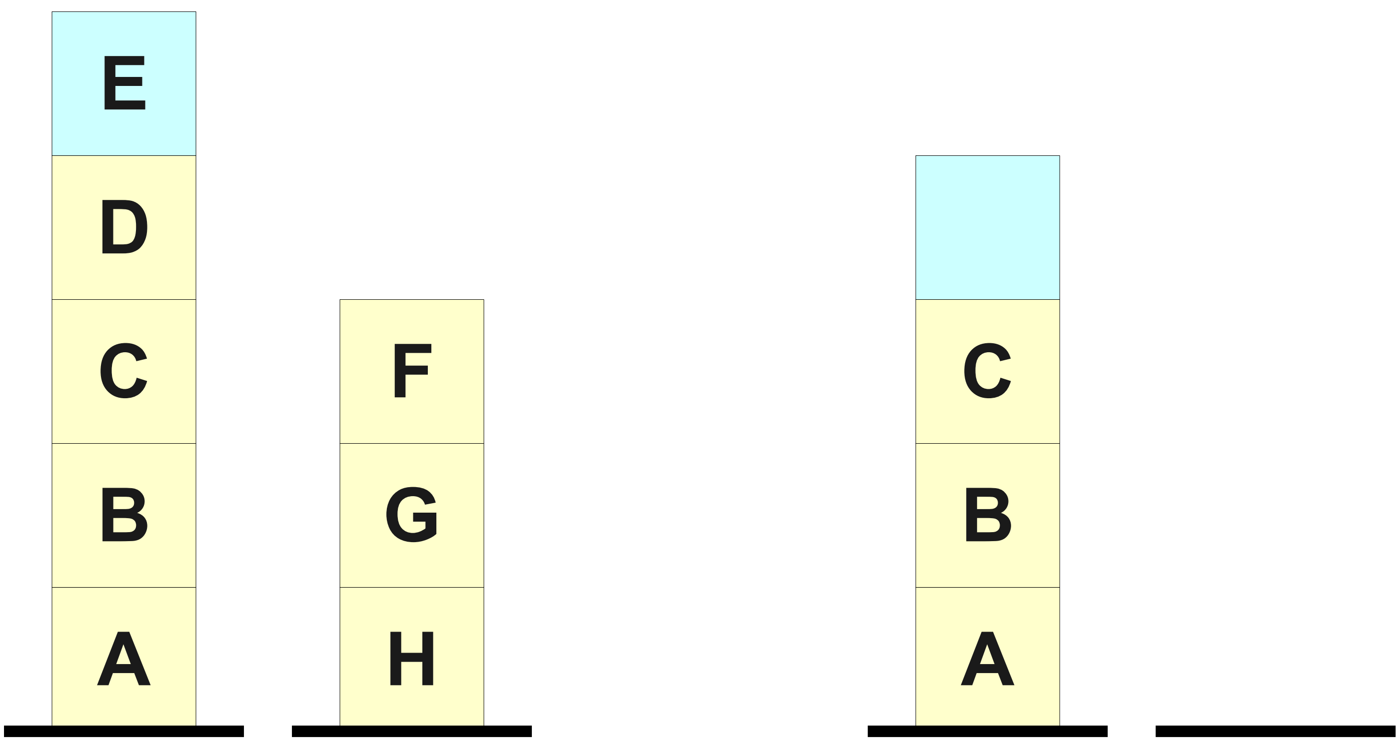


What Else Can We Add?

- Function call and return.
  - Have a stack to use as the call stack.
  - Calling a function pushes the index of the instruction to which it should return.
  - Returning pops the stack and jumps back.
- Named variables.
  - Have a tape storing a sequence of values of the form **`name: value`**.
  - Can read and write values from the tape.
- Pointers
  - Have variables hold the names of other variables.
- Primitive types and arithmetic.
  - Design subroutines for addition, subtraction, etc.
  - Apply them to named variables.
- **Pretty much any feature of any major programming language.**

The Conversion Back Down

- From **WB6** to **WB5**:
  - Add in two stacks per tape used.
  - Replace all tape operations with appropriate stack manipulations.
- From **WB5** to **WB4**:
  - Add in one track per stack, plus one extra track.
  - Replace all stack operations with appropriate manipulations of those tracks.

The Conversion Back Down

- From **WB4** to **WB3:**
  - Expand the tape alphabet to include symbols for all track combinations.
  - Replace all references to track symbols with cascading if's for each possible case.
- From **WB3** to **WB2:**
  - Replicate the code once for each possible assignment to variables.
  - Hardcode in statements referencing variables.
  - Replace variable manipulation code with code to jump to the appropriate copy.

The Conversion Back Down

- From **WB2** to **WB**:
  - Expand out **`move ... until`** statements by replacing them with cascading **`if`** statements.
- From **WB** to Turing machines:
  - Replace each statement with the appropriate Turing machine gadget.

The Conversion Back Down

- The total conversion of a WB6 program using variables, multiple tracks, multiple stacks, and multiple tapes might produce an *enormous* Turing machine!
- But that said, the result is still a Turing machine.
- **Turing machines are simple, yet have enormous computational power**.

Just how powerful **are** Turing machines?

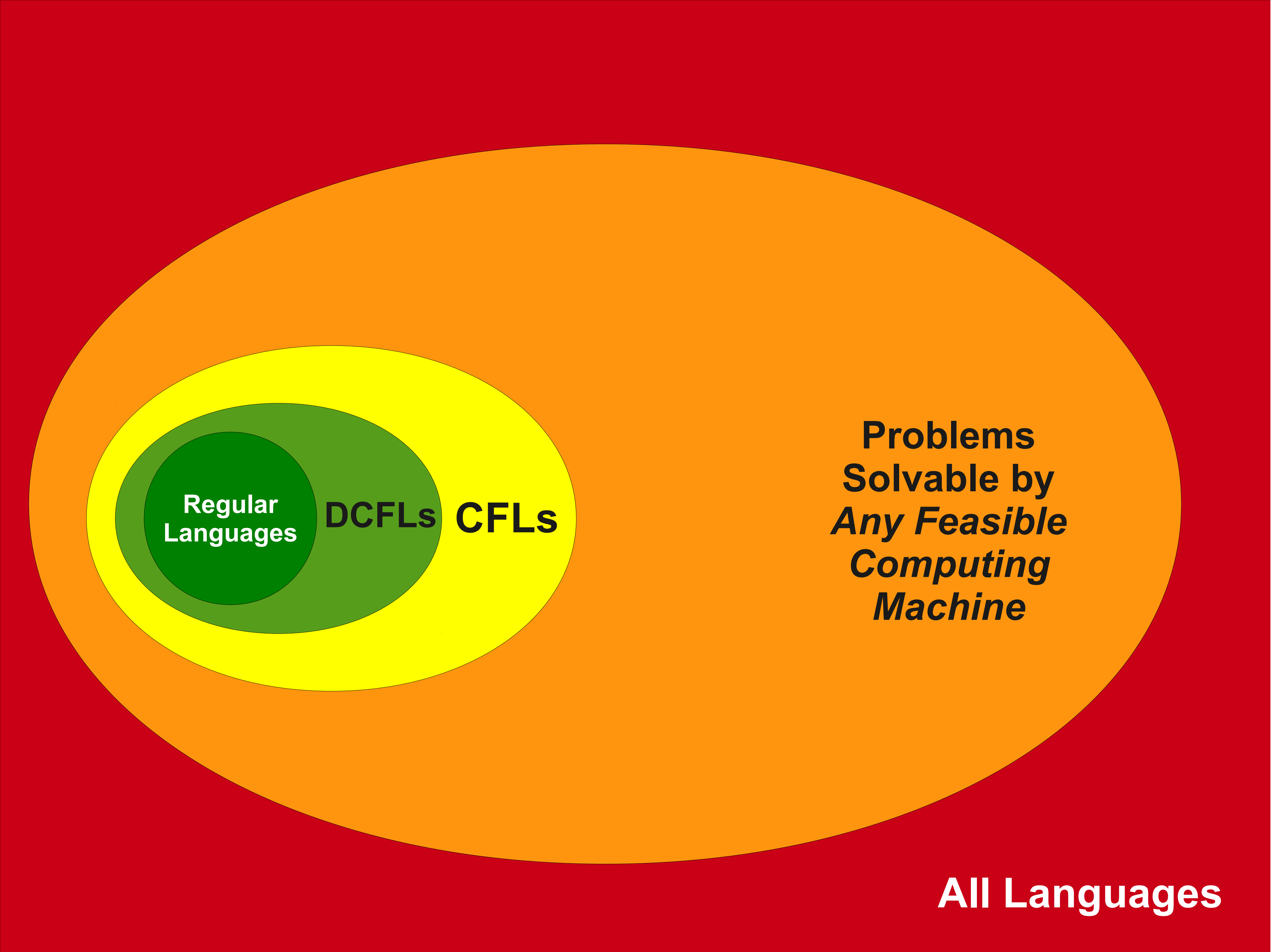
Effective Computation

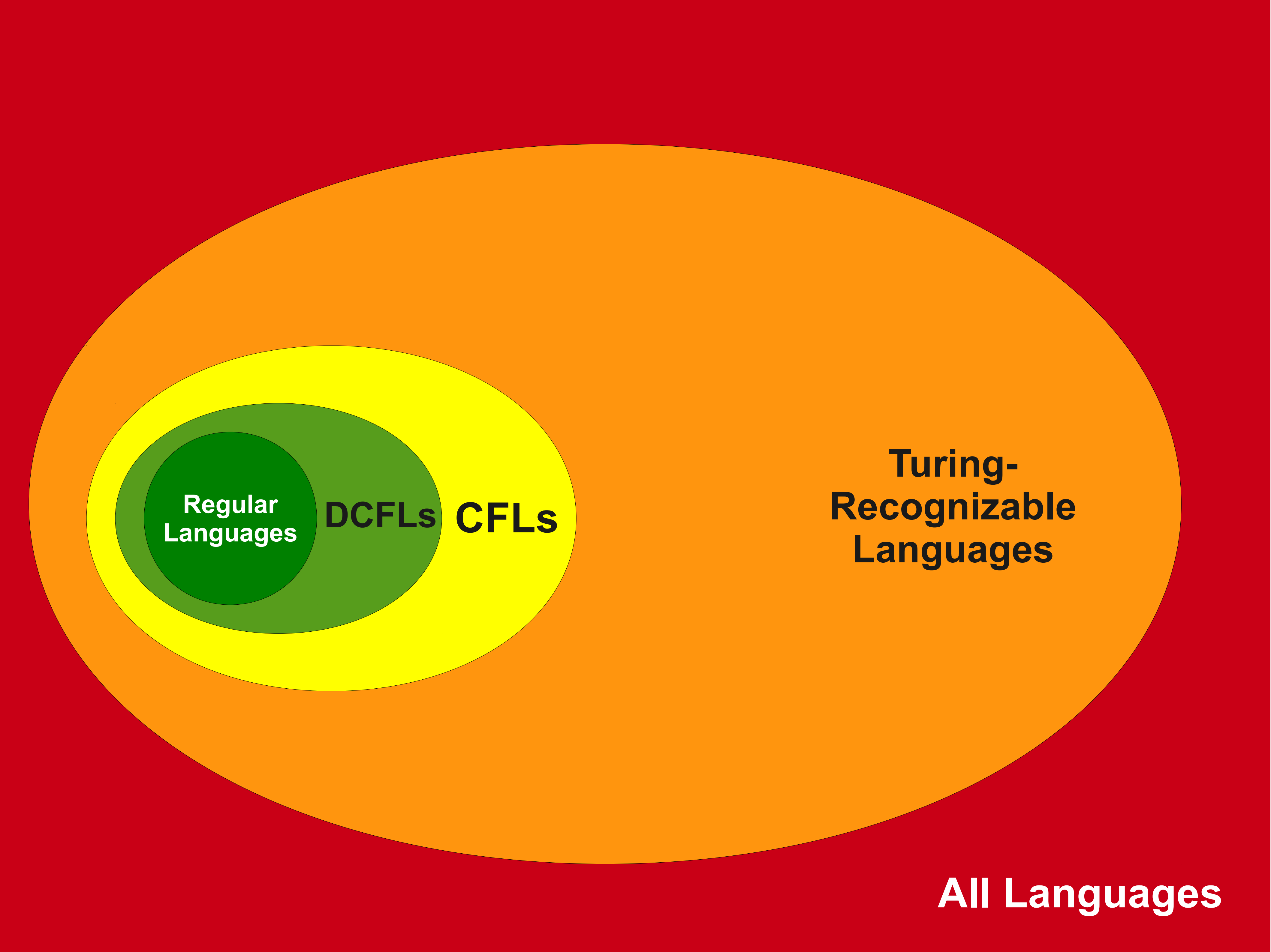
- An **effective method of computation** is a form of computation with the following properties:
  - The computation consists of a set of steps.
  - There are fixed rules governing how one step leads to the next.
  - Any computation that yields an answer does so in finitely many steps.
  - Any computation that yields an answer always yields the correct answer.

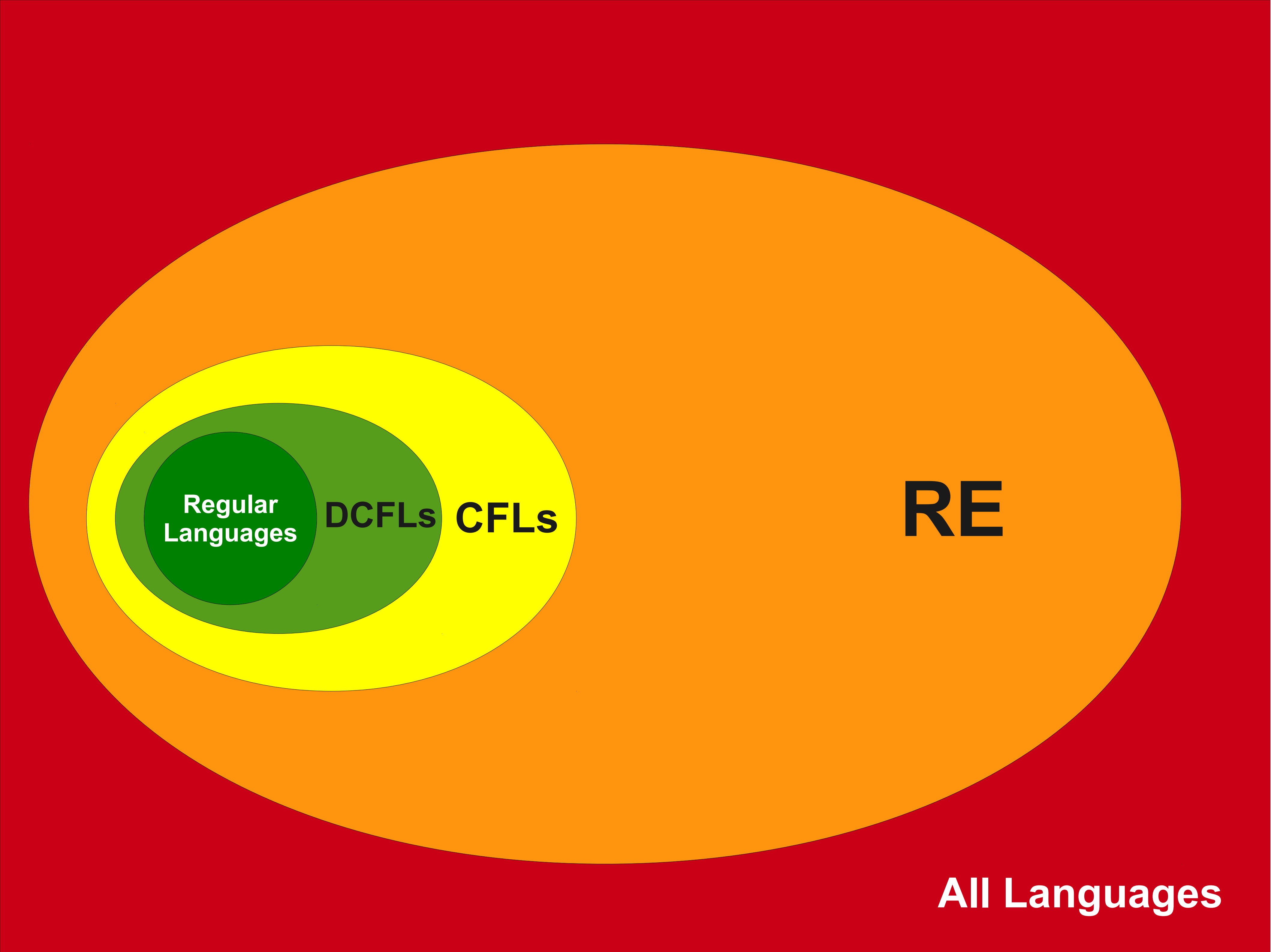
The **Church-Turing Thesis** states that

Every effective method of computation is either equivalent to or weaker than a Turing machine.

This statement cannot be proven or disproven, but is widely considered true.


Regular Languages
DCFLs
CFLs
Problems Solvable by *Any Feasible Computing Machine*
All Languages


Regular Languages
DCFLs
CFLs
Turing-Recognizable Languages
All Languages


Regular Languages
DCFLs
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RE
All Languages

Next Time

- **Encodings**
  - How do we do computations over arbitrary objects?
- **The Universal Turing Machine**
  - A Turing machine for running other Turing machines.
- **Nondeterministic Turing Machines**
  - What happens when we supercharge a TM? What does this even mean?
- **R and RE Languages**
  - A finer gradation within the **RE** languages.