Binary Relations

Problem Set Two checkpoint due in the box up front if you're using a late period.

Studying Relationships

- We have just explored the graph as a way of studying relationships between objects.
- However, graphs are not the only formalism we can use to do this.

Relationships

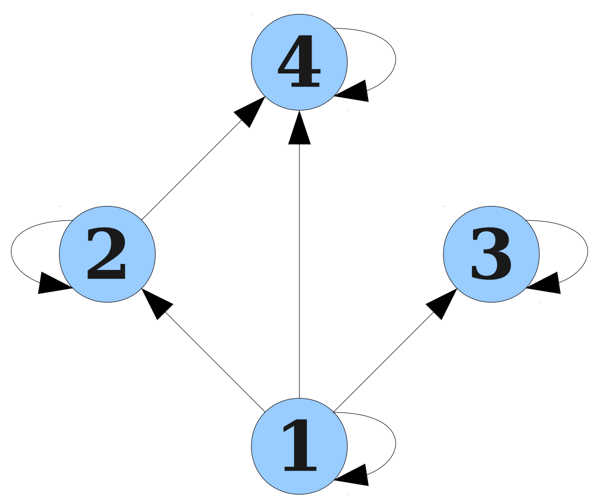
- We've seen different types of relationships
  - between sets:
    - $A \subseteq B \quad A \subset B$
  - between numbers:
    - $x < y \quad x \equiv_k y$
  - between nodes in a graph:
    - $u \leftrightarrow v$
- **Goal:** Focus on these types of relationships and study their properties.

Binary Relations

- Intuitively speaking: a **binary relation over a set *A*** is some relation R where, for every $x, y \in A$, the statement xRy is either true or false.
- Examples:
  - $<$ can be a binary relation over $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, etc.
  - $\leftrightarrow$ can be a binary relation over V for any undirected graph $G = (V, E)$.
  - $\equiv_k$ is a binary relation over $\mathbb{Z}$ for any integer k.
- We'll give a formal definition later today.

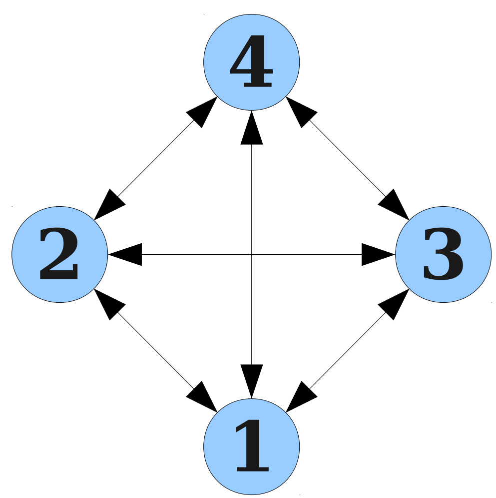
Binary Relations and Graphs

- We can visualize a binary relation R over a set A as a graph:
  - The nodes are the elements of A.
  - There is an edge from x to y iff xRy.
- Example: the relation $a \mid b$ (meaning "a divides b") over the set {1, 2, 3, 4} looks like this:



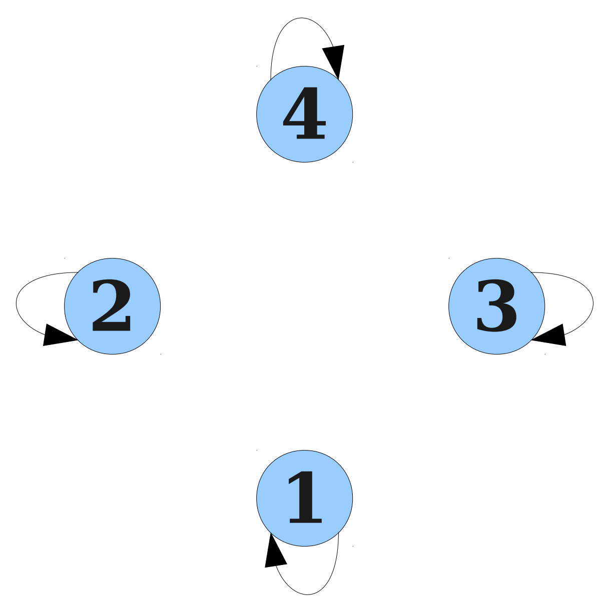
Binary Relations and Graphs

- We can visualize a binary relation R over a set A as a graph:
  - The nodes are the elements of A.
  - There is an edge from x to y iff xRy.
- Example: the relation $a \neq b$ over $\{1, 2, 3, 4\}$ looks like this:



Binary Relations and Graphs

- We can visualize a binary relation R over a set A as a graph:
  - The nodes are the elements of A.
  - There is an edge from x to y iff xRy.
- Example: the relation $a = b$ over $\{1, 2, 3, 4\}$ looks like this:



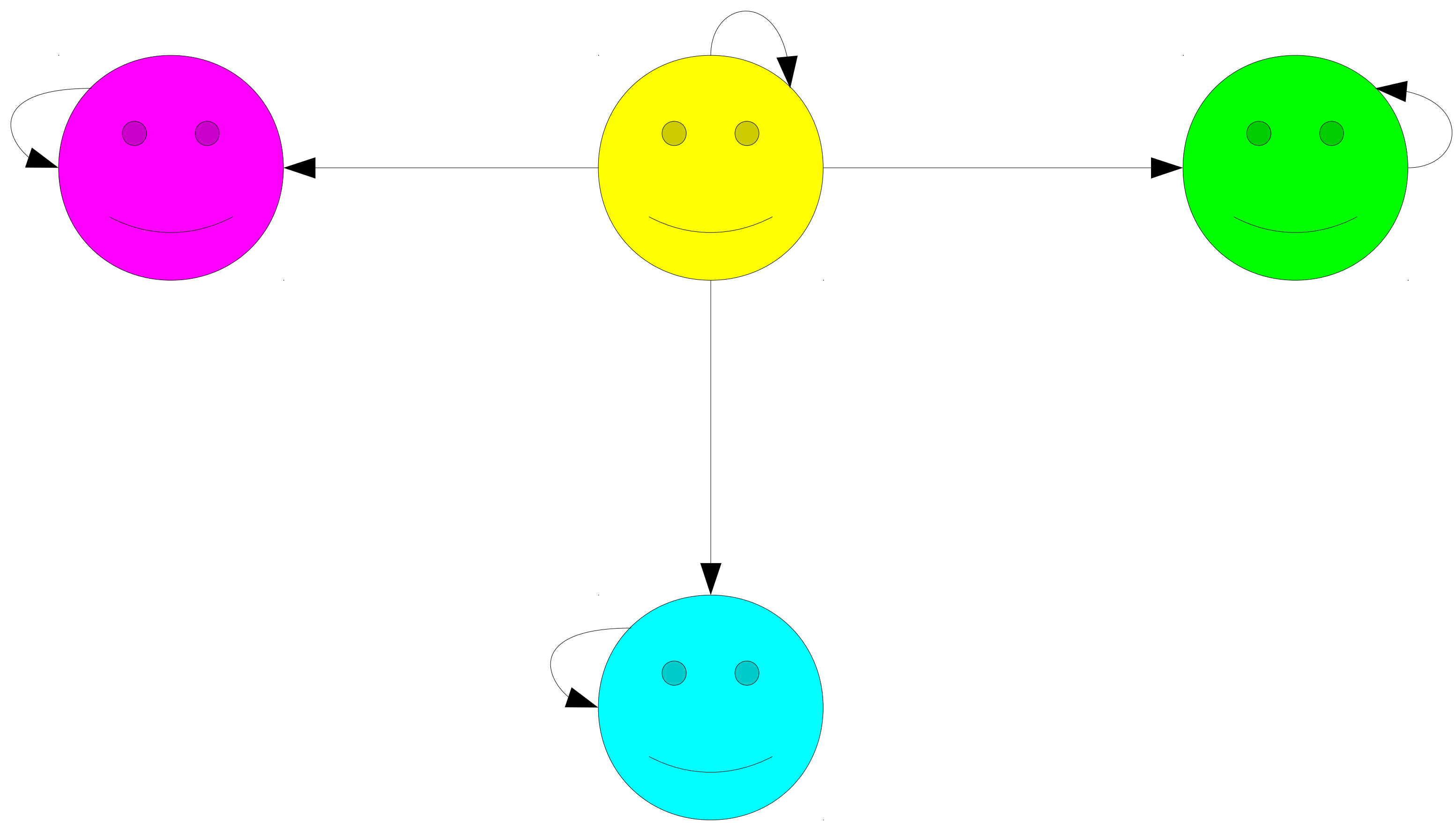
Categorizing Relations

- Collectively, there are few properties shared by all relations.
- We often categorize relations into different types to study relations with particular properties.
- General outline for today:
  - Find certain properties that hold of the relations we've seen so far.
  - Categorize relations based on those properties.
  - See what those properties entail.

Reflexivity

- Some relations always hold for any element and itself.
- Examples:
  - $x = x$ for any x.
  - $A \subseteq A$ for any set A.
  - $x \equiv_k x$ for any x.
  - $u \leftrightarrow u$ for any u.
- Relations of this sort are called **reflexive**.
- Formally: a binary relation R over a set A is **reflexive** iff for all $x \in A$, the relation xRx holds.

An Intuition for Reflexivity

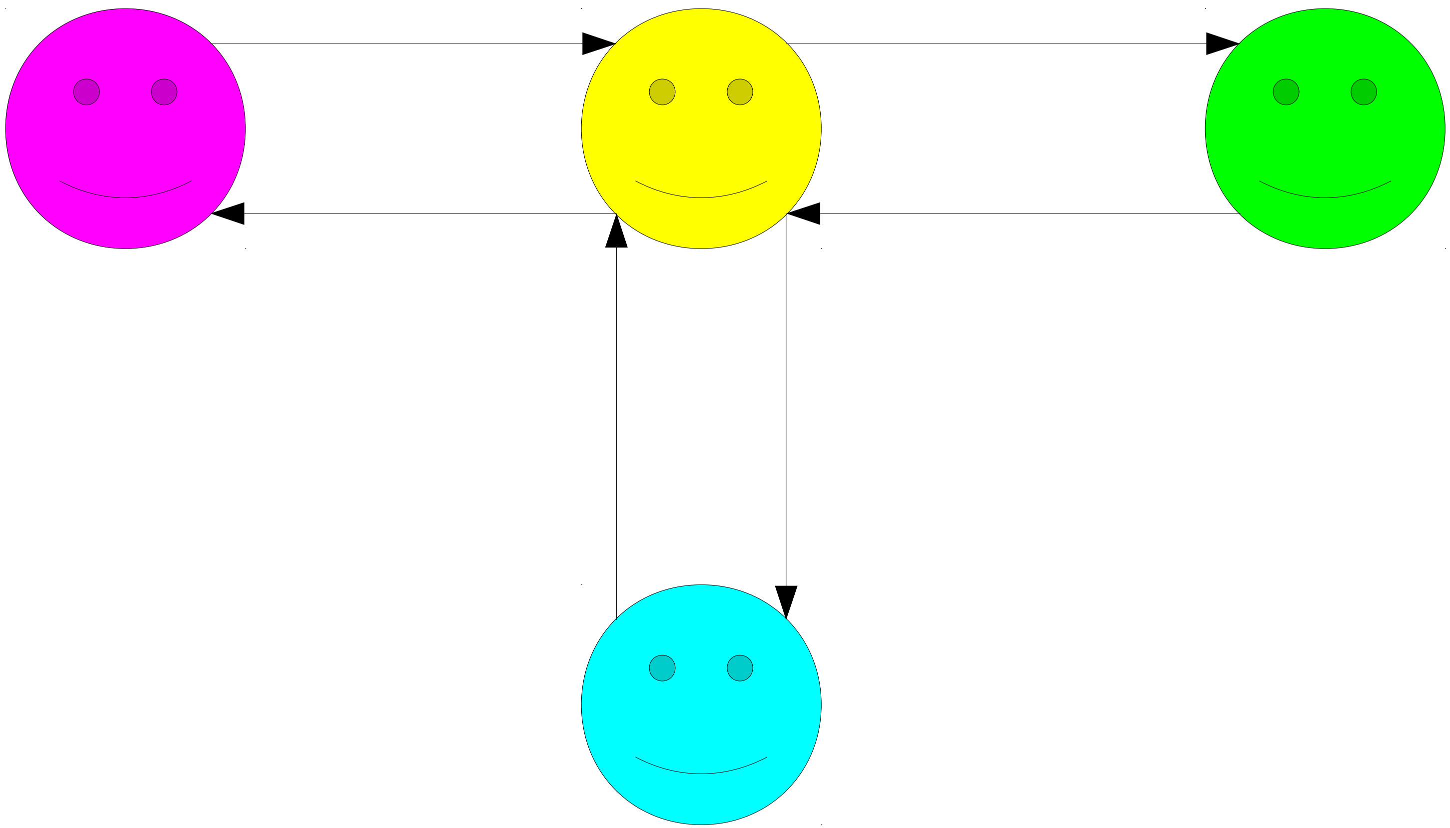


For every $x \in A$, the relation xRx holds.

Symmetry

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $x = y$, then $y = x$.
  - If $u \leftrightarrow v$, then $v \leftrightarrow u$.
  - If $x \equiv_k y$, then $y \equiv_k x$.
- These relations are called **symmetric**.
- Formally: A binary relation R over a set A is called **symmetric** iff for all $x, y \in A$, if xRy, then yRx.

An Intuition for Symmetry

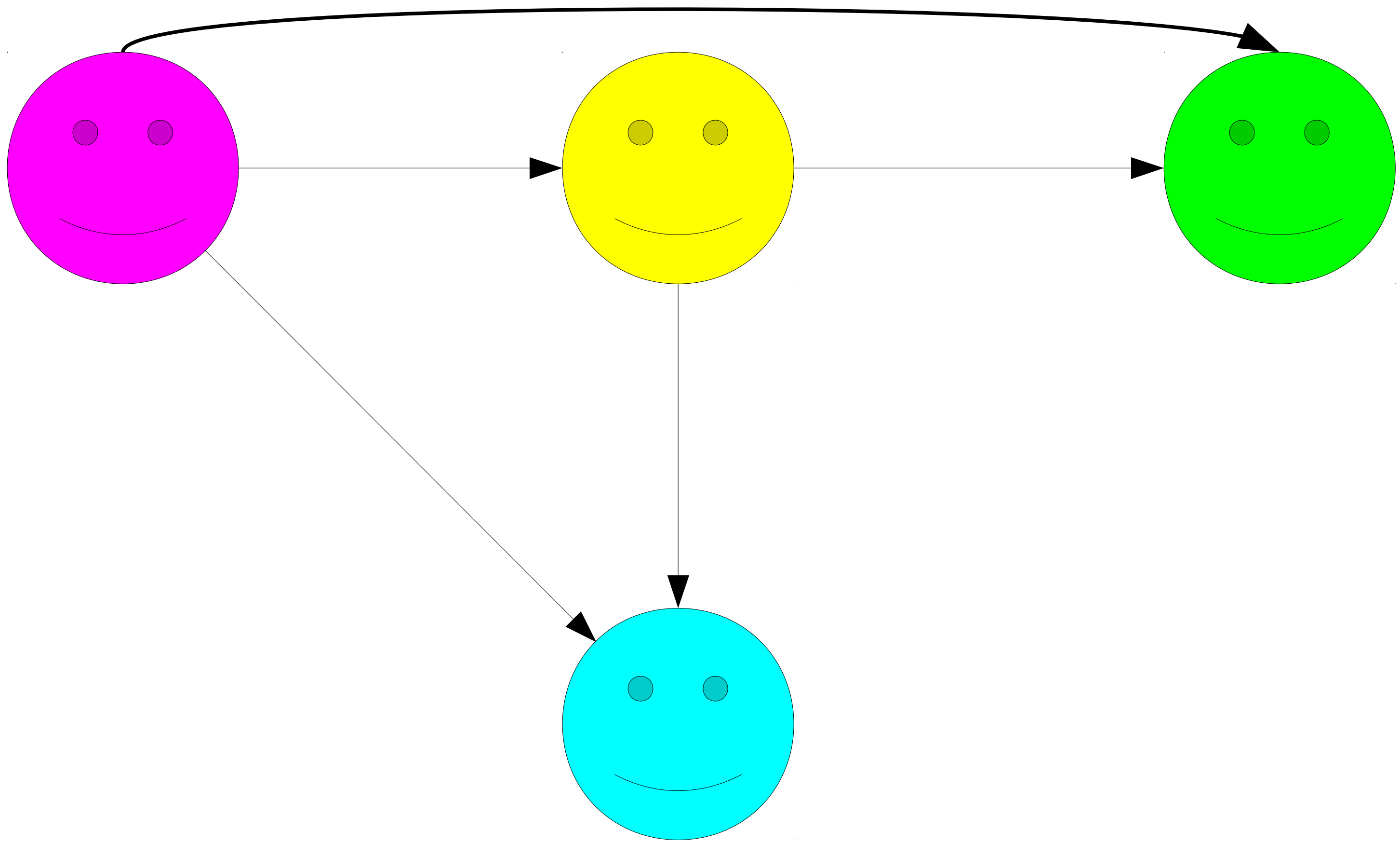


For any $x \in A$ and $y \in A$,
if xRy, then yRx.

Transitivity

- Many relations can be chained together.
- Examples:
  - If $x = y$ and $y = z$, then $x = z$.
  - If $u \leftrightarrow v$ and $v \leftrightarrow w$, then $u \leftrightarrow w$.
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
- These relations are called **transitive**.
- Formally: A binary relation R over a set A is called **transitive** iff for all $x, y, z \in A$, if xRy and yRz, then xRz.

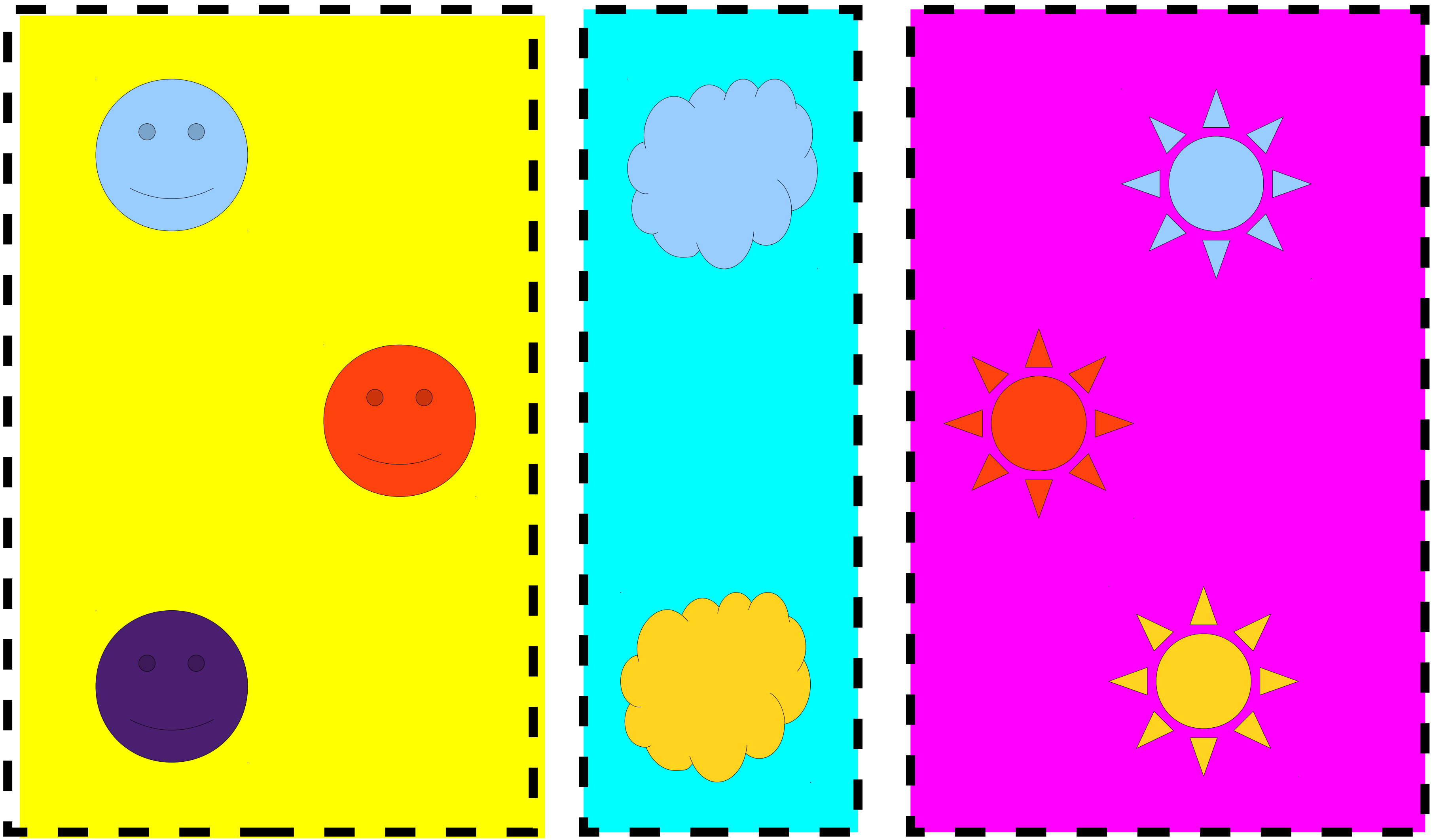
An Intuition for Transitivity



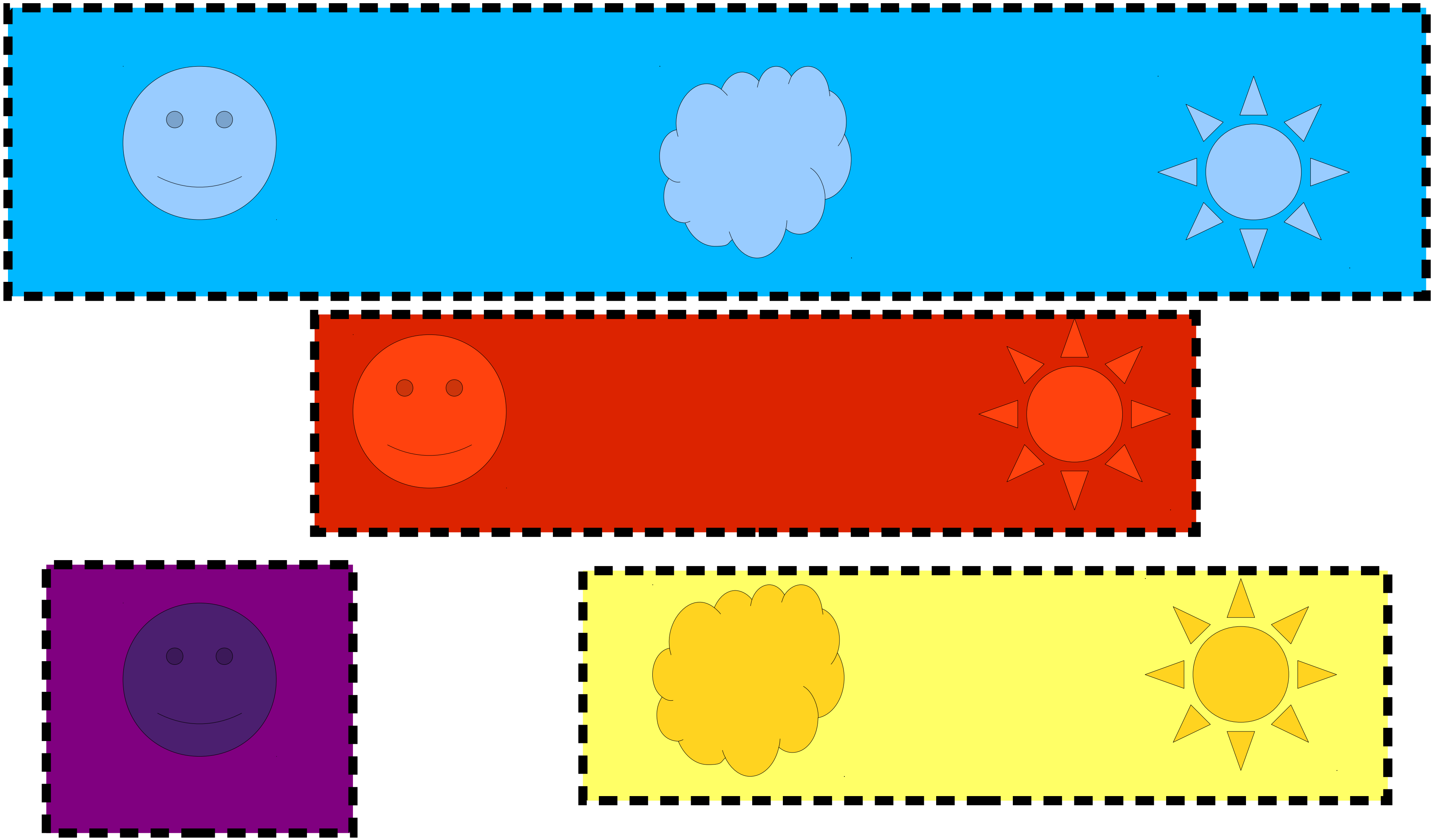
For any x, y, $z \in A$,
if xRy and yRz,
then xRz.

Equivalence Relations

- Some relations are reflexive, symmetric, and transitive:
  - $x = y$
  - $u \leftrightarrow v$
  - $x \equiv_k y$
- Definition: An **equivalence relation** is a relation that is reflexive, symmetric and transitive.



$xRy \equiv x$ and y have the same shape.



$xRy \equiv x$ and y have the same **color**.

Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the **equivalence class of x** is the set

$$[x]_R = \{\ y \in A \mid xRy\ \}$$

- $[x]_R$ is the set of all elements of A that are related to x.
- ***Theorem:*** If R is an equivalence relation over A, then every $a \in A$ belongs to exactly one equivalence class.

Closing the Loop

- In any graph $G = (V, E)$, we saw that the connected component containing a node $v \in V$ is given by

$$\{ x \in V \mid v \leftrightarrow x \}$$

- What is the equivalence class for some node $v \in V$ under the relation $\leftrightarrow$?

$$[v]_{\leftrightarrow} = \{ x \in V \mid v \leftrightarrow x \}$$

- *Connected components are just equivalence classes of* $\leftrightarrow$!

Why This Matters

- Developing the right definition for a connected component was challenging.
- Proving every node belonged to exactly one equivalence class was challenging.
- Now that we know about equivalence relations, we get both of these for free!
- **If you arrive at the same concept in two or more ways, it is probably significant!**

Your Questions

“What are practical applications of planar graphs (besides the four-color theorem)?”

"How is complete induction any better than normal induction? If you show $P(0)$ as your base case, don't both types of induction prove that $P(n)$ is true for any natural number n?"

Back to Relations!

Partial Orders

Partial Orders

- Many relations are equivalence relations:

$$x = y \qquad x \equiv_k y \qquad u \leftrightarrow v$$

- What about these sorts of relations?

$$x \leq y \qquad x \subseteq y$$

- These relations are called **partial orders**, and we'll explore their properties next.

Antisymmetry

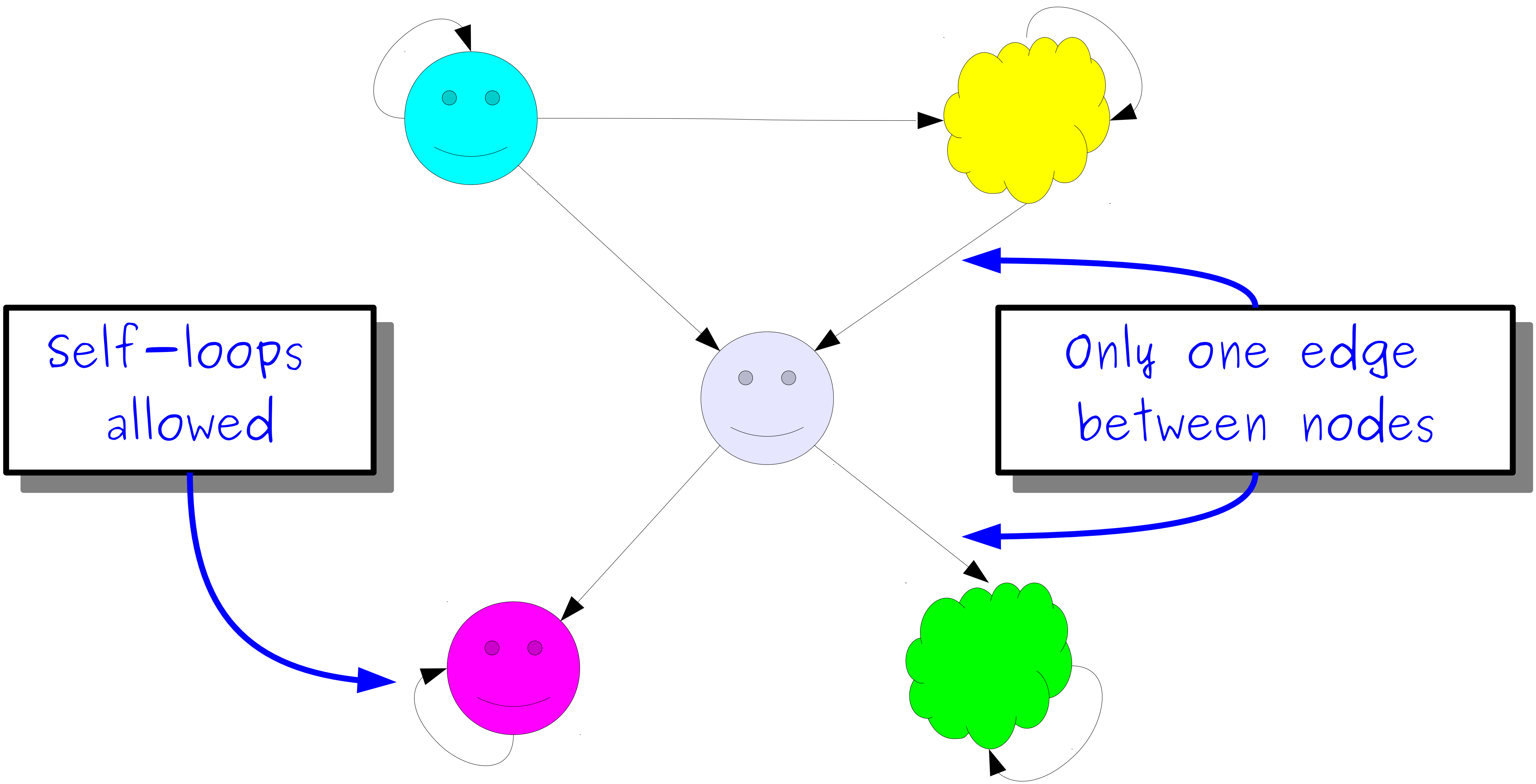
A binary relation R over a set A is called **antisymmetric** iff

For any $x \in A$ and $y \in A$,
If xRy and $x \neq y$, then $y \not R x$.

Equivalently:

For any $x \in A$ and $y \in A$,
if xRy and yRx, then $x = y$.

An Intuition for Antisymmetry



For any $x \in A$ and $y \in A$,
If xRy and $y \neq x$, then $y \not R x$.

Partial Orders

- A binary relation R is a **partial order** over a set A iff it is
  - **reflexive**,
  - **antisymmetric**, and
  - **transitive**.

Why "partial"?

2012 Summer Olympics

Gold	Silver	Bronze	Total
46	29	29	104
38	27	23	88
29	17	19	65
24	26	32	82
13	8	7	28
11	19	14	44
11	11	12	34

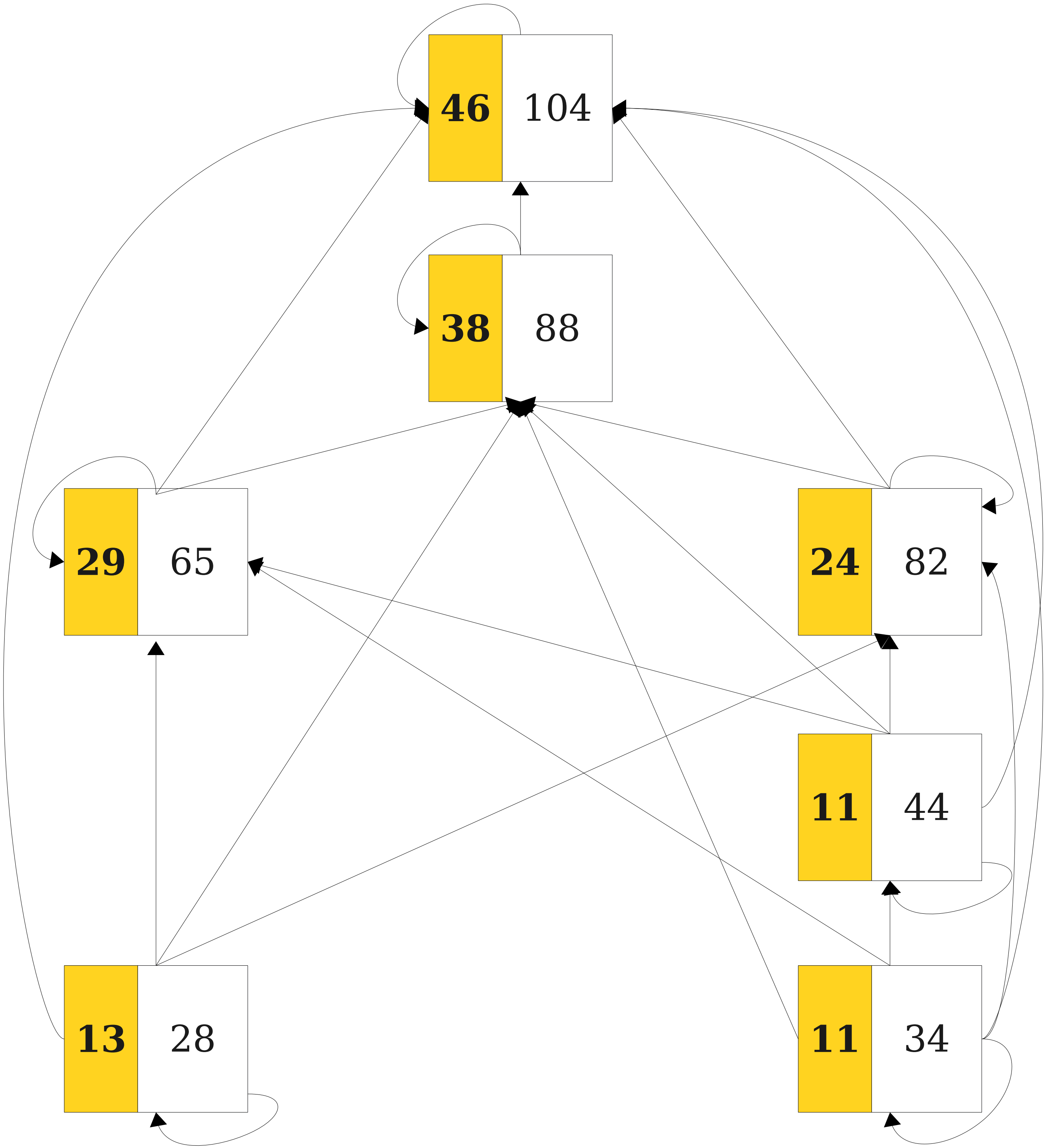
Inspired by http://tartarus.org/simon/2008-olympics-hasse/
Data from http://www.london2012.com/medals/medal-count/

Define the relationship

$$(\mathbf{gold}_0, \mathbf{total}_0)\boldsymbol{R}(\mathbf{gold}_1, \mathbf{total}_1)$$

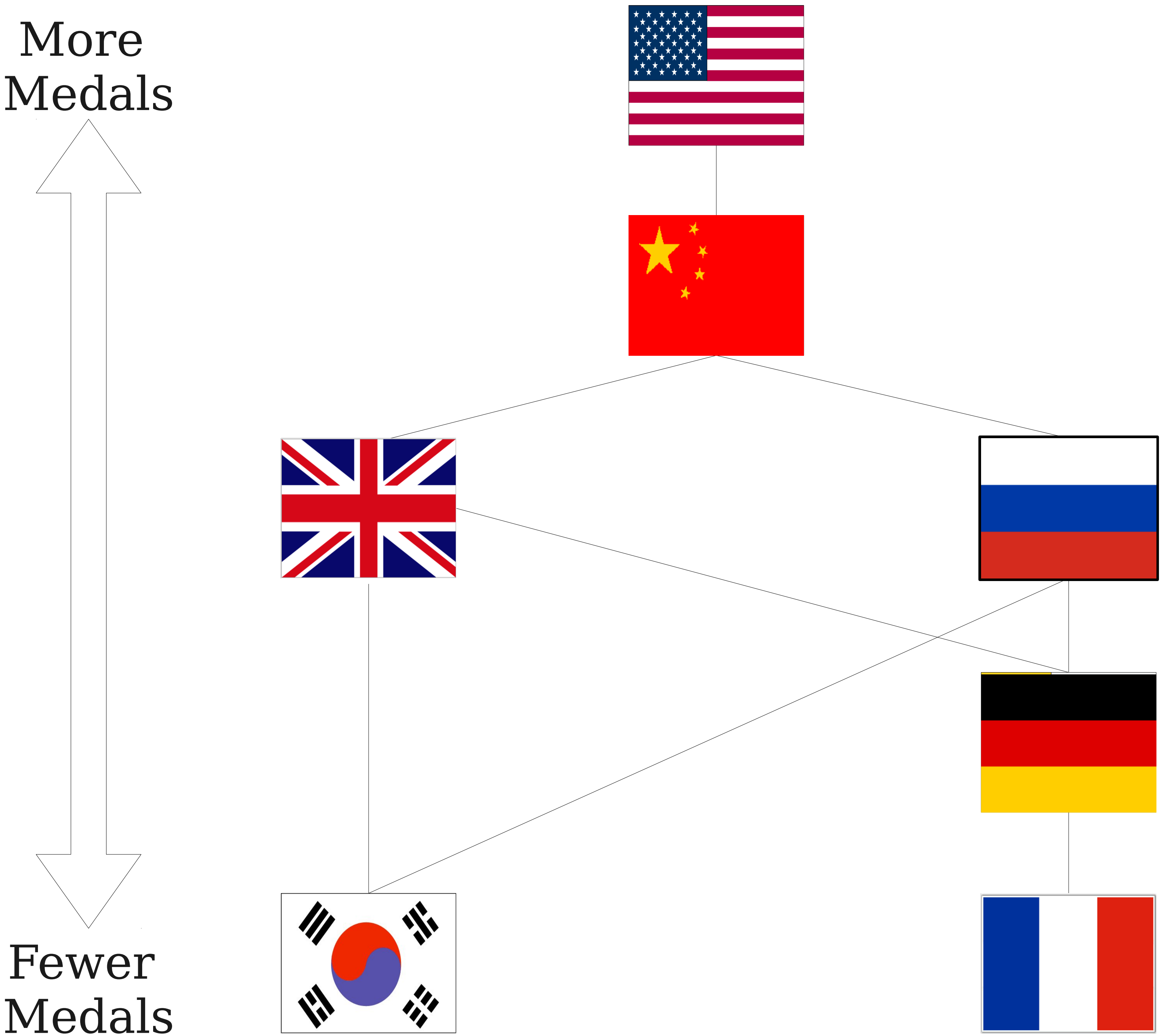
to be true when

$$\mathbf{gold}_0 \le \mathbf{gold}_1 \text{ and } \mathbf{total}_0 \le \mathbf{total}_1$$


46
104
38
88
29
65
24
82
11
44
13
28
11
34

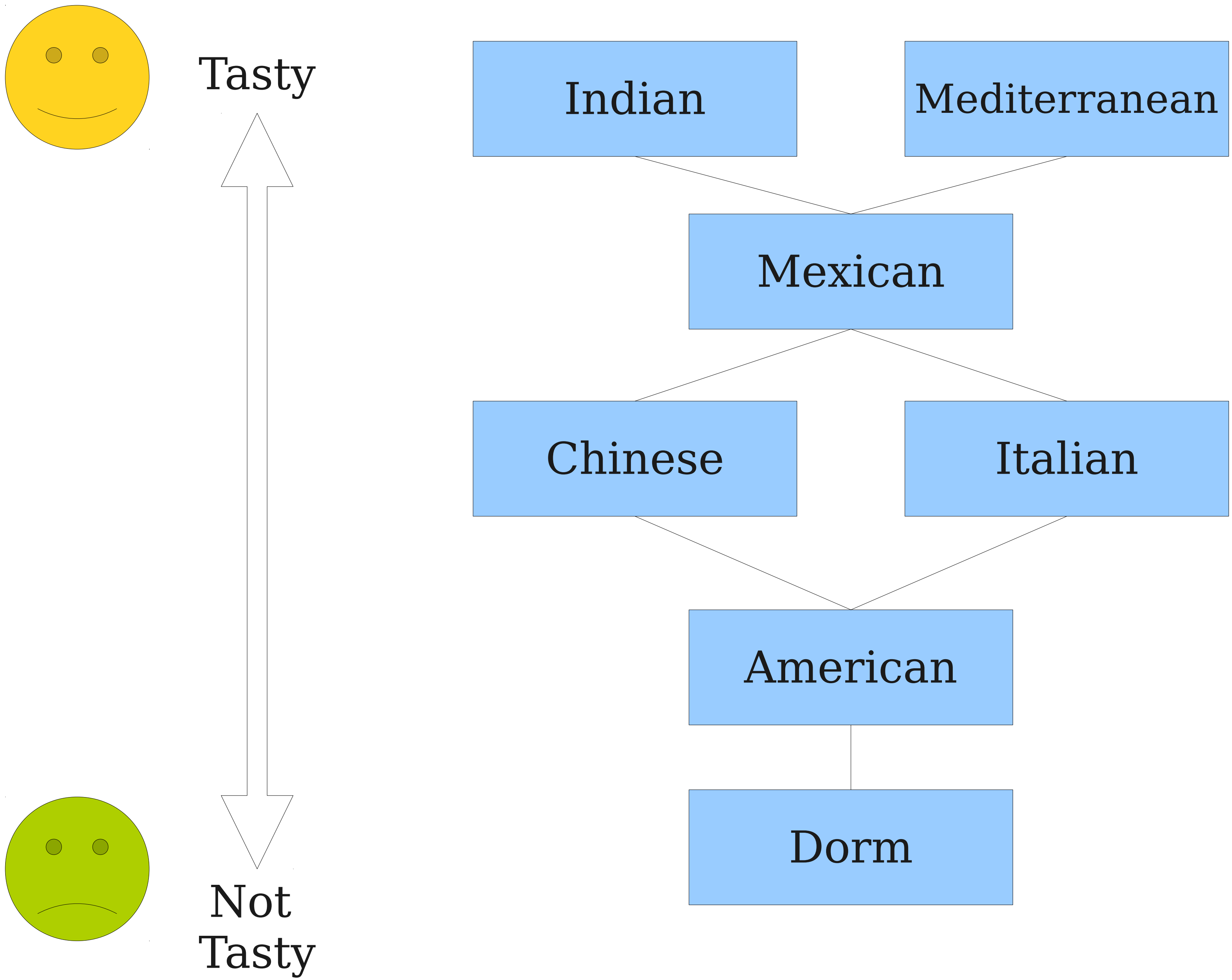
Partial and Total Orders

- A binary relation R over a set A is called **total** iff for any $x \in A$ and $y \in A$, at least one of xRy or yRx is true.
- A binary relation R over a set A is called a **total order** iff it is a partial order and it is total.
- Examples:
  - Integers ordered by $\leq$.
  - Strings ordered alphabetically.

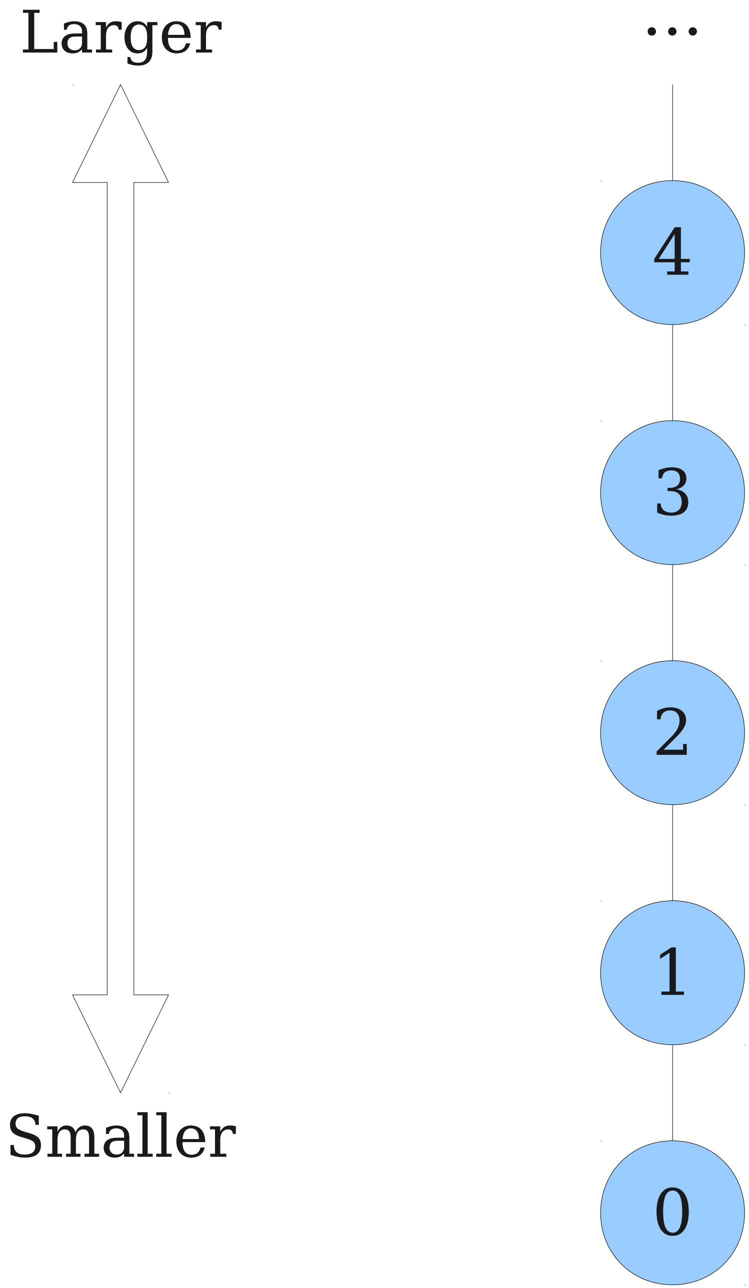

More
Medals
Fewer
Medals

Hasse Diagrams

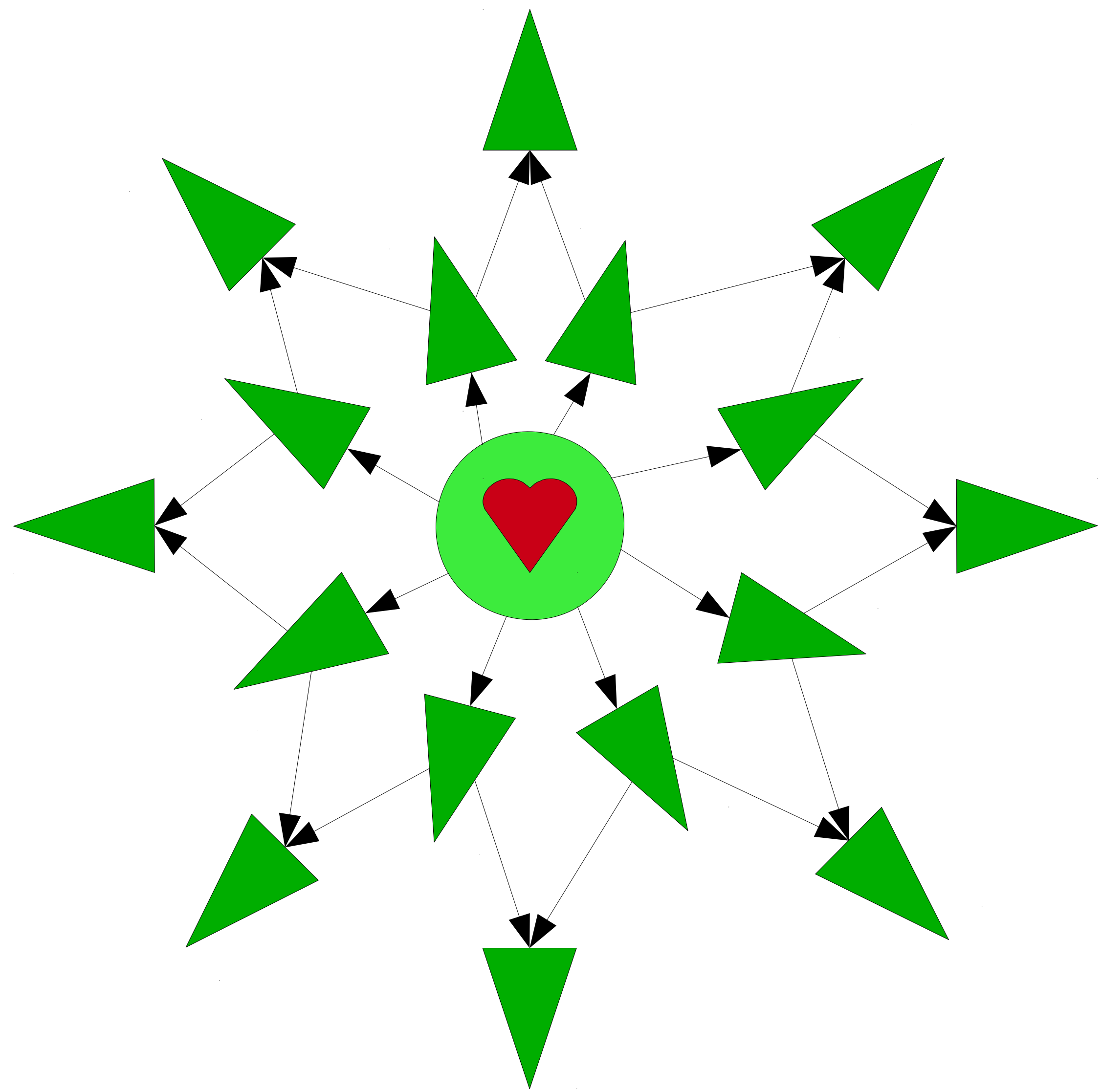
- A **Hasse diagram** is a graphical representation of a partial order.
- No self-loops: by **reflexivity**, we can always add them back in.
- Higher elements are bigger than lower elements: by **antisymmetry**, the edges can only go in one direction.
- No redundant edges: by **transitivity**, we can infer the missing edges.



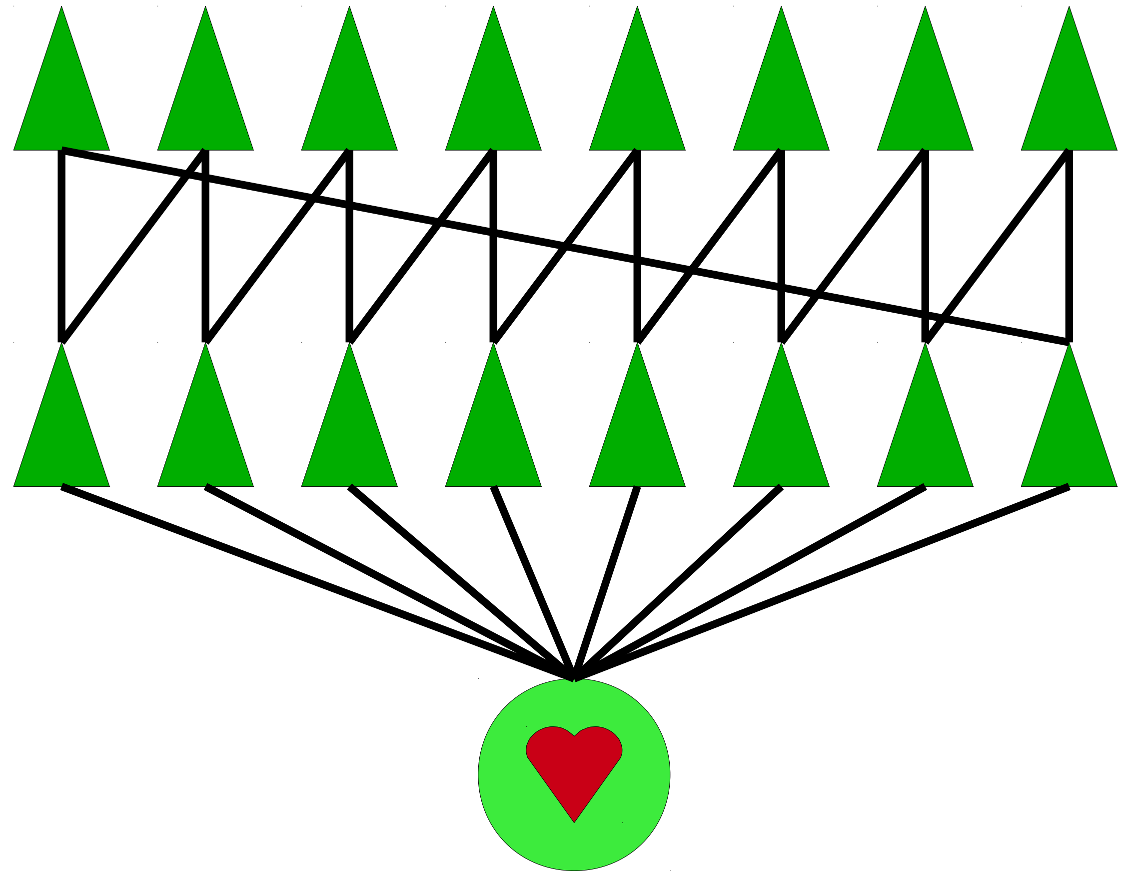
Tasty
Not
Tasty
Indian
Mediterranean
Mexican
Chinese
Italian
American
Dorm


Larger
Smaller
...
4
3
2
1
0

Hasse Artichokes



Hasse Artichokes



For More on the Olympics:

http://www.nytimes.com/interactive/2012/08/07/sports/olympics/the-best-and-worst-countries-in-the-medal-count.html

Formalizing Relations

What is a Relation?

- Up to now, we have been using an informal definition of a binary relation over a set A.
- To wrap up our treatment of relations, we'll give a formal definition.

The Cartesian Product

- The **Cartesian Product** of $A \times B$ of two sets is defined as

$$A \times B \equiv \{\ (a, b) \mid a \in A \text{ and } b \in B\ \}$$

$$\underbrace{\{\,0,\ 1,\ 2\,\}}_{A} \times \underbrace{\{\,a,\ b,\ c\,\}}_{B} = \left\{ \begin{array}{l} (0, a),(0, b),(0, c), \\ (1, a),(1, b),(1, c), \\ (2, a),(2, b),(2, c) \end{array} \right\}$$

The Cartesian Product

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$$A \times B \equiv \{\ (a, b) \mid a \in A \text{ and } b \in B\ \}$$

- We denote $A^2 \equiv A \times A$

$$\{0, 1, 2\}^2 = \left\{ \begin{matrix} (0, 0), (0, 1), (0, 2), \\ (1, 0), (1, 1), (1, 2), \\ (2, 0), (2, 1), (2, 2) \end{matrix} \right\}$$

A^2

Relations, Formally

- A binary relation R over a set A is a subset of A^2.
- xRy is shorthand for $(x, y) \in R$.
- A relation doesn't have to be meaningful; *any* subset of A^2 is a relation.
- Interesting fact:
  - Number of English sentences is equal to the number of natural numbers. *(More on that later.)*
  - Each binary relation over $\mathbb{N}$ is a subset of $\mathbb{N}^2$.
  - Number of binary relations over $\mathbb{N}$: $|\wp(\mathbb{N}^2)|$
  - ***Some binary relations over $\mathbb{N}$ are indescribable!***

Next Time

- **The Pigeonhole Principle**
  - Poignant pigeon-powered proofs!
- **Functions**
  - How do we transform objects into one another?