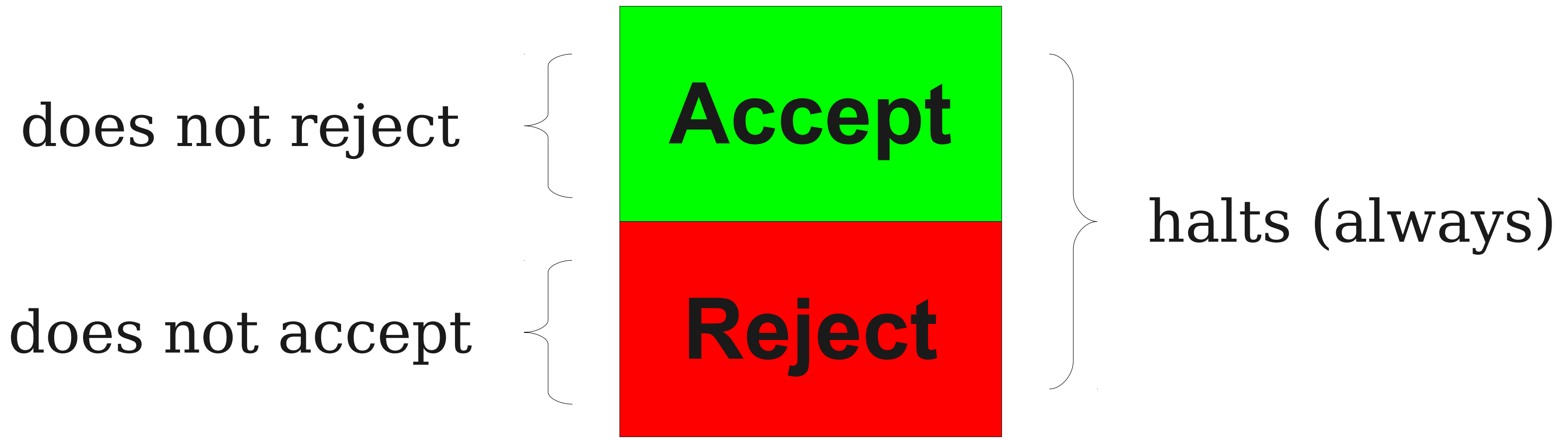
Reducibility

Part I

Deciders

- Some Turing machines always halt; they never go into an infinite loop.
- Turing machines of this sort are called **deciders**.
- For deciders, accepting is the same as not rejecting and rejecting is the same as not accepting.

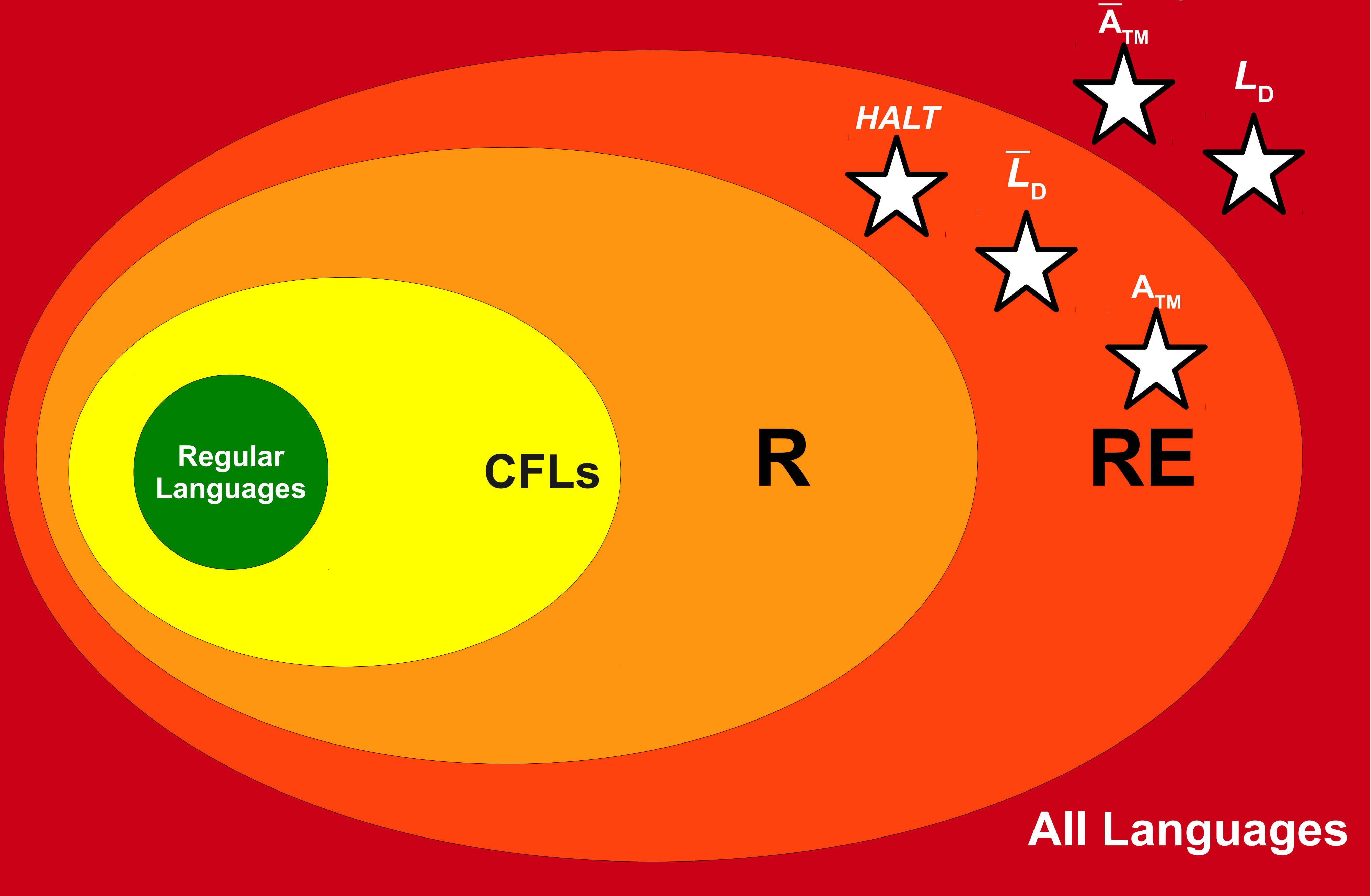


Decidable Languages

- A language L is called **decidable** iff there is a decider M such that $\mathscr{L}(M) = L$.
- Given a decider M, you *can* learn whether or not a string $w \in \mathscr{L}(M)$.
  - Run M on w.
  - Although it might take a staggeringly long time, M will eventually accept or reject w.
- The set **R** is the set of all decidable languages.

$$L \in \mathbf{R} \quad \text{iff} \quad L \text{ is decidable}$$

The Limits of Computability

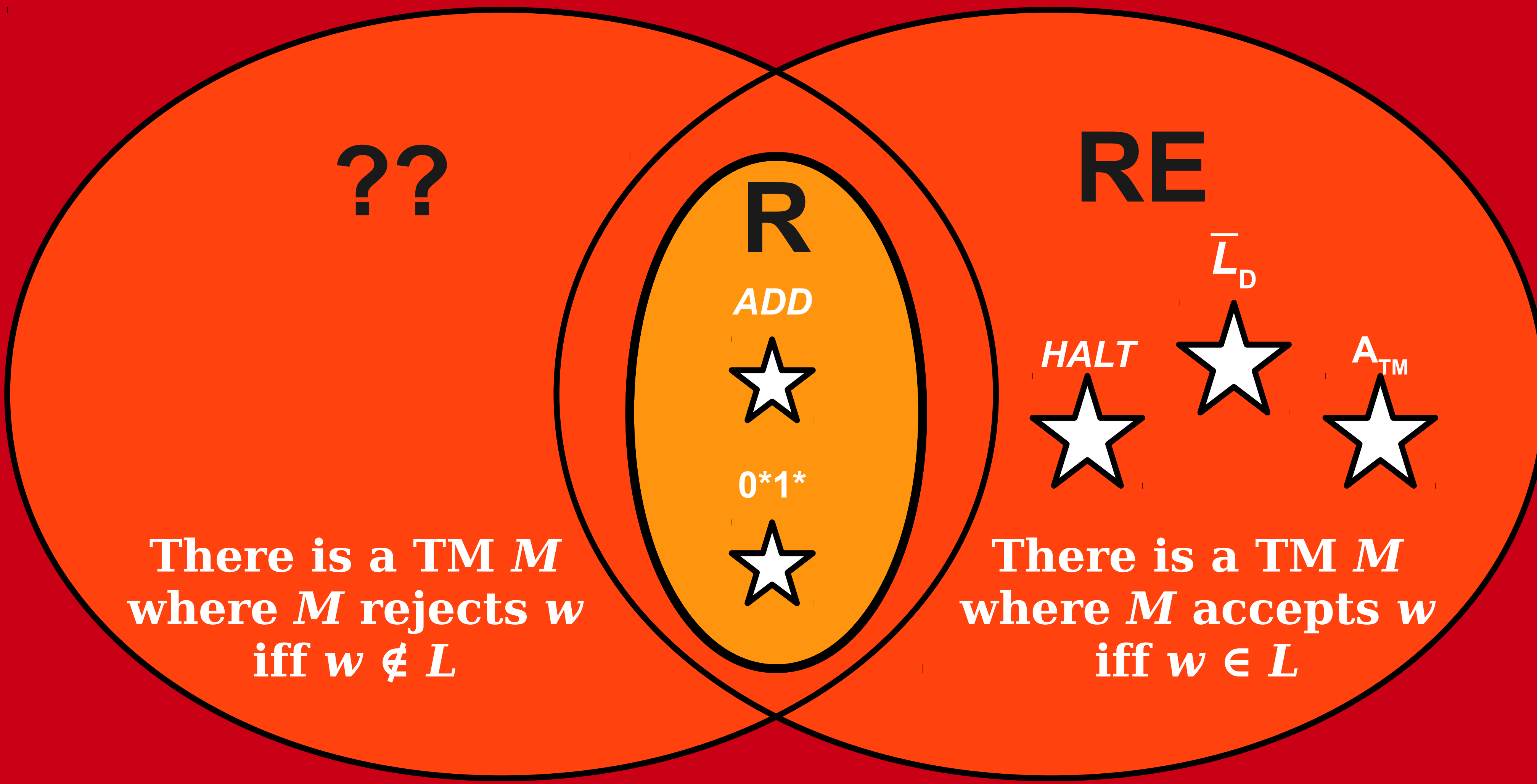


A_{TM} and *HALT*

- Both A_{TM} and *HALT* are undecidable.
  - There is no way to decide whether a TM will accept or eventually terminate.
- However, both A_{TM} and *HALT* are recognizable.
  - We can always run a TM on a string *w* and accept if that TM accepts or halts.
- **Intuition:** The only general way to learn what a TM will do on a given string is to run it and see what happens.

Resolving an Asymmetry

The Limits of Computability



A New Complexity Class

- A language L is in **RE** iff there is a TM M such that
  - if $w \in L$, then M accepts w.
  - if $w \notin L$, then M does not accept w.
- A TM M of this sort is called a *recognizer*, and L is called *recognizable*.
- A language L is in co-**RE** iff there is a TM M such that
  - if $w \in L$, then M does not reject w.
  - if $w \notin L$, then M rejects w.
- A TM M of this sort is called a ***co-recognizer***, and L is called ***co-recognizable***.

RE and co-**RE**

- Intuitively, **RE** consists of all problems where a TM can exhaustively search for **proof** that $w \in L$.
  - If $w \in L$, the TM will find the proof.
  - If $w \notin L$, the TM cannot find a proof.
- Intuitively, co-**RE** consists of all problems where a TM can exhaustively search for a **disproof** that $w \in L$.
  - If $w \in L$, the TM cannot find the disproof.
  - If $w \notin L$, the TM will find the disproof.

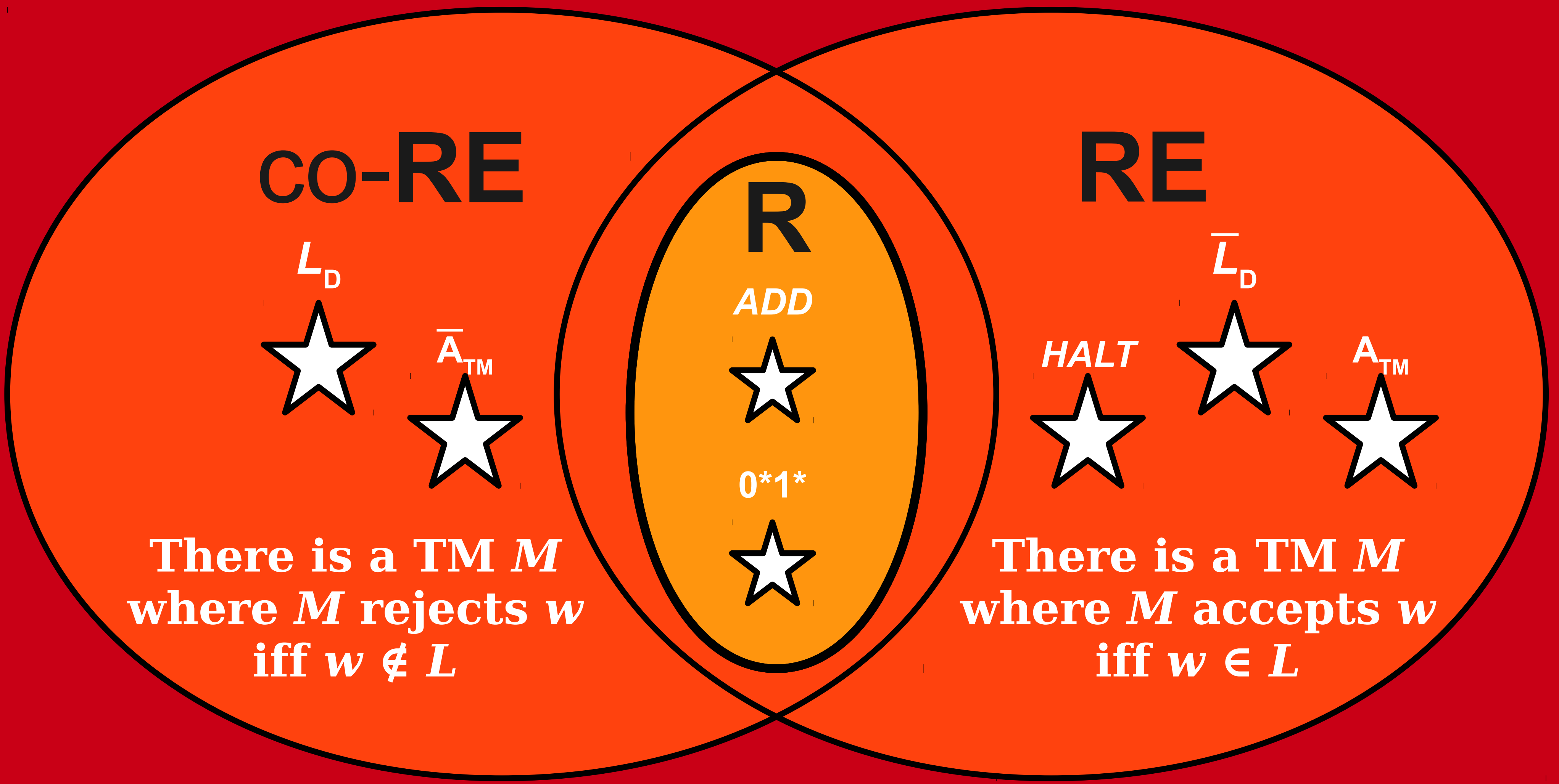
RE and co-**RE** Languages

- A_{TM} is an **RE** language:
  - Simulate the TM *M* on the string *w*.
  - If you find that *M* accepts *w*, accept.
  - If you find that *M* rejects *w*, reject.
  - (If *M* loops, we implicitly loop forever)
- $\overline{A}_{TM}$ is a co-**RE** language:
  - Simulate the TM *M* on the string *w*.
  - If you find that *M* accepts *w*, reject.
  - If you find that *M* rejects *w*, accept.
  - (If *M* loops, we implicitly loop forever)

RE and co-**RE** Languages

- $\overline{L}_D$ is an **RE** language.
  - Simulate M on $\langle M \rangle$.
  - If you find that M accepts $\langle M \rangle$, accept.
  - If you find that M rejects $\langle M \rangle$, reject.
  - (If M loops, we implicitly loop forever)
- L_D is a co-**RE** language.
  - Simulate M on $\langle M \rangle$.
  - If you find that M accepts $\langle M \rangle$, reject.
  - If you find that M rejects $\langle M \rangle$, accept.
  - (If M loops, we implicitly loop forever)

The Limits of Computability



RE and co-**RE**

Theorem: $L \in$ **RE** iff $\overline{L} \in$ co-**RE**.

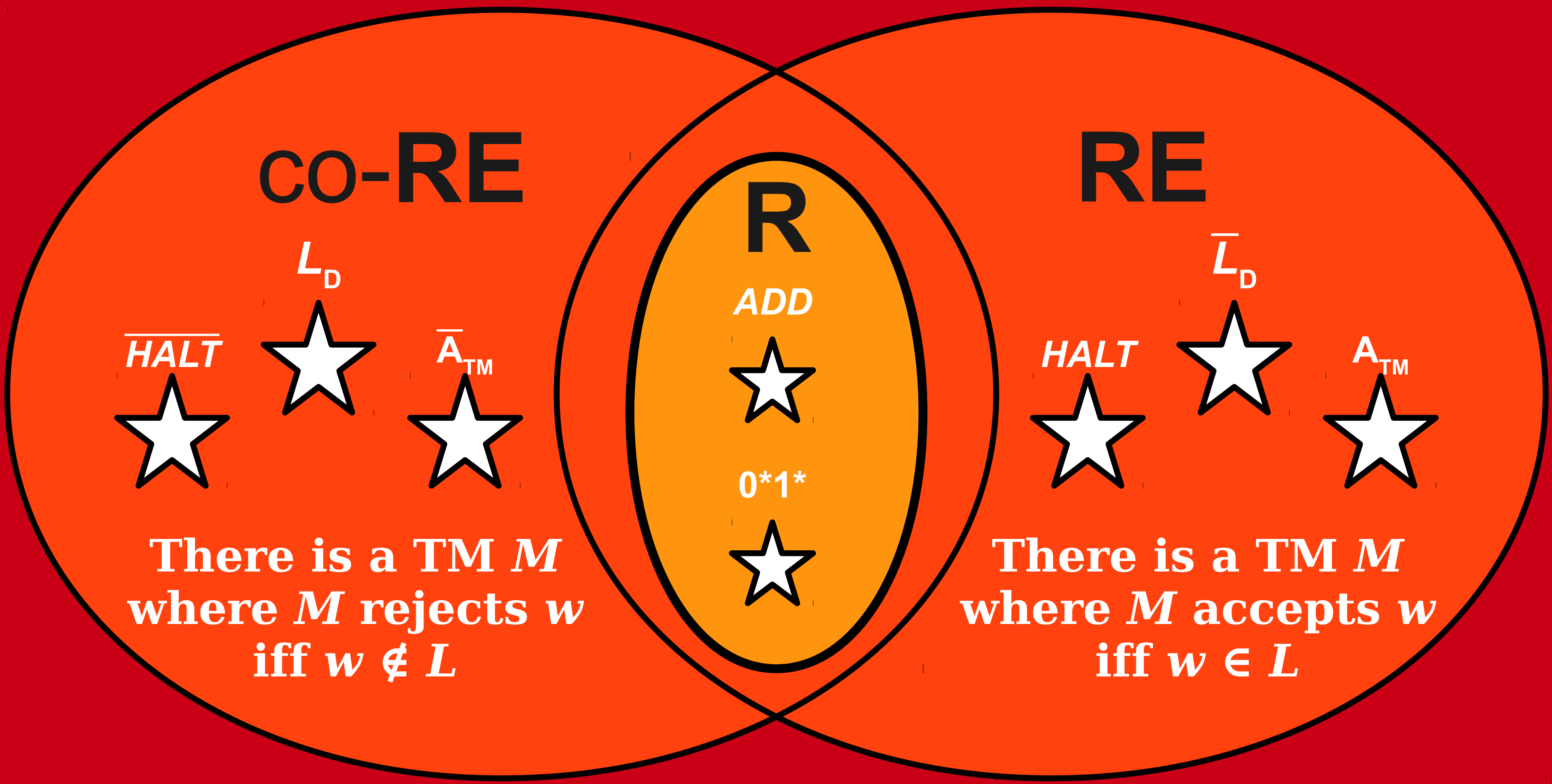
Proof Sketch: Start with a recognizer M for L. Then, flip its accepting and rejecting states to make machine M'. Then

M' rejects w
iff M accepts w
iff $w \in L$
iff $w \notin \overline{L}$.

M' does not reject w
iff M' accepts w or M' loops on w
iff M rejects w or M loops on w
iff $w \notin L$
iff $w \in \overline{L}$.

The same approach works if we flip the accept and reject states of a co-recognizer for $\overline{L}$. ■

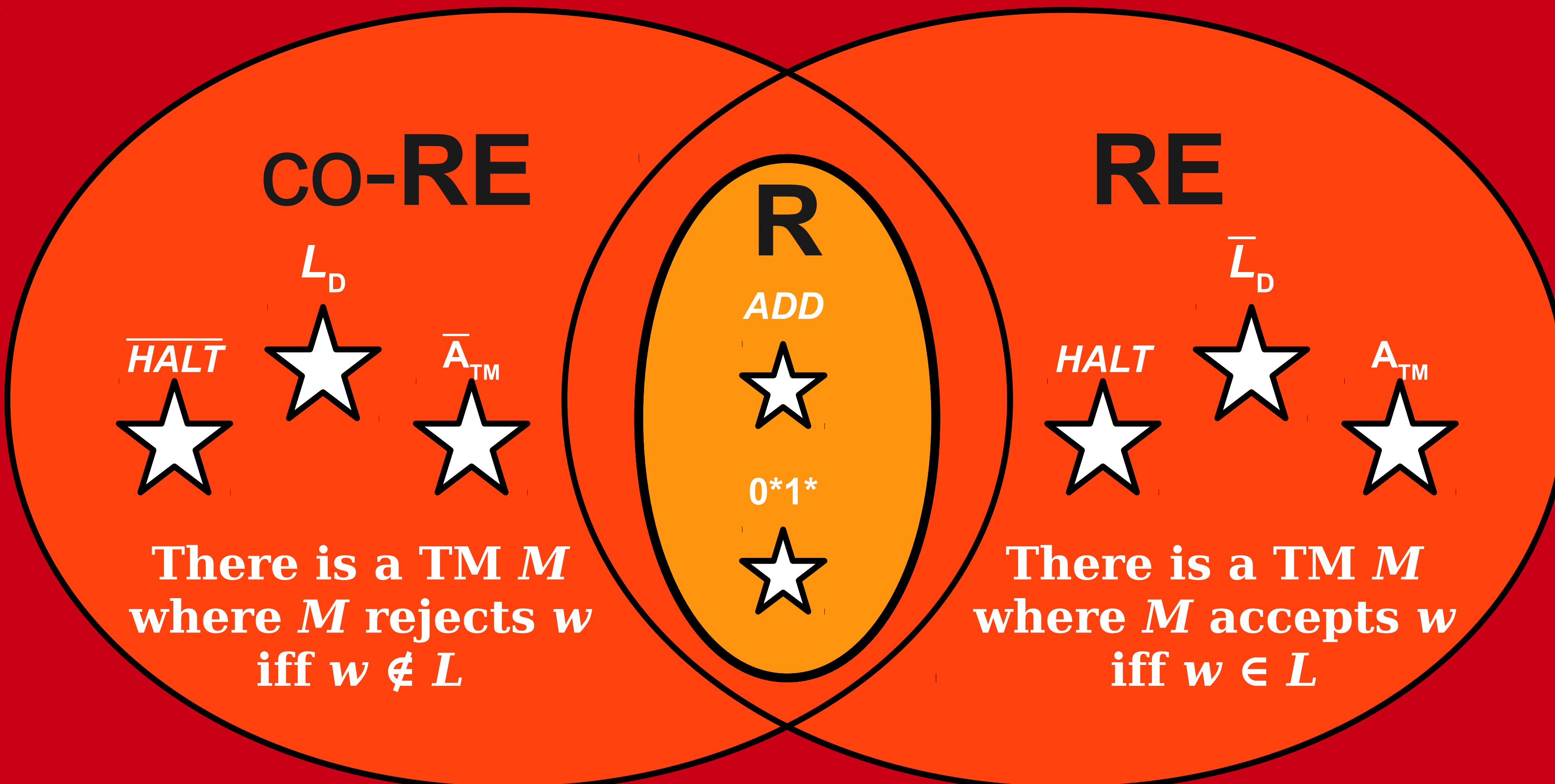
The Limits of Computability



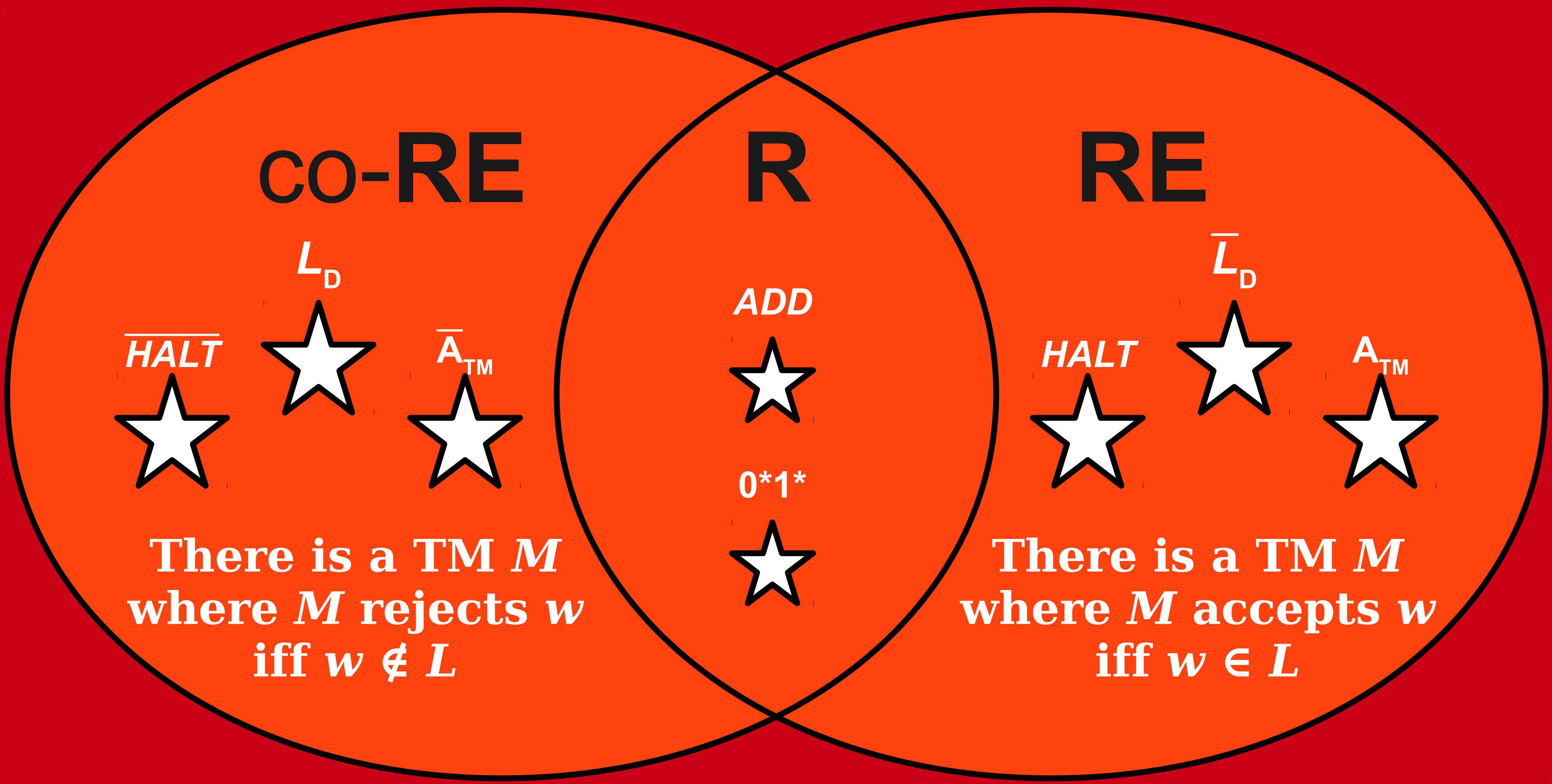
R, **RE**, and co-**RE**

- Every language in **R** is in both **RE** and co-**RE**.
- Why?
  - A decider for L accepts all $w \in L$ and rejects all $w \notin L$.
- In other words, $\mathbf{R} \subseteq \mathbf{RE} \cap \text{co-}\mathbf{RE}$.
- **Question:** Does $\mathbf{R} = \mathbf{RE} \cap \text{co-}\mathbf{RE}$?

Which Picture is Correct?



Which Picture is Correct?



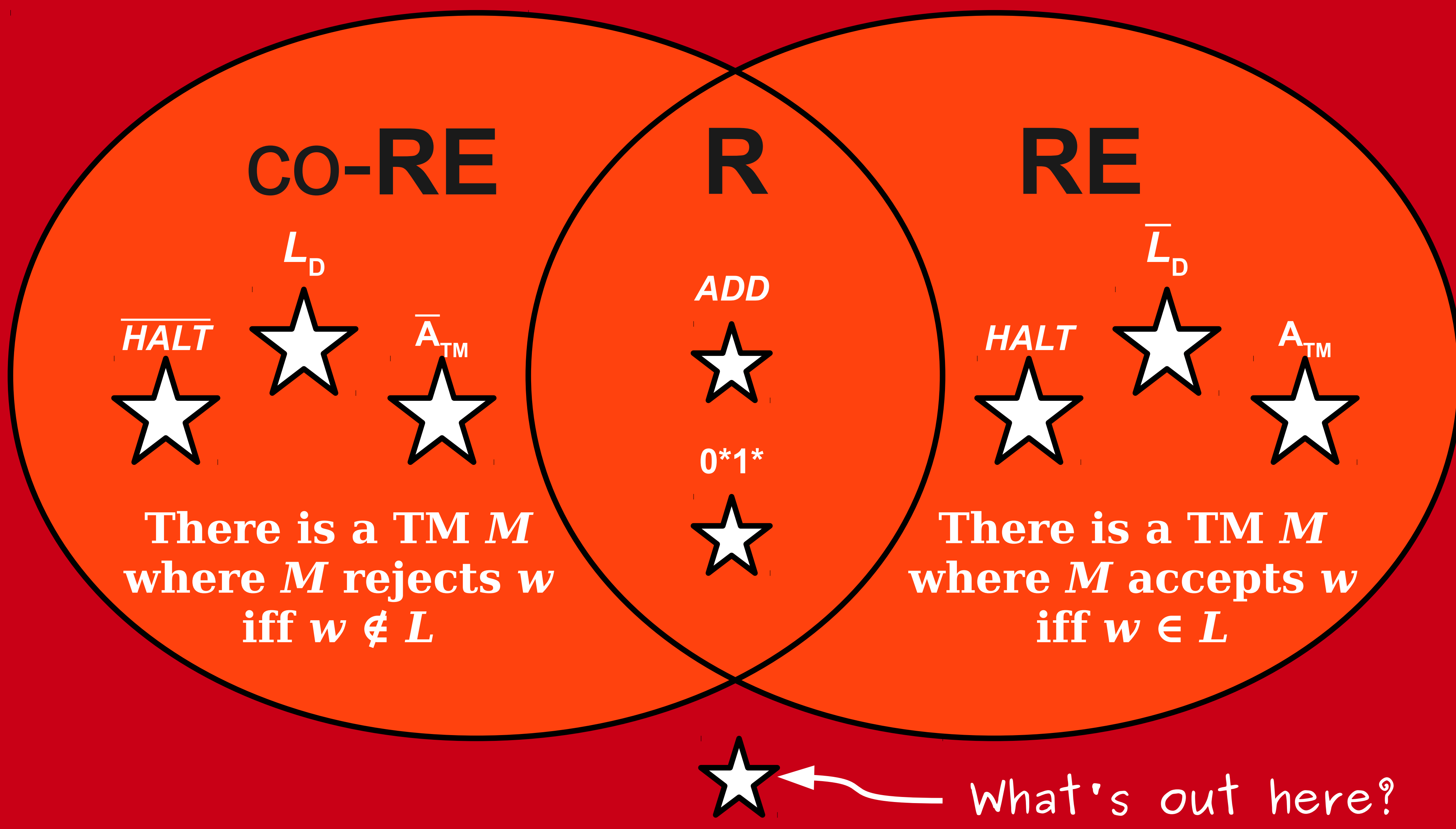
R, **RE**, and co-**RE**

- ***Theorem:*** If $L \in$ **RE** and $L \in$ co-**RE**, then $L \in$ **R**.
- ***Proof sketch:*** Since $L \in$ **RE**, there is a recognizer M for it. Since $L \in$ co-**RE**, there is a co-recognizer $\overline{M}$ for it.

  This TM D is a decider for L:

  D = "On input w:
  Run M on w and $\overline{M}$ on w in parallel.
  If M accepts w, accept.
  If $\overline{M}$ rejects w, reject.

The Limits of Computability



Time-Out For Announcements!

Friday Four Square!

Today at 4:15PM outside Gates

Two Handouts Online

- **24: Additional Proofs on TMs**
  - See alternate proofs of why various languages are or are not **R**, **RE**, or co-**RE**.
- **25: Extra Practice Problems**
  - By popular demand, extra questions on topics you'd like some more practice with!
  - Solutions released Monday.

Picking up Problem Sets

- If you pick up problem sets from the filing cabinet,

  please put all other papers back into the filing cabinet when you're done!

- If you don't:
  - they get mixed with problem sets from other classes and lost,
  - it causes a fire hazard, and
  - I get flak from the building managers about making a mess.

Your Questions

“Can you recommend software for designing and / or simulating Turing machines?”

http://www.jflap.org/

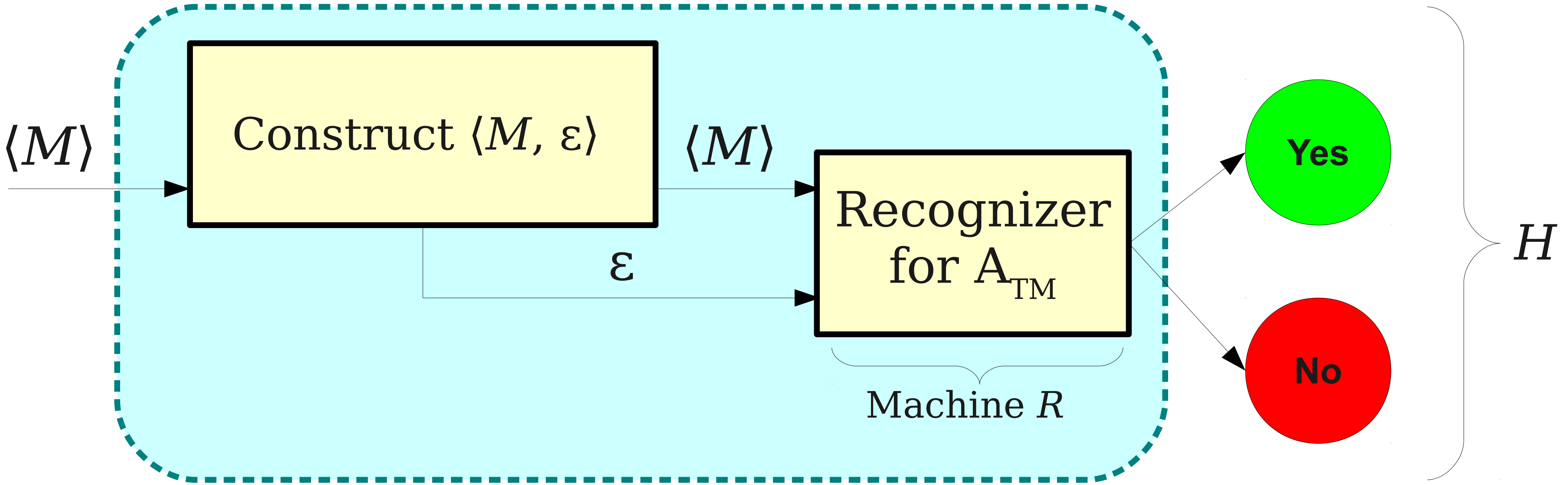
“Is there a difference between when a TM “runs” another TM as a subroutine vs. when it “simulates running” another TM?”

“Sometime my brain is stuck and I make silly and stupid mistakes [...]. What [do] you do when you are stuck on a problem?”

Back to CS103!

A Repeating Pattern

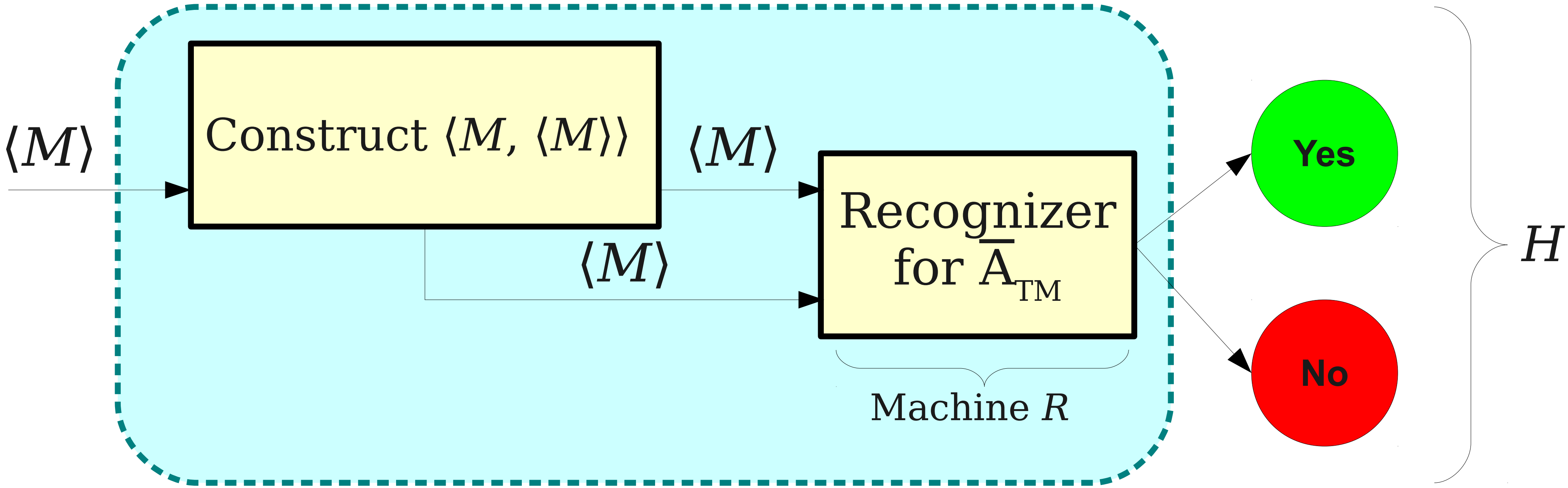
$L = \{ \langle M \rangle \mid M$ is a TM that accepts ε $\}$



H = "On input $\langle M \rangle$:

- Construct the string $\langle M, \varepsilon \rangle$.
- Run R on $\langle M, \varepsilon \rangle$.
- If R accepts $\langle M, \varepsilon \rangle$, then H accepts $\langle M, \varepsilon \rangle$.
- If R rejects $\langle M, \varepsilon \rangle$, then H rejects $\langle M, \varepsilon \rangle$."

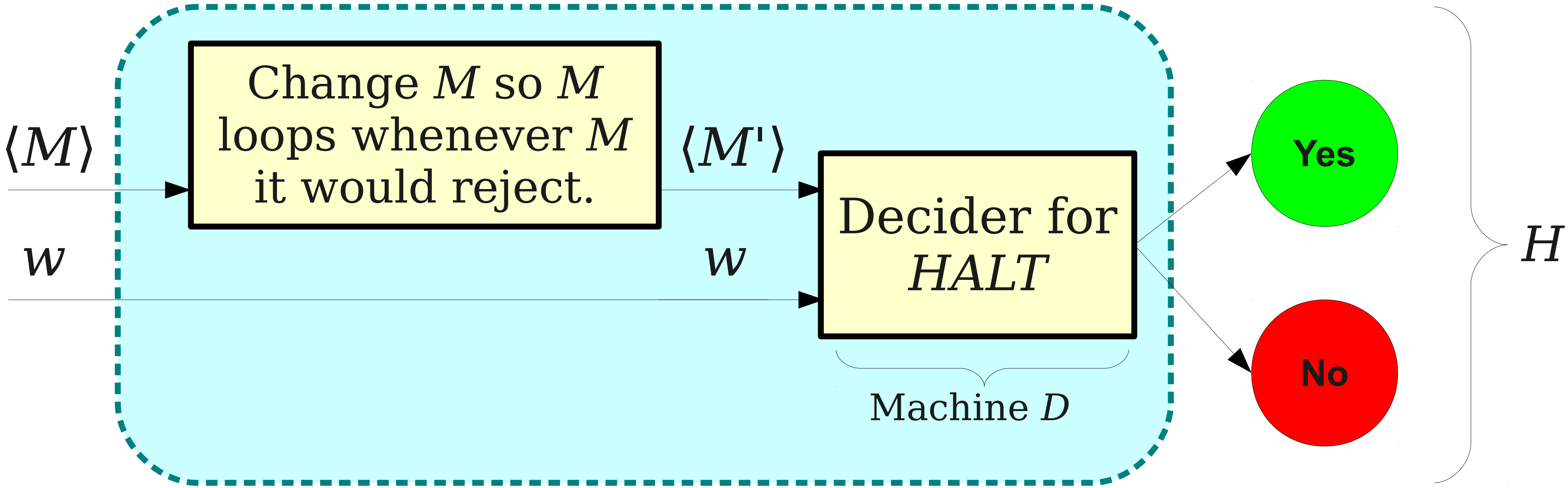
From $\overline{\text{A}}_{\text{TM}}$ to L_{D}



H = "On input $\langle M \rangle$:

- Construct the string $\langle M, \langle M \rangle \rangle$.
- Run R on $\langle M, \langle M \rangle \rangle$.
- If R accepts $\langle M, \langle M \rangle \rangle$, then H accepts $\langle M, \langle M \rangle \rangle$.
- If R rejects $\langle M, \langle M \rangle \rangle$, then H rejects $\langle M, \langle M \rangle \rangle$."

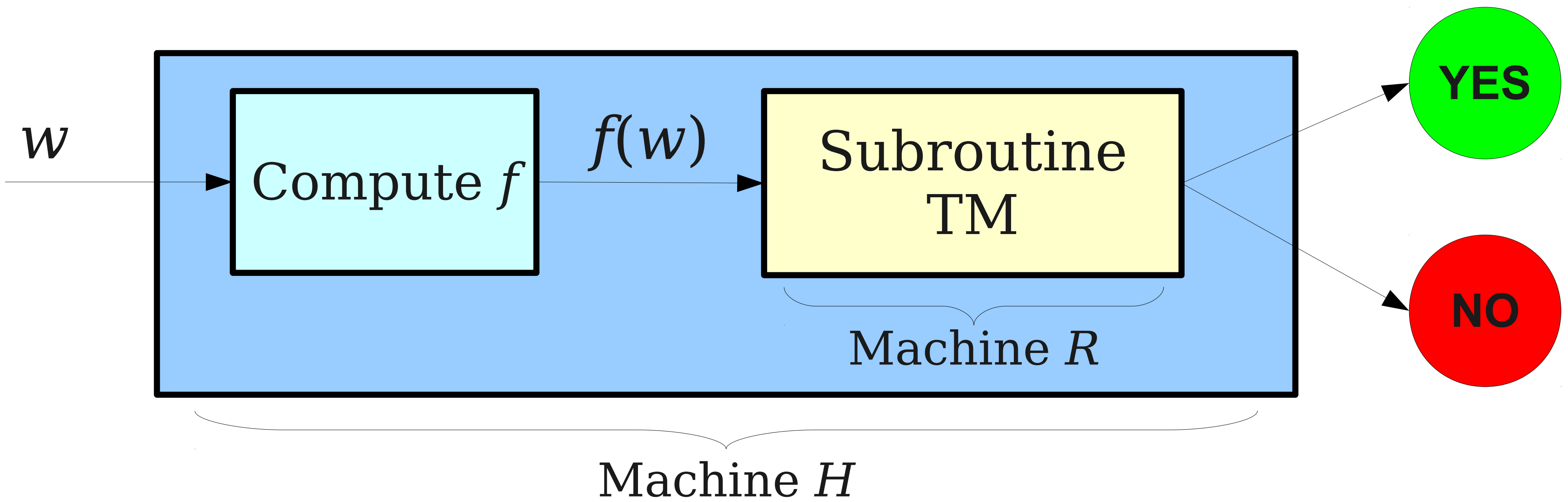
From *HALT* to A_{TM}



H = "On input $\langle M, w \rangle$:

- Build M into M' so M' loops when M rejects.
- Run D on $\langle M', w \rangle$.
- If D accepts $\langle M', w \rangle$, then H accepts $\langle M, w \rangle$.
- If D rejects $\langle M', w \rangle$, then H rejects $\langle M, w \rangle$."

The General Pattern

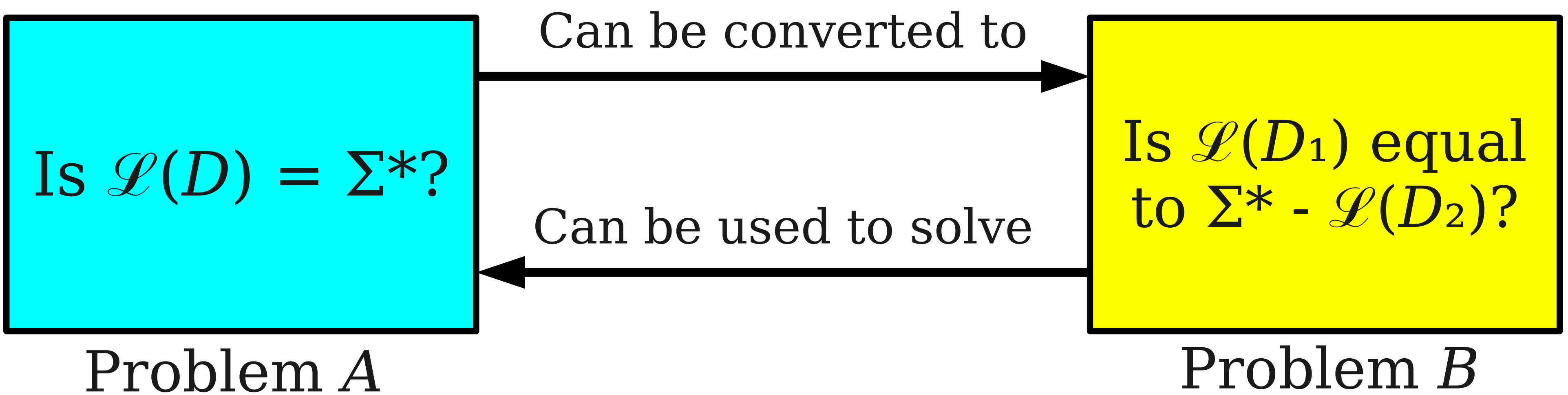


H = "On input w:

- Transform the input w into $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

Reductions

- Intuitively, problem A **reduces** to problem B iff a solver for B can be used to solve problem A.



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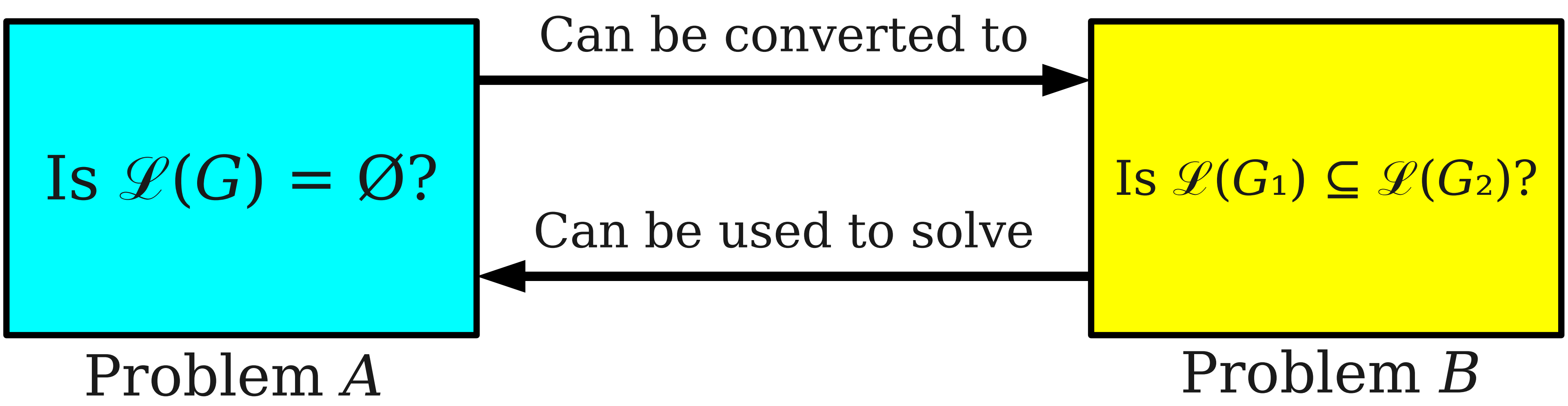

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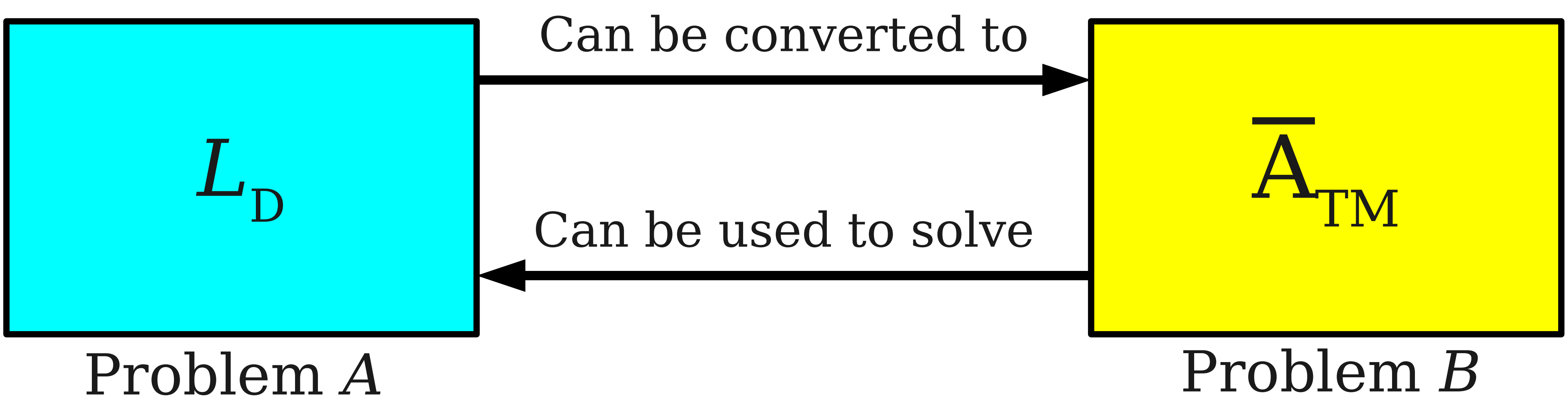
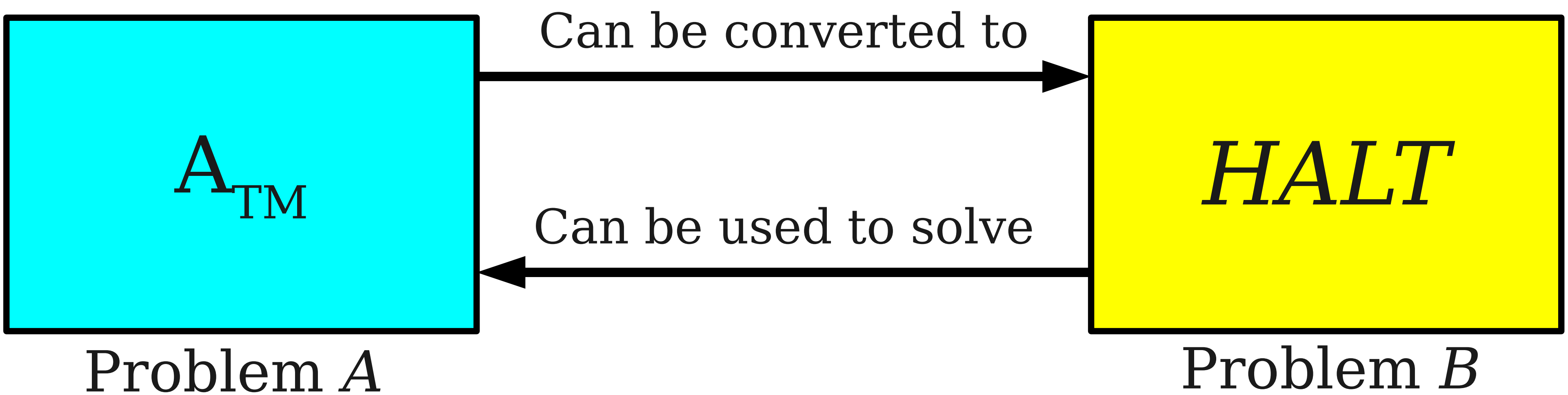


Reductions

- Intuitively, problem *A* **reduces** to problem *B* iff a solver for *B* can be used to solve problem *A*.



Reductions

- Intuitively, problem A **reduces** to problem B iff a solver for B can be used to solve problem A.
- Reductions can be used to show certain problems are "solvable:"

If *A* reduces to *B* and *B* is "solvable," then *A* is "solvable."

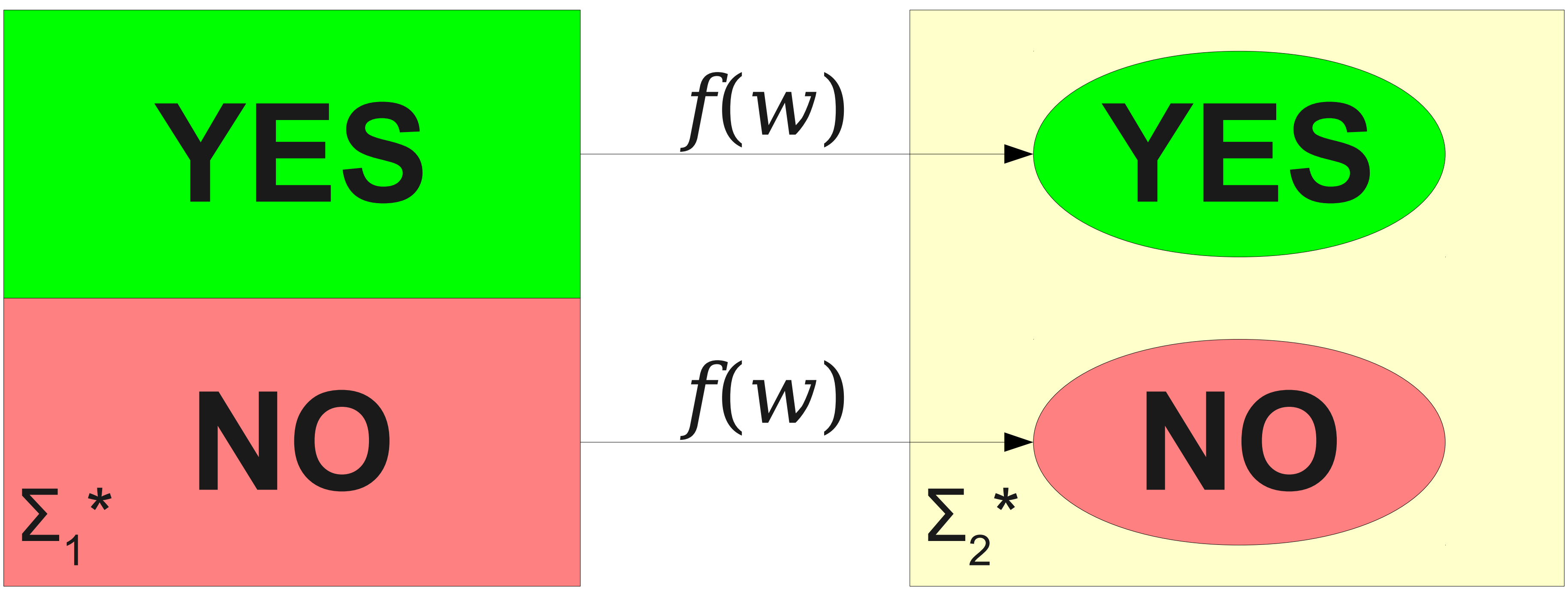
Formalizing Reductions

- In order to make the previous intuition more rigorous, we need to formally define reductions.
- There are many ways to do this; we'll explore two:
  - **Mapping reducibility** (today / Monday), and
  - **Polynomial-time reducibility** (next week).

Defining Reductions

- A **reduction** from A to B is a function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ such that

For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$



Defining Reductions

- A **reduction** from A to B is a function $f : \Sigma_1^* \to \Sigma_2^*$ such that

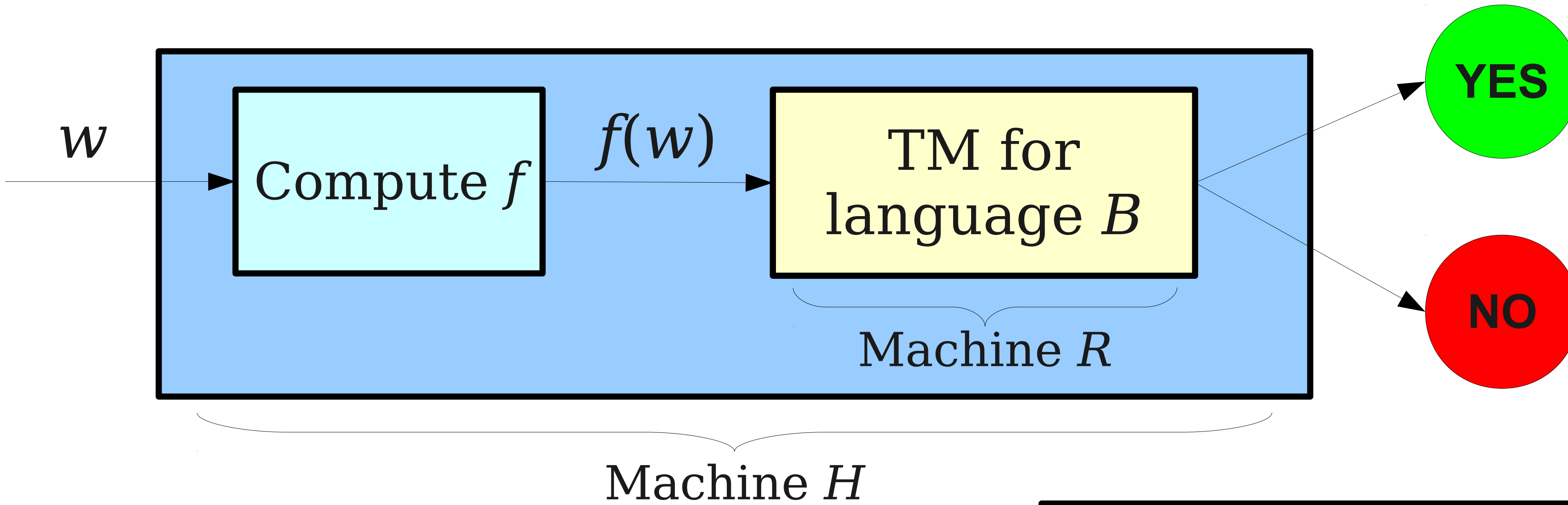
For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$

- Every $w \in A$ maps to some $f(w) \in B$.
- Every $w \notin A$ maps to some $f(w) \notin B$.
- f does not have to be injective or surjective.

Why Reductions Matter

- If language A reduces to language B, we can use a recognizer / co-recognizer / decider for B to recognize / co-recognize / decide problem A.
  - (There's a slight catch – we'll talk about this in a second).
- How is this possible?

$$\boldsymbol{w \in A \quad \text{iff} \quad f(w) \in B}$$



H = "On input *w*:

- Transform the input *w* into *f*(*w*).
- Run machine *R* on *f*(*w*).
- If *R* accepts *f*(*w*), then *H* accepts *w*.
- If *R* rejects *f*(*w*), then *H* rejects *w*."

H* accepts *w

iff

***R* accepts *f*(*w*)**

iff

$\boldsymbol{f(w) \in B}$

iff

$\boldsymbol{w \in A}$

A Problem

- Recall: f is a reduction from A to B iff

$$\boldsymbol{w \in A \quad \textbf{iff} \quad f(w) \in B}$$

- Under this definition, *any* language A reduces to *any* language B unless $B = \emptyset$ or Σ^*.
- Since $B \neq \emptyset$ and $B \neq \Sigma^*$, there is some $w_{yes} \in B$ and some $w_{no} \notin B$.
- Define $f : \Sigma_1^* \rightarrow \Sigma_2^*$ as follows:

$$f(w) = \begin{cases} w_{yes} & \text{if } w \in A \\ w_{no} & \text{if } w \notin A \end{cases}$$

- Then f is a reduction from A to B.

A Problem

- Example: let's reduce L_D to **0*****1***.
- Take w_{yes} = **01**, w_{no} = **10**.
- Then $f(w)$ is defined as

$$f(w)=\begin{cases} 01 & \text{if } w \in L_D \\ 10 & \text{if } w \notin L_D \end{cases}$$

- There is no TM that can actually evaluate the function $f(w)$ on all inputs, since no TM can decide whether or not $w \in L_D$.

Computable Functions

- This general reduction is mathematically well-defined, but might be impossible to actually compute!
- To fix our definition, we need to introduce the idea of a computable function.
- A function $f : \Sigma_1^* \to \Sigma_2^*$ is called a **computable function** if there is some TM M with the following behavior:

  “On input w:

    Compute $f(w)$ and write it on the tape.

    Move the tape head to the start of $f(w)$.

    Halt.”

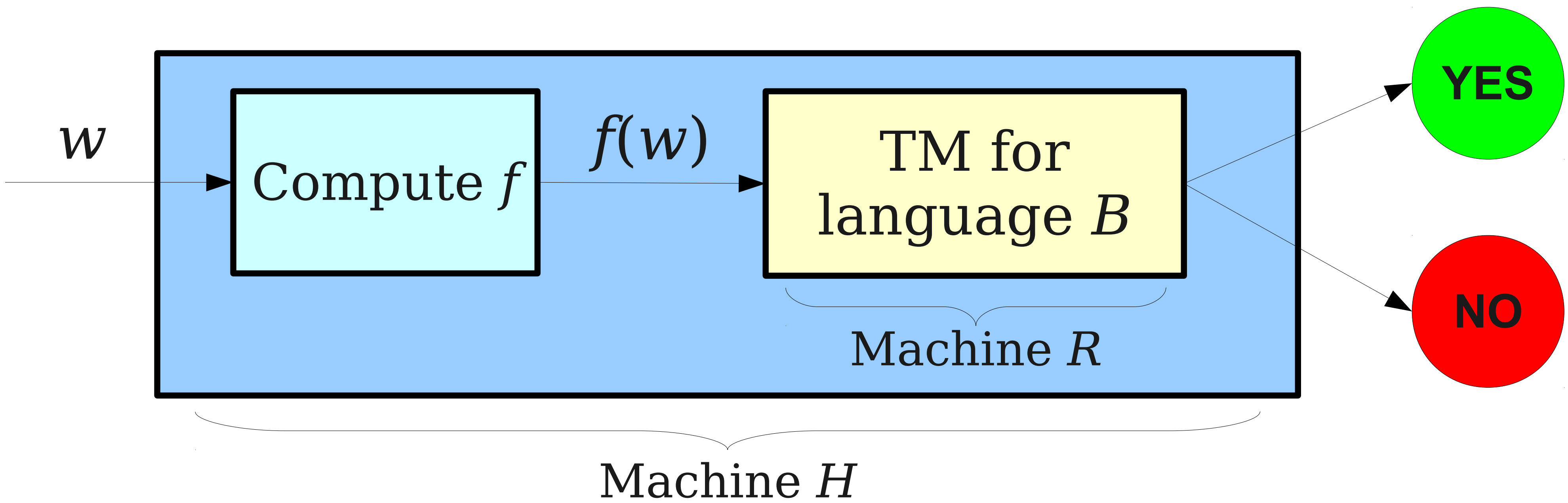
Mapping Reductions

- A function $f : \Sigma_1^* \to \Sigma_2^*$ is called a **mapping reduction** from A to B iff
  - For any $w \in \Sigma_1^*$, $w \in A$ iff $f(w) \in B$.
  - f is a computable function.
- Intuitively, a mapping reduction from A to B says that a computer can transform any instance of A into an instance of B such that the answer to B is the answer to A.

Mapping Reducibility

- If there is a mapping reduction from language *A* to language *B*, we say that language *A* is **mapping reducible** to language *B*.
- Notation: $\boldsymbol{A \leq_M B}$ iff language *A* is mapping reducible to language *B*.
- Note that we reduce ***languages***, not ***machines***.

$A \leq_M B$



H = "On input w:

- Compute $f(w)$.
- Run machine R on $f(w)$.
- If R accepts $f(w)$, then H accepts w.
- If R rejects $f(w)$, then H rejects w."

If R is a decider for B, then H is a decider for A.

If R is a recognizer for B, then H is a recognizer for A.

If R is a co-recognizer for B, then H is a co-recognizer for A.

Why Mapping Reducibility Matters

- **Theorem**: If $B \in \mathbf{R}$ and $A \leq_{\mathrm{M}} B$, then $A \in \mathbf{R}$.
- **Theorem**: If $B \in \mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $A \in \mathbf{RE}$.
- **Theorem**: If $B \in \text{co-}\mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $A \in \text{co-}\mathbf{RE}$.
- *Intuitively:* $A \leq_{\mathrm{M}} B$ means "A is not harder than B."

Why Mapping Reducibility Matters

- **Theorem**: If $A \notin \mathbf{R}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \mathbf{R}$.
- **Theorem**: If $A \notin \mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \mathbf{RE}$.
- **Theorem**: If $A \notin \text{co-}\mathbf{RE}$ and $A \leq_{\mathrm{M}} B$, then $B \notin \text{co-}\mathbf{RE}$.
- *Intuitively:* $A \leq_{\mathrm{M}} B$ means "B is at at least as hard as A."

Why Mapping Reducibility Matters

If this one is "easy" (R, RE, co-RE)…

$$A \leq_{\mathrm{M}} B$$

… then this one is "easy" (R, RE, co-RE) too.

Why Mapping Reducibility Matters

If this one is "hard" (not **R**, not **RE**, or not co-**RE**)…

$$A \leq_M B$$

… then this one is "hard" (not **R**, not **RE**, or not co-**RE**) too.