Mathematical Induction

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let P be some property. The ***principle of mathematical induction*** states that if

P is true for 0

and

For any $k \in \mathbb{N}$, if P is true for k, then P is true for $k + 1$

then

P is true for every $n \in \mathbb{N}$.

If it starts true…

…and it stays true…

…then it's always true.

Induction, Intuitively

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

Proof by Induction

- A ***proof by induction*** is a way to use mathematical induction to show that some result is true for all natural numbers n.
- In a proof by induction, there are three steps:
  - Prove that P is true for 0.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if P is true for some arbitrary natural number k, then P must also be true for $k+1$.
    - This is called the ***inductive step***.
    - The assumption that P is true for k is called the ***inductive hypothesis***.
  - Conclude, by induction, that P is true for all natural numbers n.

Some Summations

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 \qquad = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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At the start of the proof, we tell the reader what property we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly stating $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to simplify a complex expression. This is a common theme in inductive proofs.

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A Quick Aside

- This result helps explain the range of numbers that can be stored in an **`int`**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. You'll see a few over the course of this quarter.

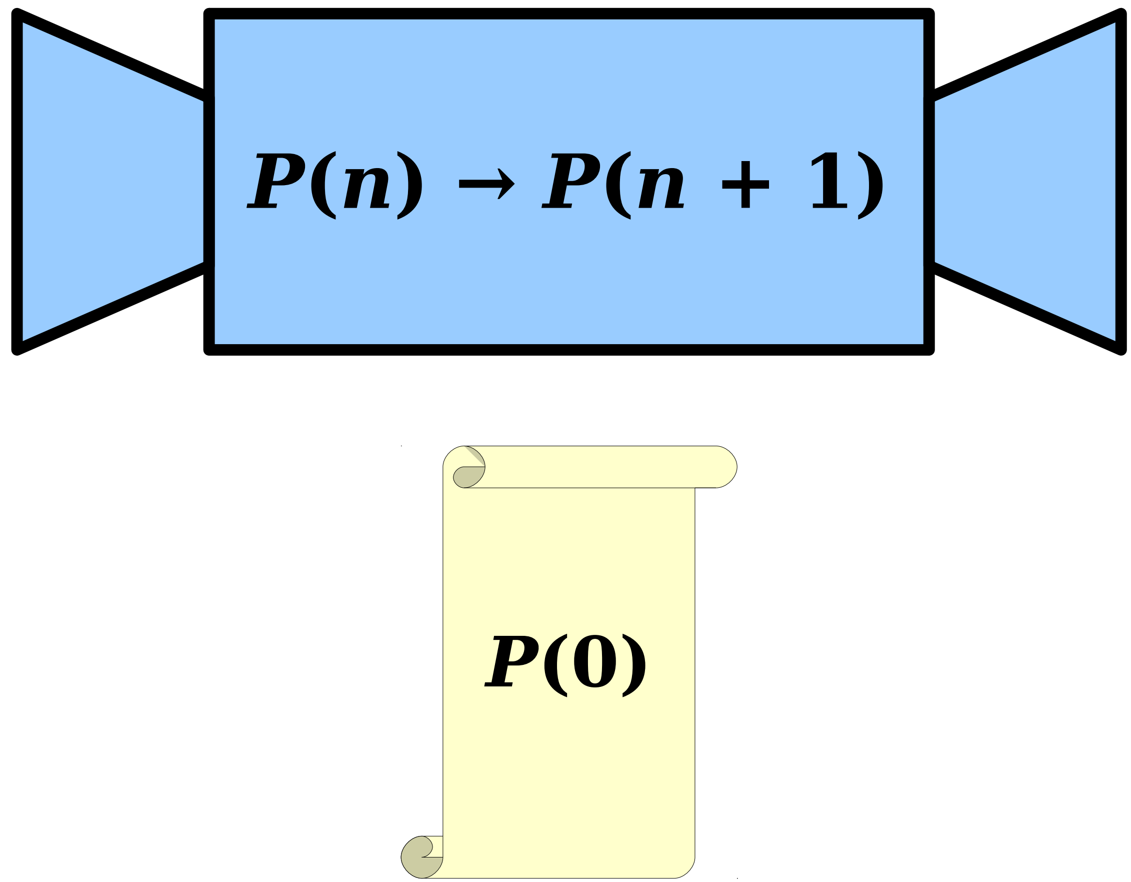
Structuring a Proof by Induction

- Define some property P that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove the property holds for 0, then go prove it.
- Prove the inductive step:
  - Say that you're assuming P is true for some natural number k, then write out exactly what that means.
  - Say that you're going to prove P is true for $k+1$, then write out exactly what that means.
  - Prove that P is true for $k+1$ using any proof technique you'd like.
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

Induction, Intuitively

- You can imagine an “machine” that turns proofs that the property holds for k into proofs that the property holds for $k + 1$.
- Starting with a proof that the property holds for 0, we can run the machine as many times as we'd like to get proofs for 1, 2, 3, … .
- The principle of mathematical induction says that this style of reasoning is a rigorous argument.

Why Induction Works



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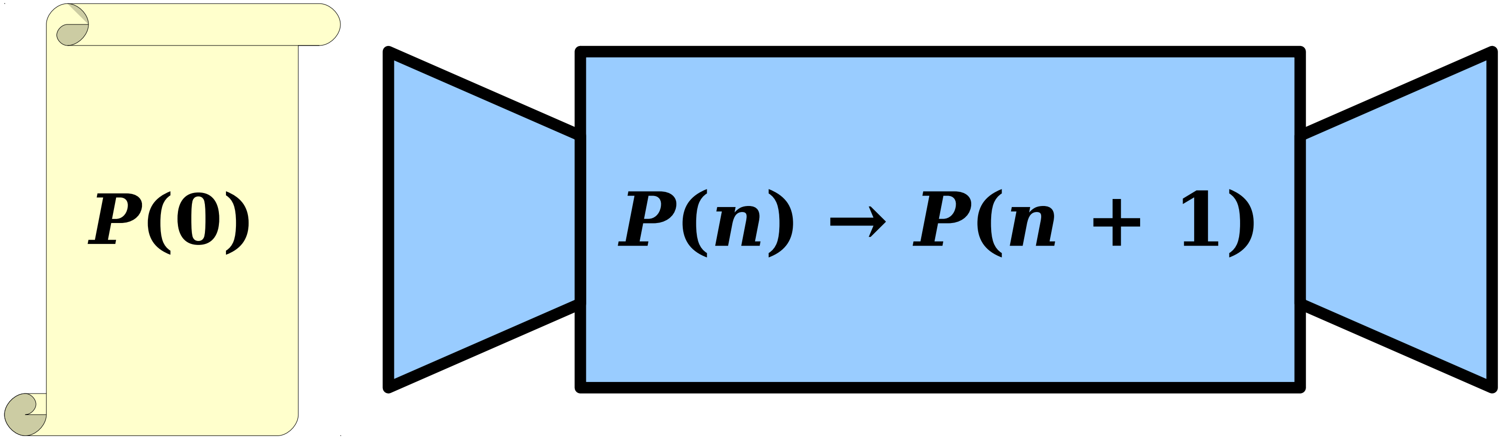

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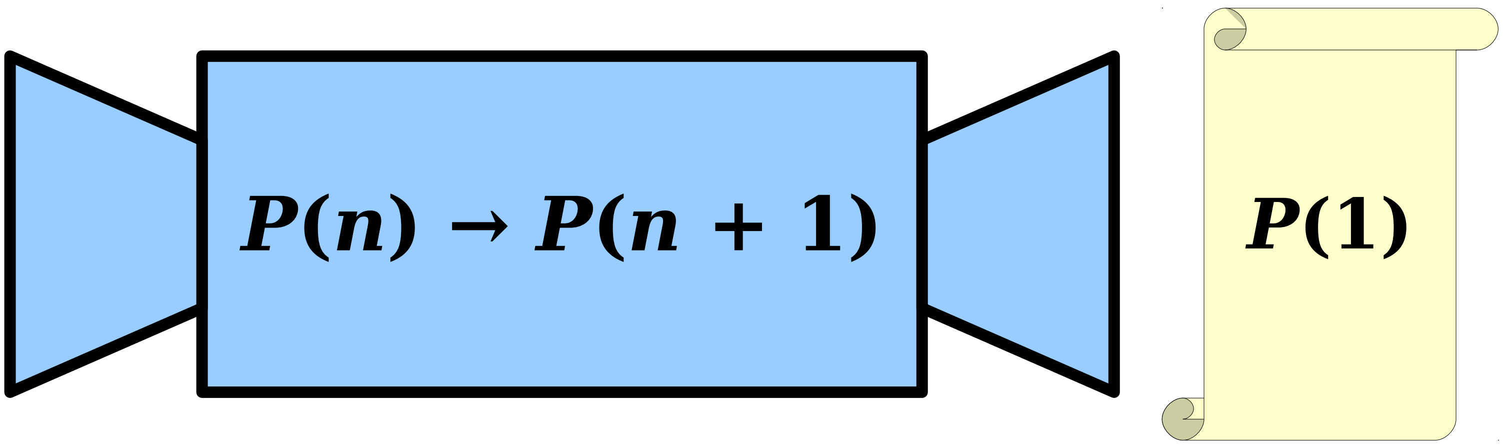

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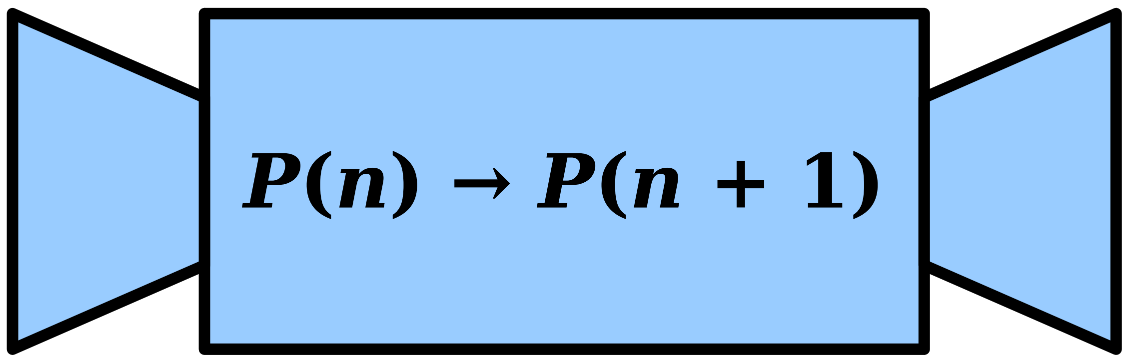

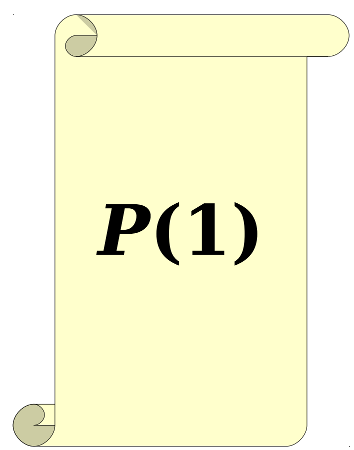

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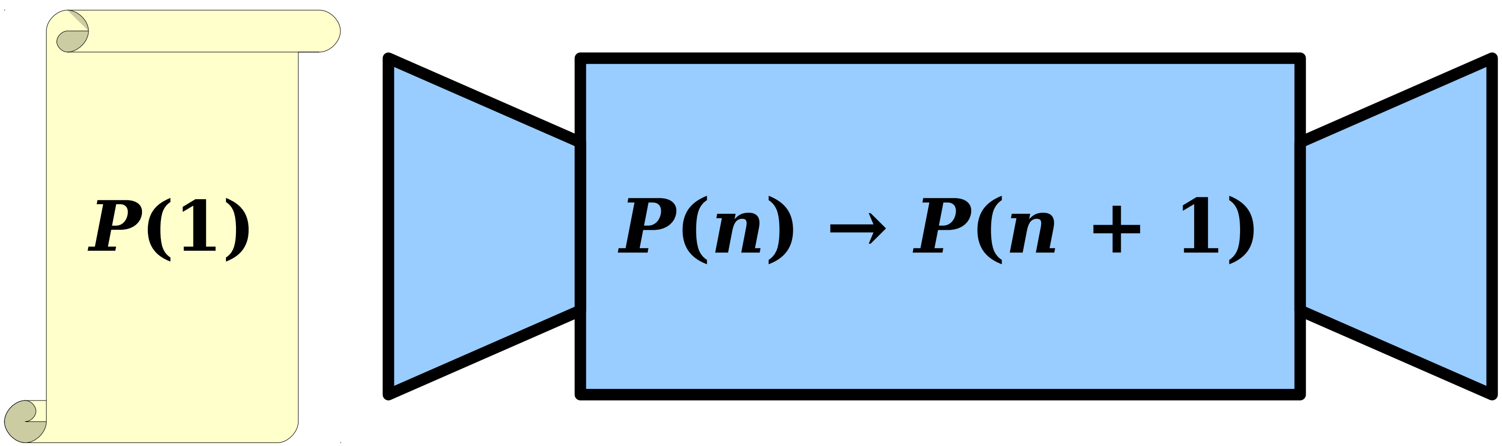

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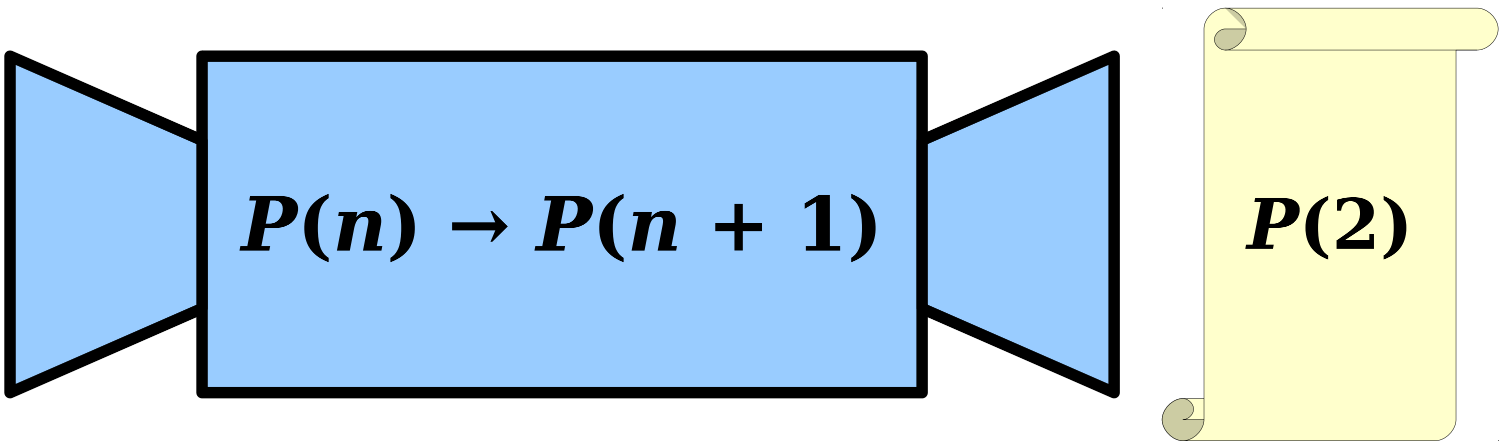

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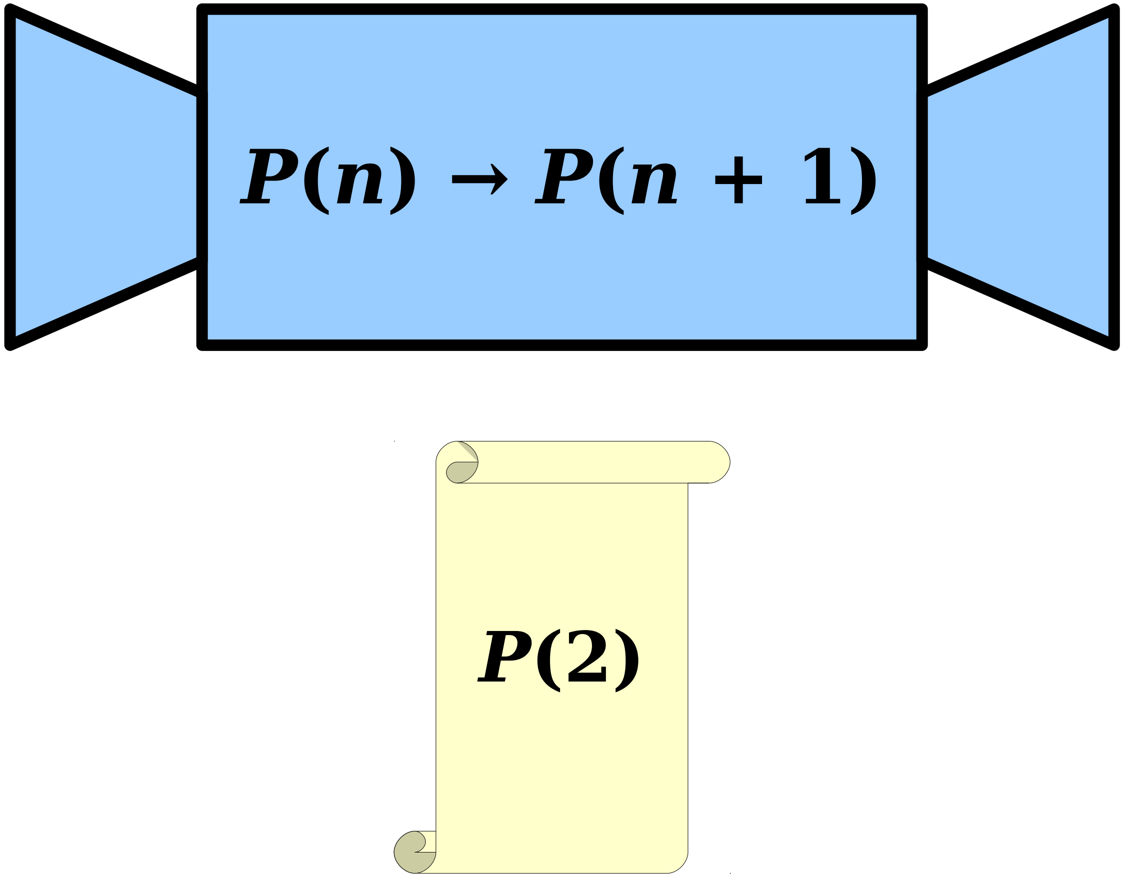

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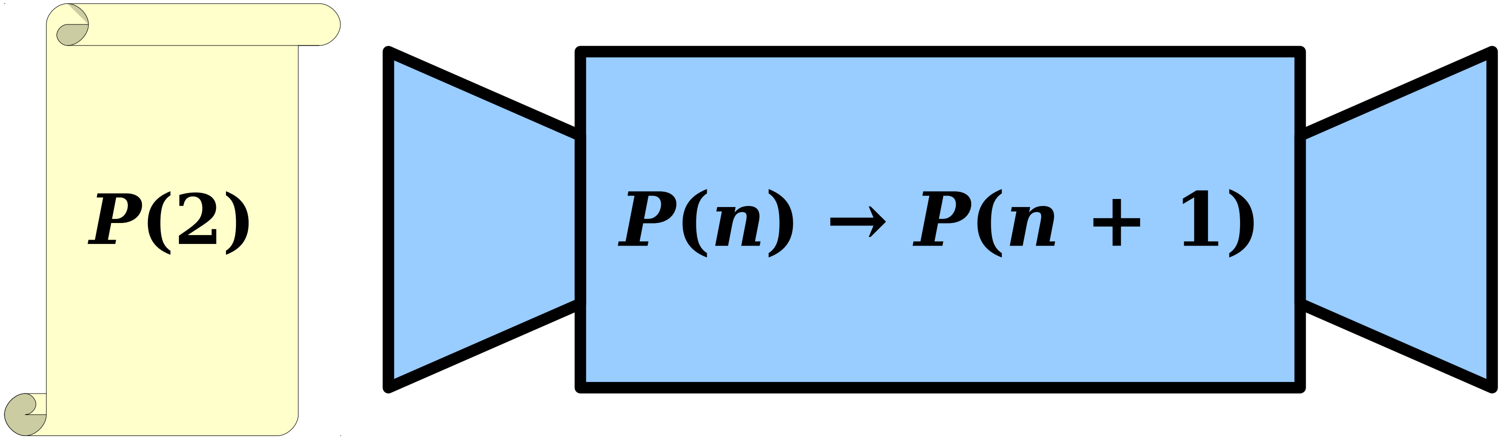

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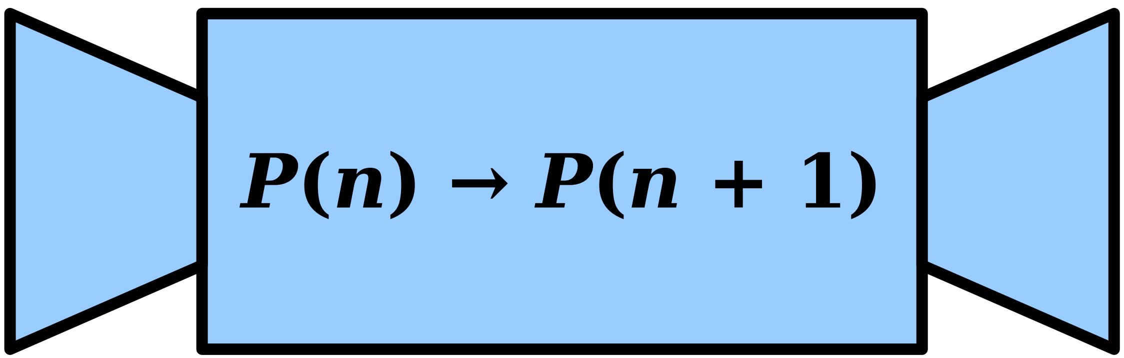
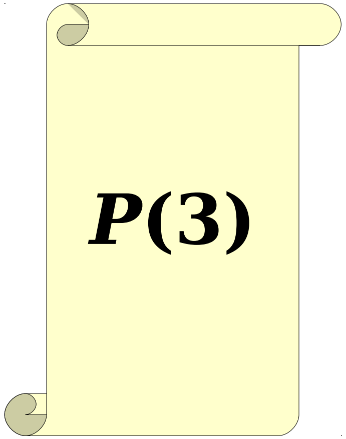


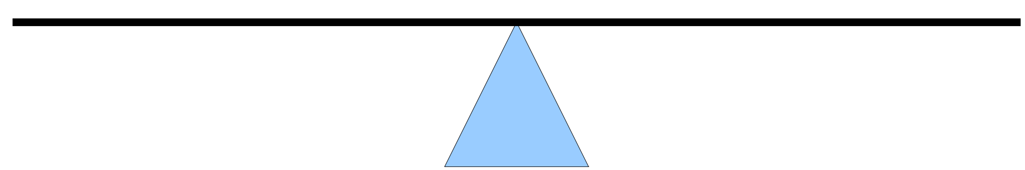


The Counterfeit Coin Problem

Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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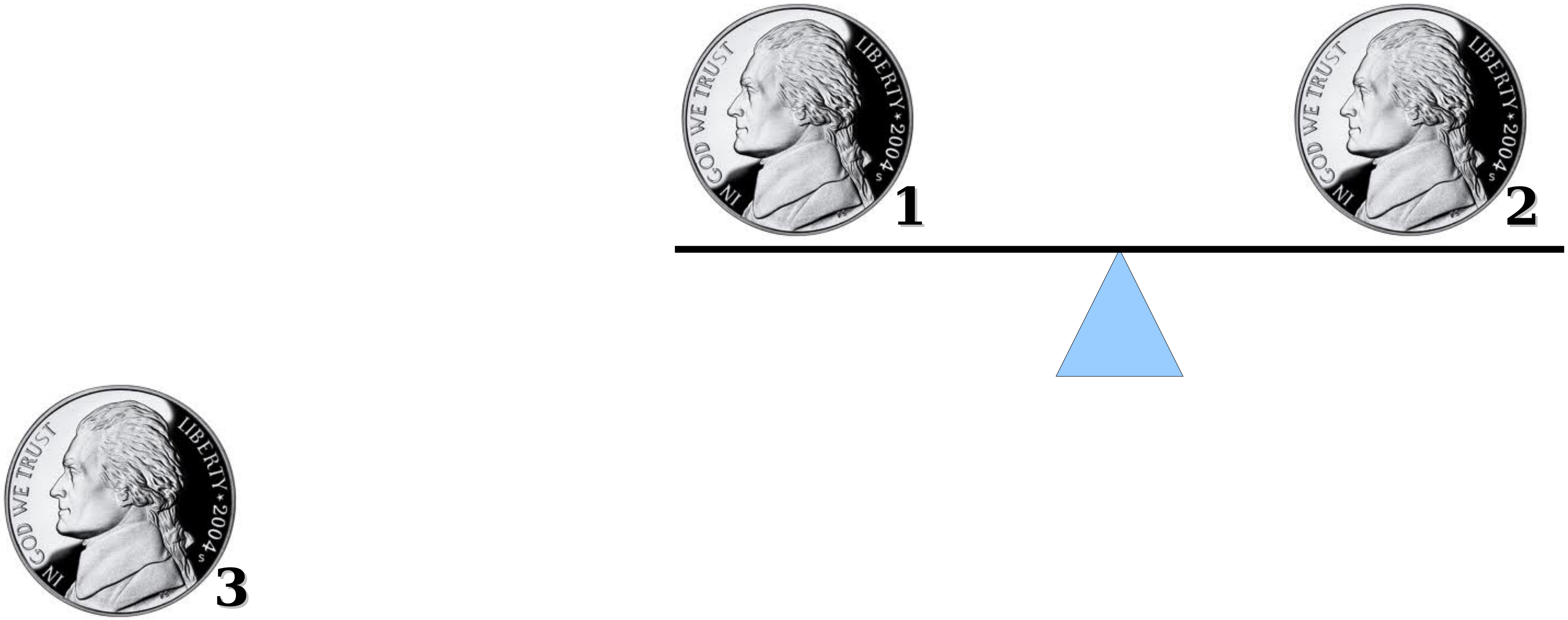

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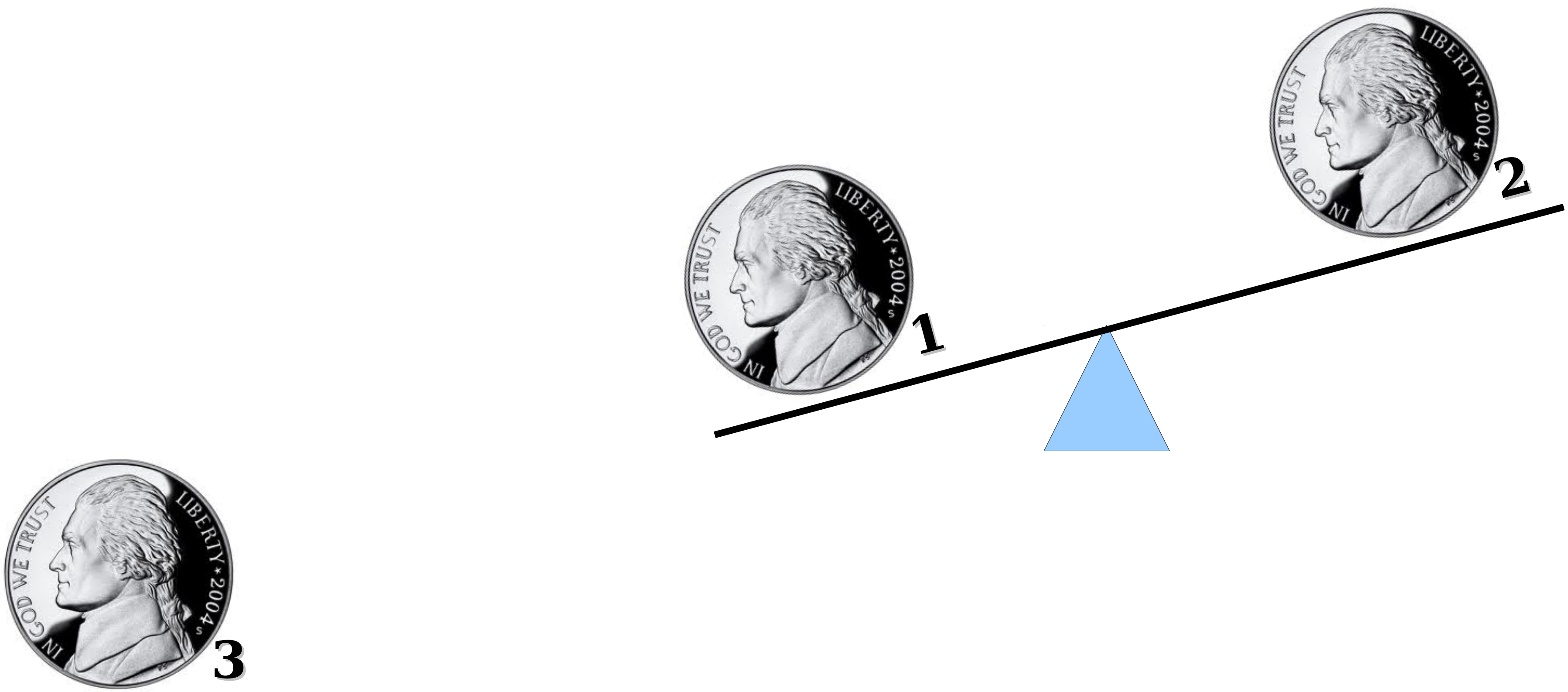

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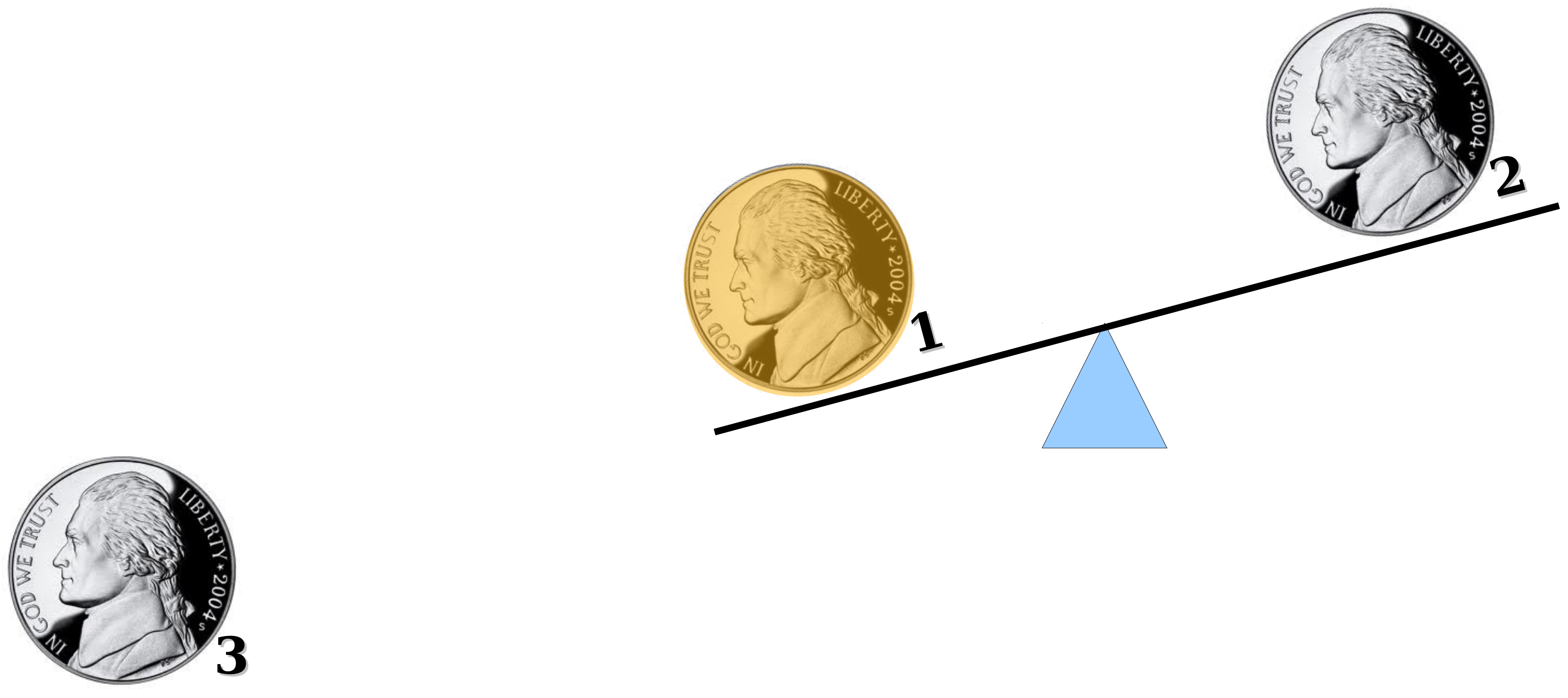

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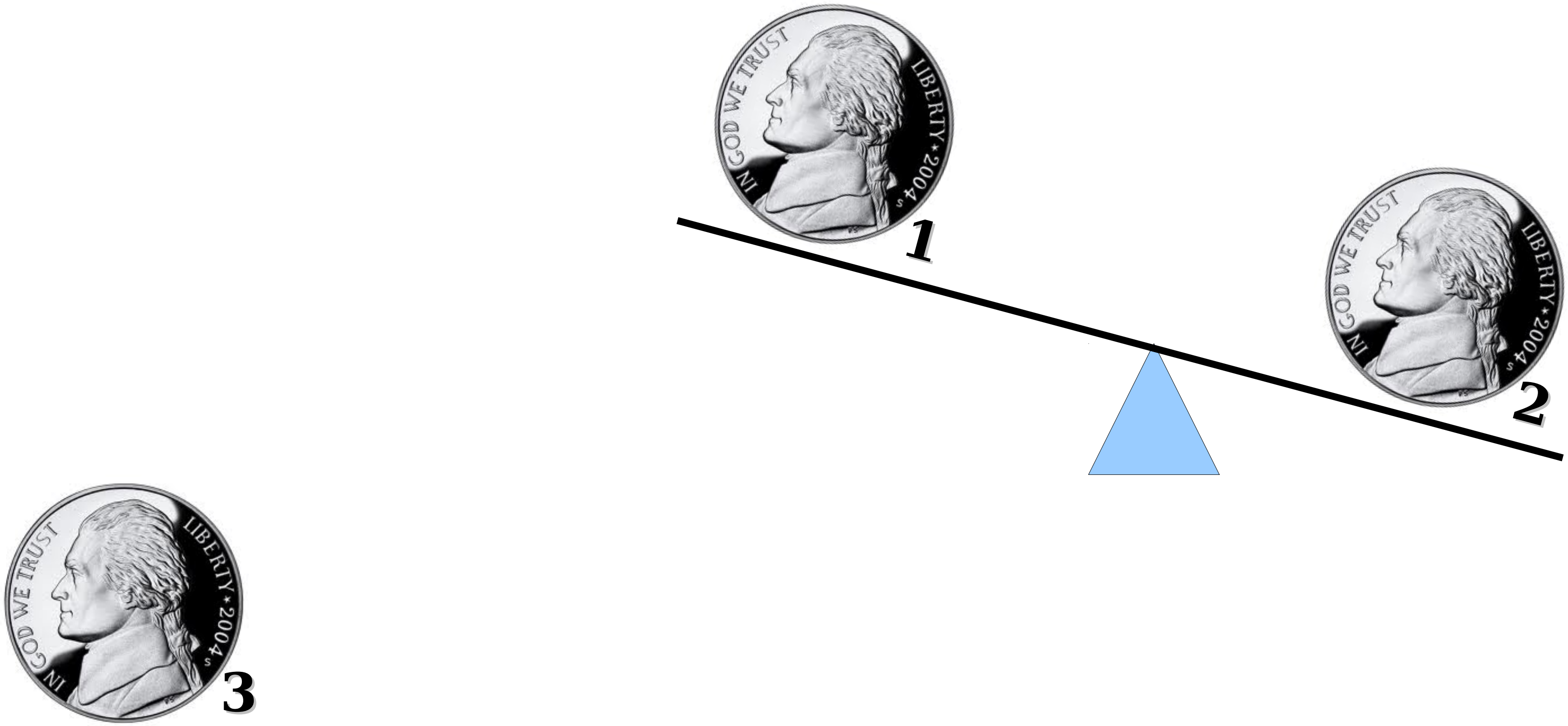

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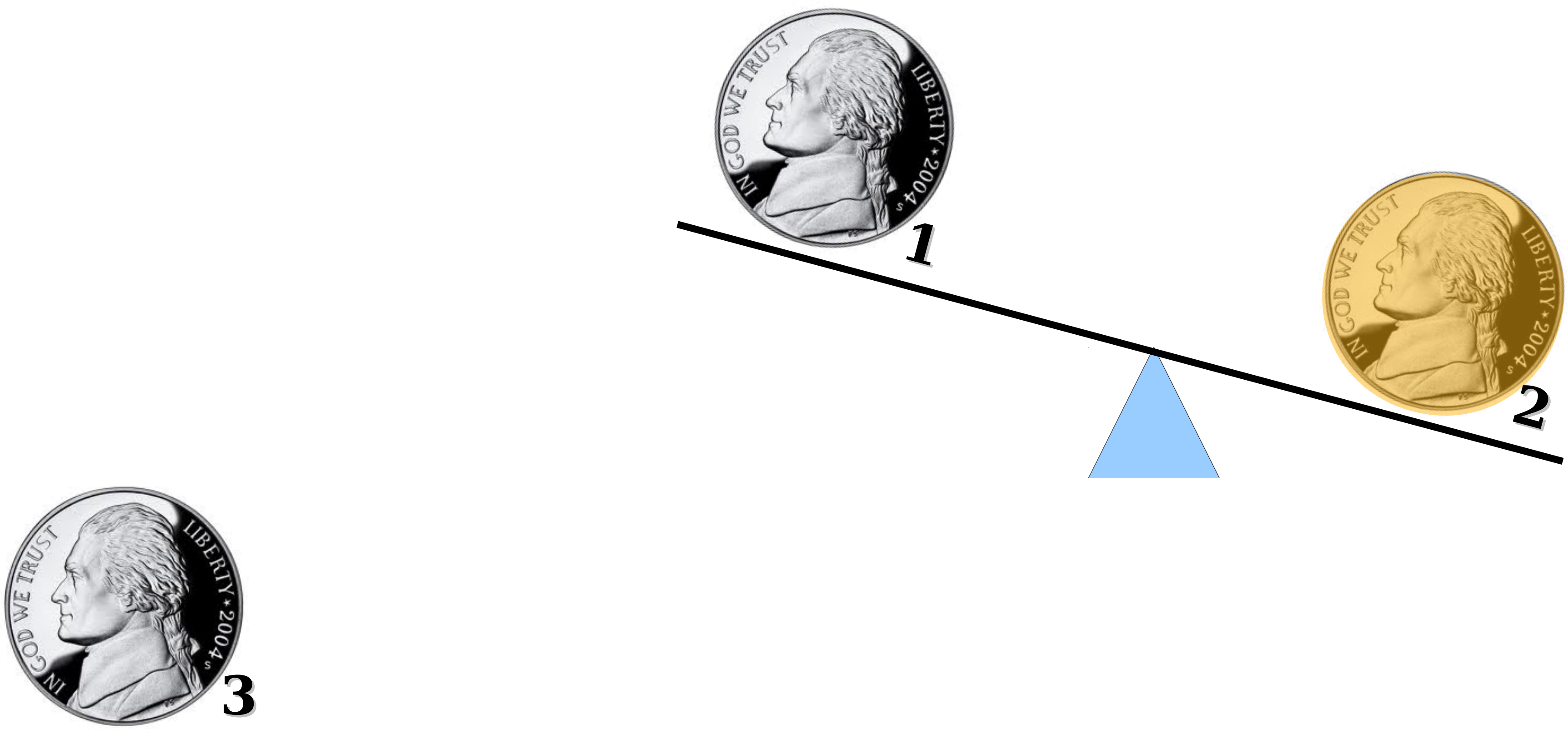

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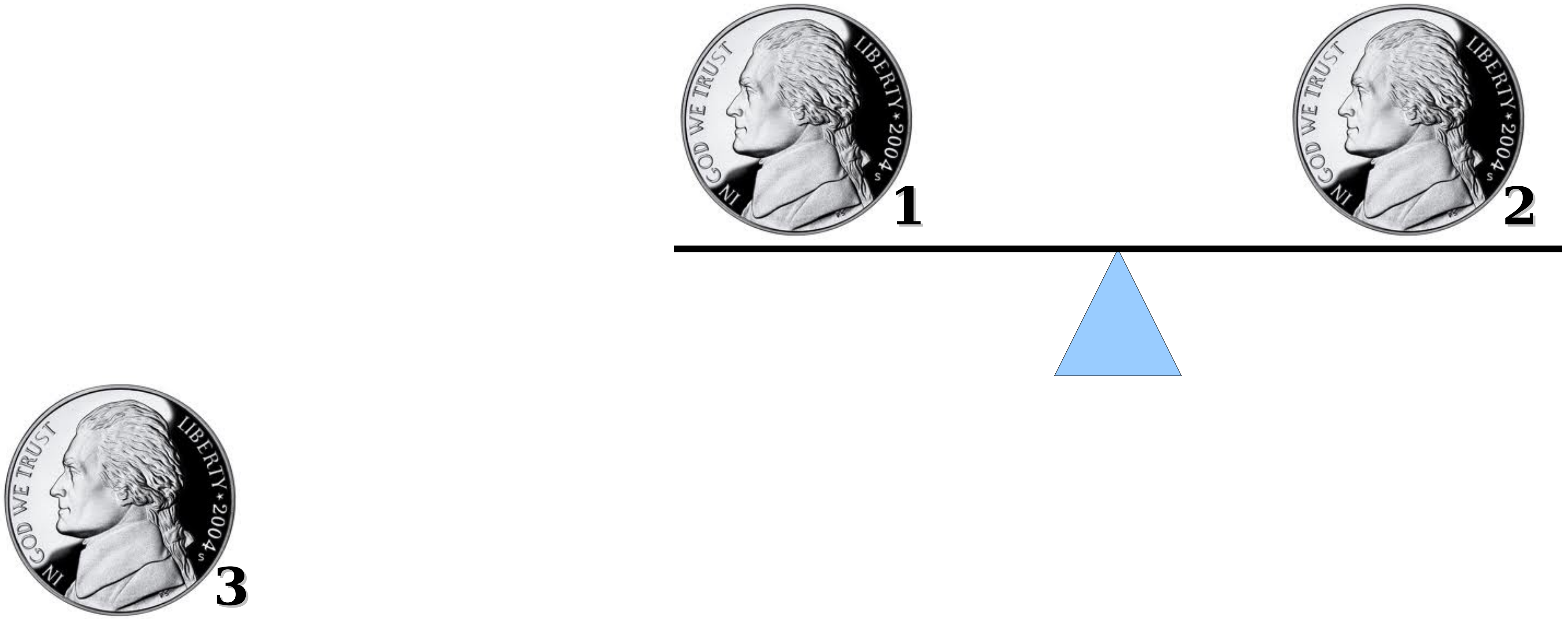

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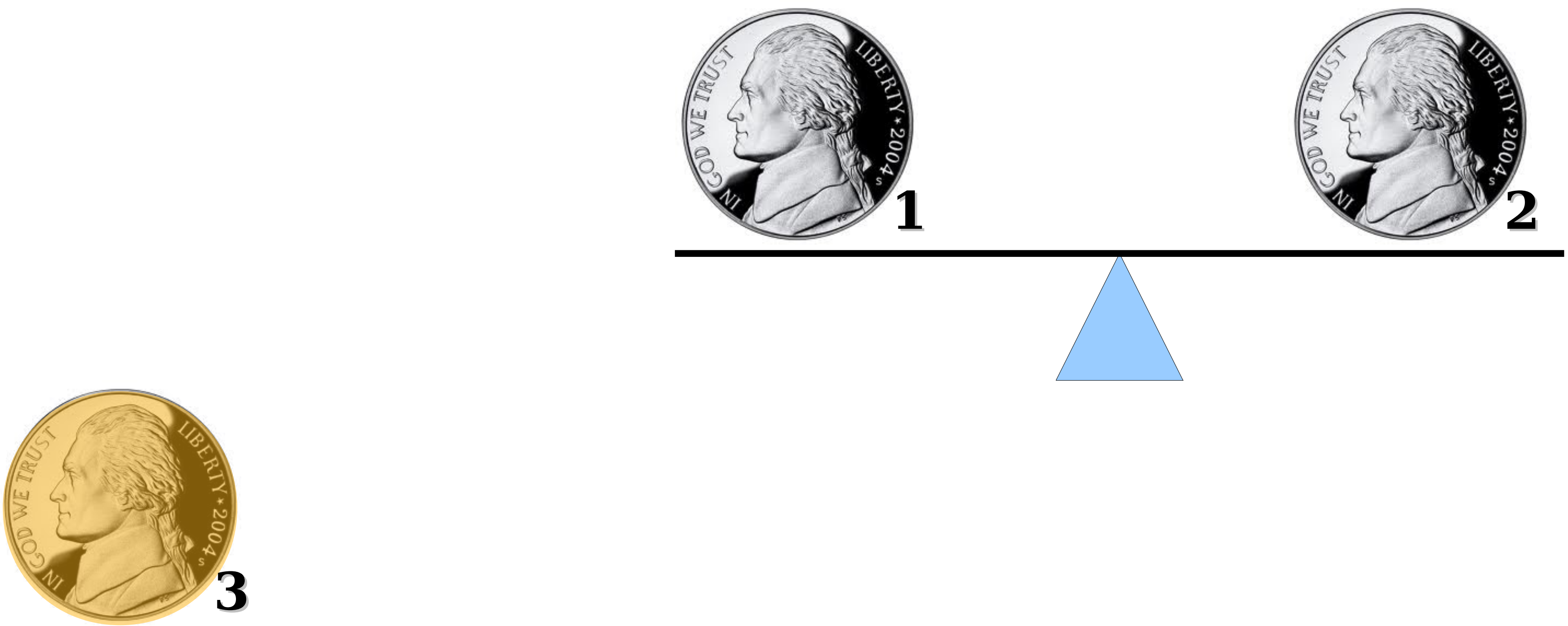
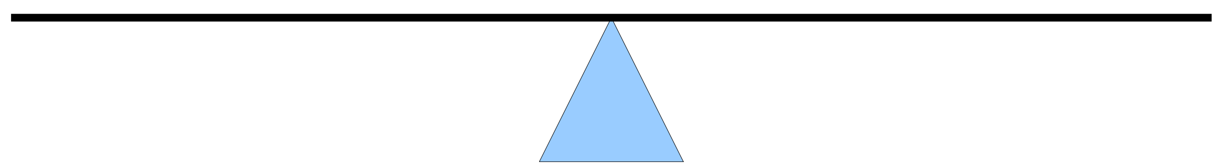


A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



Finding the Counterfeit Coin

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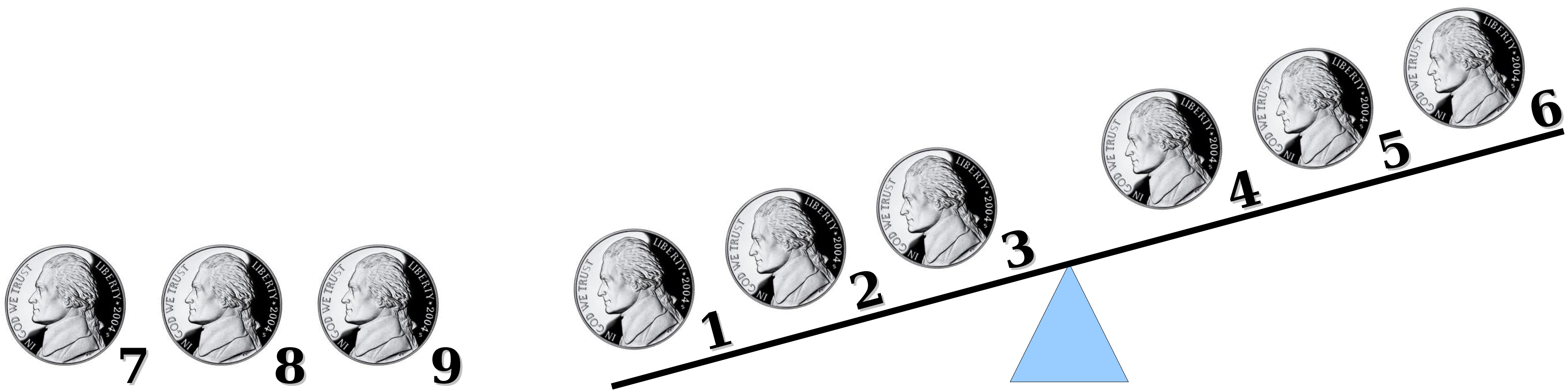

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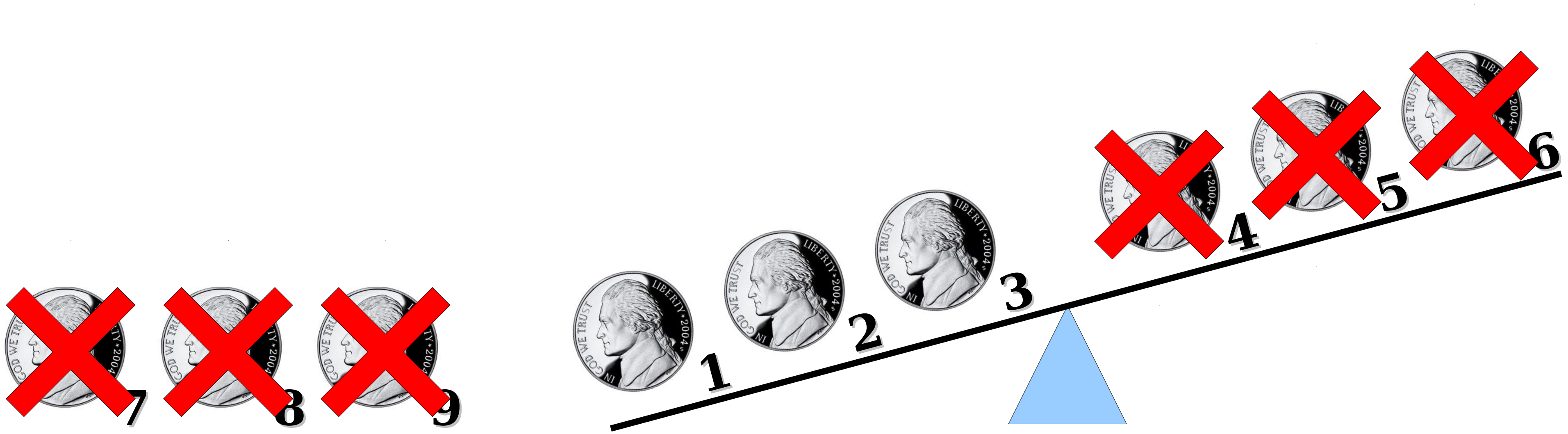
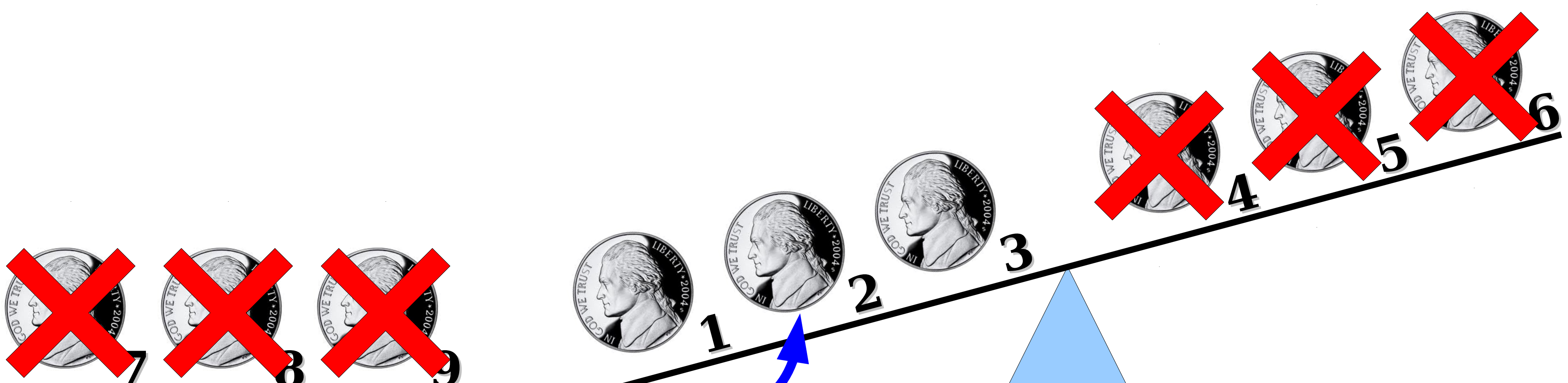
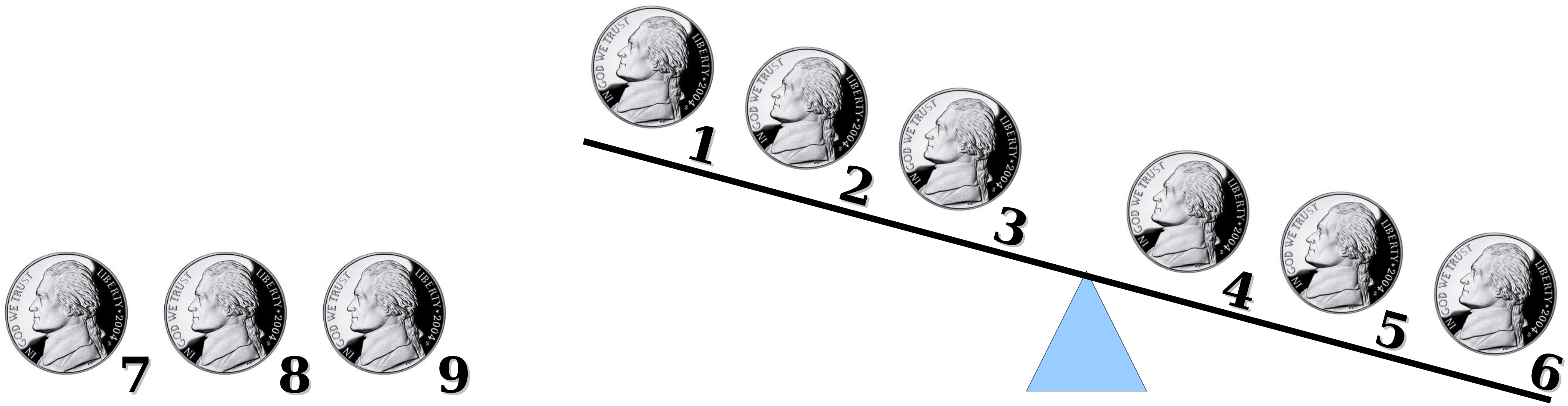


Finding the Counterfeit Coin



Now we have one weighing to find the counterfeit out of these three coins.

Finding the Counterfeit Coin



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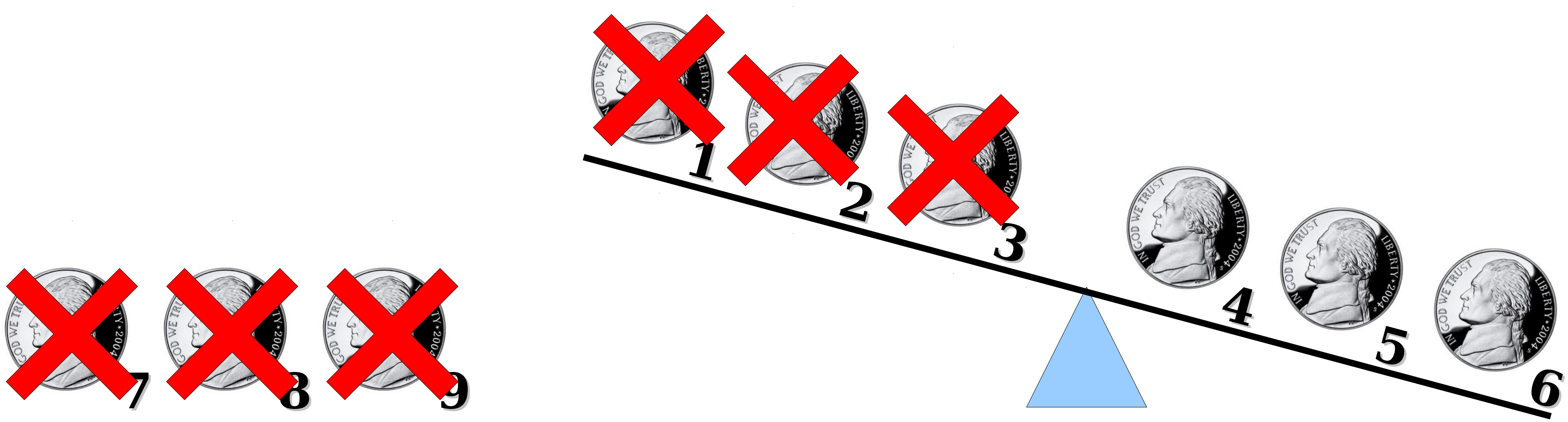

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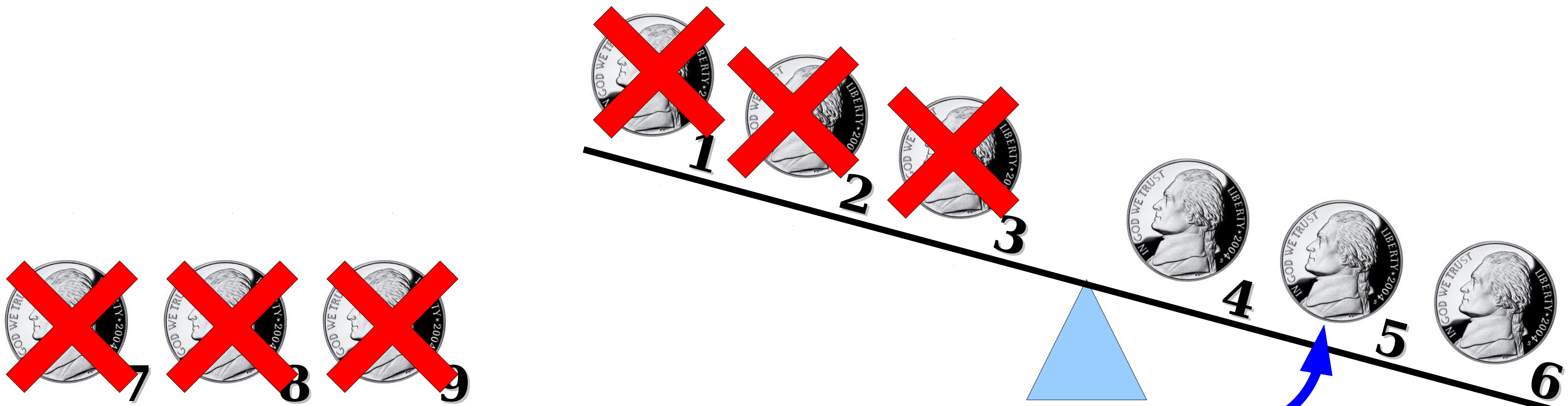

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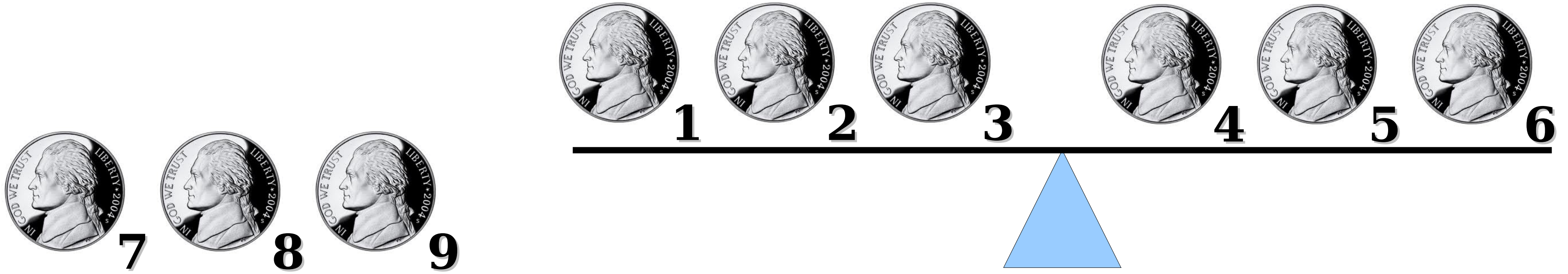

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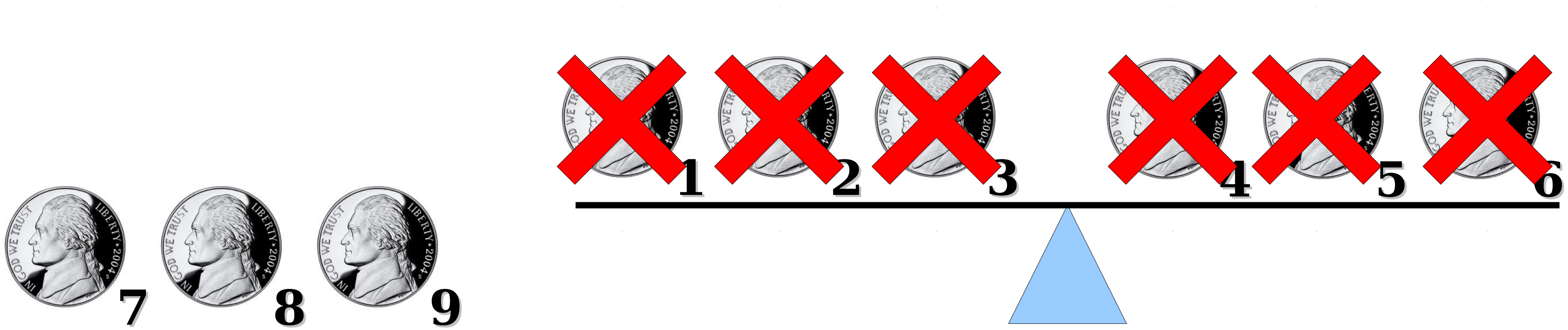

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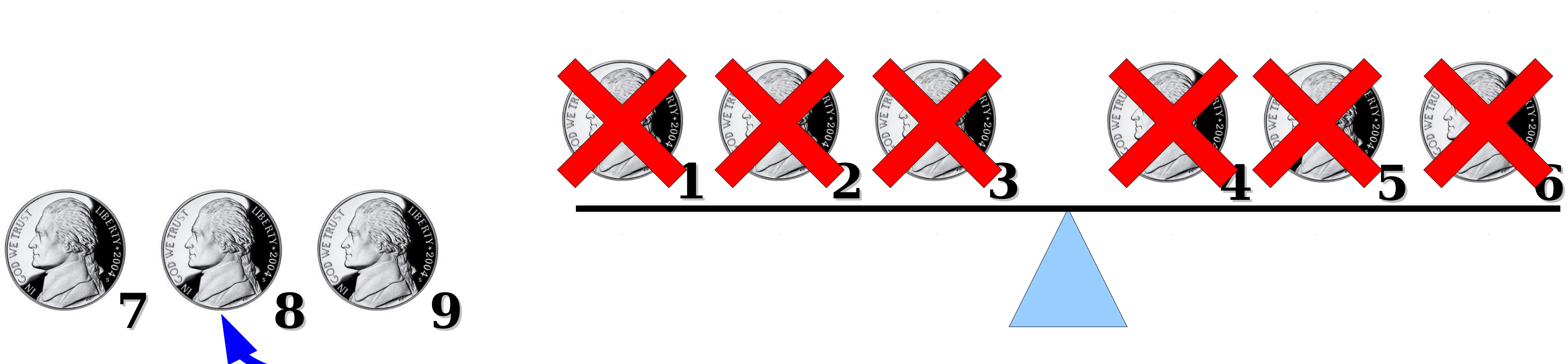

Now we have one weighing to find the counterfeit out of these three coins.

If we have n weighings on the scale, what is the largest number of coins out of which we can find the counterfeit?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One coin**, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$1, 3, 9 = 3^0, 3^1, 3^2$

Does this pattern continue?

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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At the start of the proof, we tell the reader what property we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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In a proof by induction, we need to prove that

- ☐ $P(0)$ is true
- ☐ If $P(k)$ is true, then $P(k+1)$ is true.

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As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0=1$ coins with one coin heavier than the rest, we can find that coin with zero weighings.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Suppose we have 3^{k+1} coins with one heavier than the others.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin.

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Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

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☐ If $P(k)$ is true, then $P(k+1)$ is true.

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Some Fun Problems

- Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get k weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
- What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have k weighings?

Time-Out for Announcements!

Office Hours

- Office hours start today! A schedule is available online.
- Come with questions, leave with answers.
- You can also ask questions on Piazza or by emailing the staff list.

Discussion Sections

- Each week, we have *optional* discussion sections where you can work through additional practice problems.
- We choose problems that are related to the problem set questions, so hopefully it provides extra practice.
- Check the OH schedule for details; discussion sections are marked “DS.”
- Feel free to stop by at any time during discussion sections – you don't have to stay for the whole time.
- Today's 4:15 discussion section is in room 160-317.

Problem Set Clarification

- All problem sets are designed to use only the material up to and include the lecture in which they are released.
- We'll explicitly mark any problems for which we won't have covered the requisite material.
- (In particular, you shouldn't need induction for any of the current problem set questions.)

Checkpoint Feedback

Your Questions

“Can you please repeat questions during lecture before answering them? I watch lectures online with SCPD, and can't always hear the questions that students ask."

Sure, I can try to do that. Sorry about that!

"After completing the proof by contrapositive, is it enough to write 'from this, we see that if not q then not p, so we're done?' Or, do we have to also write 'so by proving the contrapositive we have proved if p then q?'"

You don't need to justify why proving the contrapositive proves the overall result. As long as you've told the reader that you're going to be proving the contrapositive at the start of the proof, you're done once you've proven it

“I haven't figured out how to write proofs for 'a typical CS103 student' yet. Can you give some examples of proofs that are too verbose or prove steps that should be obvious, or are too concise or assume things that you don't consider obvious?”

Sure! I'll go update the guide to proofs with some examples.

"I'd still love a CS themed musical. Thoughts?"

I can try to put some songs together later in the quarter. Ask me again after we start covering regular languages. ☺

Back to our regularly scheduled programming…

Back to our regularly scheduled ~~programming~~...

math

How Not To Induct

Something's Wrong...

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Theorem: The sum of the first n powers of two is 2^n.

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… + …

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When writing a proof by induction,

make sure to show the base case!

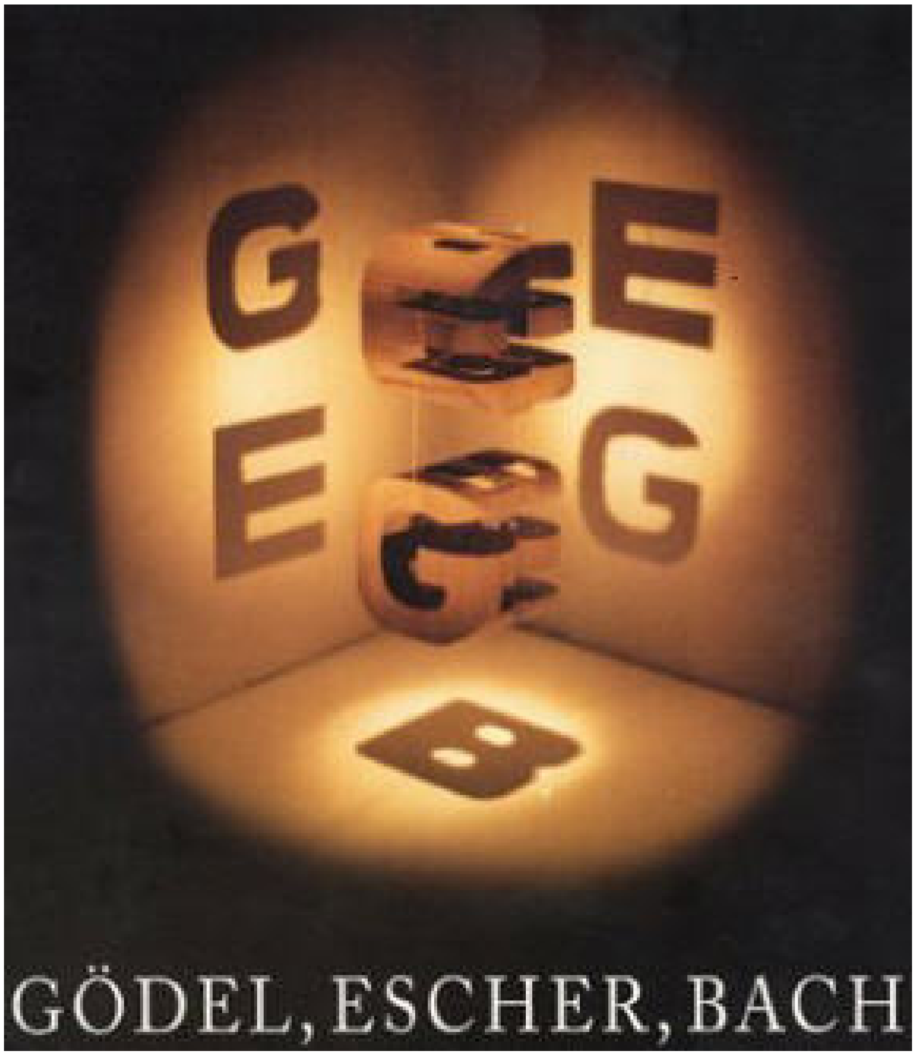
Otherwise, your argument is invalid!

Why This Worked

- The math internally checked out because we made an incorrect assumption!
- Induction requires both the base case and the inductive step.
  - The base case shows that the property initially holds true.
  - The inductive step shows how each step influences the next.

The **MU** Puzzle

Gödel, Escher Bach: An Eternal Golden Braid



- Pulitzer-Prize winning book exploring recursion, computability, and consciousness.
- Written by Douglas Hofstadter, cognitive scientist at Indiana University.
- A great (but dense!) read.

The `MU` Puzzle

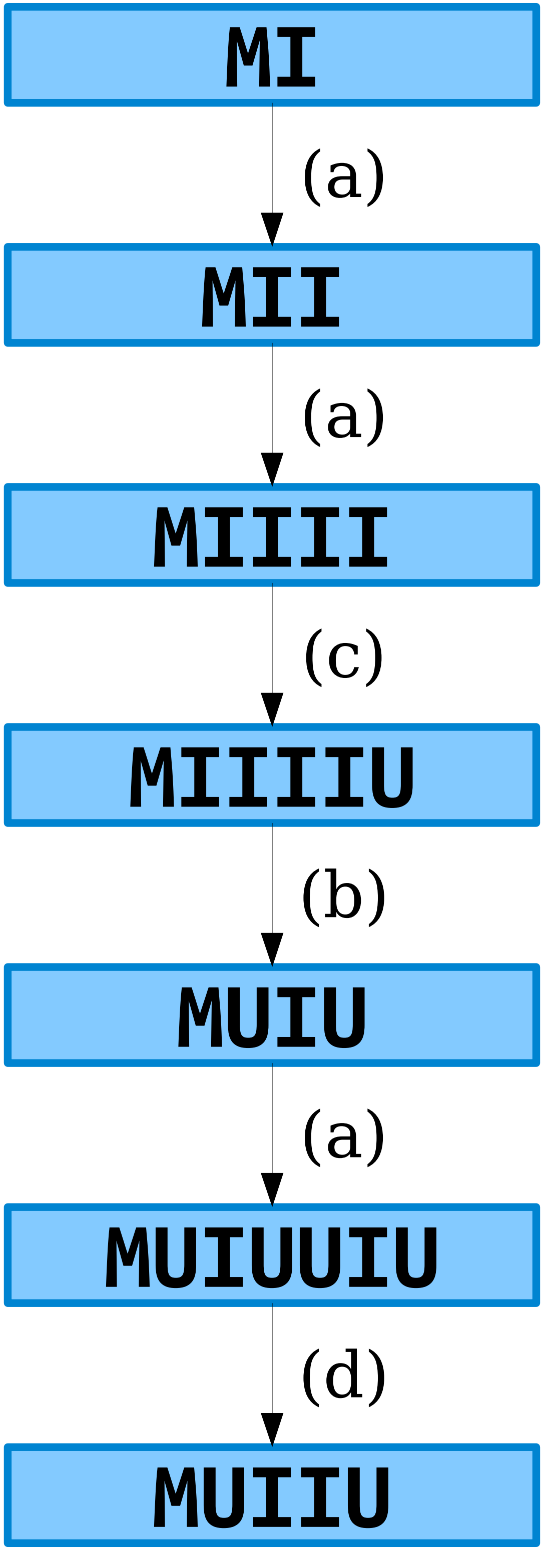
- Begin with the string `MI`.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the `M`: for example, `MIIU` becomes `MIIUIIU`, or `MI` becomes `MII`.
  - Replace `III` with `U`: `MIIII` becomes `MUI` or `MIU`.
  - Append `U` to the string if it ends in `I`: `MI` becomes `MIU`.
  - Remove any `UU`: `MUUU` becomes `MU`.
- **Question**: How do you transform `MI` to `MU`?

(a) Double the string after an `M`.

(b) Replace `III` with `U`.

(c) Append `U`, if the string ends in `I`.

(d) Delete `UU` from the string.



Try It!

Starting with `MI`, apply these operations to make `MU`:

(a) Double the string after an `M`.

(b) Replace `III` with `U`.

(c) Append `U`, if the string ends in `I`.

(d) Delete `UU` from the string.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting **I**'s


MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

The Key Insight

- Initially, the number of **`I`**'s is *not* a multiple of three.
- To make **`MU`**, the number of **`I`**'s must end up as a multiple of three.
- Can we *ever* make the number of **`I`**'s a multiple of three?

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

Proof: By contrapositive; we'll prove that if $n - 3$ is a multiple of three, then n is also a multiple of three. Because $n - 3$ is a multiple of three, we can write $n - 3 = 3k$ for some integer k. Then $n = 3(k+1)$, so n is also a multiple of three, as required. ■

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Proof: Let n be a number that isn't a multiple of three. If n is congruent to one modulo three, then $n = 3k + 1$ for some integer k. This means $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$, so $2n$ is not a multiple of three. Otherwise, n must be congruent to two modulo three, so $n = 3k + 2$ for some integer k. Then $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$, and so $2n$ is not a multiple of three. ■

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Case 1: Double the string after the `M`.

Case 2: Replace `III` with `U`.

Case 3: Either append `U` or delete `UU`.

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Therefore, no sequence of $k+1$ moves ends with a multiple of three `I`'s.

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Therefore, no sequence of $k+1$ moves ends with a multiple of three `I`'s. Thus $P(k+1)$ is true, completing the induction.

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Theorem: The **`MU`** puzzle has no solution.

Proof: Assume for the sake of contradiction that the **`MU`** puzzle has a solution and that we can convert **`MI`** to **`MU`**. This would mean that at the very end, the number of **`I`**'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of **`I`**'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the **`MU`** puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a **loop invariant**.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Next Time

- **Variations on Induction**
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.