CS103 Final Exam

This final exam is open-book and open-note. You may use a computer only to look at notes that you yourself have written, to access the course website and the tools there, and to read an online copy of one of the recommended readings. No other use of the computer is permitted during this exam. You must hand-write all of your solutions on this physical copy of the exam. No electronic submissions will be considered without prior consent of the course staff.

SUNetID: ____________________________
Last Name: __________________________
First Name: __________________________

I accept both the letter and the spirit of the honor code. I have not received any unpermitted assistance on this test, nor will I give any. My answers are my own work and are not adapted from any unpermitted sources. I will not use a computer except in the ways specified at the top of the exam, and I understand that the rules governing computer usage on this exam are not the same as those on the midterm.

(signed) __________________________________

You have three hours to complete this exam. There are 180 total points, and this exam is worth 25% of your total grade in this course. You may find it useful to read through all the questions to get a sense of what this exam contains. As a rough sense of the difficulty of each question, there is one point on this exam per minute of testing time.

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It has been a pleasure teaching CS103 this quarter. Good luck on the final exam!
Problem One: Discrete Mathematics  

There can be many functions from one set $A$ to a second set $B$. This question explores how many functions of this sort there are.

For any set $S$, we will denote by $2^S$ the following set:

$$2^S = \{ f \mid f : S \to \{0, 1\} \}$$

That is, $2^S$ is the set of all functions whose domain is $S$ and whose codomain is the set $\{0, 1\}$. Note that $2^S$ does not mean “two raised to the $S$th power.” It's just the notation we use to denote the set of all functions from $S$ to $\{0, 1\}$.

Prove that for any nonempty set $S$, we have $|2^S| = \mathcal{P}(S)$. You may find the following definition useful: given two functions $f : A \to B$ and $g : A \to B$, we have $f = g$ iff for all $a \in A$, $f(a) = g(a)$. Your proof should work for all sets $S$, including infinite sets.
(Extra space for Problem 1, if you need it)
Problem Two: Regular Languages  

(i) Rock, Paper, Scissors  

The number of characters in a regular expression is defined to be the total number of symbols used to write out the regular expression. For example, \((a|b)^*\) is a six-character regular expression, and \(ab\) is a two-character regular expression.

Let \(\Sigma = \{a, b\}\). Find examples of all of the following:

- A regular language over \(\Sigma\) with a one-state NFA but no one-state DFA.
- A regular language over \(\Sigma\) with a one-state DFA but no one-character regular expression.
- A regular language over \(\Sigma\) with a one-character regular expression but no one-state NFA.

Prove that all of your examples have the required properties.
(Extra space for Problem 2.i, if you need it)
(ii) Nonregular Languages

A natural number $n > 1$ is called composite iff it can be written as $n = rs$ for natural numbers $r$ and $s$, where $r \geq 2$ and $s \geq 2$. A natural number $n > 1$ is called prime iff it is not composite.

Let $\Sigma = \{ a \}$ and consider the language $L = \{ a^n | n \text{ is prime} \}$. For example:

- $\varepsilon \notin L$
- $a \notin L$
- $a^2 \in L$
- $a^3 \not\in L$  
- $a^4 \not\in L$  
- $a^5 \in L$
- $a^6 \not\in L$  
- $a^7 \in L$
- $a^8 \not\in L$

Prove that $L$ is not regular. You may want to use the fact that for every natural number $n$, there is a prime number $p$ such that $p > n$. (Hint: Use the pumping lemma. Try finding an expression for $|xy^i z|$ in terms of $i$, then see if you can find an $i$ such that $|xy^i z|$ is composite.)
(Extra space for Problem 2.ii, if you need it)
Problem Three: Context-Free Languages (30 Points Total)

(i) Context-Free Grammars (20 Points)

Let $\Sigma = \{a, b\}$ and let $L = \{a^n b^n \mid n \in \mathbb{N}\}$. The complement of this language is the language $\bar{L}$. For example:

- $abb \in \bar{L}$
- $aab \in \bar{L}$
- $baab \in \bar{L}$
- $abab \in \bar{L}$
- $\varepsilon \notin \bar{L}$
- $ab \notin \bar{L}$
- $aabb \notin \bar{L}$
- $aaabbb \notin \bar{L}$

Write a context-free grammar that generates $\bar{L}$, then give derivations for the four strings listed in the left-hand column.

(Hint: There are four separate cases you need to consider. You might want to design the grammar to consider each of these four cases independently of one another.)
(extra space for Problem 3.i, if you need it)
(ii) Disjoint Unions (10 Points)

Let $\Sigma = \{0, 1\}$ and let $L_1$ and $L_2$ be arbitrary context-free languages over $\Sigma$. Prove that $L_1 \cup L_2$ is context-free as well. As a reminder,

$$L_1 \cup L_2 = \{0w \mid w \in L_1\} \cup \{1w \mid w \in L_2\}$$
Problem Four: R, RE, and co-RE Languages (55 Points Total)

(i) The Halting Problem (15 Points)
Prove or disprove: for any TMs $H$ and $M$ and any string $w$, if $H$ is a recognizer for $HALT$ and $M$ loops on $w$, then $H$ loops on $\langle M, w \rangle$. 
(ii) RE Languages

A palindrome number is a number whose base-10 representation is a palindrome. For example, 1 is a palindrome number, as is 14941 and 7897987.

Consider the following language:

\[ L = \{ \langle n \rangle | n \in \mathbb{N} \text{ and there is a number } k \in \mathbb{N} \text{ where } k > 0 \text{ and } nk \text{ is a palindrome number} \} \]

For example, \( \langle 106 \rangle \in L \) because \( 106 \times 2 = 212 \), which is a palindrome number. Also, \( \langle 29 \rangle \in L \), because \( 29 \times 8 = 232 \), which is a palindrome number.

Prove or disprove: \( L \in \text{RE} \).
(iii) Unsolvable Problems

Consider the following language \textit{DECIDER}:

\[ \text{DECIDER} = \{ \langle M \rangle \mid M \text{ is a decider} \} \]

Prove that \textit{DECIDER} \∉ \textit{RE} and \textit{DECIDER} \∉ \textit{co-RE}. We recommend using a mapping reduction involving the language \textit{A}_{\text{ALL}} from Problem Set 8, which is neither \textit{RE} nor \textit{co-RE}. For reference:

\[ \text{A}_{\text{ALL}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^* \} \]
(Extra space for Problem 4.iii, if you need it)
Problem Five: P and NP Languages (30 Points Total)

(i) Non-NPC Languages (15 Points)
There are exactly two languages in NP that we currently know are not NP-complete: \( \emptyset \) and \( \Sigma^* \).
Prove that \( \Sigma^* \) is not NP-complete.
(ii) **Resolving \( P \neq NP \)**  

(15 Points)

Suppose that we can prove the following statement:

For every pair of \( NP \) languages \( A \) and \( B \) (where neither \( A \) nor \( B \) is \( \emptyset \) or \( \Sigma^* \)), we have \( A \leq_P B \).

Under this assumption, decide which of the following is true, then prove your choice is correct.

- \( P \) is necessarily equal to \( NP \).
- \( P \) is necessarily not equal to \( NP \).
- \( P \) may or may not be equal to \( NP \).