Mathematical Induction

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

$P(0)$ is true

If it starts …

and

… and it keeps going …

For any $n \in \mathbb{N}$, we have $P(n) \to P(n+1)$

Then

… then it's always true.

For any $n \in \mathbb{N}$, $P(n)$ is true.

Another Example of Induction

Human Dominoes

- Everyone (except that last guy) eventually fell over.
- Why is that?
  - Someone fell over.
  - Once someone fell over, the next person fell over as well.

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

$P(0)$ is true

and

For any $n \in \mathbb{N}$, we have $P(n) \to P(n+1)$

Then

For any $n \in \mathbb{N}$, $P(n)$ is true.

Induction, Intuitively

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- ...

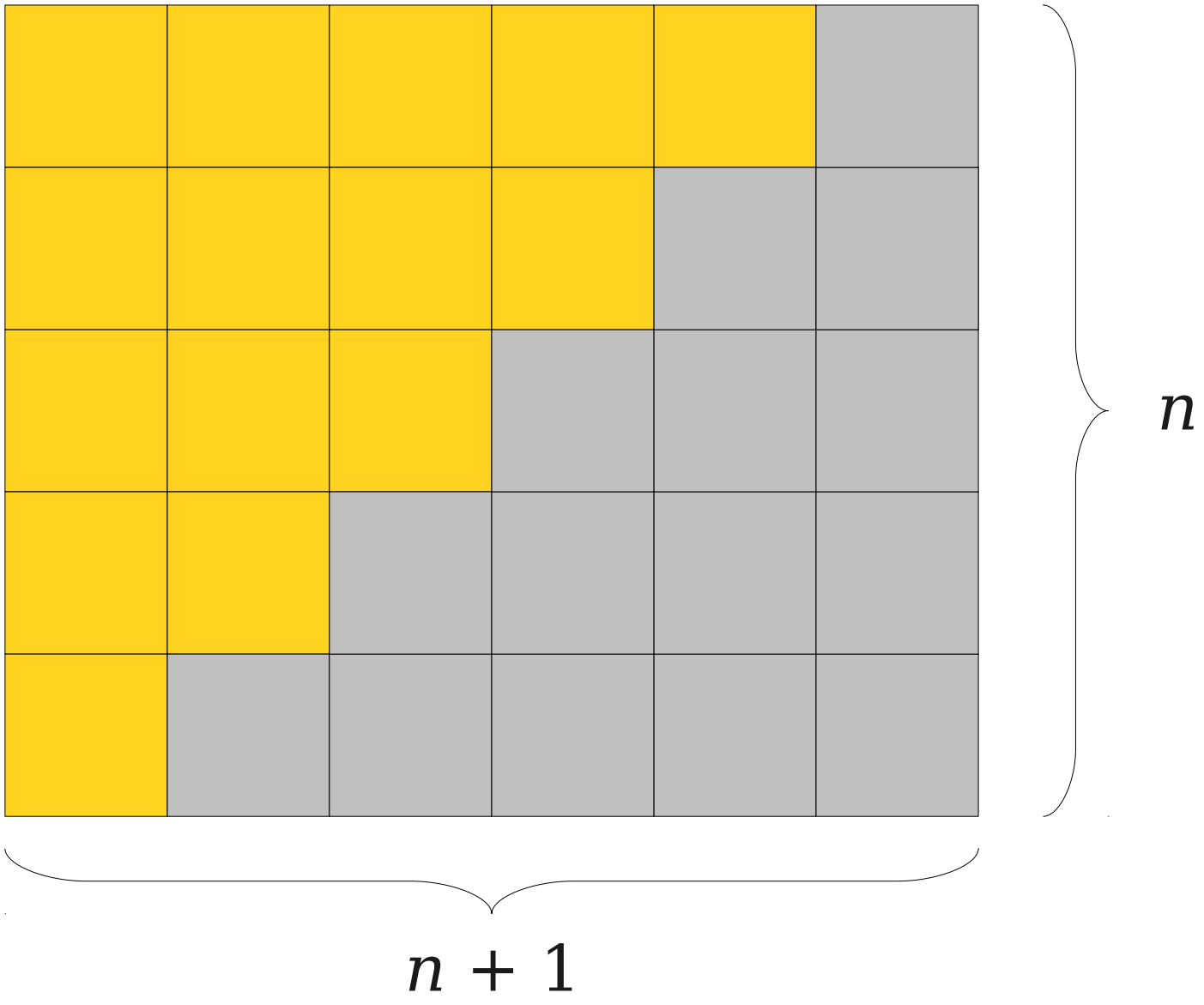
Proof by Induction

- Suppose that you want to prove that some property $P(n)$ holds of all natural numbers. To do so:
  - Prove that $P(0)$ is true.
    - This is called the **basis** or the **base case**.
  - Prove that for all $n \in \mathbb{N}$, that if $P(n)$ is true, then $P(n + 1)$ is true as well.
    - This is called the **inductive step**.
    - $P(n)$ is called the **inductive hypothesis**.
  - Conclude by induction that $P(n)$ holds for all n.

Some Sums

$1 = 1$
$1 + 2 = 3$
$1 + 2 + 3 = 6$
$1 + 2 + 3 + 4 = 10$
$1 + 2 + 3 + 4 + 5 = 15$

$$1 + 2 + \ldots + (n - 1) + n = n(n + 1) / 2$$



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Some Sums

$1 = 1$ **$= 1(1 + 1) / 2$**

$1 + 2 = 3$ **$= 2(2 + 1) / 2$**

$1 + 2 + 3 = 6$ **$= 3(3 + 1) / 2$**

$1 + 2 + 3 + 4 = 10$ **$= 4(4 + 1) / 2$**

$1 + 2 + 3 + 4 + 5 = 15$ **$= 5(5 + 1) / 2$**

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Just as in a proof by contradiction or contrapositive, we should mention this proof is by induction.

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Now, we state what property $P(n)$ we are going to prove holds for all $n \in \mathbb{N}$.

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Proof: By induction. Let $P(n)$ be "the sum of the first n positive natural numbers is $n(n + 1) / 2$." We show that $P(n)$ is true for all $n \in \mathbb{N}$.

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The first step of an inductive proof is to show $P(0)$. We explicitly state what $P(0)$ is, then try to prove it. We can prove $P(0)$ using any proof technique we'd like.

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For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero positive natural numbers is $0(0 + 1)/2$. Since the sum of the first zero positive natural numbers is $0 = 0(0 + 1)/2$, $P(0)$ is true.

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For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $1 + 2 + \ldots + n = n(n + 1) / 2$.

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For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $1 + 2 + \ldots + n = n(n + 1) / 2$.

The goal of this step is to prove

"For any $n \in \mathbb{N}$, if $P(n)$, then $P(n + 1)$"

To do this, we'll choose an arbitrary n, assume that $P(n)$ holds, then try to prove $P(n + 1)$.

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Here, we're explicitly stating $P(n + 1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Consider the sum of the . This is the sum of the first n . By the inductive hypothesi

We're assuming that $P(n)$ is true, so we can replace this sum with the value $\boldsymbol{n(n + 1) / 2}$.

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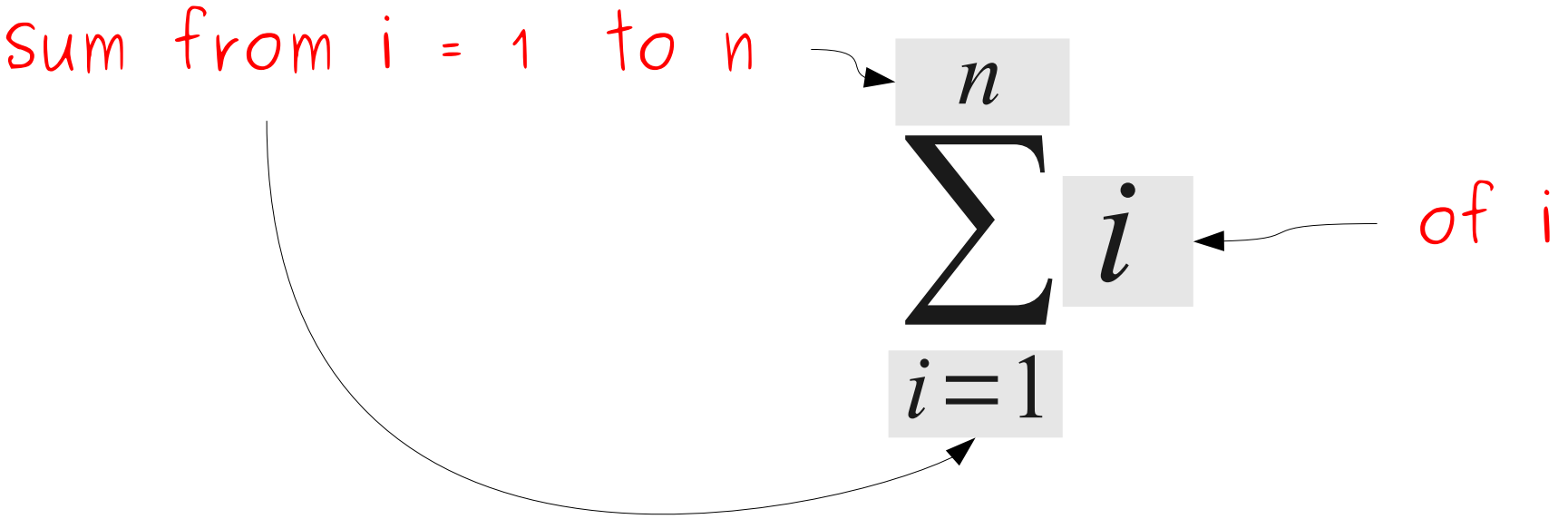
Structuring a Proof by Induction

- State that your proof works by induction.
- State your choice of $P(n)$.
- Prove the base case:
  - State what $P(0)$ is, then prove it using any technique you'd like.
- Prove the inductive step:
  - State that for some arbitrary $n \in \mathbb{N}$ that you're assuming $P(n)$ and mention what $P(n)$ is.
  - State that you are trying to prove $P(n + 1)$ and what $P(n + 1)$ means.
  - Prove $P(n + 1)$ using any technique you'd like.
- This is very rigorous, so as we gain more familiarity with induction we will start being less formal in our proofs.

Notation: Summations

- Instead of writing 1 + 2 + 3 + … + n, we write



Summation Examples

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

$$\sum_{i=0}^{2} \left(i^2 - i\right) = \left(0^2 - 0\right) + \left(1^2 - 1\right) + \left(2^2 - 2\right) = 2$$

The Empty Sum

- A sum of no numbers is called the **empty sum** and is defined to be zero.
- Examples:

$$\sum_{i=1}^{0} 2^i = 0 \qquad \sum_{i=137}^{42} i^i = 0 \qquad \sum_{i=0}^{-1} i = 0$$

- Why do you think it's defined to be zero as opposed to some other number?

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Since the empty sum is defined to be 0, this claim is true.

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$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

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Here, we're "peeling off" the last term of the sum. Many inductive proofs on sums will use this trick.

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Sums of Powers of Two

(empty sum) $= 0$

$2^0 = 1 = 1$

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$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$

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$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

A Quick Aside

- This result helps explain the range of numbers that can be stored in an **`int`**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. We'll see one in a week.

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A Brief Interlude for Announcements

Recitation Sections

- Handout #04 contains several discussion questions for this week.
- Stop by a recitation section (marked on the OH schedule) to work through them.
- All sections cover the same material.
- Solutions distributed at recitation sections and online on Tuesday night.

Back to our regularly
scheduled programming...

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How Not To Induct

An Incorrect Proof

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$$\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$$

We want to show that $P(n + 1)$ is true, which means that we want to show

$$\sum_{i=1}^{n+1} i = \frac{1}{2}\left(n + 1 + \frac{1}{2}\right)^2 = \frac{1}{2}\left(n + \frac{3}{2}\right)^2$$

To see this, note that

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + n + 1 = \frac{1}{2}\left(n + \frac{1}{2}\right)^2 + n + 1 = \frac{\left(n + \frac{1}{2}\right)^2}{2} + \frac{2(n+1)}{2} = \frac{\left(n + \frac{1}{2}\right)^2 + 2(n+1)}{2}$$

$$= \frac{n^2 + n + \frac{1}{4} + 2n + 2}{2} = \frac{n^2 + 3n + \frac{9}{4}}{2} = \frac{\left(n + \frac{3}{2}\right)^2}{2}$$

So $P(n + 1)$ holds, completing the induction. ■

An Incorrect Proof

Theorem: For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$

Proof: By induction. Let $P(n)$ be defined as $P(n) \equiv \sum_{i=1}^{n}$

Where did we prove the base case?

Now, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, i.e.

$$\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$$

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An Incorrect Proof

$\equiv \sum_{i=}^{n}$

olds,

Where did we prove the base case?

means that we want to show

$\left(\frac{3}{2}\right)^2$

$\frac{2(n+1)}{2} = \frac{\left(n+\frac{1}{2}\right)^2 + 2(n+1)}{2}$

■

When proving $P(n)$ is true
for all $n \in \mathbb{N}$ by induction,

make sure to show the base case!

Otherwise, your argument is invalid!

The Counterfeit Coin Problem

Problem Statement

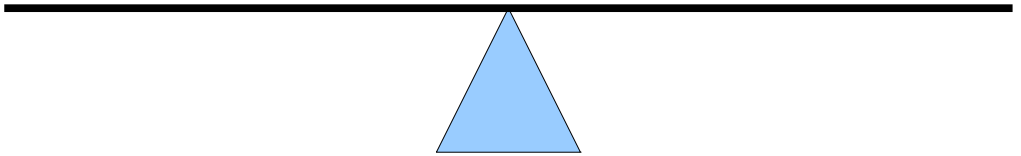
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin

1

2

3



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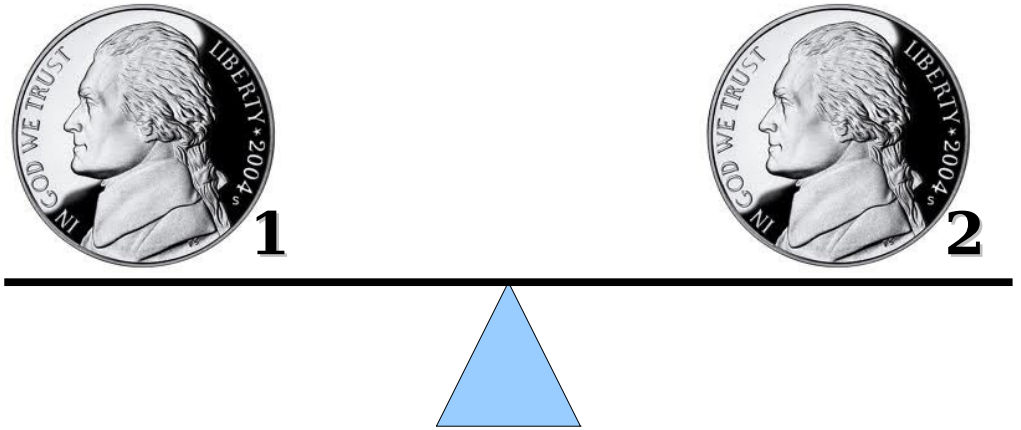

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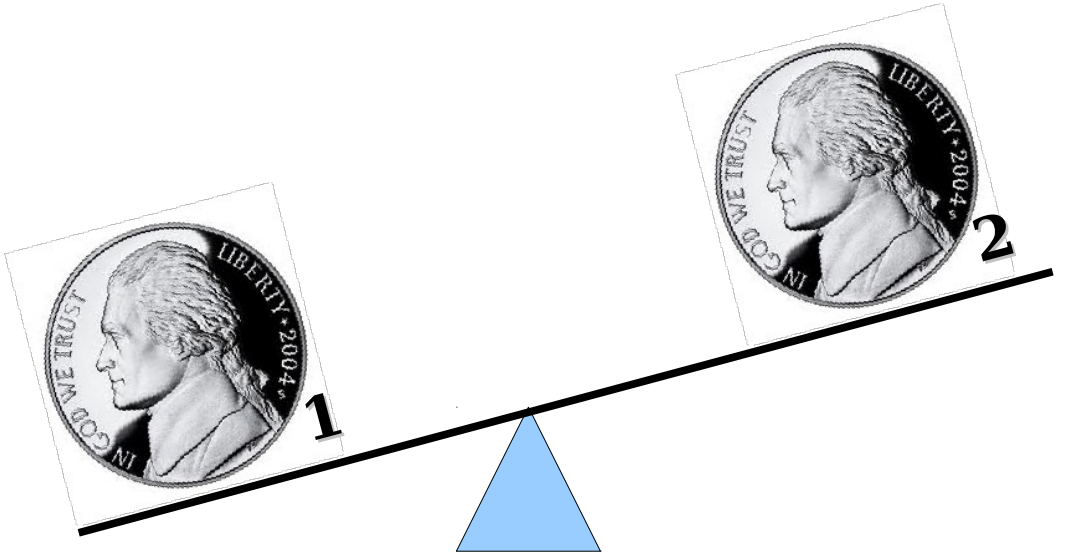

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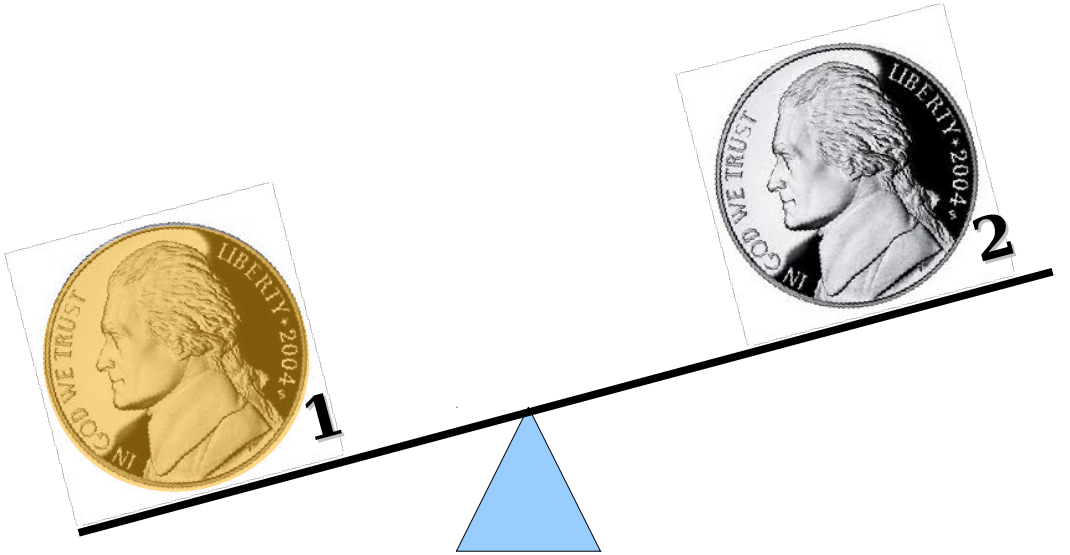

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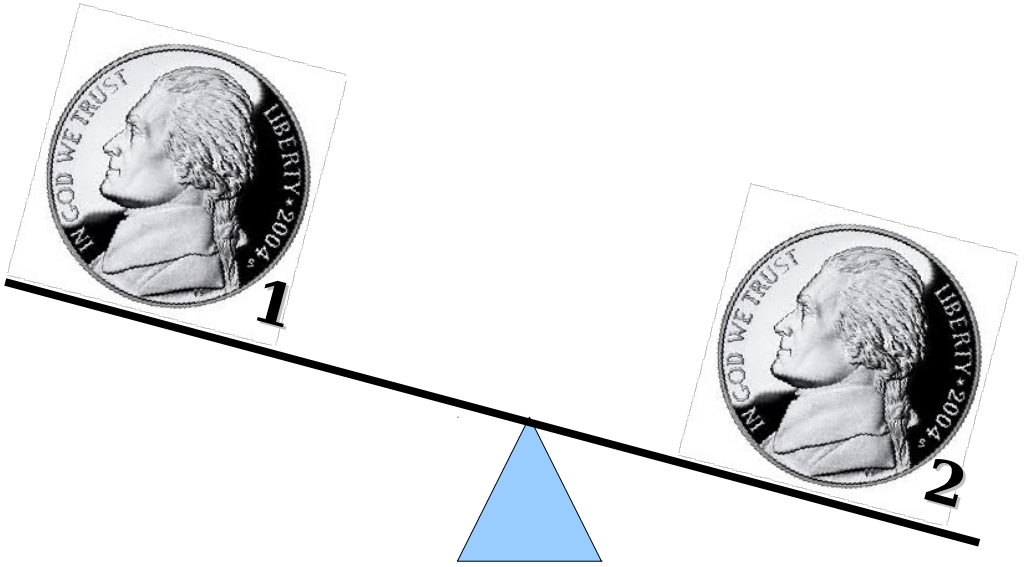

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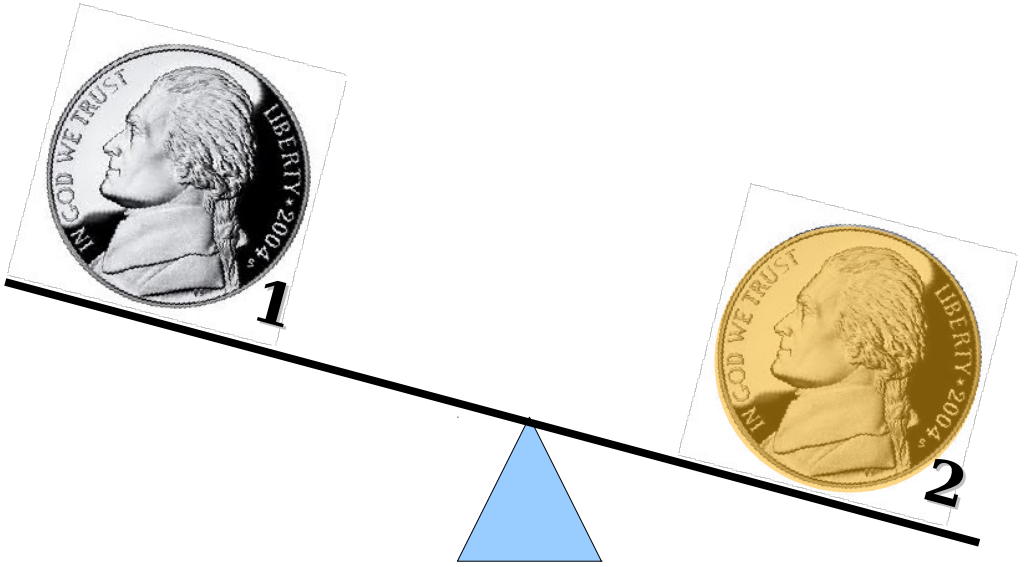

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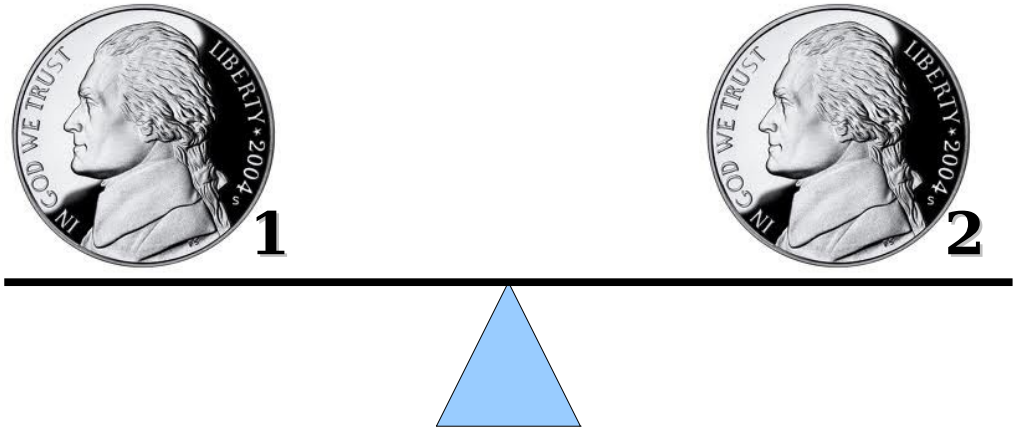

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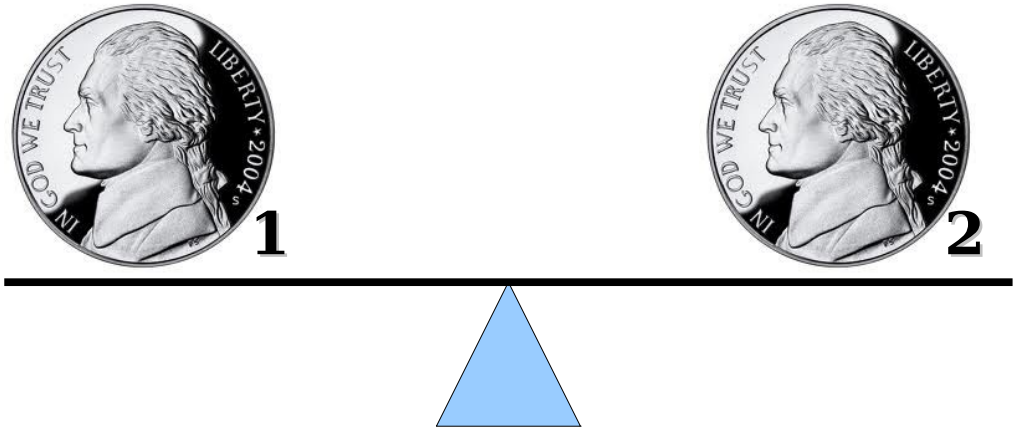
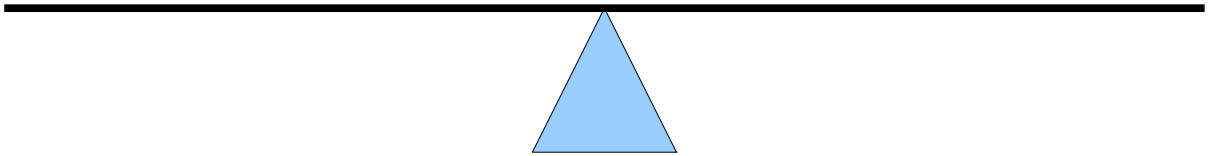


A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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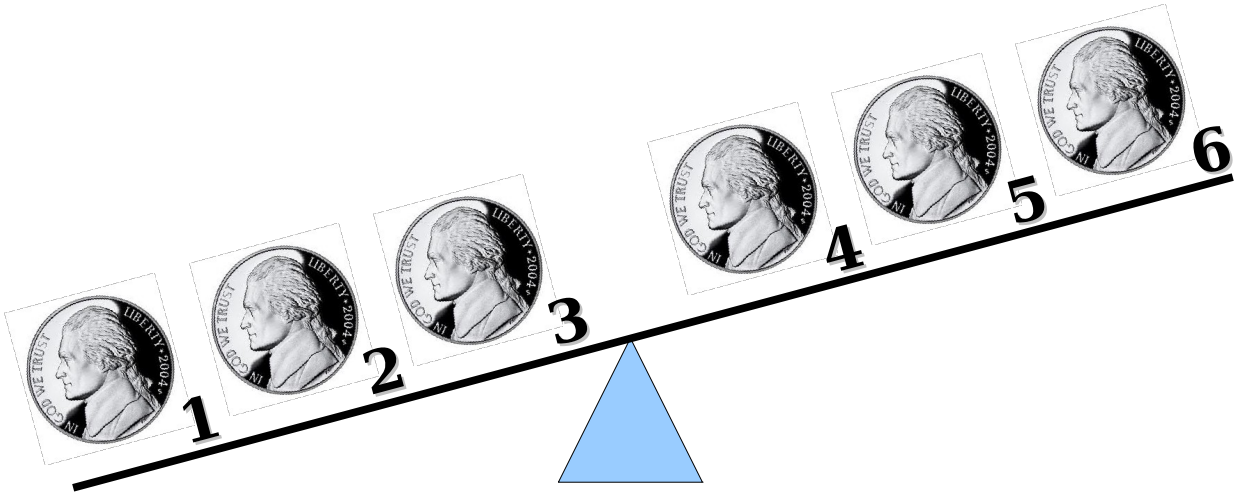

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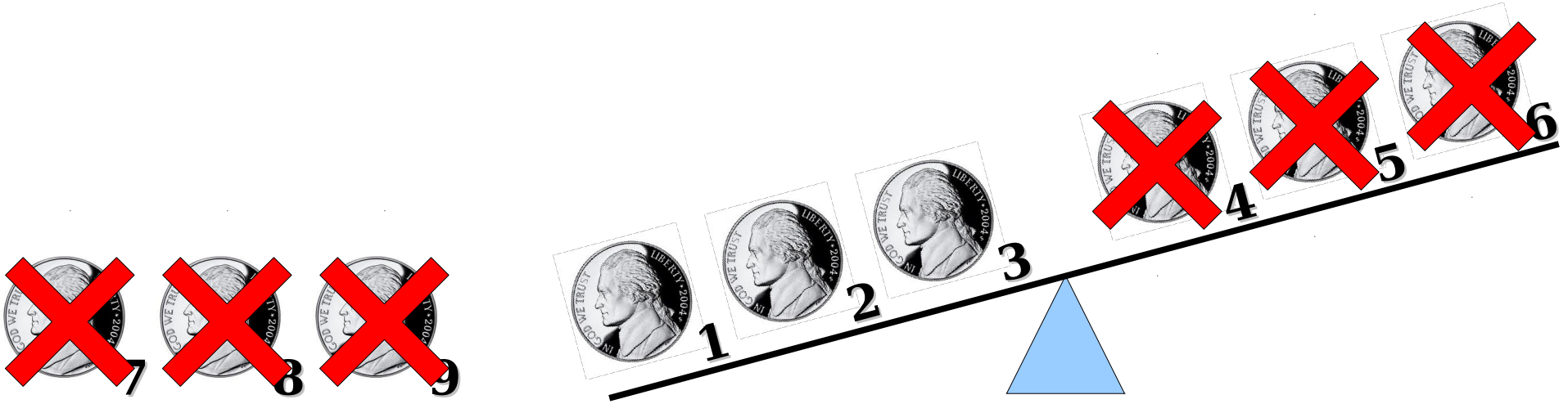

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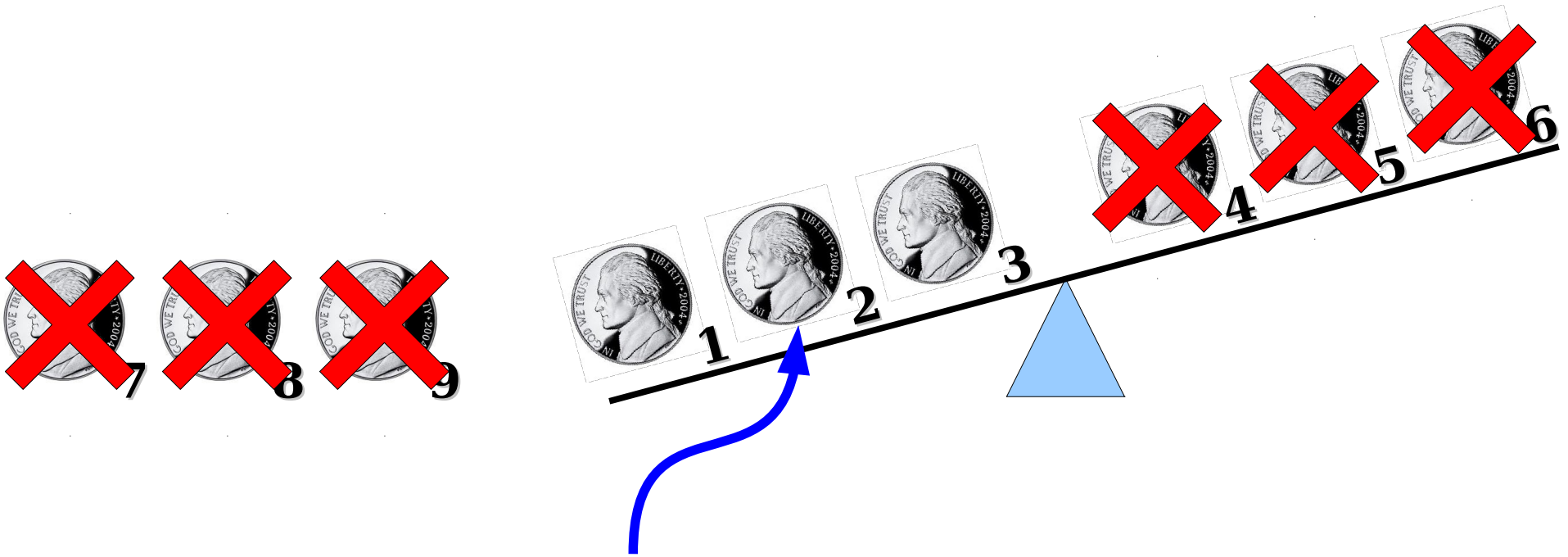
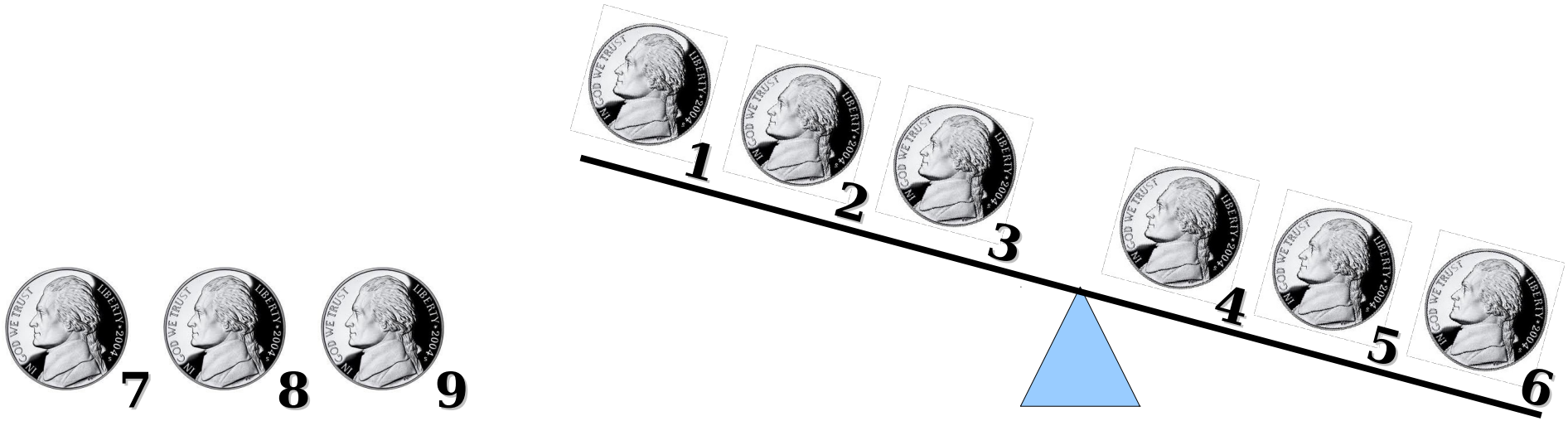


Now we have one weighing to find the counterfeit out of these three

Finding the Counterfeit Coin



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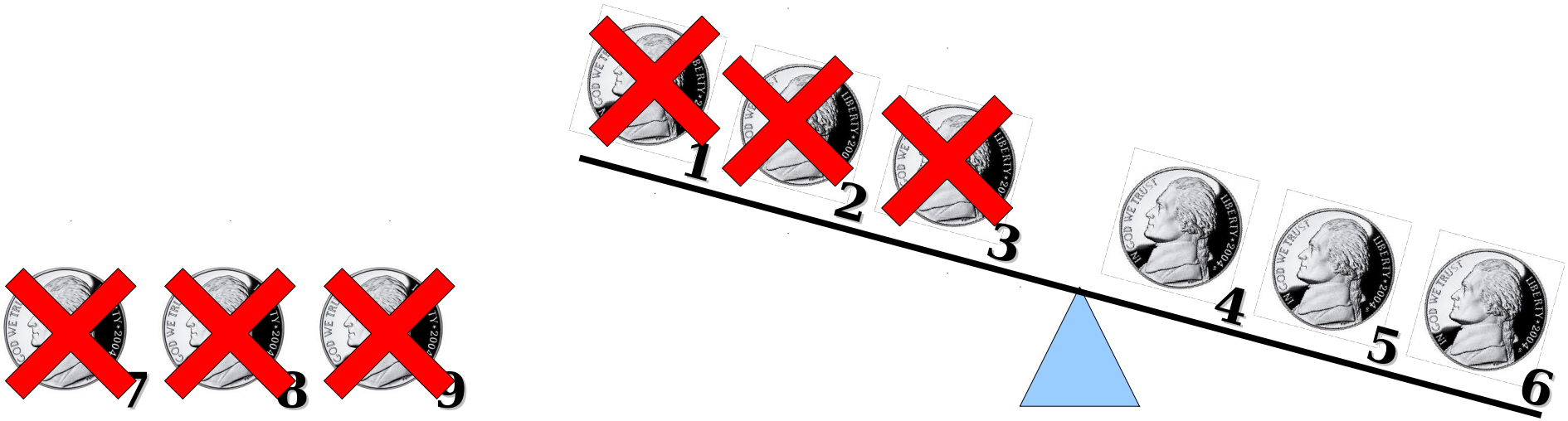

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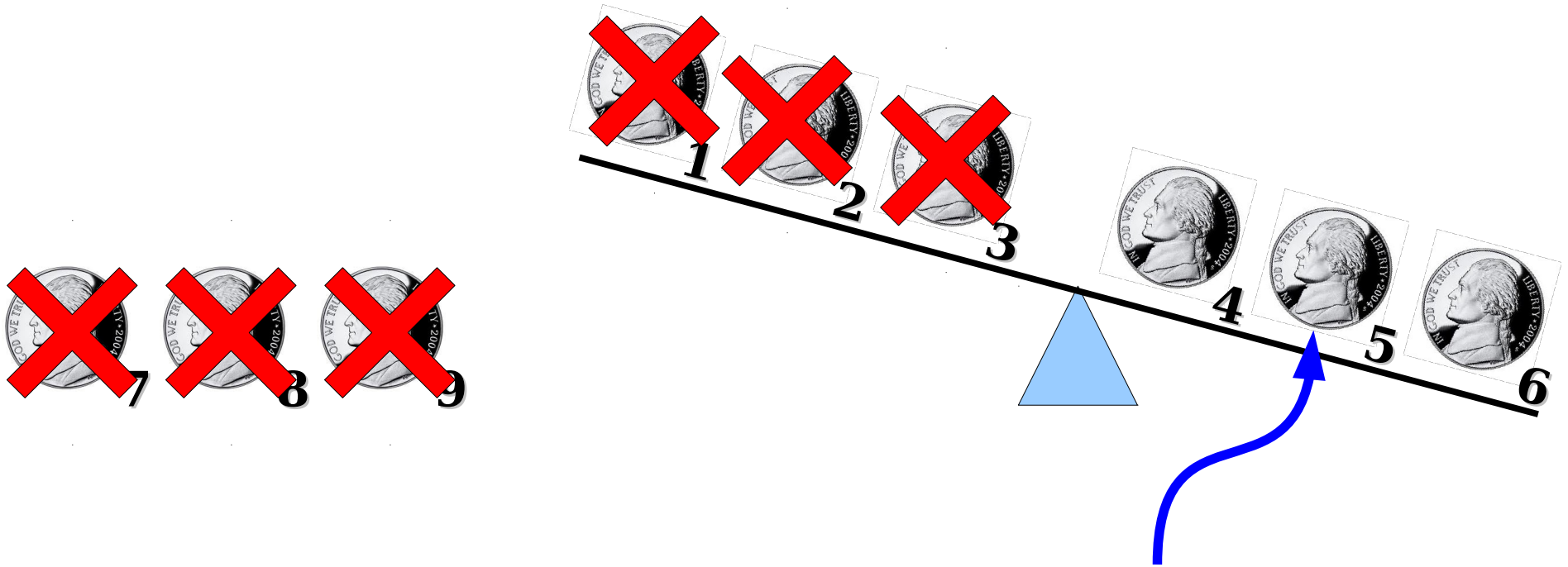

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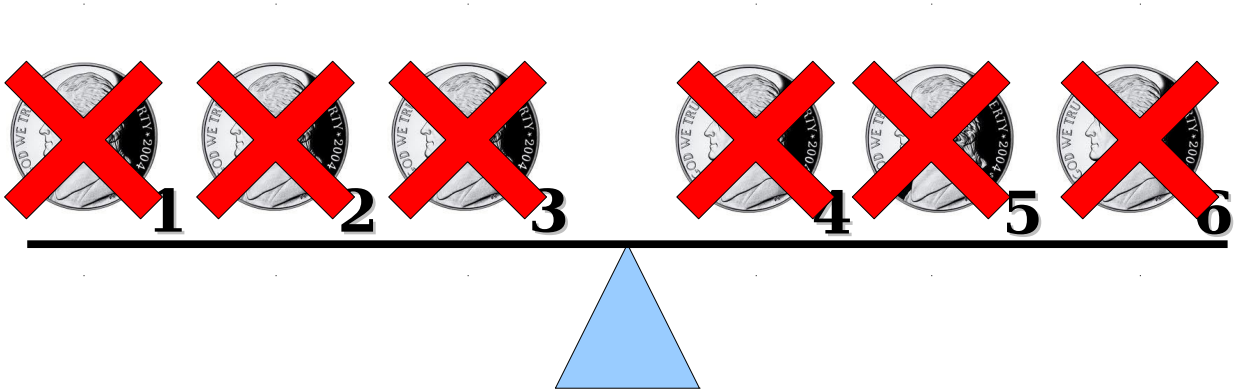

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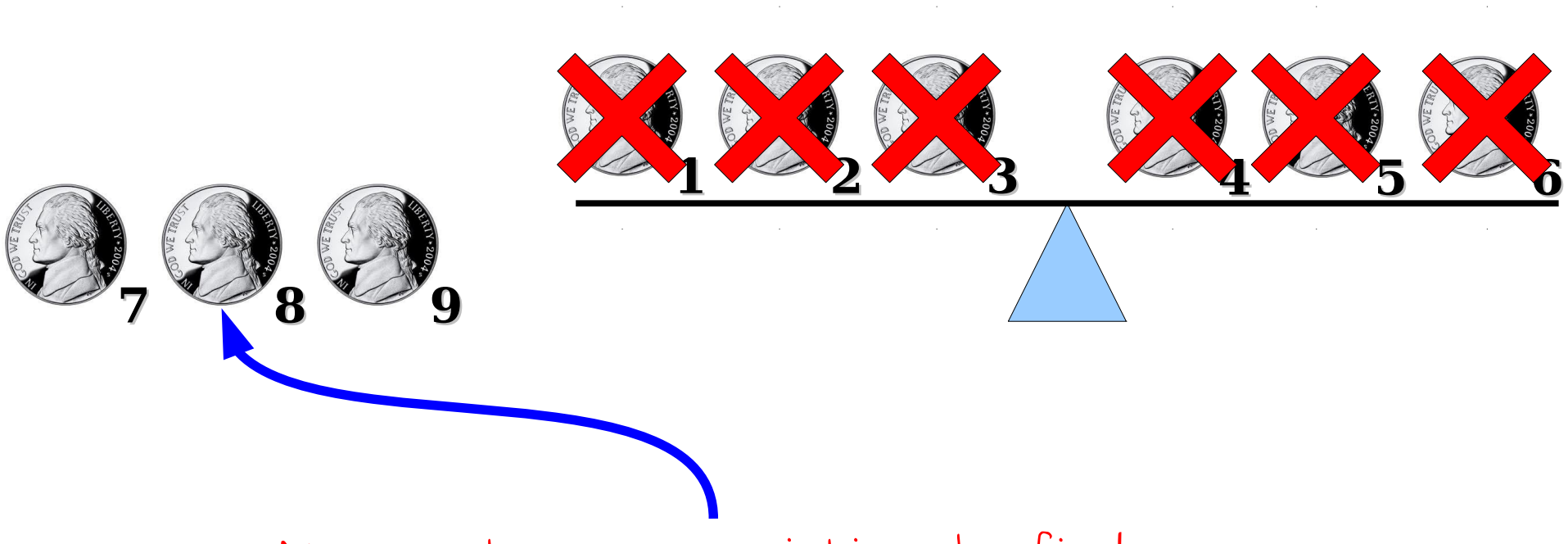

Now we have one weighing to find the counterfeit out of these three

If we have n weighings on the scale, what is the largest number of coins out of which we can find the counterfeit?

A Pattern

- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One coin**, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$1, 3, 9 = 3^0, 3^1, 3^2$

Does this pattern continue?

Theorem: Given n weighings, we can detect which of 3^n coins is counterfeit.

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Proof: By induction. Let $P(n)$ be "Given n weighings, we can detect which of the 3^n coins is counterfeit." We prove that $P(n)$ is true for all $n \in \mathbb{N}$.

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For the inductive step, suppose that for some n, $P(n)$ holds, so we can detect which of 3^n coins is counterfeit using n weighings.

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In any case, we can use one weighing to find a group of 3^n coins that contains the counterfeit coin.

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In any case, we can use one weighing to find a group of 3^n coins that contains the counterfeit coin. By the inductive hypothesis, we can use n more weighings to find which of these 3^n coins is counterfeit. Combined with our original weighing, this means that we can find the counterfeit of 3^{n+1} coins in $n + 1$ weighings. Thus $P(n + 1)$ holds, completing the induction.

Theorem: Given n weighings, we can detect which of 3^n coins is counterfeit.

Proof: By induction. Let $P(n)$ be "Given n weighings, we can detect which of the 3^n coins is counterfeit." We prove that $P(n)$ is true for all $n \in \mathbb{N}$.

For the base case, we show $P(0)$ holds, which means that we can detect which of $3^0 = 1$ coins is counterfeit in no weighings. This is trivial – if there is only one coin, it must be the counterfeit.

For the inductive step, suppose that for some n, $P(n)$ holds, so we can detect which of 3^n coins is counterfeit using n weighings. We will show $P(n + 1)$ holds, meaning we can detect which of 3^{n+1} coins is counterfeit using $n + 1$ weighings.

Given 3^{n+1} coins, split them into three equal groups of size 3^n; call the groups A, B, and C. Put the coins in set A on one side of the scale and the coins in set B on the other side. There are three cases to consider:

Case 1: Side A is heavier. Then the counterfeit coin must be in group A.
Case 2: Side B is heavier. Then the counterfeit coin must be in group B.
Case 3: The scale is balanced. Then the counterfeit coin must be in group C, since it isn't in groups A or B.

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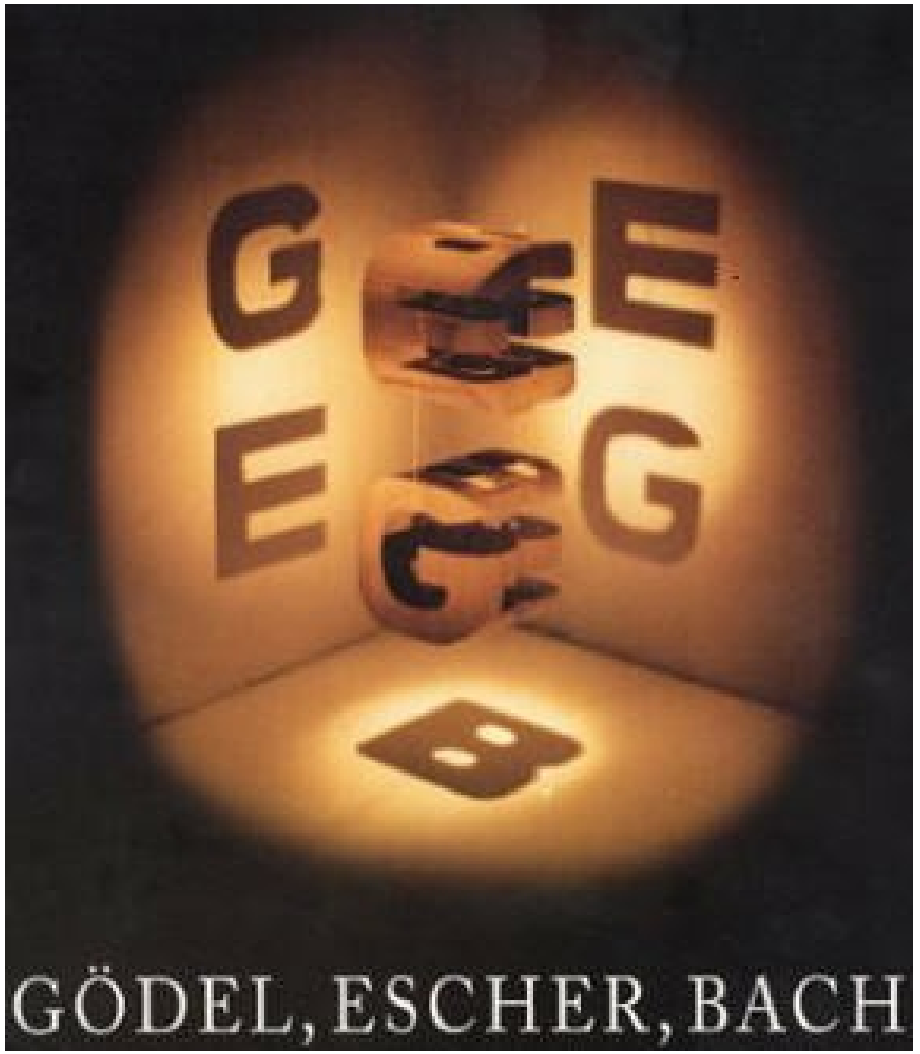
The **MU** Puzzle

Gödel, Escher Bach: An Eternal Golden Braid



- Pulitzer-Prize winning book exploring recursion, computability, and consciousness.
- Written by Douglas Hofstadter, computer scientist at Indiana University.
- A great (but dense!) read.

The **MU** Puzzle

- Begin with the string **MI**.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the **M**: for example, **MIIU** becomes **MIIUIIU** or **MI** becomes **MII**.
  - Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**
  - Append **U** to the string if it ends in **I**: **MI** becomes **MIU**
  - Remove any **UU**: **MUUU** becomes **MU**
- **Question**: How do you transform **MI** to **MU**?

A) Double the contents of the string after `M`.

B) Replace `III` with `U`.

C) Remove `UU`

D) Append `U` if the string ends in `I`.

MI

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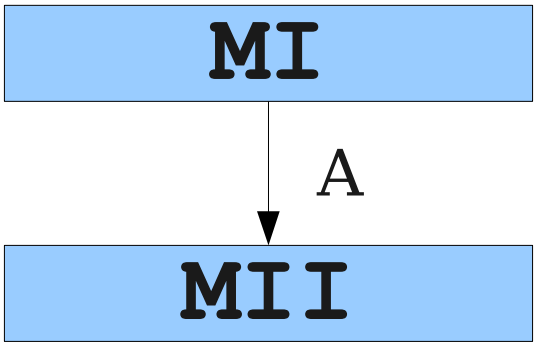

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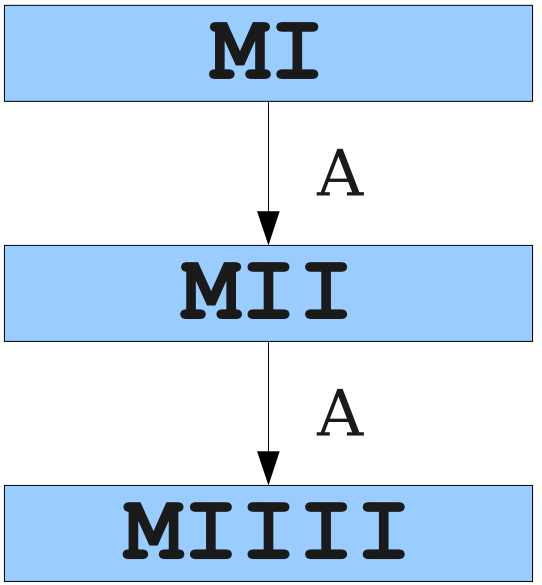

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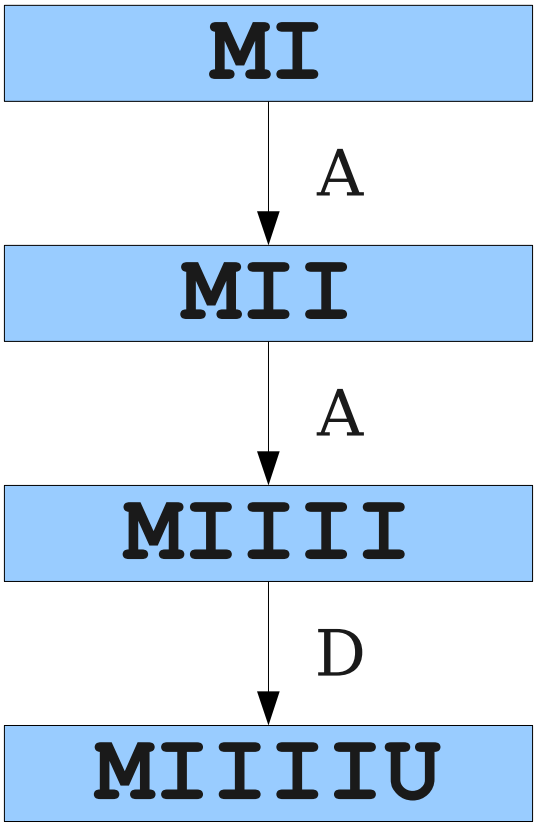

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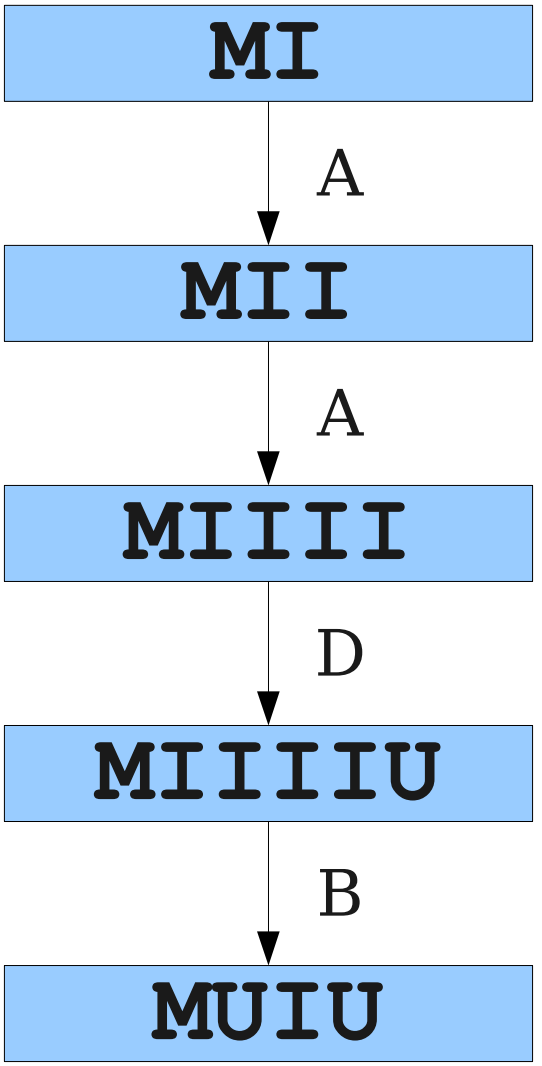

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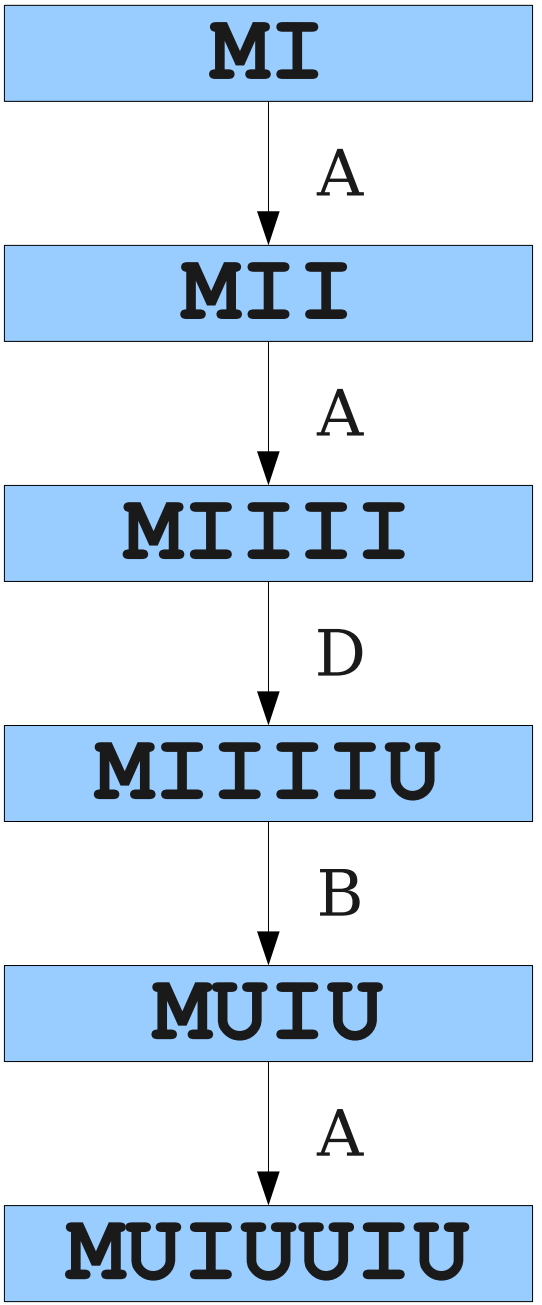

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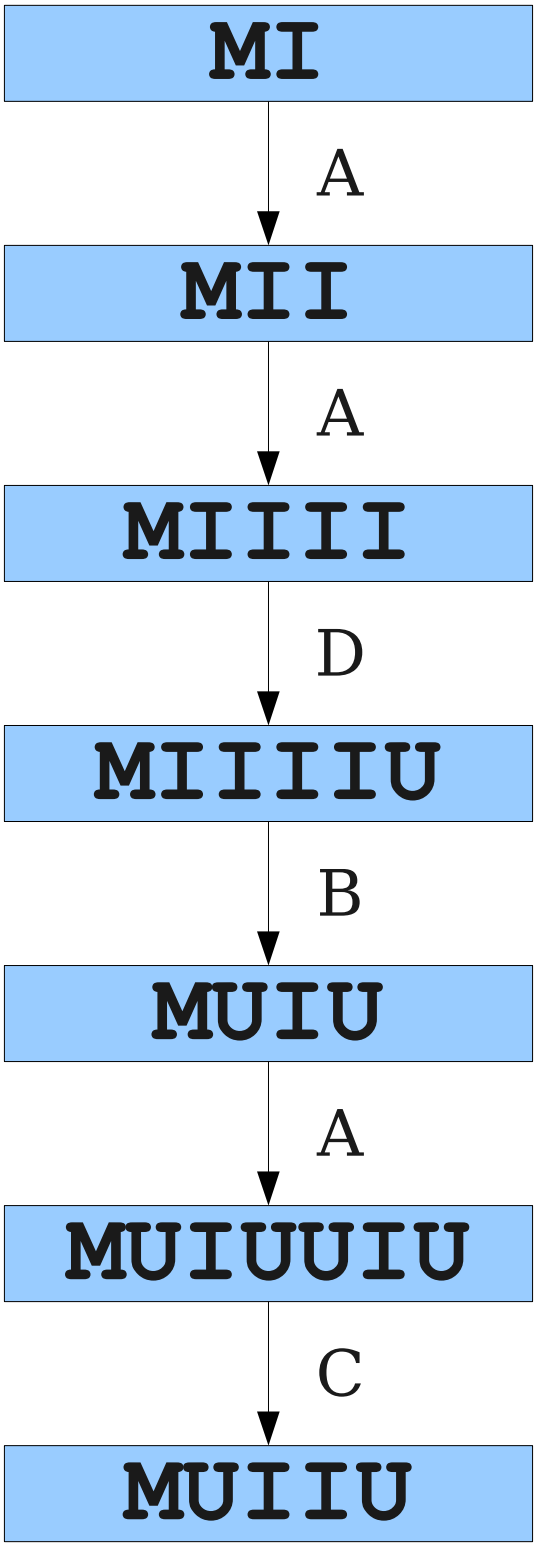

Try It!

Starting with **`MI`**, apply these operations to make **`MU`**:

A) Double the contents of the string after **`M`**.

B) Replace **`III`** with **`U`**.

C) Remove **`UU`**

D) Append **`U`** if the string ends in **`I`**.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting `I`'s

Counting **I**'s

MI
MII
MIIII
MIIIIU
MIIIIUIIIIU
MIIIIUUIU
MIIIIUUIUIIIIUUIU
MUIUUIUIIIIUUIU

Counting **I**'s

MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

Counting **I**'s

MI
MII
MIIII
MIIIIU
MIIIIUIIIIU
MIIIIUUIU
MIIIIUUIUIIIIUUIU
MUIUUIUIIIIUUIU
1
2
4
4
8
5
10
7
None of these are multiples of three...

The Key Insight

- Initially, the number of `I`'s is **not** a multiple of three.
- To make `MU`, the number of `I`'s must end up as a multiple of three.
- Can we *ever* make the number of `I`'s a multiple of three?

Lemma: Beginning with **`MI`** and applying any legal sequence of moves, the number of **`I`**'s is never a multiple of 3.

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Proof: By induction. Let $P(n)$ be "After making n legal moves starting with string **MI**, the number of **I**'s is not a multiple of 3." We prove $P(n)$ holds for all $n \in \mathbb{N}$.

As a base case, to prove $P(0)$, we show that after making no moves the number of **I**'s is not a multiple of 3. **MI** has one **I** in it, which is not a multiple of 3.

For the inductive step, assume for some $n \in \mathbb{N}$ that $P(n)$ holds and that after any sequence of n operations, the number of **I**'s is not a multiple of 3. We prove $P(n + 1)$, that after $n + 1$ operations, the number of **I**'s is not a multiple of 3.

To see this, note that any sequence of $n + 1$ operations is formed from a sequence of n operations followed by one final operations. By the inductive hypothesis, after the first n operations, the number of **I**'s is not a multiple of 3. Thus before performing the $(n + 1)$st operation, the number of **I**'s either has the form $3k + 1$ or $3k + 2$ for some $k \in \mathbb{N}$. Now, consider the $(n + 1)$st operation:

Case 1: It's "double the string after the **M**." Then we either end up with either $2(3k + 1) = 6k + 2 = 3(2k) + 2$ or $2(3k + 2) = 6k + 4 = 3(2k + 1) + 1$ copies of **I**, neither of which is a multiple of 3.

Case 2: It's "delete **UU**" or "append **U**." Then the number of **I**'s is unchanged.

Case 3: It's "delete **III**." Then we either go from $3k + 1$ to $3k + 1 - 3 = 3(k - 1) + 1$ **I**'s, or from $3k + 2$ to $3k + 2 - 3 = 3(k - 1) + 2$ **I**'s, neither of which is a multiple of 3.

Thus any sequence of $n + 1$ moves starting with **MI** ends with the number of **I**'s not a multiple of three. Thus $P(n + 1)$ holds, completing the induction. ■

Theorem: The **MU** puzzle has no solution.

Proof: By contradiction; assume it has a solution. By our lemma, the number of **I**'s in the final string must not be a multiple of 3. However, for the solution to be valid, the number of **I**'s must be 0, which is a multiple of 3. We have reached a contradiction, so our assumption was wrong and the **MU** puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a **loop invariant**.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Why does induction work?

One Definition of $\mathbb{N}$

- The set $\mathbb{N}$ is often defined as follows:
  - $0 \in \mathbb{N}$.
  - If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
  - No other objects are in $\mathbb{N}$.
- Under this definition, induction makes sense: every natural number is either 0 or one plus some other natural number.

Next Time

- **Strong Induction**
  - A useful variant of induction.