Direct Proofs

New TA Introductions

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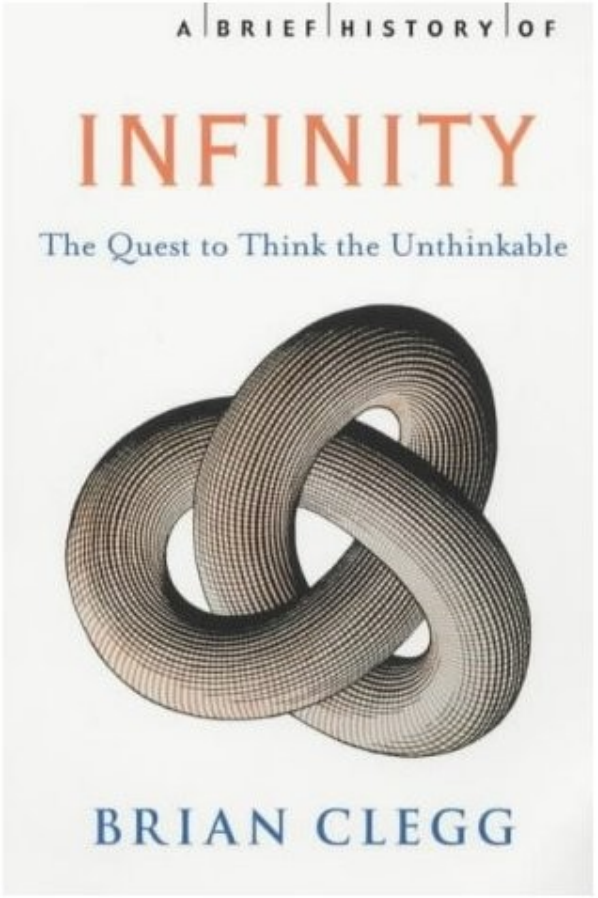
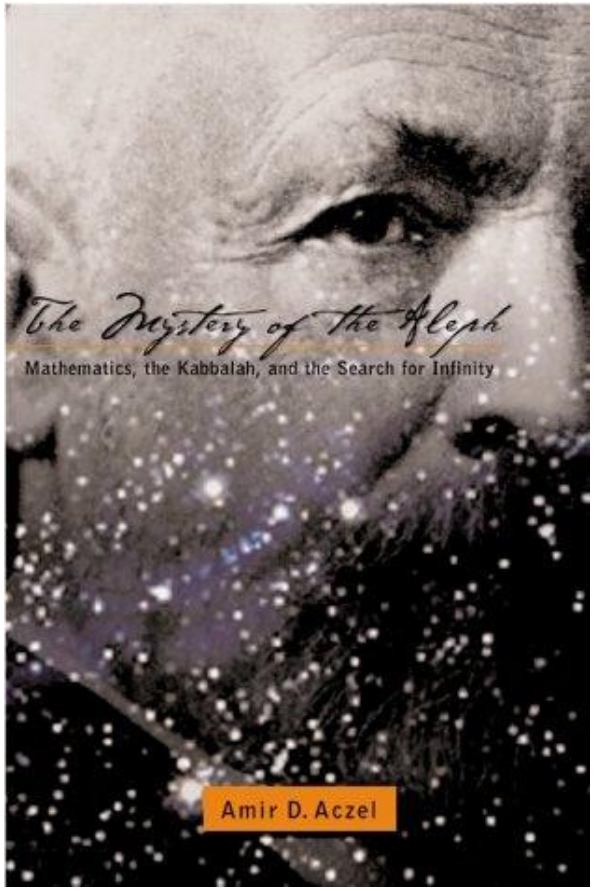
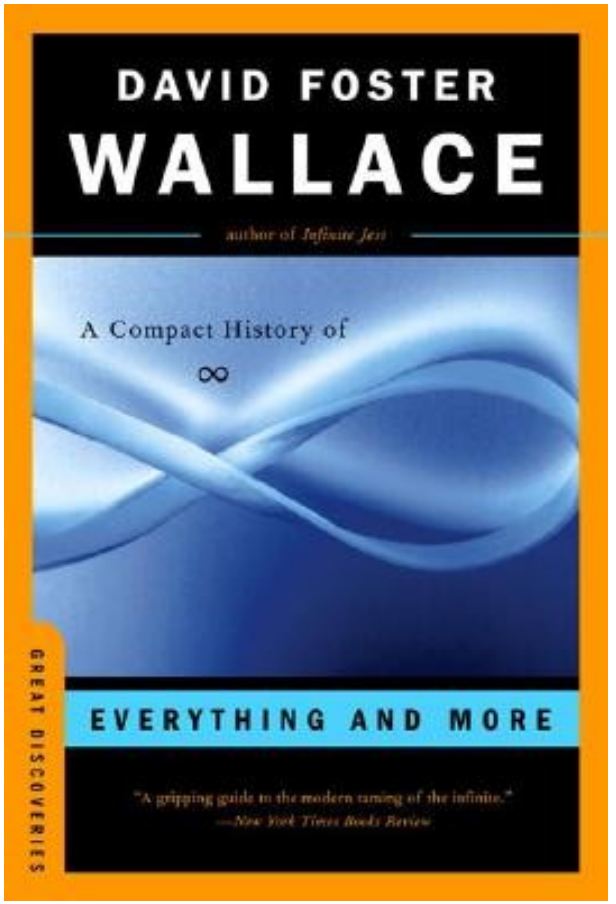
Recommended Reading



A Brief History of Infinity



The Mystery of the Aleph



Everything and More

Recommended Courses

Math 161: Set Theory

What is a Proof?

Induction and Deduction

- In the sciences, much reasoning is done **inductively**.
  - Conduct a series of experiments and find a rule that explains all the results.
  - Conclude that there is a general principle explaining the results.
  - Even if all data are correct, the conclusion might be incorrect.
- In mathematics, reasoning is done **deductively**.
  - Begin with a series of statements assumed to be true.
  - Apply logical reasoning to show that some conclusion necessarily follows.
  - If all the starting assumptions are correct, the conclusion necessarily must be correct.

Structure of a Mathematical Proof

- Begin with a set of initial assumptions.
- Apply logical reasoning to derive the final result from those initial assumptions.
- Assuming that all intermediary steps are sound logical reasoning, the proof is valid.
- It is a secondary question whether the initial assumptions are correct; that's the domain of the *philosophy of mathematics*.

Direct Proofs

Direct Proofs

- A **direct proof** is the simplest type of proof.
- Starting with an initial set of assumptions, apply simple logical steps to derive the result.
  - *Directly* proving that the result is true.
- Contrasts with **indirect proofs**, which we'll see on Friday.

Two Quick Definitions

- An integer n is **even** if there is some integer k such that $n = 2k$.
  - This means that 0 is even.
- An integer n is **odd** if there is some integer k such that $n = 2k + 1$.
- We'll assume the following for now:
  - Every integer is either even or odd.
  - No integer is both even and odd.

A Simple Direct Proof

Theorem: If n is an even integer, then n^2 is even.

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Since $2k^2$ is an integer, this means that there is some integer m (namely, $2k^2$) such that $n^2 = 2m$.

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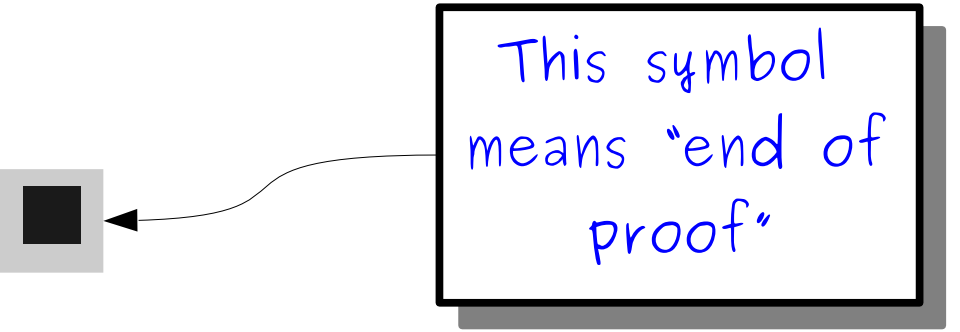

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This is the definition of an even integer. When writing a mathematical proof, it's common to call back to the definitions.

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uch

Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.

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Our ultimate goal is to prove that n^2 is even. This means that we need to find some m such that $n^2 = 2m$. Here, we're explicitly showing how we can do that.

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Hey, that's what we were trying to show! We're done now.

Thus n^2 is even. ■

An Important Result

- Set equality is defined as follows

$A = B$ precisely when
for every $x \in A$, we have $x \in B$,
and vice-versa.

- This definition makes it a bit tricky to prove that two sets are equal.
- It's often easier to use the following result to show that two sets are equal:

For any sets A and B,
if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

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How do we prove that this is true for *any* choice of sets?

Proving Something Always Holds

- Many statements have the form

  For any *X*, P(*X*) is true.

- Examples:

  For all integers n, if n is even, n^2 is even.

  For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

  For all sets S, $|S| < |\wp(S)|$.

  Everybody's looking forward to the weekend, weekend.

- How do we prove these statements when there are (potentially) infinitely many cases to check?

Arbitrary Choices

- To prove that P(x) is true for all possible x, show that no matter what choice of x you make, P(x) must be true.
- Start the proof by making an **arbitrary choice** of x:
  - "Let x be chosen arbitrarily."
  - "Let x be an arbitrary even integer."
  - "Let x be an arbitrary set containing 137."
  - "Consider any x."
- Demonstrate that P(x) holds true for this choice of x.
- Conclude that since the choice of x was arbitrary, P(x) must hold true for all choices of x.

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We're showing here that regardless of what ***A*** and ***B*** you pick, the result will still be true.

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Proof: Consider an arbitrary even natural number, say, 16. 16 is even, and it has no odd divisors. Since our choice was arbitrary, for any arbitrary n, if n is even, n has no odd divisors. ■

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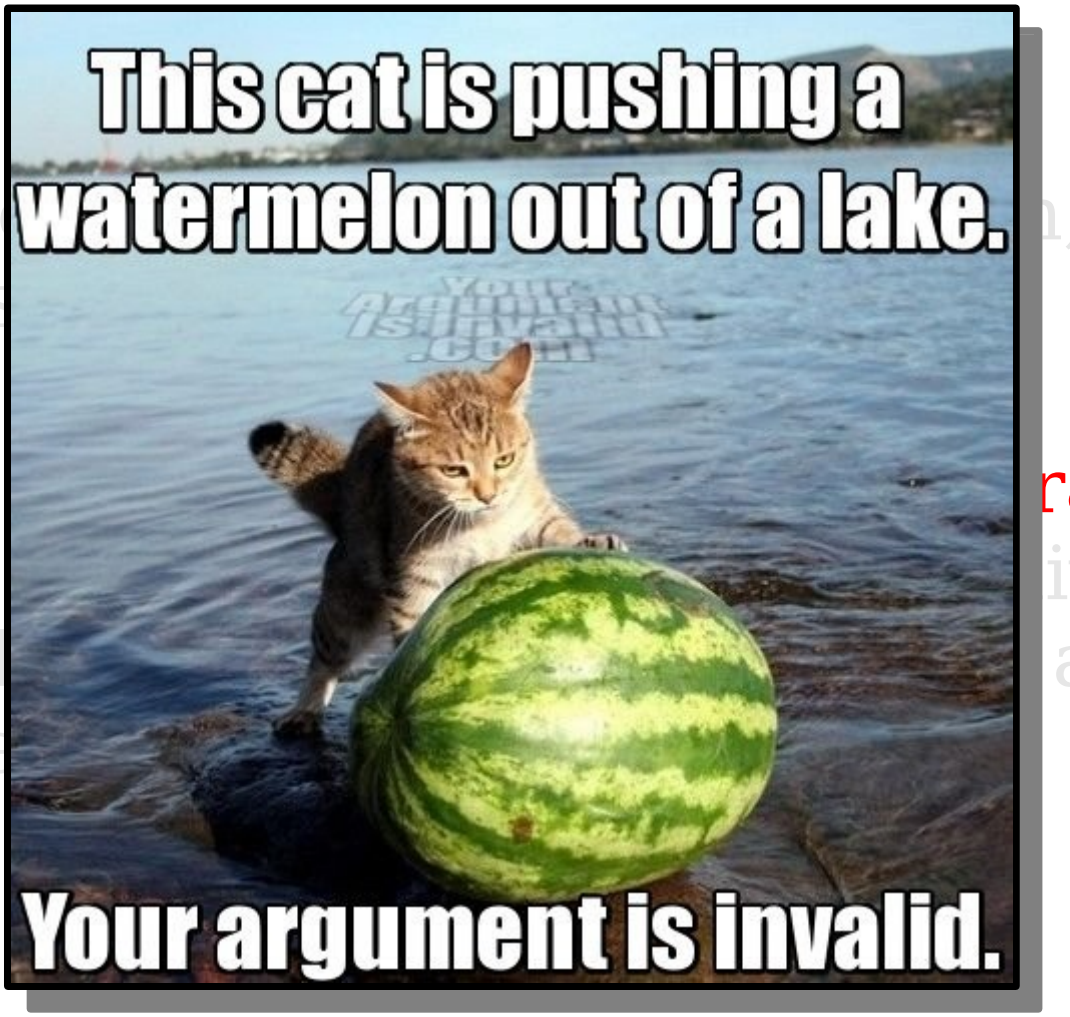

ar·bi·trar·y
adjective /'ärbi,trerē/

1. Based on random choice or personal whim, rather than any reason or system - *“his mealtimes were entirely arbitrary”*

2. *(of power or a ruling body)* Unrestrained and autocratic in the use of authority - *“arbitrary rule by King and bishops has been made impossible”*

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Not this one!

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To prove something is true for all x, **do not** choose an x and base the proof off of your choice!

Instead, leave x unspecified and show that no matter what x is, the specified property must hold.

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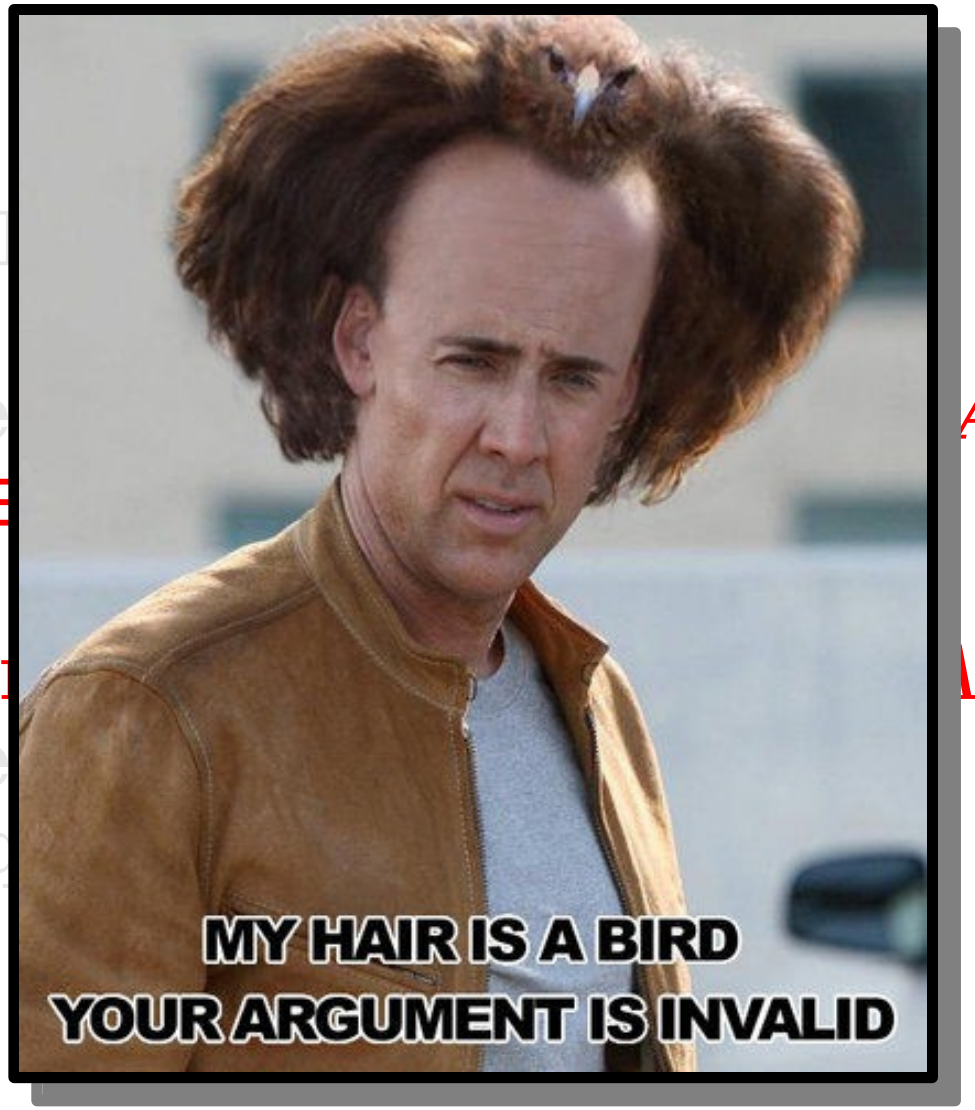

If you want to prove that P implies Q,
assume P and prove Q.

Don't assume Q and then prove P!

An Entirely Different Proof

Theorem: There exists a natural number $n > 0$ such that the sum of all natural numbers less than n is equal to n.

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This is a fundamentally different type of proof that what we've done before. Instead of showing that *every* object has some property, we want to show that *some* object has a given property.

Universal vs. Existential Statements

- A **universal statement** is a statement of the form

  For all *x*, P(*x*) is true.

- We've seen how to prove these statements.

Universal vs. Existential Statements

- A **universal statement** is a statement of the form

  For all *x*, P(*x*) is true.

- We've seen how to prove these statements.
- An **existential statement** is a statement of the form

  There exists an *x* for which P(*x*) is true.

- How do you prove an existential statement?

Proving an Existential Statement

- We will see several different ways to prove “there is some x for which P(x) is true.”
- Simple approach: Just go and find some x for which P(x) is true!
  - In our case, we need to find a positive natural number n such that that sum of all smaller natural numbers is equal to n.
  - Can we find one?

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An Entirely Different Proof

Theorem: There exists a natural number $n > 0$ such that the sum of all natural numbers less than n is equal to n.

Proof: Take $n = 3$.

There are three natural numbers smaller than 3: 0, 1, and 2.

We have $0 + 1 + 2 = 3$.

Thus 3 is a natural number greater than zero equal to the sum of all smaller natural numbers. ■

Extended Example: **XOR**

Logical Operators

- A **bit** is a value that is either 0 or 1.
- The set $\mathbb{B} = \{0, 1\}$ is the set of all bits.
- A **logical operator** is an operator that takes in some number of bits and produces a new bit as output.
- Example: Logical NOT, denoted $\neg x$:

$$\neg 0 = 1 \qquad \neg 1 = 0$$

Logical XOR

- The **exclusive OR** operator (**XOR**) operates on two bits and produces 0 if the bits are the same and 1 if they are different.
  - Since XOR operates on two values, it is called a **binary operator**.
- We denote the XOR of a and b by $\boldsymbol{a \oplus b}$.
- Formally, XOR is defined as follows:

$$0 \oplus 0 = 0 \qquad 0 \oplus 1 = 1$$

$$1 \oplus 0 = 1 \qquad 1 \oplus 1 = 0$$

Fun with XOR

- The XOR operator has numerous uses throughout computer science.
  - Applications in cryptography, data structures, error-correcting codes, networking, machine learning, etc.
- XOR is useful because of four key properties:
  - XOR has an **identity element**.
  - XOR is **self-inverting**.
  - XOR is **associative**.
  - XOR is **commutative**.

Identity Elements

- An **identity element** for a binary operator ★ is some value z such that for any a:

$$a \bigstar z = z \bigstar a = a$$

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In math-speak, the term "***for any a***" is synonymous with "for every a" or "***for every possibly choice of a.***" It does not mean "***for some specific choice of a.***"

Identity Elements

- An **identity element** for a binary operator ★ is some value z such that for any a:

$$a \star z = z \star a = a$$

- Example: 0 is an identity element for +:

$$a + 0 = 0 + a = a$$

- Example: 1 is an identity element for ×:

$$a \times 1 = 1 \times a = a$$

Theorem: 0 is an identity element for $\oplus$.

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Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$. To do this, consider an arbitrary $b \in \mathbb{B}$.

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Case 1: $b = 0$.

Case 2: $b = 1$.

Theorem: 0 is an identity element for $\oplus$.

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This is called a proof by cases (alternatively, a proof by exhaustion) and works by showing that the theorem is true regardless of what specific outcome arises.

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Case 1: $b = 0$. Then we have

$$b \oplus 0 = 0 \oplus 0$$

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Proof: We will prove that for any $b \in \mathbb{B}$ that $b \oplus 0 = b$ and that $0 \oplus b = b$. To do this, consider an arbitrary $b \in \mathbb{B}$. We consider two cases:

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Case 2:

$b \oplus$ $\qquad$ 1

$= b$ $\qquad$ $= b$

In a proof by cases, after demonstrating each case, you should summarize the cases afterwards to make your point clearer.

In both cases, we find $b \oplus 0 = 0 \oplus b = b$.

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Self-Inverting Operators

- A binary operator ★ with identity element z is called **self-inverting** when for any a, we have

$$a \star a = z$$

- Is + self-inverting?
- Is – self-inverting?

XOR is Self-Inverting

Theorem: $\oplus$ is self-inverting.

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Associative Operators

- A binary operator ★ is called **associative** when for any a, b and c, we have

$$a \star (b \star c) = (a \star b) \star c$$

- Is + associative?
- Is – associative?
- Is × associative?

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Case 2: $c = 1$. Then we have that

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus (b \oplus 1) \\ &= \textbf{?} \end{aligned}$$

When You Get Stuck

- When writing proofs, you are bound to get stuck at some point.
- When this happens, it can mean multiple things:
  - What you're proving is incorrect.
  - You are on the wrong track.
  - You're on the right tack, but you need to prove an additional result to get to your goal.
- Unfortunately, there is no general way to determine which case you are in.
- You'll build this intuition through experience.

Where We're Stuck

- Right now, we have the expression

  $$a \oplus (b \oplus 1)$$

  and we don't know how to simplify it.
- Let's focus on the $(b \oplus 1)$ part and see what we find:
  - **0** $\oplus$ 1 = **1**
  - **1** $\oplus$ 1 = **0**
- It seems like $b \oplus 1 = \neg b$. Could we prove it?

Relations Between Proofs

- Proofs often build off of one another: large results are almost often accomplished by building off of previous work.
  - Like writing a large program – split the work into smaller methods, across different classes, etc. instead of putting the whole thing into **`main`**.
- A result that is proven specifically as a stepping stone toward a larger result is called a **lemma**.
- Our result that $b \oplus 1 = \neg b$ serves as a lemma in our larger proof that $\oplus$ is associative.

Lemma: For any $b \in \mathbb{B}$, we have $b \oplus 1 = \neg b$.

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Lemma: For any $b \in \mathbb{B}$, we have $b \oplus 1 = \neg b$.
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(using our first lemma)

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Commutative Operators

- A binary operator ★ is called **commutative** when the following is always true:

$$a \star b = b \star a$$

- Is + commutative?
- Is – commutative?

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To do this, let $x = a \oplus b$. Then

$x = a \oplus b$	
$x \oplus b = (a \oplus b) \oplus b$	
$x \oplus b = a \oplus (b \oplus b)$	*(since $\oplus$ is associative)*
$x \oplus b = a \oplus 0$	*(since $\oplus$ is self-inverting)*
$x \oplus b = a$	*(since 0 is an identity of $\oplus$)*
$x \oplus (x \oplus b) = x \oplus a$	
$(x \oplus x) \oplus b = x \oplus a$	*(since $\oplus$ is associative)*
$0 \oplus b = x \oplus a$	*(since $\oplus$ is self-inverting)*
$b = x \oplus a$	*(since 0 is an identity of $\oplus$)*
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This means that $a \oplus b = x = b \oplus a$.

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$$
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b \oplus a &= x \oplus (a \oplus a) \\
b \oplus a &= x \oplus 0 \\
b \oplus a &= x
\end{aligned}
$$

This means that $a \oplus b = x$ [illegible] commutative. ■

The only properties of $\oplus$ that we used here are that it is associative, has an identity, and is self-inverting. This same proof works for any operator with these three properties!

Binary operators that have this property give rise to boolean groups (but you don't need to know that for this class).

Application: **Encryption**

Bitstrings

- A **bitstring** is a finite sequence of 0s and 1s.
- Internally, computers represent all data as bitstrings.
  - For details on how, take CS107 or CS143.

Bitstrings and ⊕

- We can generalize the ⊕ operator from working on individual bits to working on bitstrings.
- If A and B are bitstrings of length n, then we'll define $A \oplus B$ to be the bitstring of length n formed by applying ⊕ to the corresponding bits of A and B.
- For example:

$$\begin{array}{r} \mathtt{110110} \\ \oplus\ \mathtt{011010} \\ \hline \mathtt{101100} \end{array}$$

Encryption

- Suppose that you want to send me a secret bitstring M of length n.
- You should be able to read the message, but anyone who intercepts the secret message should not be able to read it.
- How might we accomplish this?

⊕ and Encryption

- In advance, you and I share a randomly-chosen bitstring K of length n (called the **key**) and keep it secret.
- To send me message M secretly, you send me the string $C = M \oplus K$.
  - C is called the **ciphertext**.
- To decrypt the ciphertext C, I compute the string $C \oplus K$. This is

$$\begin{aligned} C \oplus K &= (M \oplus K) \oplus K \\ &= M \oplus (K \oplus K) \\ &= M \end{aligned}$$

⊕ and Encryption

- Suppose that you don't have the key and get the message $M \oplus K$.
- If K is chosen to be truly random, then every bit in $M \oplus K$ appears to be truly random.
- Intuition: Let b be a original bit from the message and k be the corresponding bit in the key.
  - If $k = 0$, then $b \oplus k = b \oplus 0 = b$.
  - If $k = 1$, then $b \oplus k = b \oplus 1 = \neg b$.
- Since the key bit is truly random, the bits in the original string are flipped totally randomly.
- Can formalize the math; take CS109 for details!

An Example

PUPPIES

M	01010000010101010101000001010000010010010100010101010011
K	11011100101110111100010011010101111001101111011111000010
C	10001100111011101001010010000101101011111011001010010001

Œî”…©²‘

An Example

Œî”…©²‘

C	**1000110011101110100101001000010110101111101100101001000 1**
K	**1101110010111011110001001101010111100110111101111000010**
M	**0101000001010101010100000101000001001001010001010101001 1**

PUPPIES

An Example

Œî”…©²‘

C	**10001100111011101001010010000101101011111011001010010001**
K?	**01011100010101010101000001010000010010010100010101010011**
M?	**01001100010011110100110001000110010000010100100101001100**

LOLFAIL

One Catch

- This scheme is **very insecure** if you encrypt multiple messages using the same key.
- You'll determine why this is in the problem set.

Next Time

- **Indirect Proofs**
  - Proof by contradiction.
  - Proof by contrapositive.