Programming Turing Machines

Turing Machines are Hard

	1			×			=			B		
q_s	B	R	q_1	B	R	$q_\times$	B	R	$q_=$	B	R	q_{s2}
q_1	1	R	q_1	1	R	$q_\times$	1	R	$q_=$	1	L	q_R
$q_\times$	×	R	q_1	×	R	$q_\times$	×	R	$q_=$	×	L	q_R
$q_=$	=	R	q_1	=	R	$q_\times$	=	R	$q_=$	=	L	q_R
q_R	1	L	q_R	×	L	q_R	=	L	q_R	B	R	q_{s2}
q_{s2}	1	R	q_{s2}	×	R	$q_{\times 2}$	Reject			Reject		
$q_{\times 2}$	1	R	$q_{\times 2}$	Reject			=	R	$q_{=2}$	Reject		
$q_{=2}$	1	R	$q_{=2}$	Reject			Reject			B	L	q_{L2}
q_{L2}	1	L	q_{L2}	×	L	q_{L2}	=	L	q_{L2}	B	R	**D**

Outline for Today

- **A programming language for Turing machines.**
- Design a simple programming language that “compiles” down to Turing machines.
- Keep extending our language to see just how powerful the Turing machine is.

Our Initial Language: **WB**

- Programming language **WB** ("Wang B-machine") controls a tape head over a singly-infinite tape, as in a normal Turing machine.
- Language has six commands:
  - **`Move`** ***direction***
    - Moves the tape head the specified direction (either left or right)
  - **`Write`** ***s***
    - Writes symbol ***s*** to the tape.
  - **`Go to`** ***N***
    - Jumps to instruction number *N* (all instructions are numbered)
  - **`If reading`** ***s*****`, go to`** ***N***
    - If the current tape symbol is ***s***, jump to the instruction numbered ***N***.
  - **`Accept`** and **`Reject`**
    - Ends the program.
- Statements in **WB** are executed in the order in which they appear, unless control flow changes.

A Simple Program in **WB**

```
0: If reading B, go to 4.
1: If reading 1, go to 5.
2: Move right.
3: Go to 0.
4: Accept.
5: Reject.
```

A **WB** Program for Even Palindromes

- Suppose we want to test if a string is an even-length palindrome.
- Idea: Cross off the first symbol and match it with the symbol on the far side of the tape.
- If it matches, great! Repeat.
- Otherwise, we should reject.

A **WB** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.

// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
13: If reading 1, go to 11.
14: Move left.
15: If reading 1, go to Next.
16: Reject.

// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
20: If reading 1, go to 18
21: Move right
22: Go to Start.
```

WB and Turing Machines

- *Recall:* A language L is **recursively enumerable** iff there is a TM for it.
- **Theorem:** A language L is recursively enumerable iff there is a **WB** program for it.
- Need to show the following:
  - Any TM can be converted into an equivalent **WB** program.
  - Any **WB** program can be converted into an equivalent TM.

From Turing Machines to **WB**

- Basic idea: Construct a small **WB** program for each state that simulates that state.
- Combine all programs together to get an overall **WB** program that simulates the Turing machine.

A State in a Turing Machine

- There are three kinds of states in a Turing machine:
  - Accepting states,
  - Rejecting states, and
  - "Working" states.
- We can easily build **WB** programs for the first two:

```
// q_acc
0: Accept
```

```
// q_rej
0: Reject
```

Working States

- At a given working state in a Turing machine, we will do exactly the following, in this order:
  - Read the current symbol.
  - Write back a new symbol based on this choice of symbol.
  - Transition to some destination state.
- Could we build a **WB** program for this?

Working States

	0			1			B		
q_0	B	R	q_1	0	L	q_0	B	R	q_{acc}

```
// q₀
 0: If reading 0, go to 0q₀.
 1: If reading 1, go to 1q₀.
 2: If reading B, go to Bq₀.

// 0q₀
 3: Write B.
 4: Move right.
 5: Go to q₁

// 1q₀
 6: Write 0
 7: Move left.
 8: Go to q₀

// Bq₀
 9: Write B
10: Move right.
11: Go to q_acc
```

A Complete Construction

	0			1			B		
q_0	0	R	q_1	1	R	q_{rej}	1	R	q_{acc}
q_1	0	R	q_{rej}	1	R	q_0	1	R	q_{acc}

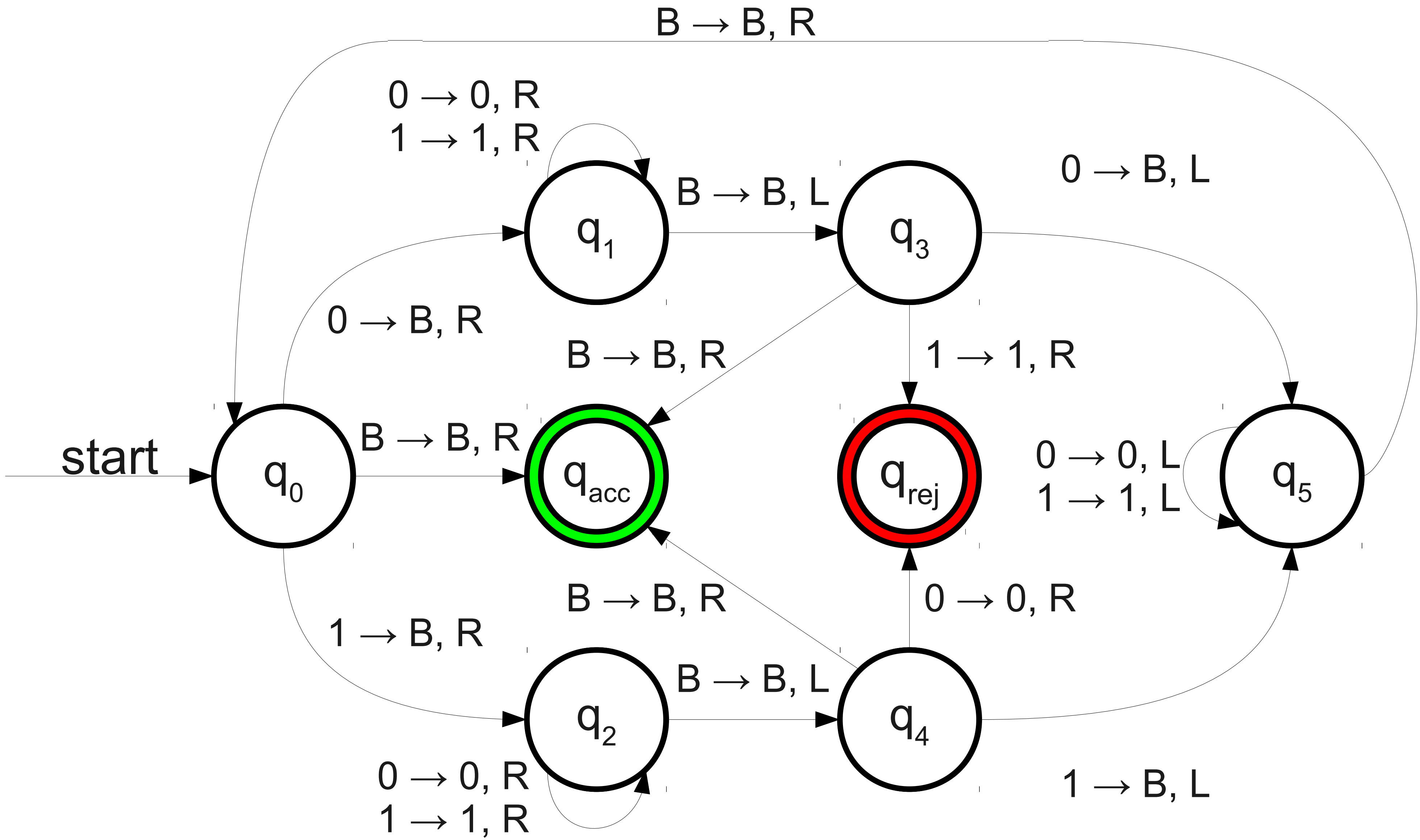
```
// q_0
 0: If reading 0, go to 3.
 1: If reading 1, go to 6.
 2: If reading B, go to 9.
 3: Write 0.
 4: Move right.
 5: Go to q_1.
 6: Write 1.
 7: Move right.
 8: Go to q_rej.
 9: Write 1.
10: Move right.
11: Go to q_acc.

// q_acc
12: Accept.
```

```
// q_1
13: If reading 0, go to 16.
14: If reading 1, go to 19.
15: If reading B, go to 22.
16: Write 0.
17: Move right.
18: Go to q_rej.
19: Write 1.
20: Move right.
21: Go to q_0.
22: Write 1.
23: Move right.
24: Go to q_acc.

// q_rej
25: Reject.
```

From **WB** to Turing Machines

- We now need a way to convert a **WB** program into a Turing machine.
- Construction sketch:
  - Create a state in the TM for each line of the **WB** program.
  - Introduce extra "helper" states to implement some of the trickier instructions.
  - Connect the states by transitions that simulate the **WB** program.
- We will show how to translate each **WB** command into a collection of states plus transitions.

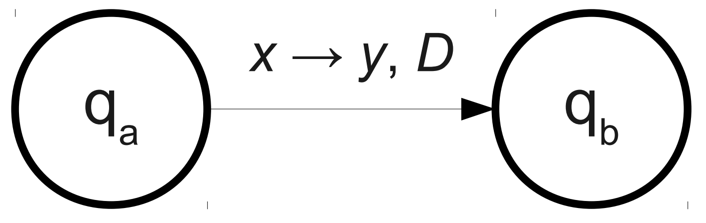
Refresher: Turing Machine Notation



Refresher: Turing Machine Notation

- The accept and reject states are denoted

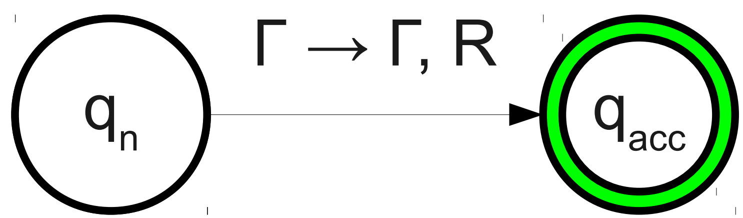
- A transition of the form



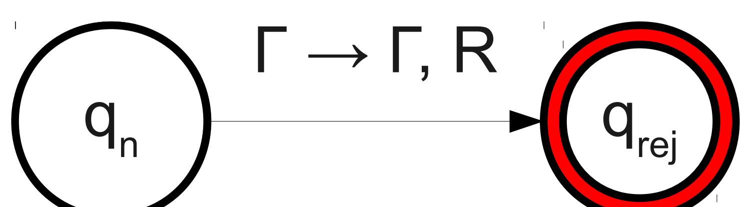
means "on seeing x, write y and move direction D."

`Accept` and **`Reject`**

- The **`Accept`** and **`Reject`** commands are the easiest to translate.
- To translate **`N: Accept`** into TM states, construct the following:

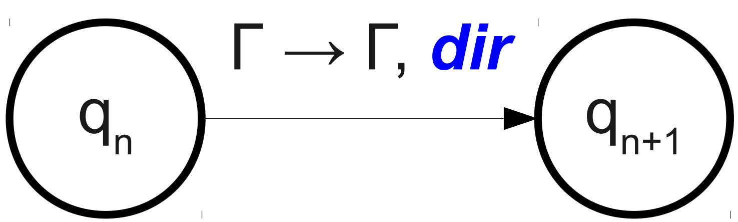


- To translate **`N: Reject`** into TM states, construct the following:



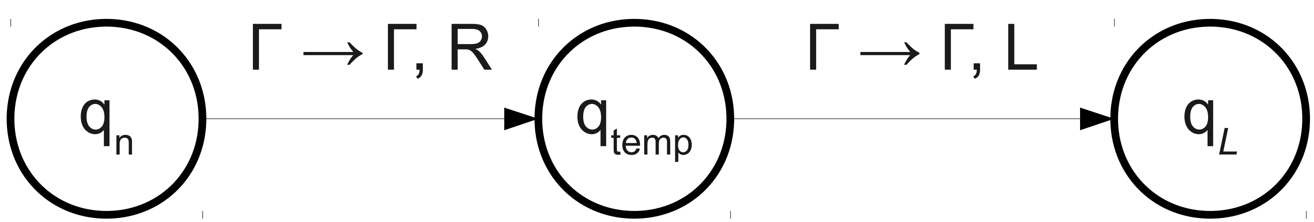
`Move left` and `Move right`

- We can translate **`N: Move left`** and **`N: Move right`** by having the TM do the following:
  - Write back the same symbol that was already on the tape (ensuring that we don't change the tape).
  - Move in the indicated direction.
  - Transition into the state representing line **`N`** + 1.



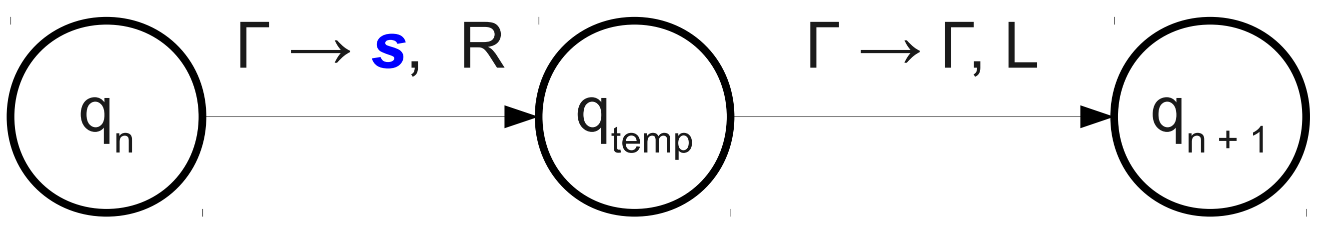
Go to *L*

- The line **`N: Go to M`** needs to change into the state for line **`M`** without moving the tape head.
- All TM transitions move the tape head; how might we address this?
- Move right and change into a new state that then moves back to the left.



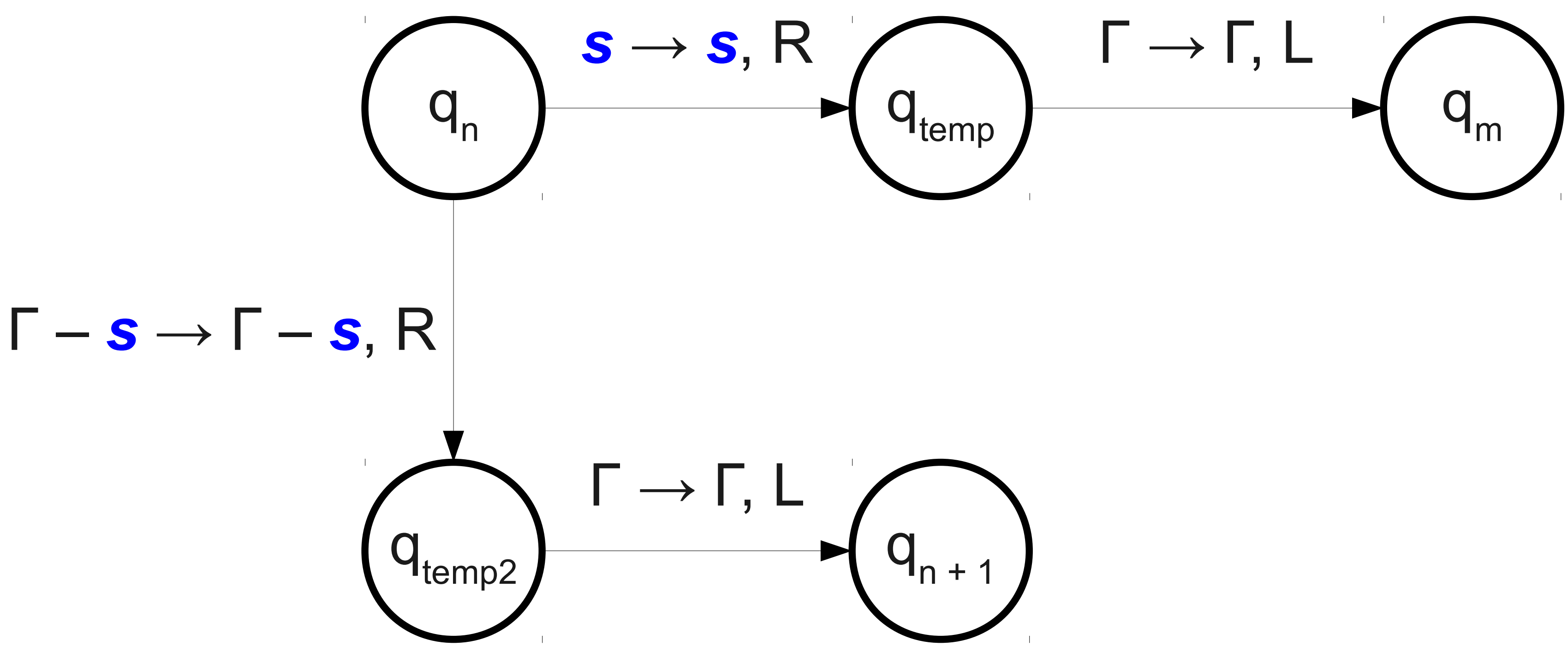
`Write` s

- The line **`N: Write`** s needs to
  - Write the symbol s,
  - Leave the tape head where it is, and
  - Move to line **`N`** + 1.
- We use a similar trick as before:

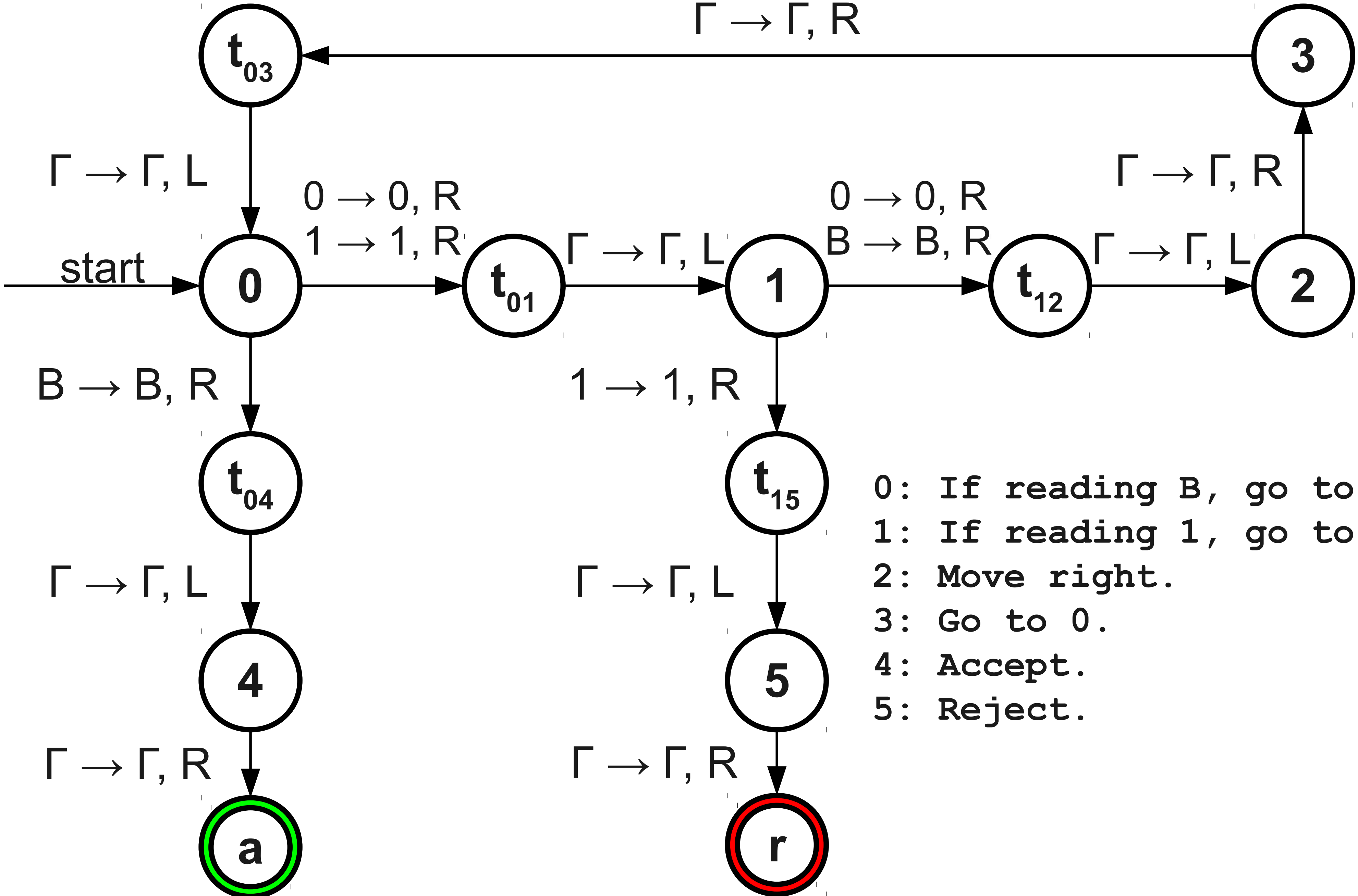


If reading *s*, go to M

- The line **N: If reading *s*, go to M** either
  - Executes a "go to **M**" step as before if reading ***s***, or
  - Does nothing and transitions to state **N** + 1.



A Complete Conversion




```
0: If reading B, go to 4.
1: If reading 1, go to 5.
2: Move right.
3: Go to 0.
4: Accept.
5: Reject.
```

The Story So Far

- We have just built a simple programming language that is equivalent in power to a Turing machine.
- This language, however, makes for some very complicated programs.
- Let's add some new features to our programming language to make it a bit easier to work with.

Revisiting Even Palindromes

```
// M0
3: Write B.
4: Move right.
5: If reading 0, go to 4.
6: If reading 1, go to 4.
7: Move left.
8: If reading 0, go to Next.
9: Reject.
```

```
// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
20: If reading 1, go to 18
21: Move right.
22: Go to Start.
```

- Steps 4 – 6 essentially say “move right, then move right until you read a blank.”
- Steps 18 – 20 essentially say “move left, then move left until you read a blank.”
- Is it really necessary to write this out each time?

Introducing **WB2**

- The programming language **WB2** is the language **WB** with two new commands:
  - **`Move left until {`**$\boldsymbol{s_1, s_2, \ldots, s_n}$**`}.`**
    - Moves the tape head left until we read one of $\boldsymbol{s_1, s_2, s_3, \ldots, s_n}$.
  - **`Move right until {`**$\boldsymbol{s_1, s_2, \ldots, s_n}$**`}.`**
    - Moves the tape head right until we read one of $\boldsymbol{s_1, s_2, s_3, \ldots, s_n}$.
- Both commands are no-ops if we're already reading one of the specified symbols.
- We can write programs in **WB2** that are much easier to read than in **WB**.

A **WB** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: If reading 0, go to 4.
 6: If reading 1, go to 4.
 7: Move left.
 8: If reading 0, go to Next.
 9: Reject.

// M1
10: Write B.
11: Move right.
12: If reading 0, go to 11.
13: If reading 1, go to 11.
14: Move left.
15: If reading 1, go to Next.
16: Reject.

// Next
17: Write B.
18: Move left.
19: If reading 0, go to 18
20: If reading 1, go to 18
21: Move right
22: Go to Start.
```

A **WB2** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: Move right until {B}.
 6: Move left.
 7: If reading 0, go to Next.
 8: Reject.
```

```
// M1
 9: Write B.
10: Move right.
11: Move right until {B}.
12: Move left.
13: If reading 1, go to Next.
14: Reject.

// Next
15: Write B.
16: Move left.
17: Move left until {B}.
18: Move right.
19: Go to Start.
```

A **WB2** Program for *BALANCE*

- Let $\Sigma = \{$ `0`, `1` $\}$ and consider the language *BALANCE*:

  $\{ w \in \Sigma^* \mid w$ has the same number of `0`s and `1`s. $\}$

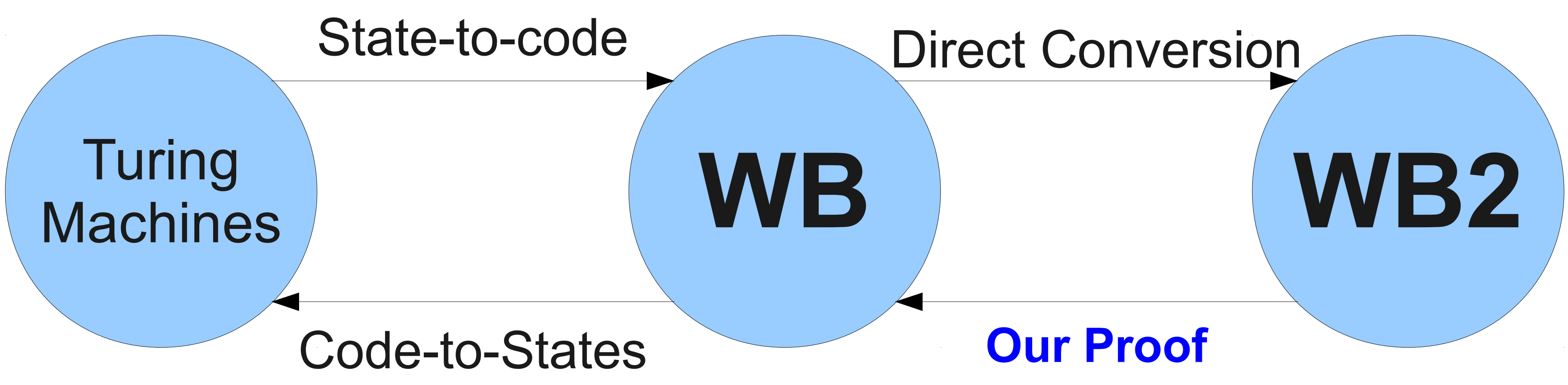
- Let's write a **WB2** program for *BALANCE*.

A **WB2** Program for *BALANCE*

```
// Start
 0: Move right until {0, 1, B}.
 1: If reading 0, go to Match0.
 2: If reading 1, go to Match1.
 3: Accept.

// Match0
 4: Write B.
 5: Move right.
 6: Move right until {1, B}.
 7: If reading 1, go to Found.
 8: Reject.

// Match1
 9: Write B.
10: Move right.
11: Move right until {0, B}.
12: If reading 0, go to Found.
13: Reject.

// Found
14: Write x.
15: Move left until {B}.
16: Move right.
17: Go to Start.
```

WB2 and Turing Machines

- **Theorem:** A language is recursively enumerable iff there is a **WB2** program for it.
- We could directly prove this again by showing equivalence with Turing machines.
- Instead, we'll connect it to **WB**:



From **WB2** to **WB**

- We will show how to turn any **WB2** program into an equivalent **WB** program.
- All old instructions are still valid.
- We need to show how to implement the new **`Move`** … **`until`** commands using just **WB**.

Implementing `Move ... until`

- Replace `N: Move` *dir* `until {`s_1`, ..., `s_n`}` as follows:

```
N+0:      If reading s1, go to N+n+2.
N+1:      If reading s2, go to N+n+2.
N+2:      If reading s3, go to N+n+2.
...
N+(n-1):  If reading sn, go to N+n+2.
N+n:      Move dir.
N+n+1:    Go to N
```

- Renumber other lines as appropriate.

Why This Matters

- We are starting to move more and more away from the Turing machine with from we started.
- The structure of our approach is
  - Find some simple programming language that can be directly translated into a Turing machine (and vice-versa).
  - Add new features to the language, and show how to implement those new features using the old language.
  - Add new features to *that* language, and show how to implement those features using the previous language.
  - (etc.)
  - Conclude that the final language is equivalent to a Turing machine.

A Repeating Pattern

```
// Match0
 4: Write B.
 5: Move right.
 6: Move right until {1, B}.
 7: If reading 1, go to Found.
 8: Reject.
```

```
// Match1
 9: Write B.
10: Move right.
11: Move right until {0, B}.
12: If reading 0, go to Found.
13: Reject.
```

A Simple Memory

- Right now, our programming language **WB2** has no variables in it.
- To solve larger classes of problems, let's invent a new language **WB3** that has support for variables.
- We will severely limit the scope of our variables:
  - Only **finitely many** total variables throughout the program.
  - Each variable can only hold a single tape symbol.
  - Each variable initially holds the blank symbol.

Our New Commands

- We will define **WB3** as **WB2** with the following extra commands:
  - **`Load`** ***s*** **`into`** ***v*****.**
    - Sets the variable ***v*** equal to tape symbol ***s***.
  - **`Load current into`** ***v*****.**
    - Sets the variable ***v*** equal to the currently-scanned tape symbol.
  - **`If`** v_1 **`=`** v_2**`, go to`** ***L*****.**
    - If v_1 and v_2 have the same value, go to instruction ***L***.
    - These may be constants or variables.
- Additionally, any command that referenced a tape symbol (for example, **`write`**, **`if reading`**, **`move`** … **`until`**) can refer to variables in addition to constants.

A **WB2** Program for Even Palindromes

```
// Start
 0: If reading 0, go to M0.
 1: If reading 1, go to M1.
 2: Accept

// M0
 3: Write B.
 4: Move right.
 5: Move right until {B}.
 6: Move left.
 7: If reading 0, go to Next.
 8: Reject.
```

```
// M1
 9: Write B.
10: Move right.
11: Move right until {B}.
12: Move left.
13: If reading 1, go to Next.
14: Reject.

// Next
15: Write B.
16: Move left.
17: Move left until {B}.
18: Move right.
19: Go to Start.
```

A **WB3** Program for Even Palindromes

```
// Start
 0: Read current into X.
 1: If X = B, go to Acc.
 2: Write B.
 3: Move right.
 4: Move right until {B}.
 5: Move left.
 6: If reading X, go to Match.
 7: Reject.

// Match
 8: Write B.
 9: Move left.
10: Move left until B.
11: Move right.
12: Go to Start.

// Acc:
13: Accept.
```

A **WB2** Program for *BALANCE*

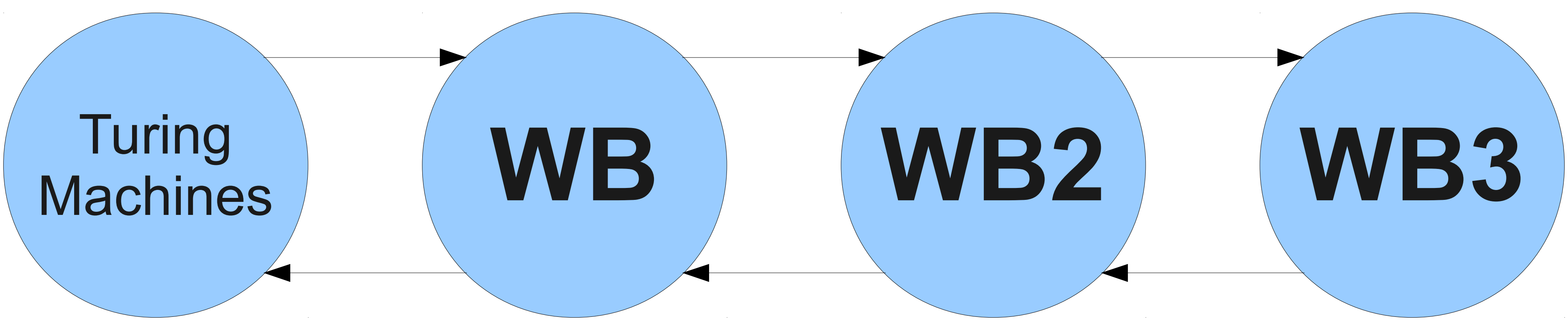
```
// Start
 0: Move right until {0, 1, B}.
 1: If reading 0, go to Match0.
 2: If reading 1, go to Match1.
 3: Accept.

// Match0
 4: Write B.
 5: Move right.
 6: Move right until {1, B}.
 7: If reading 1, go to Found.
 8: Reject.

// Match1
 9: Write B.
10: Move right.
11: Move right until {0, B}.
12: If reading 0, go to Found.
13: Reject.

// Found
14: Write x.
15: Move left until {B}.
16: Move right.
17: Go to Start.
```

A **WB3** Program for *BALANCE*

```
// Start
 0: Move right until {0, 1, B}.
 1: If reading B, go to Acc.
 2: If reading 0, go to 5.
 3: Load 0 into Y.
 4: Go to Scan.
 5: Load 1 into Y.
 6: Go to Scan.

// Scan
 8: Write B.
 9: Move right.
10: Move right until {Y, B}
11: If reading Y, go to 13.
12: Reject.
13: Write x.
14: Move left until B.
15: Move right.
16: Go to Start.

// Acc:
17: Accept.
```

Equivalence of **WB2** and **WB3**

- **Theorem:** A language is recursively enumerable iff there is a **WB3** program for it.
- Adding in these sorts of variables adds *no power* to our model of computation!
- To prove the theorem, we will show
  - Any WB2 program can be converted to a WB3 program, and
  - Any WB3 program can be converted to a WB2 program.



The Proof: An Intuition

- Our programs allow only finitely many variables holding only one of finitely many different values (tape symbols).
- We could just **replicate the program** for each possible assignment to the variables, then hardcode in the behavior in each of these cases.
- Could make the program staggeringly huge, but it will still be finite!

The Transformation, Part I

- Let V_1, V_2, …, V_n be the variables referenced in the program.
  - We can just look at the source code to determine this.
- Make $|\Gamma|^n$ copies of the initial program, one for each possible assignment of tape symbols to the variables V_i.
- Order the copies arbitrarily, but make the version where all variables hold B come first.

The Transformation, Part II

- We now have a whole bunch of copies of **WB3** programs.
- We need to convert them into legal **WB2** programs.
- This works in two steps:
  - Removing variables from older **WB2** commands like **`Write`**, **`If reading`** ..., and **`Move`** ... **`while`**.
    - For example: "**`Write`** ***`X`***," where ***`X`*** is a variable.
  - Rewriting all new **WB3** commands that reference variables to use only **WB2** commands.
    - For example: "**`Load current into`** ***`X`***."

Eliminating Variables from **WB2**

- Removing variables from purely **WB2** statements is easy because we've copied the program so many times.
- For each copy, replace all variables in **WB2** statements with the value that the variable has in that copy.

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

```
0: Load 0 into Y.
1: Write B.
2: Accept
```

Y = B

```
3: Load 0 into Y.
4: Write 0.
5: Accept
```

Y = 0

```
6: Load 0 into Y.
7: Write 1.
8: Accept
```

Y = 1

Eliminating Variables from **WB3**

- We can eliminate commands that manipulate variables by replacing them with **`Go to`**s.
- There are three commands to eliminate:
  - **`Load`** ***s*** **`into`** ***v*****`.`**
  - **`Load current into`** ***v*****`.`**
  - **`If`** v_1 **`=`** v_2**`, go to`** ***L*****`.`**

`If` v_1 `=` v_2`, go to` L

- We can eliminate this statement by just hardcoding the jump in place.
- If in the current copy of the program v_1 and v_2 have the same values, replace with

  `Go to` L

  where L is the corresponding version of L in this copy.
- Otherwise, replace with

  `Go to` N

  where N is the number of the next line in the program.

Load *s* into *v*

- To simulate the effect of loading ***s*** into ***v***, we can jump out of the current copy of the program into the copy where ***v*** has value ***s***.

```
0: Load 0 into Y.
1: Write Y.
2: Accept
```

```
0: Go to 4.
1: Write B.
2: Accept
```

Y = B

```
3: Go to 4.
4: Write 0.
5: Accept
```

Y = 0

```
6: Go to 4.
7: Write 1.
8: Accept
```

Y = 1

Load current into *v*

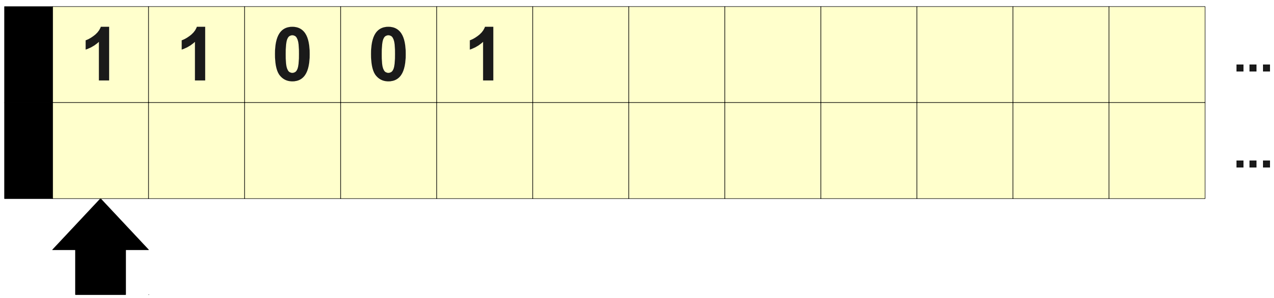
- We can simulate this instruction using a similar trick to before.
- Replace this instruction as follows:

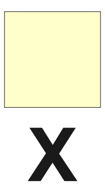
```
If reading s₁, go to LoadS₁.
If reading s₂, go to LoadS₂.
…
If reading sₙ, go to LoadSₙ.
// LoadS₁:
Load s₁ into v.
Go to Done.
…
// LoadSₙ
Load sₙ into v.
Go to Done.
// Done:
```

Souping up our Tape

- Up to this point, we've been improving our **WB** programming language by adding in new ways of scanning over the tape.
- What if we made changes to the tape itself?

A Multitrack Tape





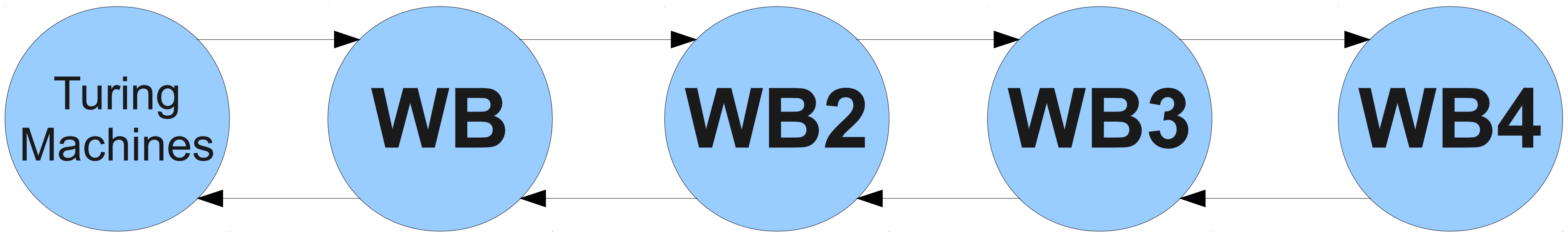

```
// Start
 0: Read track 1 into X.
 1: Move right.
 2: Write X into track 2
 3: If reading B on track 1, go to 5.
 4: Go to 0
 5: /* … */
```

Introducing **WB4**

- Let's define **WB4** to be **WB3** with the introduction of finitely many **tracks** on the tape.
- The tape head still moves as a unit to the left or right, but we can now issue read and write commands to any cell in the current track.
- All previous commands updated to specify which track is to be read or written.

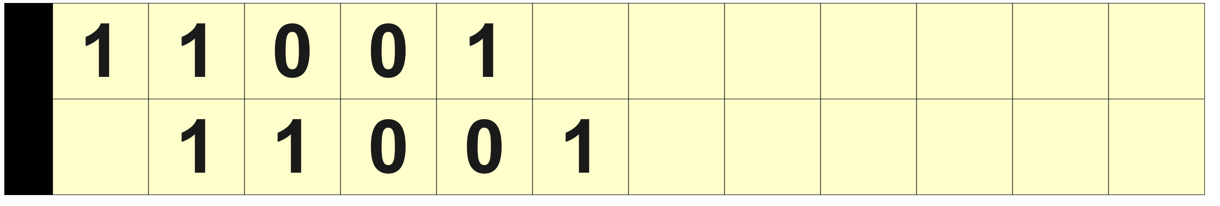
A Surprising Theorem

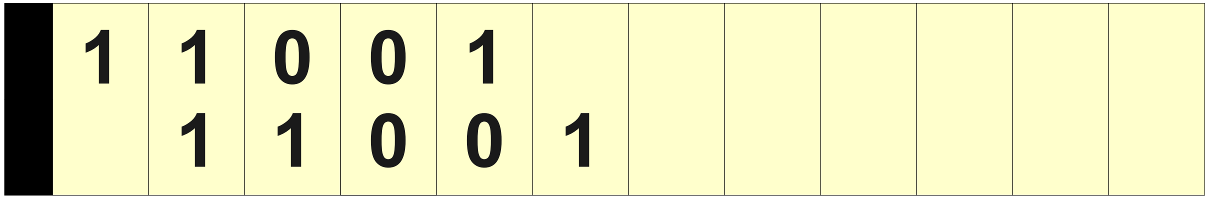
- **Theorem:** A language is recursively enumerable iff there is a **WB4** program for it.
- This is not obvious… it seems like adding in more tracks should increase the power of our programming language!
- As with before, will prove that all **WB4** programs are equivalent to **WB3** programs.



The Intuition

- Treat a single tape as a “fat tape” where each tape symbol encodes the contents of the cells of all four tracks.
- Each read or write to a specific location replaces the entire tape cell with a new symbol representing the change.



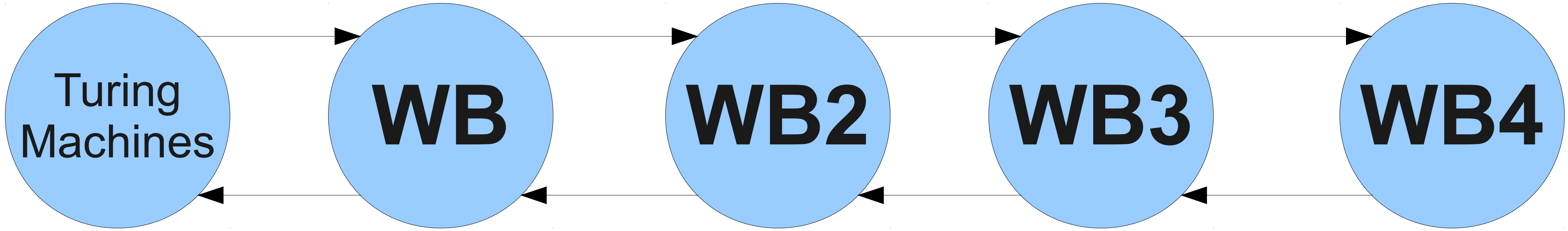


A Sketch of the Construction

- Replace each instruction that reads or writes a track with a huge cascading “if” that checks for every possible tape symbol and reacts accordingly.
- Can make the program enormously bigger, but it still ends up finite.
- I'm not even going to attempt to fit something like that onto these slides.

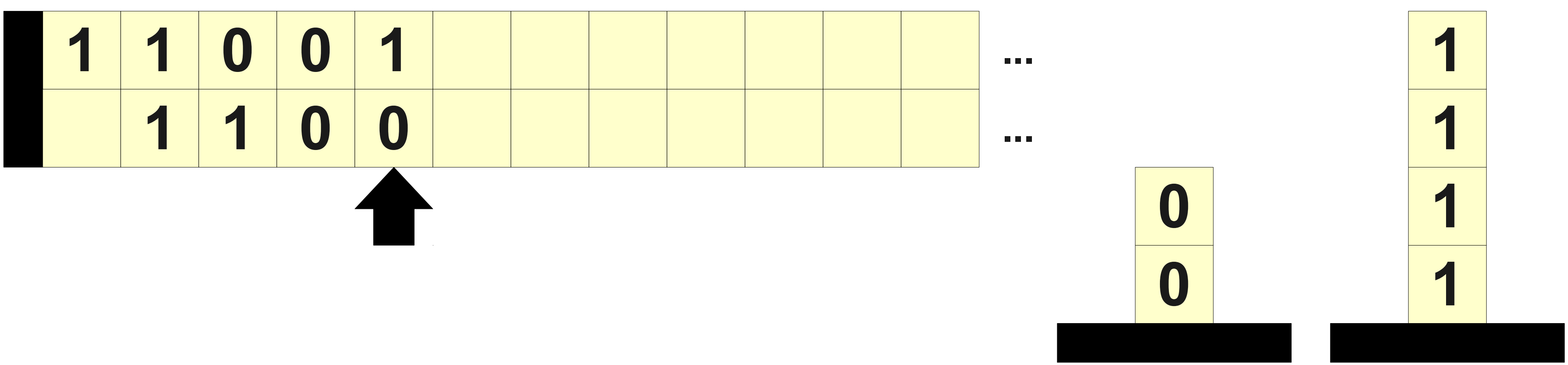
Where We Are Now

- Starting with **WB**, we have added
  - Loops to search for a value. (**WB2**)
  - Variables with finite storage. (**WB3**)
  - Multiple tracks. (**WB4**)
- Yet we still accept exactly the same set of languages.
- Every **WB*n*** program can be converted back to a TM.



Making Things Crazier

- What do you get when you combine a PDA and a **WB4** program?
- A program with an infinite tape, plus multiple stacks!



Introducing **WB5**

- The programming language **WB5** is the programming language **WB4** with the addition of a finite number of stacks.
- We add three extra commands:
  - **`Push` *s* `onto stack` *v*`.`**
    - Pushes the symbol ***s*** onto the stack named ***v***.
  - **`If stack` *v* `is empty, go to` *L*`.`**
    - If stack ***v*** is empty, go to instruction ***L***.
  - **`Pop stack` *v* `into` *w*`.`**
    - If stack ***v*** is nonempty, pops ***v*** and puts the top into ***w***.

The Multiplication Language

- Let $\Sigma = \{$ `0`, `1`, `2` $\}$ and consider the language *01MULT* defined as

  { $w \in \Sigma^*$ | the number of `2`'s in w is the product of the number of `1`'s and the number of `0`'s. }

- For example:
  - `00112222` $\in$ *01MULT*
  - `22001122122` $\in$ *01MULT*
- This language is neither context-free nor regular.
- How could we write a **WB5** program for it?

WB5 Program for *01MULTI*

```
// Start
 0: If reading 0, go to Load0.
 1: If reading 1, go to Load1.
 2: If reading 2, go to Load2.
 3: Go to Check.

// Load0
 4: Push 0 onto Stack 0.
 5: Move right.
 6: Go to Start.
```

```
// Load1
 7: Push 1 onto Stack 1.
 8: Move right.
 9: Go to Start.

// Load2
10: Push 2 onto Stack 2.
11: Move right.
12: Go to Start.
```

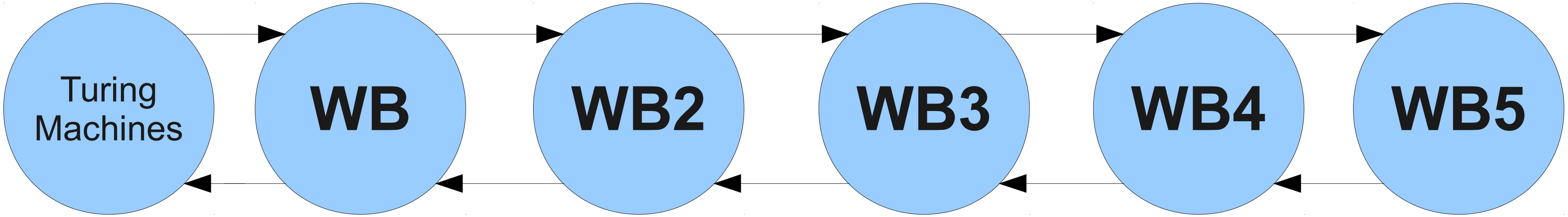
WB5 Program for *01MULTI*

```
// Check:
13: If Stack 0 is empty, go to Ver.
14: Pop Stack 0.
15: If Stack 1 is empty, go to Fix.
16: Pop Stack 1.
17: Push 1 onto Stack 1T.
18: If Stack 2 is empty, go to Rej.
19: Pop Stack 2.
20: Go to 15.

// Rej:
21: Reject.
```

```
// Fix:
22: If St 1T is empty, go to Check.
23: Pop Stack 1T.
24: Push 1 onto Stack 1.
25: Go to Fix.

// Ver:
26: If Stack 2 is empty, go to Acc.
27: Reject.

// Acc:
28: Accept.
```

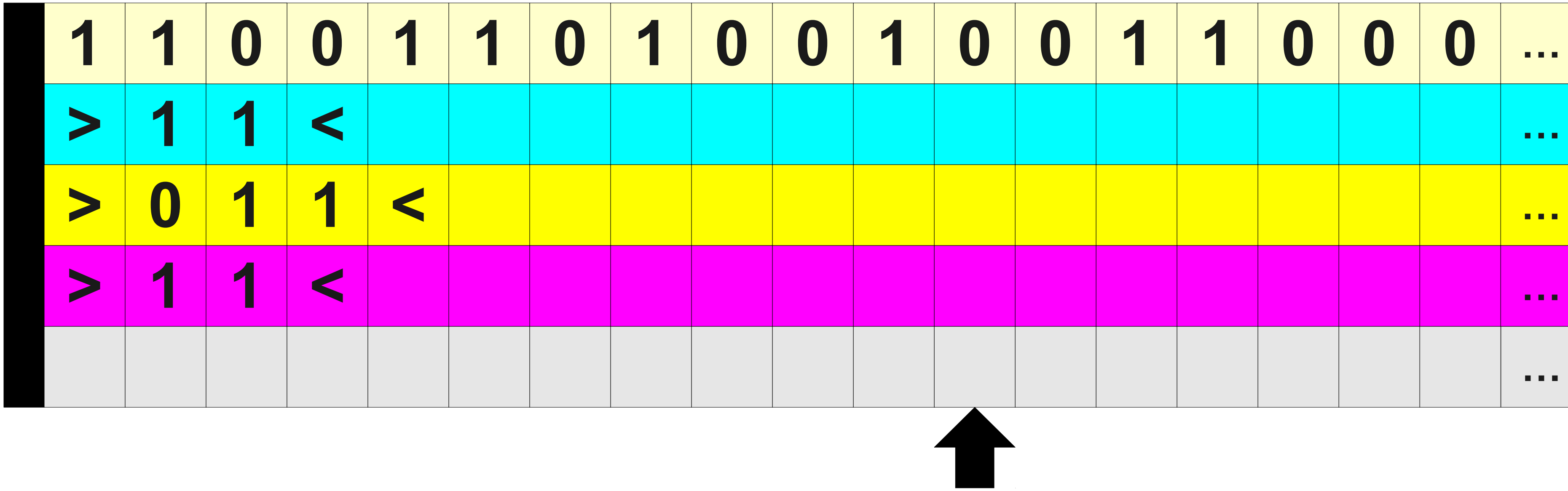
A Pretty Ridiculous Theorem

- **Theorem:** A language is recursively enumerable iff there is a **WB5** program for it.
- So adding in finitely many infinite stacks doesn't give us any more expressive power!
- As with before, will prove that all **WB5** programs are equivalent to **WB4** programs.

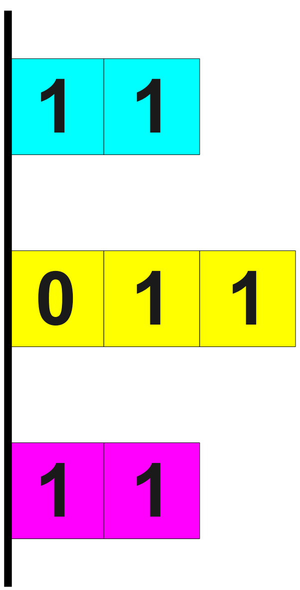


From Stacks to Tracks

- The key idea behind the construction for converting **WB5** programs into **WB4** programs is to represent each stack with its own track.
- If there are n stacks in the program, we will add $n + 1$ tracks:
  - One track for each of the n stacks, and
  - One track for bookkeeping.
- If the **WB5** program was using any tracks, we'll keep them as well and add these new ones in separately.



0: Push 1 onto Stack 3.



0: Write × on track 5.

1: Move left until {>} on track 4.

2: Move right until {<} on track 4.

3: Write 1 on track 4.

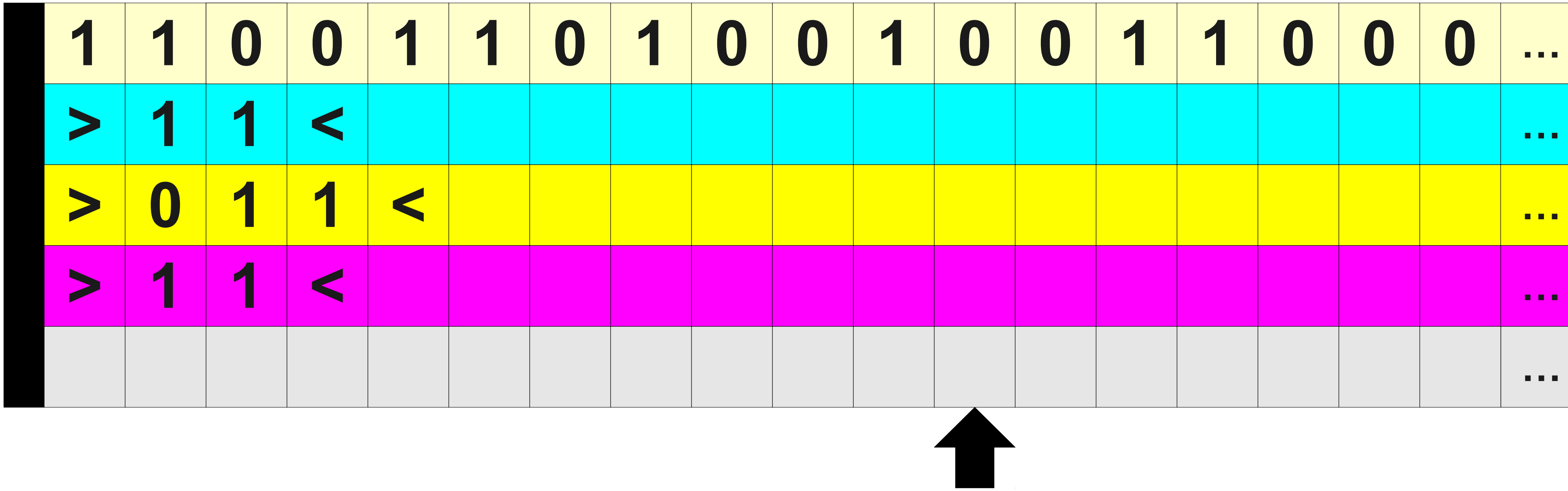
4: Move right.

5: Write < on track 4

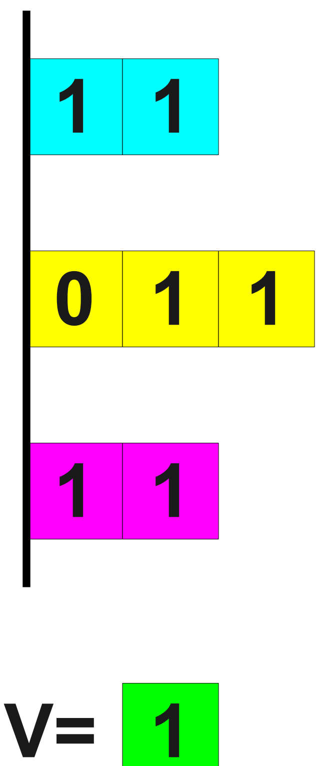
6: Move left until {>} on track 4.

7: Move right until {×} on track 5.

8: Write B on track 5.



1: If Stack 1 is empty, go to L



0: Write × on track 5.

1: Move left until {>} on track 2.

2: Move right.

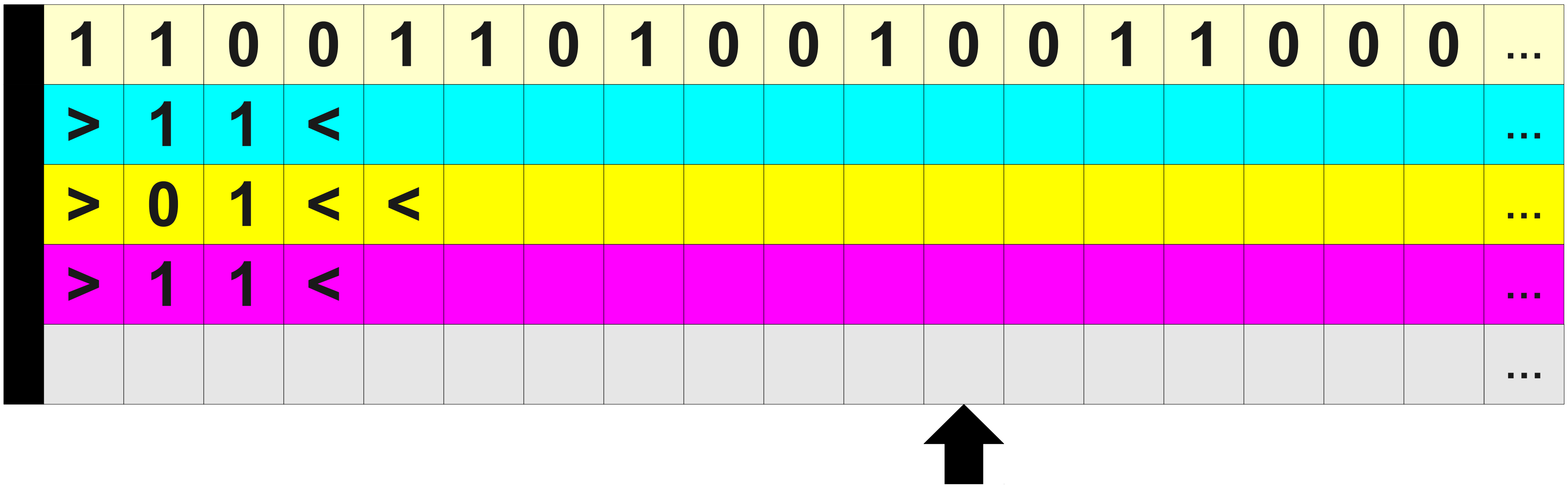
3: Load current on track 2 into V

4: Move left.

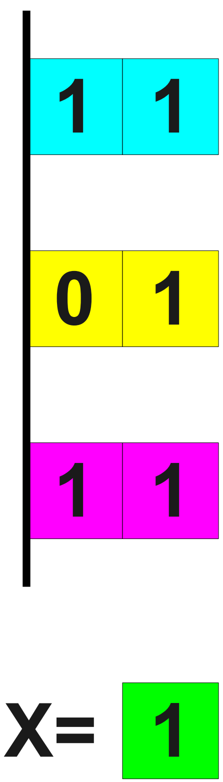
5: Move right until {×} on track 5.

6: Write B on track 5.

7: If V = <, go to L.



2: Pop Stack 2 into X.




```
0: Write × on track 5.
1: Move left until {>} on track 3.
2: Move right until {<} on track 3.
3: Move left.
4: Load current on track 3 into X.
5: If X = >, go to 7.
6: Write < on track 3
7: Move left until {>} on track 3.
8: Move right until {×} on track 5.
9: Write B on track 5.
```

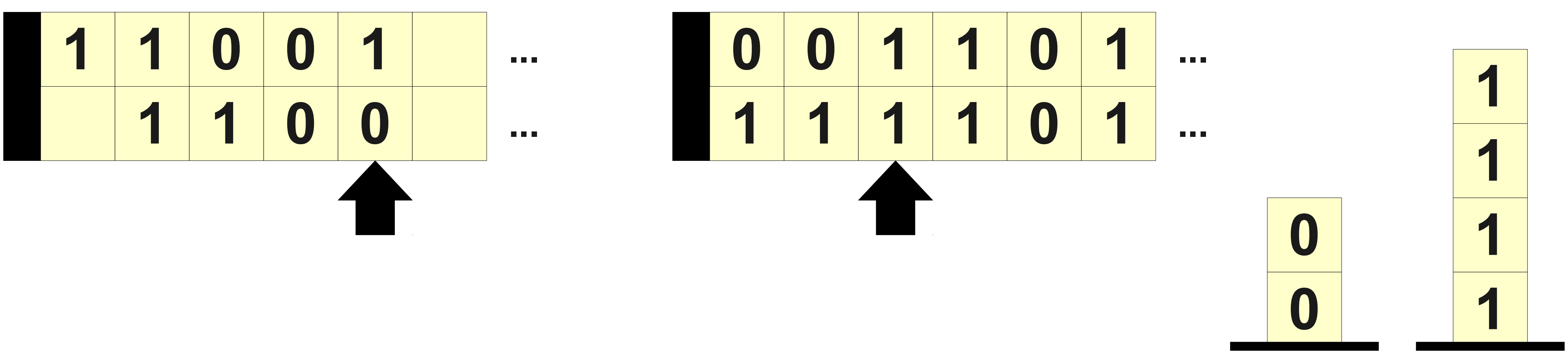
Completing the Construction

- We've seen how to convert the new **WB5** stack commands into **WB4** code.
- For this to work, the extra tracks must be set up correctly.
- Add preamble code to the generated **WB4** program to do this:

```
Write > to track 2.
...
Write > to track n.
Move right.
Write < to track 2.
...
Write < to track n.
Move left.
```

But Why Stop There?

- Adding finitely many stacks to **WB** doesn't increase its expressive power.
- What if we added finitely many **tapes** to **WB**?
- We now have a programming language controlling
  - Multiple tracks per tape,
  - Finitely many stacks, and
  - Finitely many tapes.



Introducing **WB6**

- The programming language **WB6** is **WB5** with the addition of multiple tapes.
- All tape commands have been updated to specify which tape they apply to.
- If tape unspecified, it's assumed that it's tape 1.

A **WB6** Program for *SEARCH*

- Recall from Problem Sets 5 and 6 that the language *SEARCH* over $\Sigma = \{\texttt{0}, \texttt{1}, \texttt{?}\}$ is the language

$$\{\ p\texttt{?}t \mid p, t \in \{\ \texttt{0}, \texttt{1}\ \}^* \text{ and } p \text{ is a substring of } t\ \}$$

- How would we write a **WB6** program for *SEARCH*?
- (For simplicity, we'll assume that the input is properly formatted).

A **WB6** Program for *SEARCH*

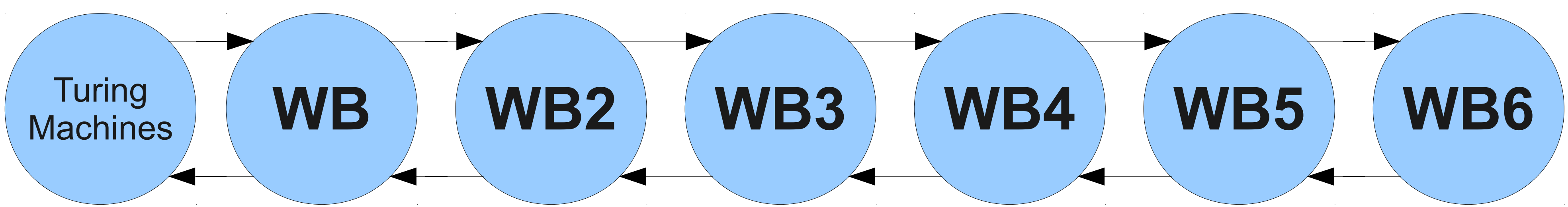
```
// Start
 0: Move tape 2 right.
 1: If reading ? on tape 1.1, go to Match.
 2: Load curr on tape 1.1 into X.
 3: Write X to tape 2.
 4: Move tape 1 right.
 5: Move tape 2 right.
 6: Go to 1.
```

A **WB6** Program for *SEARCH*

```
// Match
 7: Move tape 2 left until {B}
 8: Move tape 2 right.
 9: Move tape 1 right.
10: Write $ to tape 1, track 2.
11: If B on tape 2, go to Acc.
12: If B on tape 1, go to Rej.
13: Load tape 1, track 1 into X.
14: Load tape 2 into Y.
15: If X = Y, go to 17.
16: Go to Mismatch.
17: Move tape 1 right.
18: Move tape 2 right.
19: Go to 11.

// Mismatch
20: Move tape 1.2 left until {$}
21: Go to Match.

// Acc
22: Accept.

// Rej
23: Reject.
```

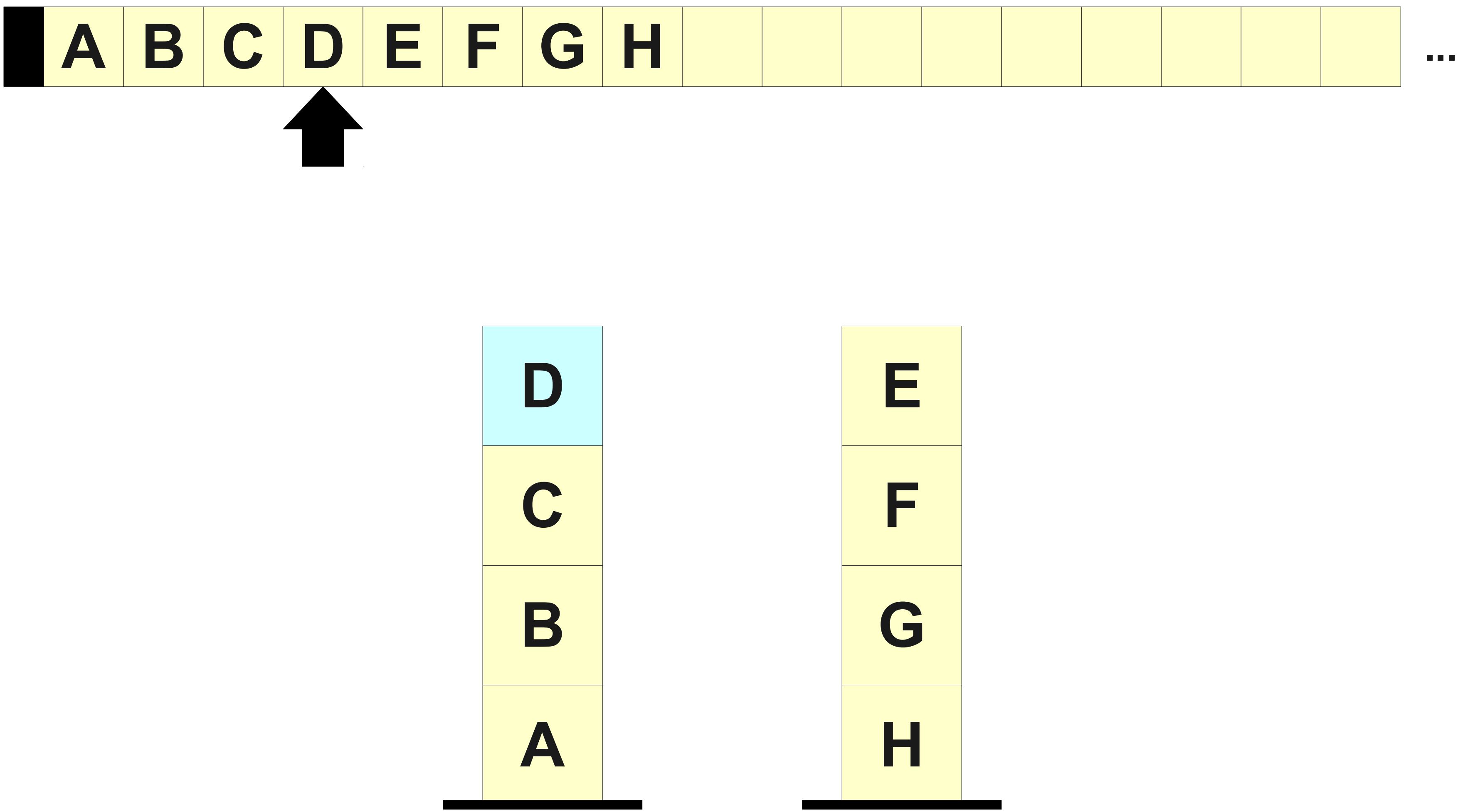
Oh, Come On Already...

- **Theorem:** A language is recursively enumerable iff there is a **WB6** program for it.
- We can really supercharge these languages without increasing our power!
- As with before, the construction will convert **WB6** programs into **WB5** programs.



The Key Idea

- Represent an infinite tape with two stacks.

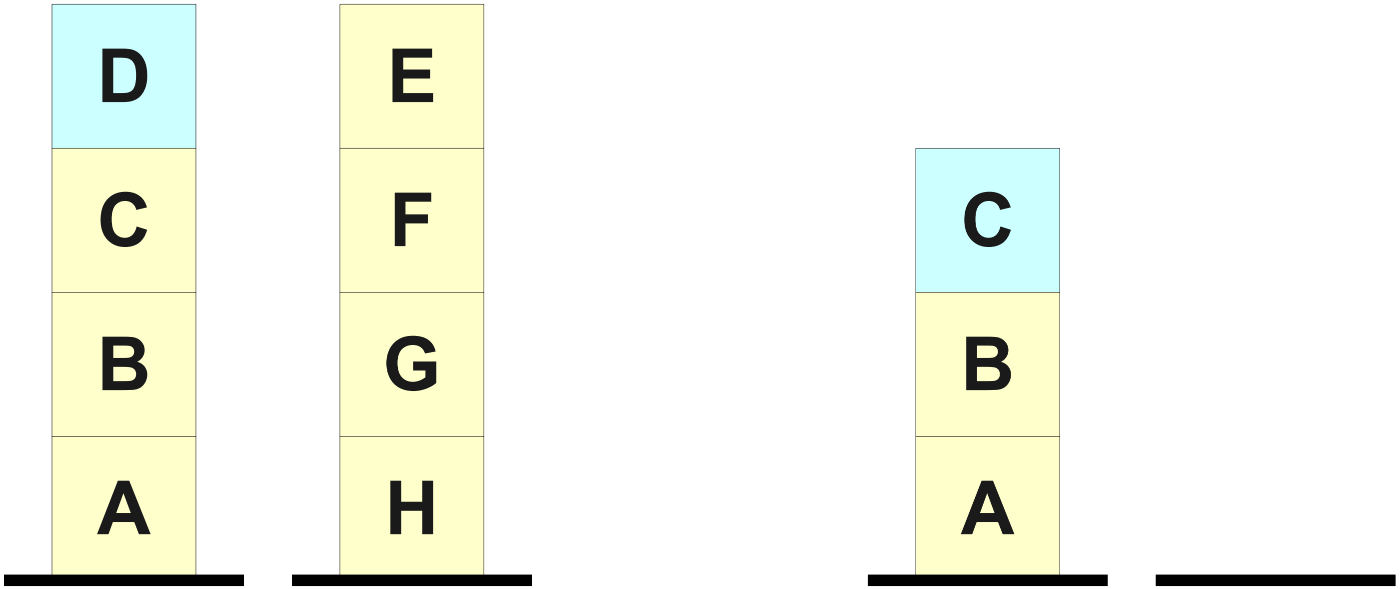


A Sketch of the Construction

- At the start of the program, copy the contents of the initial tape into a pair of stacks that will henceforth represent the first tape.
- Convert all motion operations into stack manipulation operations to push and pop values from the appropriate stacks.
- Use variables to hold temporary values (for example, when moving the top of one stack to another).

```
0: Move tape 1 right.
```

```
0: If stack 1R is empty, go to 2.
1: Go to 3.
2: Push B onto stack 1R.
3: Pop stack 1R into X.
4: Push X onto stack 1L.
```



What Else Can We Add?

- Function call and return.
  - Have a stack to use as the call stack.
  - Calling a function pushes the index of the instruction to which it should return.
  - Returning pops the stack and jumps back.
- Named variables.
  - Have a tape storing a sequence of values of the form **`name: value`**.
  - Can read and write values from the tape.
- Pointers
  - Have variables hold the names of other variables.
- Primitive types and arithmetic.
  - Design subroutines for addition, subtraction, etc.
  - Apply them to named variables.
- **Pretty much any feature of any major programming language.**

The Conversion Back Down

- From **WB6** to **WB5**:
  - Add in two stacks per tape used.
  - Replace all tape operations with appropriate stack manipulations.
- From **WB5** to **WB4**:
  - Add in one track per stack, plus one extra track.
  - Replace all stack operations with appropriate manipulations of those tracks.

The Conversion Back Down

- From **WB4** to **WB3:**
  - Expand the tape alphabet to include symbols for all track combinations.
  - Replace all references to track symbols with cascading if's for each possible case.
- From **WB3** to **WB2:**
  - Replicate the code once for each possible assignment to variables.
  - Hardcode in statements referencing variables.
  - Replace variable manipulation code with code to jump to the appropriate copy.

The Conversion Back Down

- From **WB2** to **WB**:
  - Expand out `move ... until` statements by replacing them with cascading `if` statements.
- From **WB** to Turing machines:
  - Replace each statement with the appropriate Turing machine gadget.

The Conversion Back Down

- The total conversion of a WB6 program using variables, multiple tracks, multiple stacks, and multiple tapes might produce an *enormous* Turing machine!
- But that said, the result is still a Turing machine.
- **Turing machines are simple, yet have enormous computational power**.

Just how powerful **are** Turing machines?

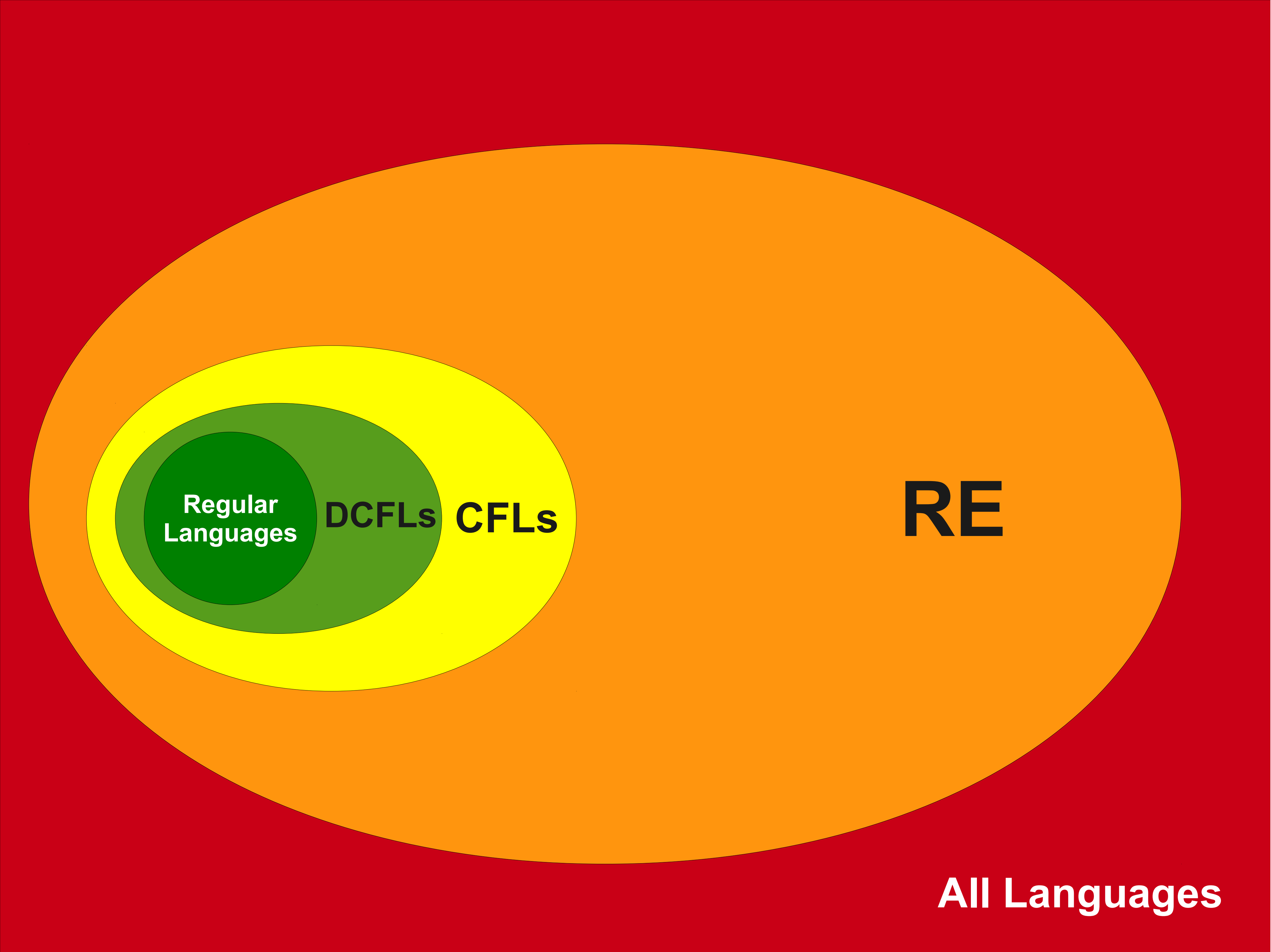
Effective Computation

- An **effective method of computation** is a form of computation with the following properties:
  - The computation consists of a set of steps.
  - There are fixed rules governing how one step leads to the next.
  - Any computation that yields an answer does so in finitely many steps.
  - Any computation that yields an answer always yields the correct answer.

The **Church-Turing Thesis** states that

Every effective method of computation is either equivalent to or weaker than a Turing machine.

This statement cannot be proven or disproven, but is widely considered true.


Regular Languages
DCFLs
CFLs
RE
All Languages

Next Time

- **Encodings**
  - How do we do computations over arbitrary objects?
- **The Universal Turing Machine**
  - A Turing machine for running other Turing machines.
- **Nondeterministic Turing Machines**
  - What happens when we supercharge a TM? What does this even mean?
- **R and RE Languages**
  - A finer gradation within the **RE** languages.