# Regular Expressions

# Recap from Last Time

# Regular Languages

- A language $L$ is a ***regular language*** if there is a DFA $D$ such that $\mathscr{L}(D) = L$.
- ***Theorem:*** The following are equivalent:
  - $L$ is a regular language.
  - There is a DFA for $L$.
  - There is an NFA for $L$.

# Language Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, then $wx$ is the ***concatenation*** of $w$ and $x$.
- If $L_1$ and $L_2$ are languages over $\Sigma$, the ***concatenation*** of $L_1$ and $L_2$ is the language $L_1L_2$ defined as

$$L_1L_2 = \{ wx \mid w \in L_1 \text{ and } x \in L_2 \}$$

# Language Exponentiation

- If $L$ is a language over $\Sigma$, the language $L^n$ is the concatenation of $n$ copies of $L$ with itself.
  - Special case: $L^0 = \{\varepsilon\}$.
- The ***Kleene closure*** of a language $L$, denoted $L^*$, is defined as

$$L^* = \{ w \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Intuitively, all strings that can be formed by concatenating any number of strings in $L$ with one another.

# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet $\Sigma$, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - $L_1^*$
- These properties are called ***closure properties of the regular languages***.

# Another View of Regular Languages

# Rethinking Regular Languages

- We currently have several tools for showing a language is regular.
  - Construct a DFA for it.
  - Construct an NFA for it.
  - Apply closure properties to existing languages.
- We have not spoken much of this last idea.

# Constructing Regular Languages

- **Idea:** Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *A bottom-up approach to the regular languages.*

# Constructing Regular Languages

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  - Start with a small set of simple languages we already
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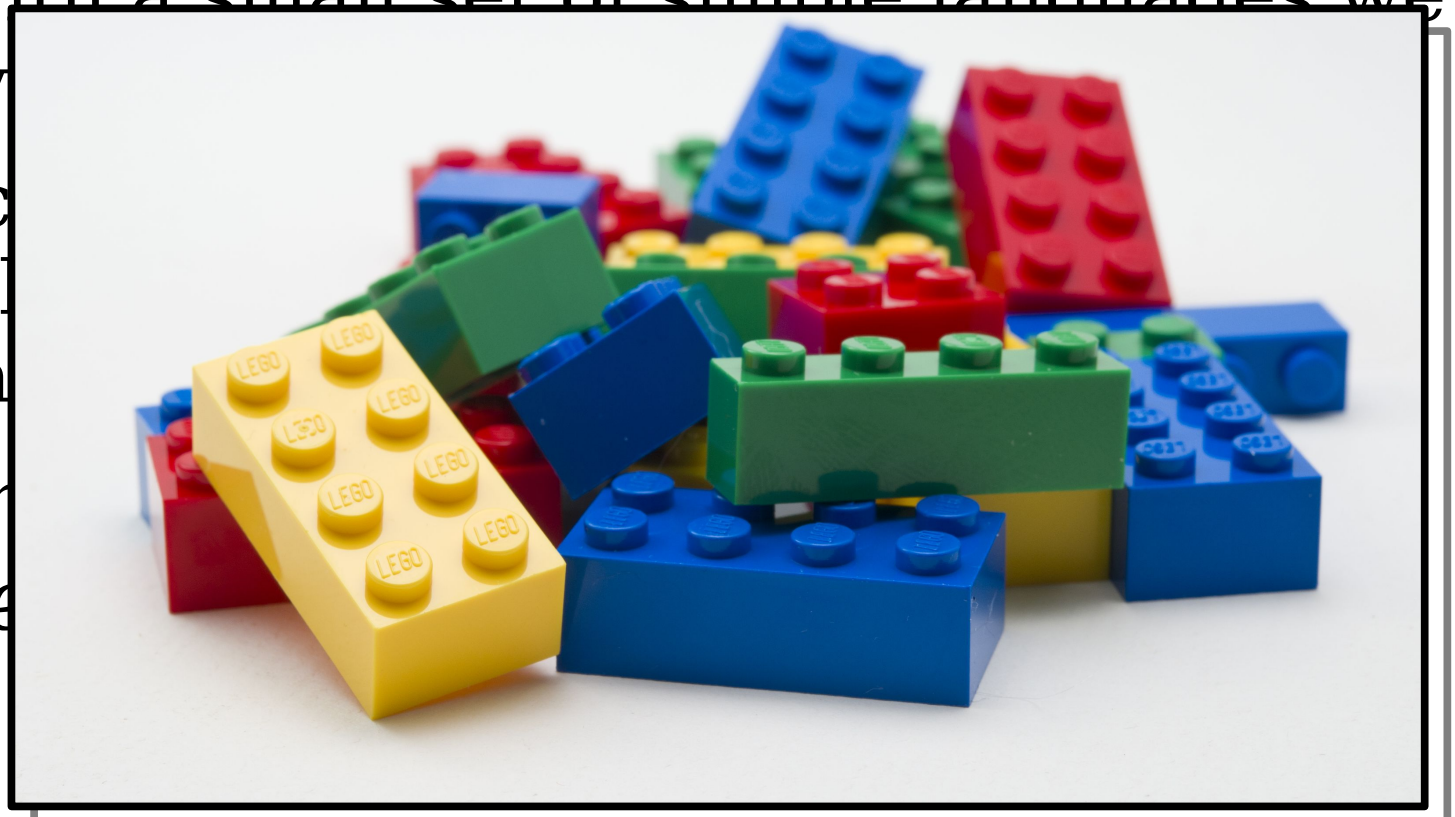
- *A bottom*
  *language*



# Regular Expressions

- ***Regular expressions*** are a way of describing a language via a string representation.
- Used extensively in software systems for string processing and as the basis for tools like `grep` and `flex`.
- Conceptually: regular languages are strings describing how to assemble a larger language out of smaller pieces.

# Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- The symbol **ε** is a regular expression that represents the language {ε}
  - ***This is not the same as Ø!***
  - ***This is not the same as ε!***
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language {**a**}

# Compound Regular Expressions

- If $R_1$ and $R_2$ are regular expressions, $\boldsymbol{R_1 R_2}$ is a regular expression for the *concatenation* of the languages of $R_1$ and $R_2$.
- If $R_1$ and $R_2$ are regular expressions, $\boldsymbol{R_1 \mid R_2}$ is a regular expression for the *union* of the languages of $R_1$ and $R_2$.
- If $R$ is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the *Kleene closure* of the language of $R$.
- If $R$ is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as $R$.

# Operator Precedence

- Regular expression operator precedence:

$$(R)$$

$$R^*$$

$$R_1 R_2$$

$$R_1 \mid R_2$$

- So `ab*c|d` is parsed as `((a(b*))c)|d`

# Regular Expression Examples

- The regular expression **`trick|treat`** represents the regular language { **`trick`**, **`treat`** }
- The regular expression **`booo*`** represents the regular language { **`boo`**, **`booo`**, **`boooo`**, ... }
- The regular expression **`candy!(candy!)*`** represents the regular language { **`candy!`**, **`candy!candy!`**, **`candy!candy!candy!`**, ... }

# Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\varepsilon) = \{\varepsilon\}$
  - $\mathscr{L}(\emptyset) = \emptyset$
  - $\mathscr{L}(\mathtt{a}) = \{\mathtt{a}\}$
  - $\mathscr{L}(R_1 R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \mid R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^*) = \mathscr{L}(R)^*$
  - $\mathscr{L}((R)) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

**a(b|c)((d))**

and see what you get.

# Regular Expressions are Awesome

- Let $\Sigma = \{$`0`, `1`$\}$
- Let $L = \{ w \in \Sigma^* \mid w$ contains `00` as a substring $\}$

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**11011100101**
**0000**
**11111011110011111**

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$$\Sigma^*00\Sigma^*$$

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The length of a string w is denoted |w|

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**1*0?1***

**11110111**
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- Regular expression for email addresses:

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$$\mathbf{a}^+ \ (.\mathbf{a}^+)^* \ @ \quad \mathbf{a}^+ \quad (.\mathbf{a}^+)^+$$

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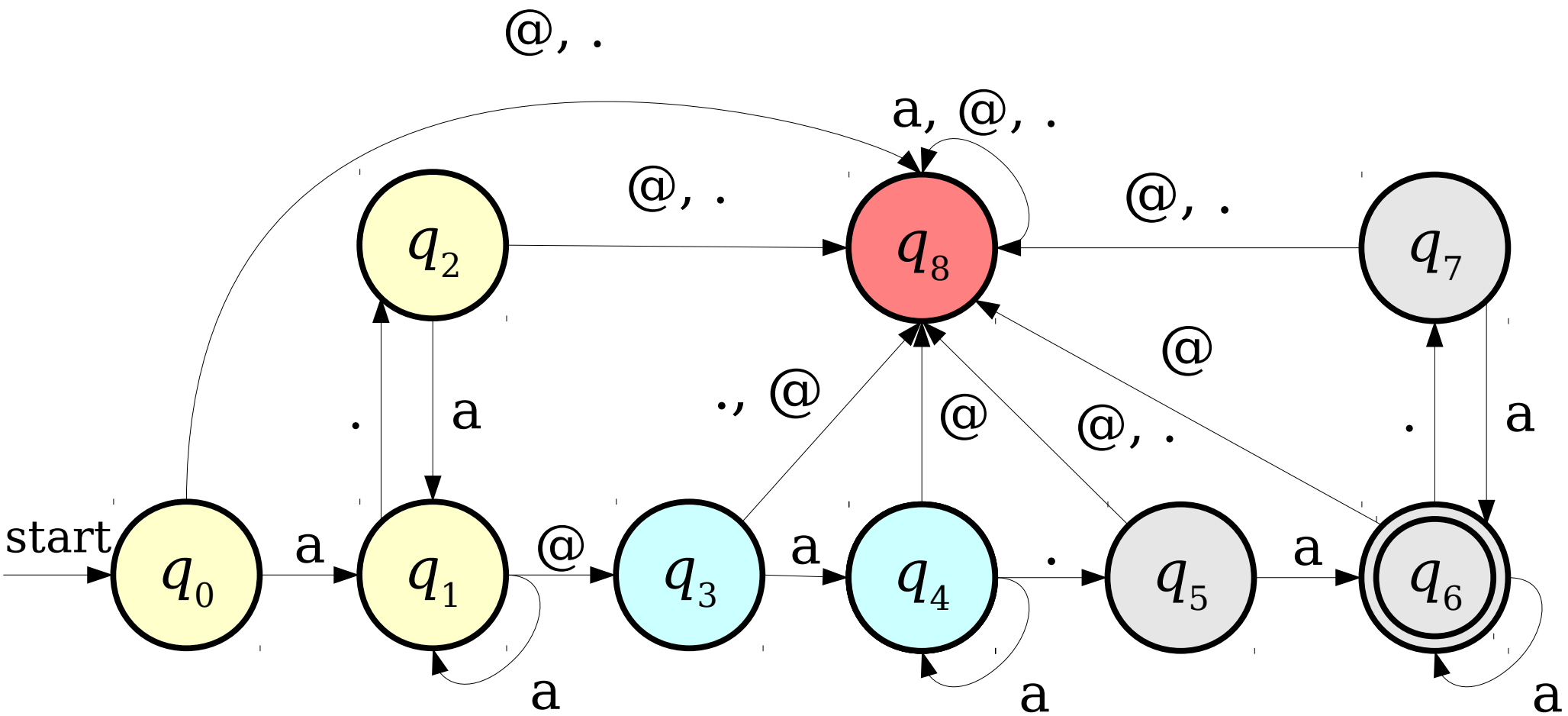


# Shorthand Summary

- $R^n$ is shorthand for $RR \ldots R$ ($n$ times).
  - Edge case: define $R^0 = \varepsilon$.
- $\Sigma$ is shorthand for "any character in $\Sigma$."
- $R$? is shorthand for ($R$ | $\varepsilon$), meaning "zero or one copies of $R$."
- $R^+$ is shorthand for $RR^*$, meaning "one or more copies of $R$."

Time-Out for Announcements!

# Problem Set Five

- Problem Set Four was due at the start of class today.
  - Need more time? Turn it in by Tuesday at 12:50 with one late day or Wednesday at 12:50 with two.
- Problem Set Five goes out today. It's due next Monday at the start of class.
  - Play around with DFAs, NFAs, regular expressions, and properties of regular languages!
  - We have online tools for developing finite automata and regular expressions; we hope you find them useful!

# Your Questions

“Why don't you use a Mac? Also, what are your thoughts on Disney making more Star Wars films?”

I only really use my laptop for email, word processing, slides, and development, so I didn't think it was worth shelling out the extra to get a Mac.

Also, I have absolutely no opinion on the new Star Wars movies.

“Why is there such a huge emphasis on women in CS or women in STEM in general? If the university is 50/50 at large, some departments would inevitably have more or fewer men or women. I have never understood this besides *en fait* assuming discrimination”

1. Random distributions alone would not account for the gender skew in CS.
2. The world as a whole benefits when everyone gets a fair shot at creating technology.
3. The assumption that technology is a meritocracy doesn't hold up to experimental evidence.
4. There are definite gender/racial/socioeconomic biases in the tech industry and in CS as a whole.

“sup”

Not much! I learned a really cool theorem and put it as the extra credit problem on the problem set! And I recently found an enchilada recipe that I (vegetarian) and my relatives (dairy allergy) can both enjoy! And I'm thinking deep thoughts about what I want to do with my life. You?

Back to CS103!

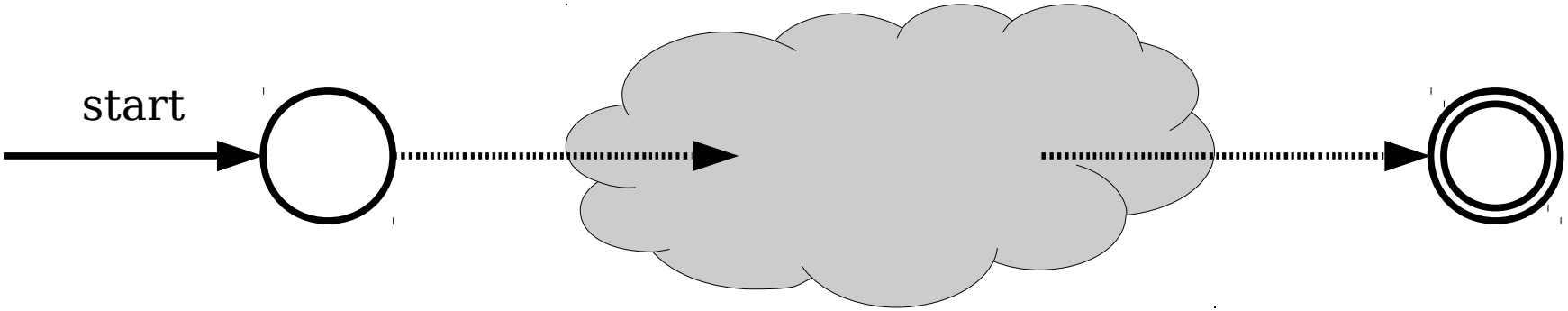
# The Power of Regular Expressions

***Theorem:*** If $R$ is a regular expression, then $\mathscr{L}(R)$ is regular.

***Proof idea:*** Show how to convert a regular expression into an NFA.

# A Marvelous Construction

- The following theorem proves the language of any regular expression is regular:
- ***Theorem:*** For any regular expression $R$, there is an NFA $N$ such that
  - $\mathscr{L}(R) = \mathscr{L}(N)$
  - $N$ has exactly one accepting state.
  - $N$ has no transitions into its start state.
  - $N$ has no transitions out of its accepting state.



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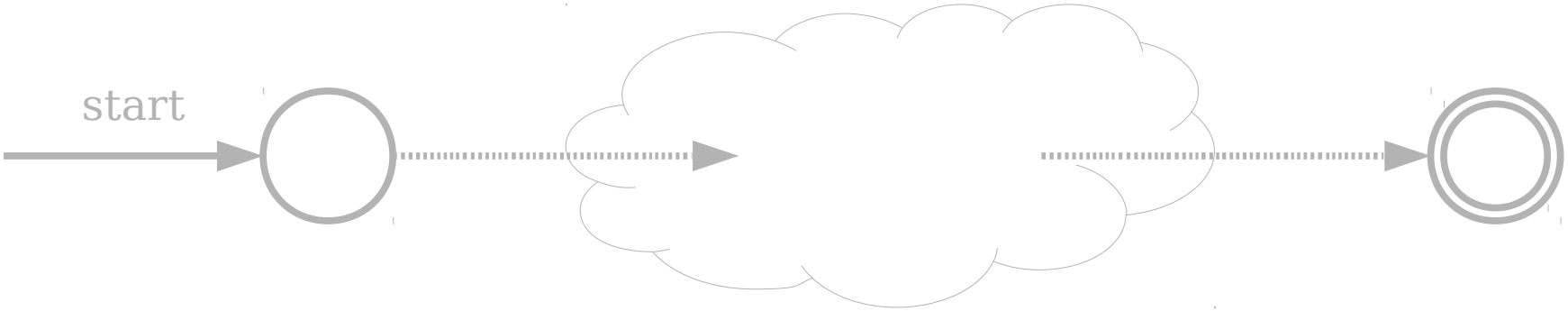

# A Marvelous Construction

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***Theorem:*** For any regu
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These are stronger requirements than are necessary for a normal NFA. We enforce these rules to simplify the construction.

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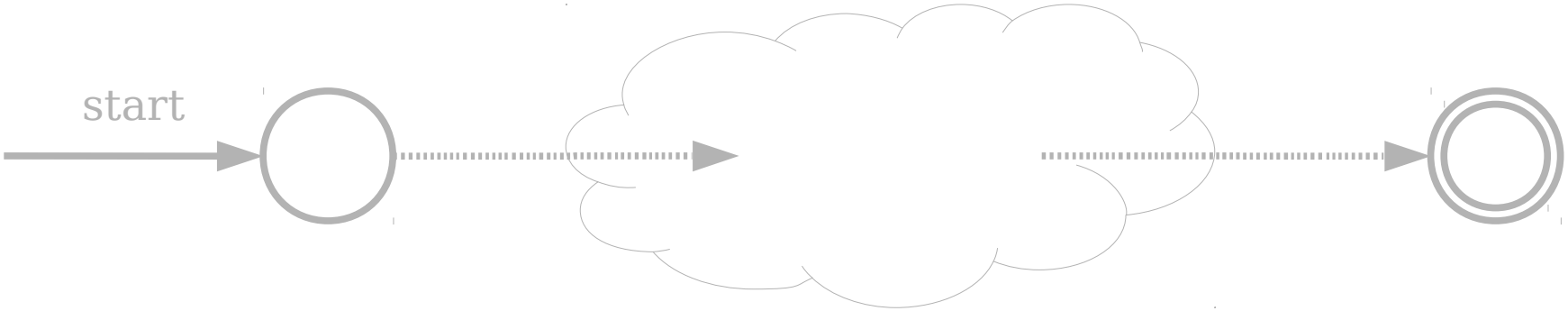
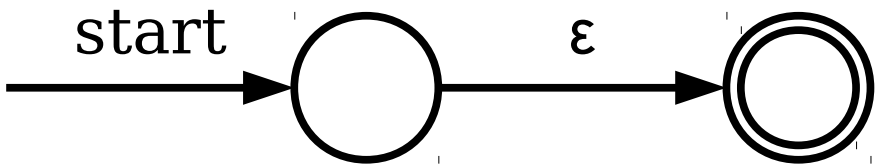
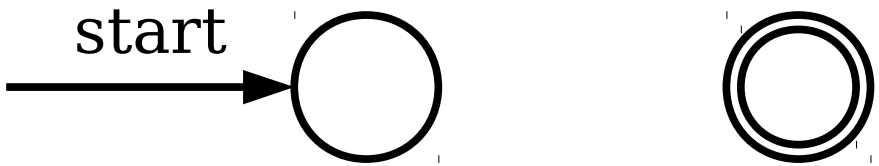


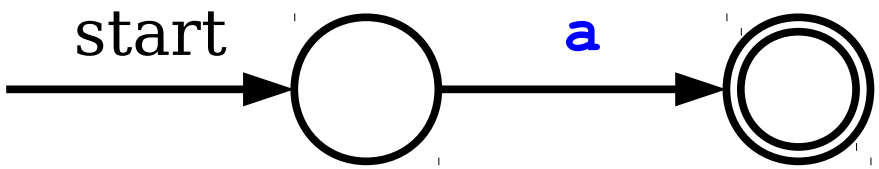
# Base Cases




Automaton for ε




Automaton for Ø




Automaton for single character **a**

# Construction for $R_1 R_2$

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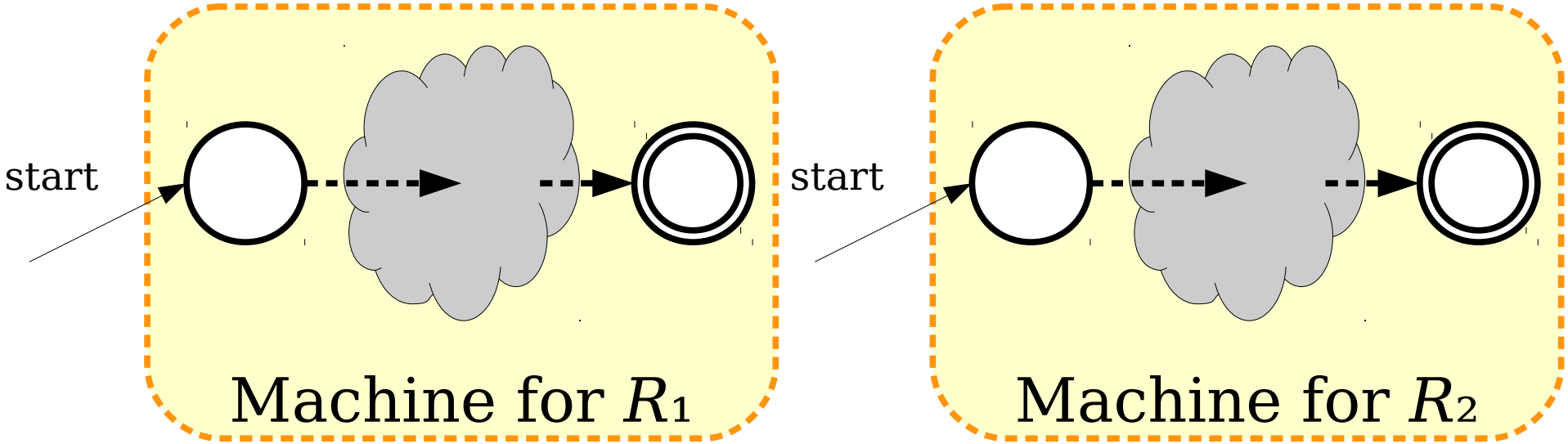

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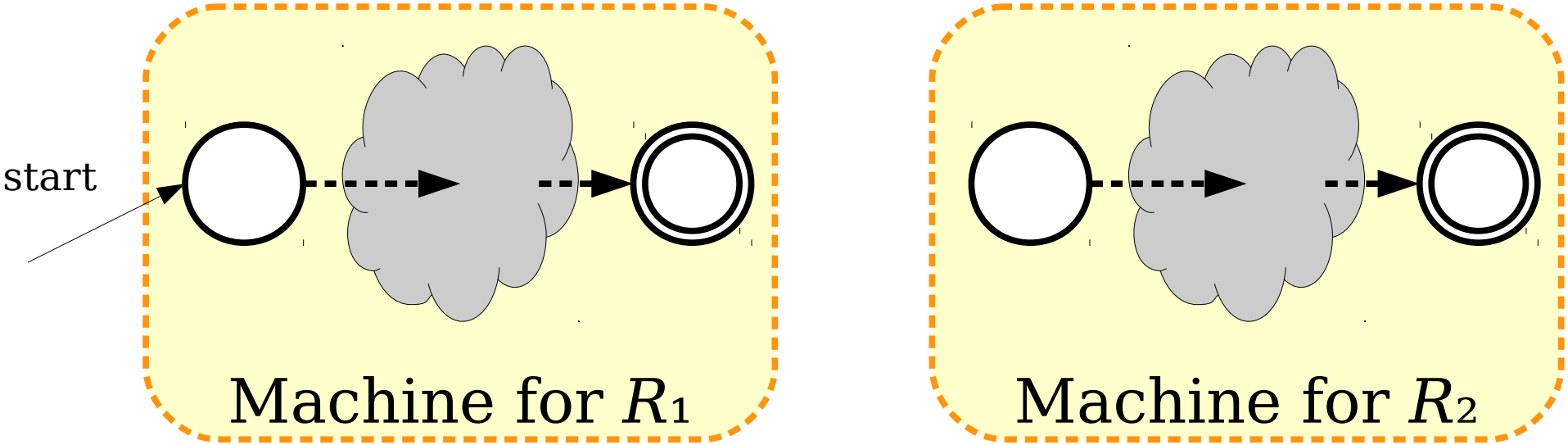

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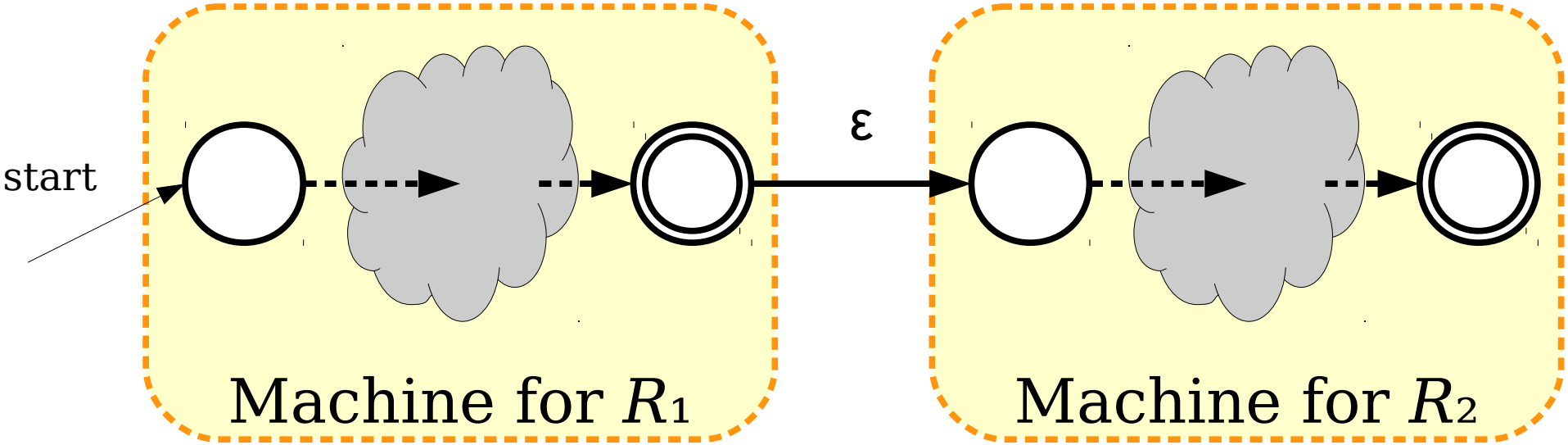

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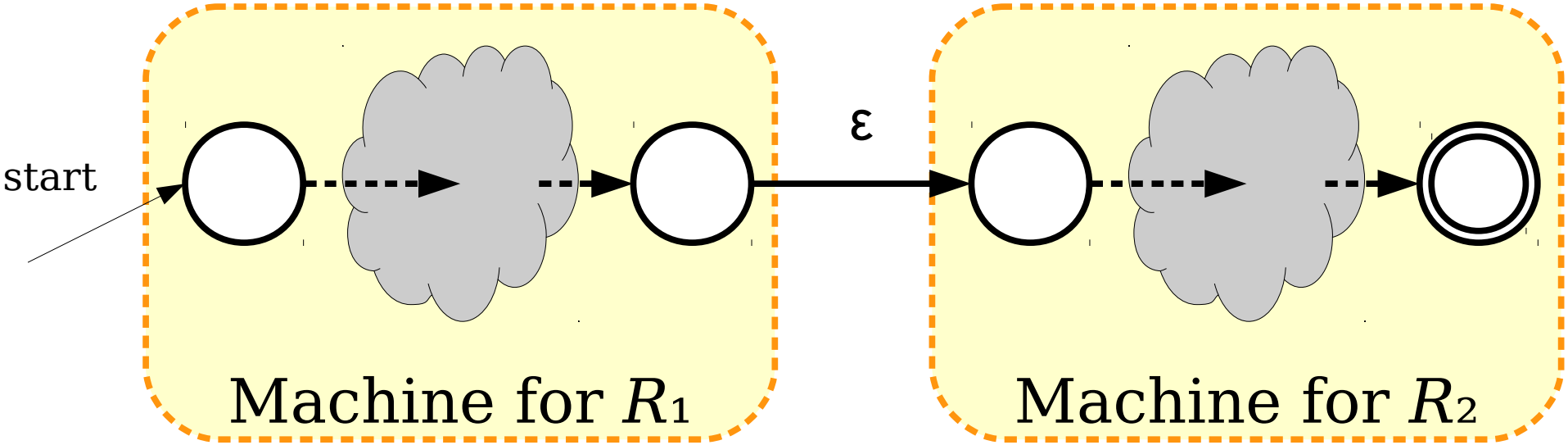

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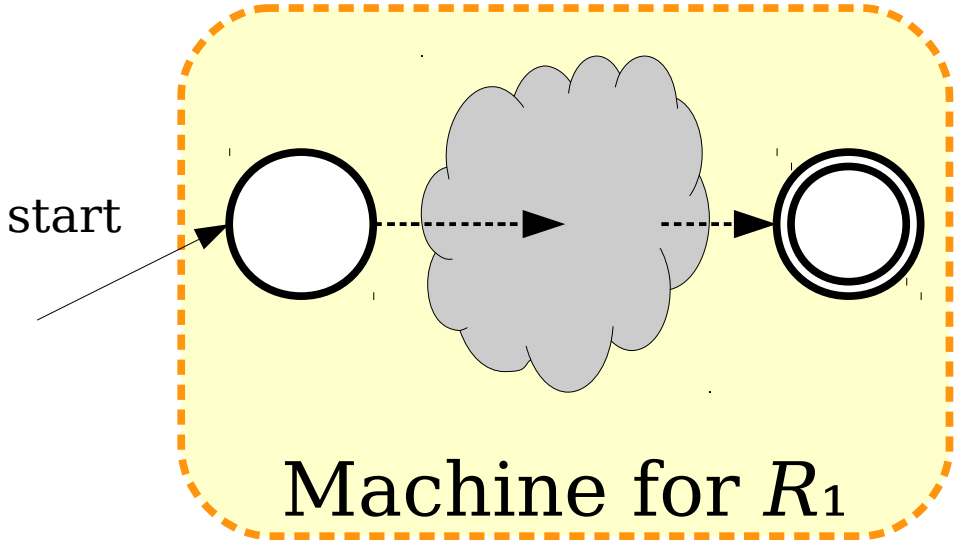


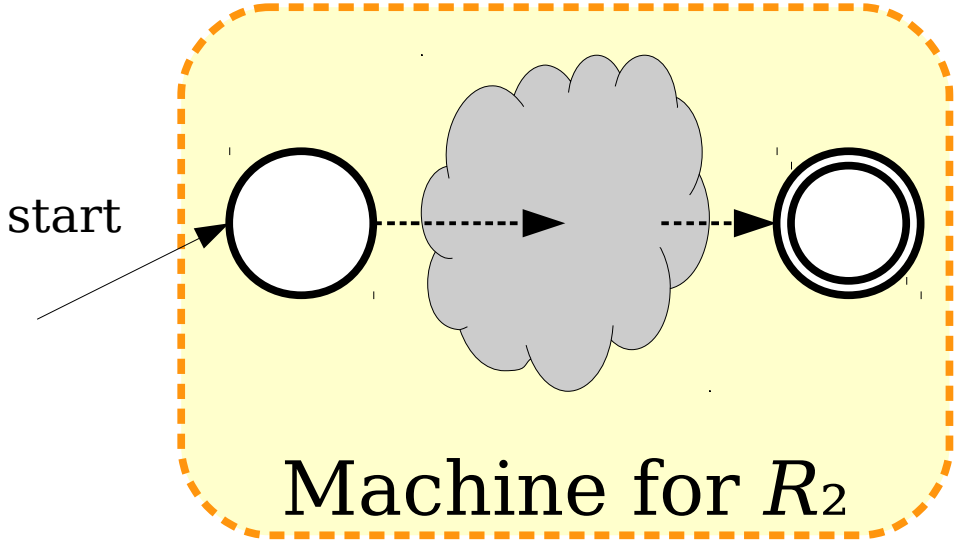

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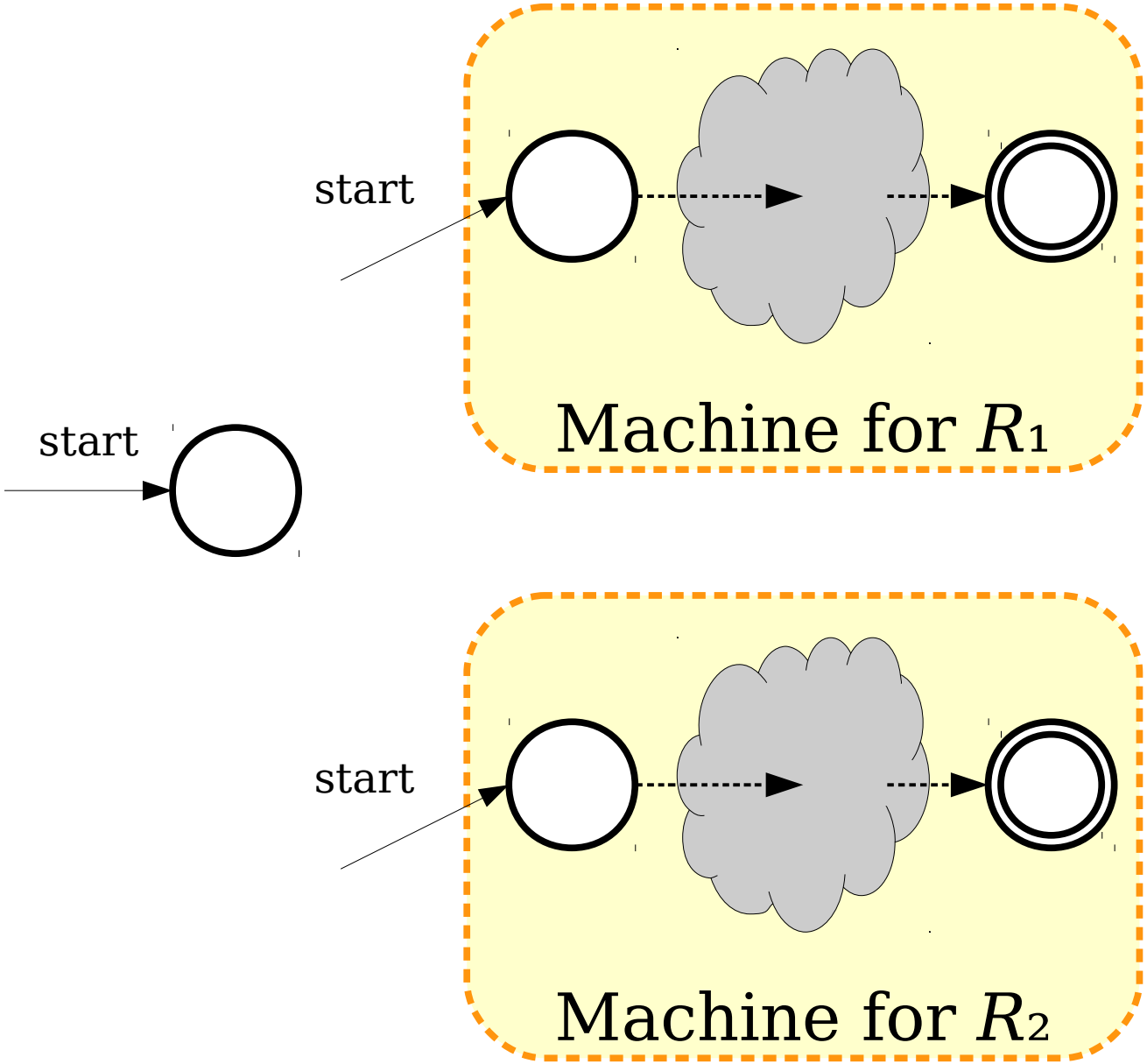

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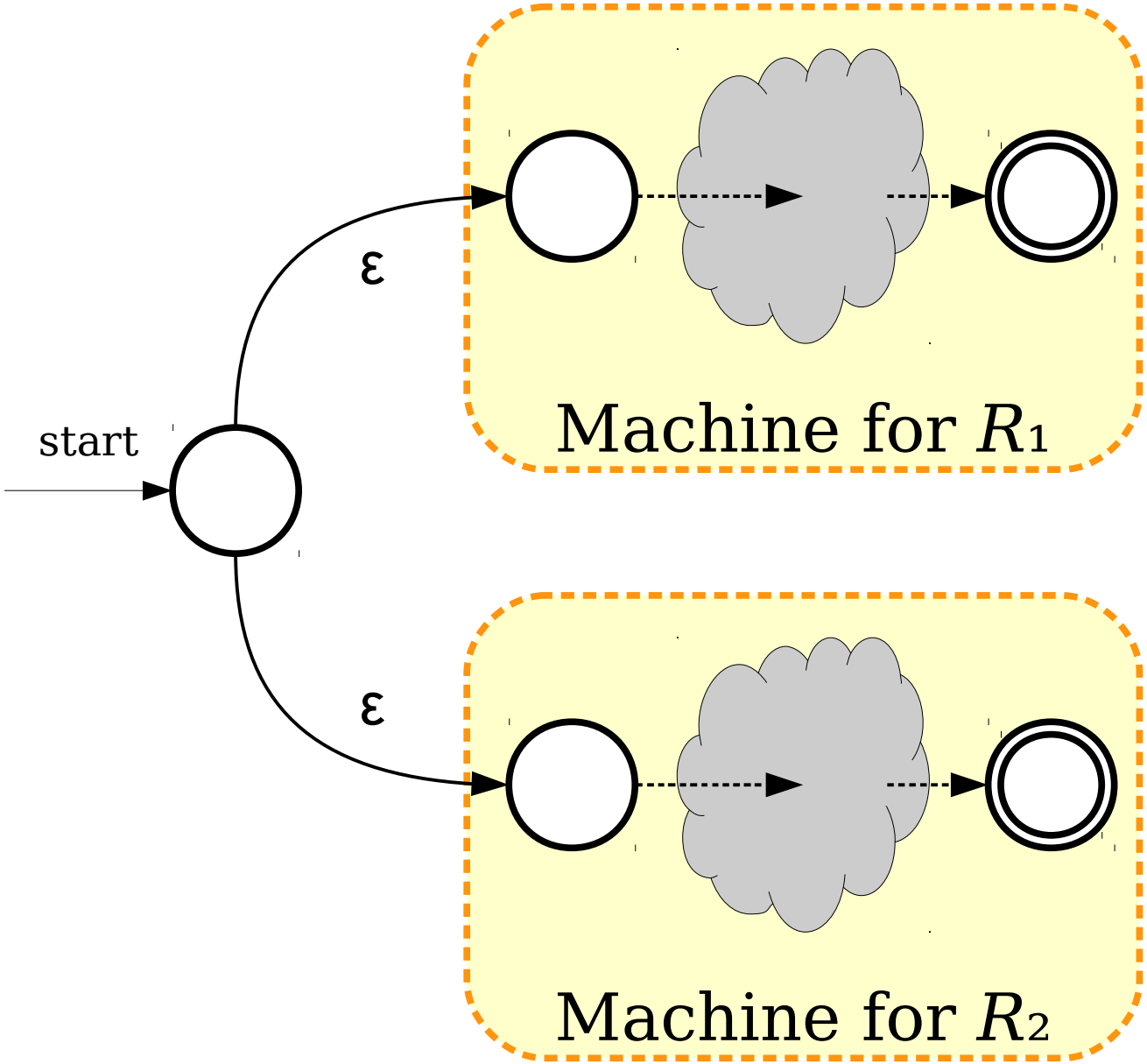

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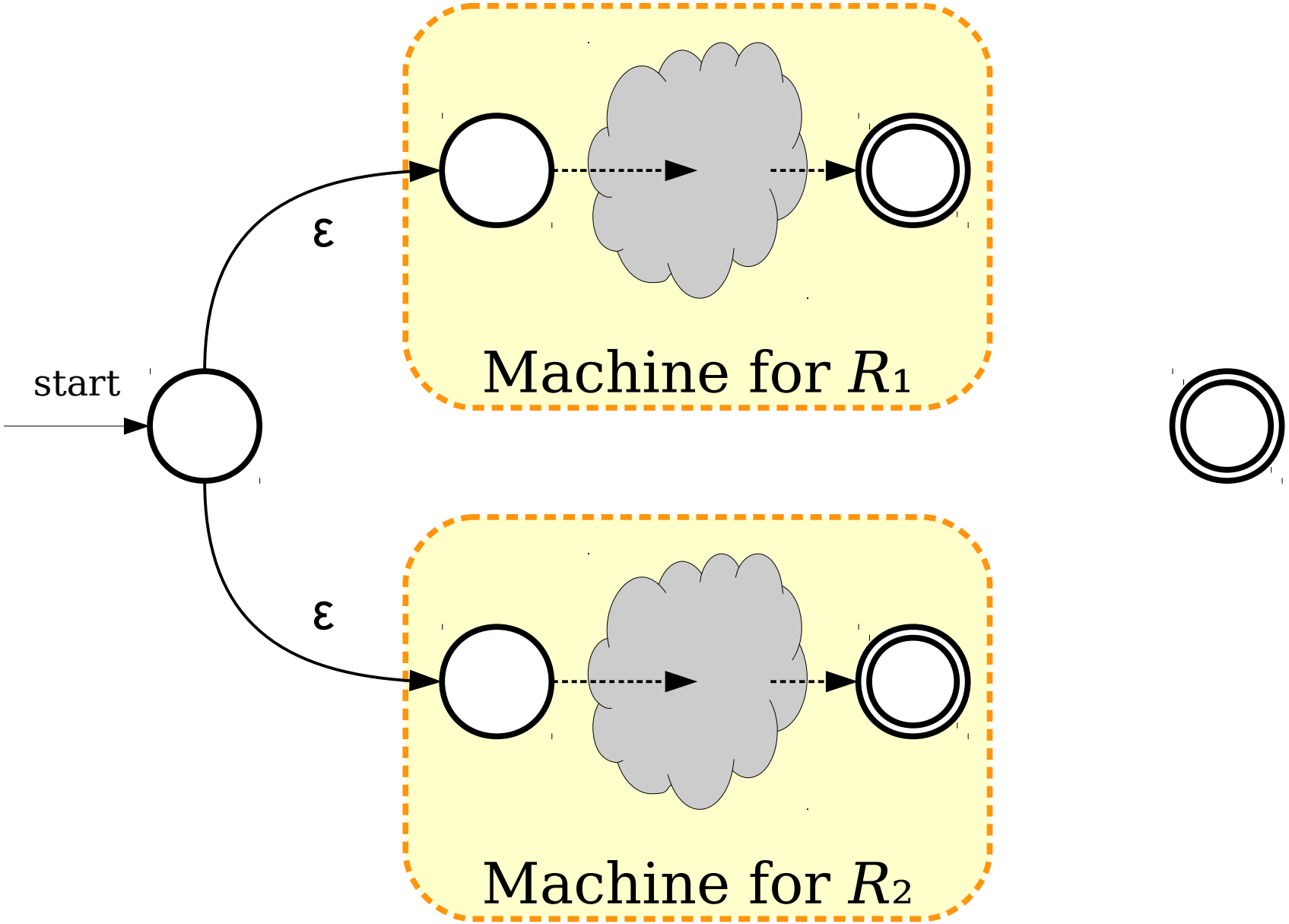

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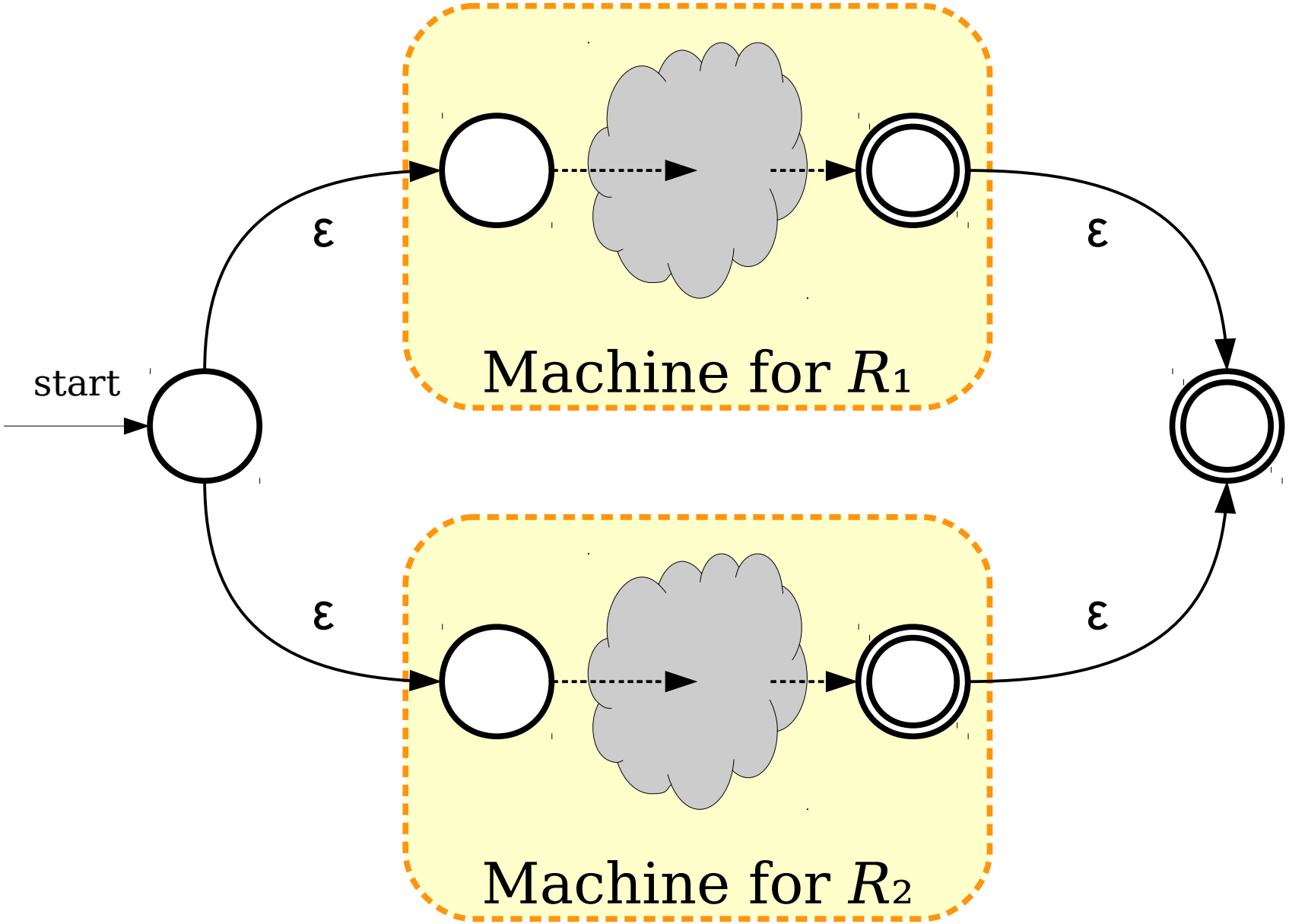

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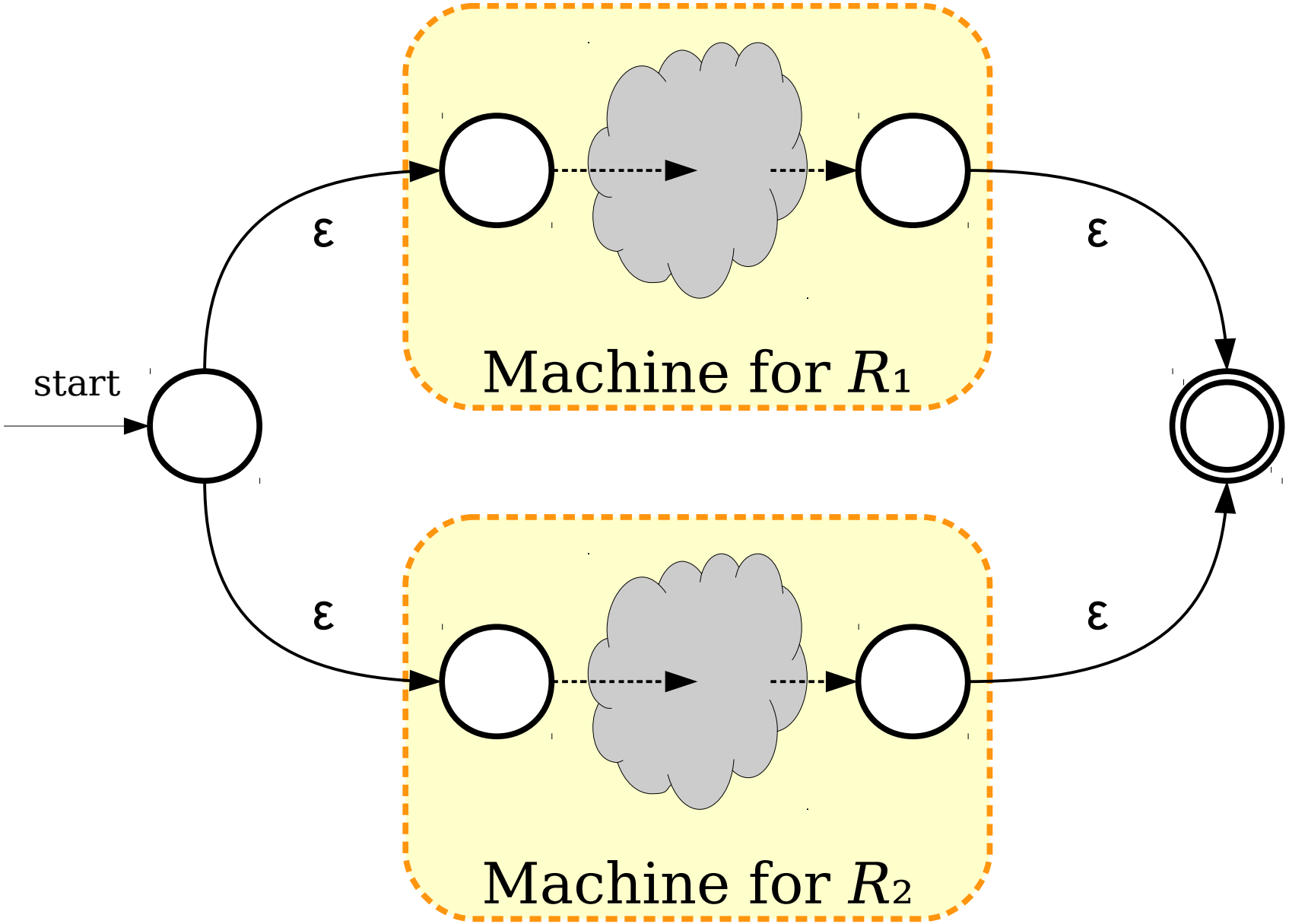

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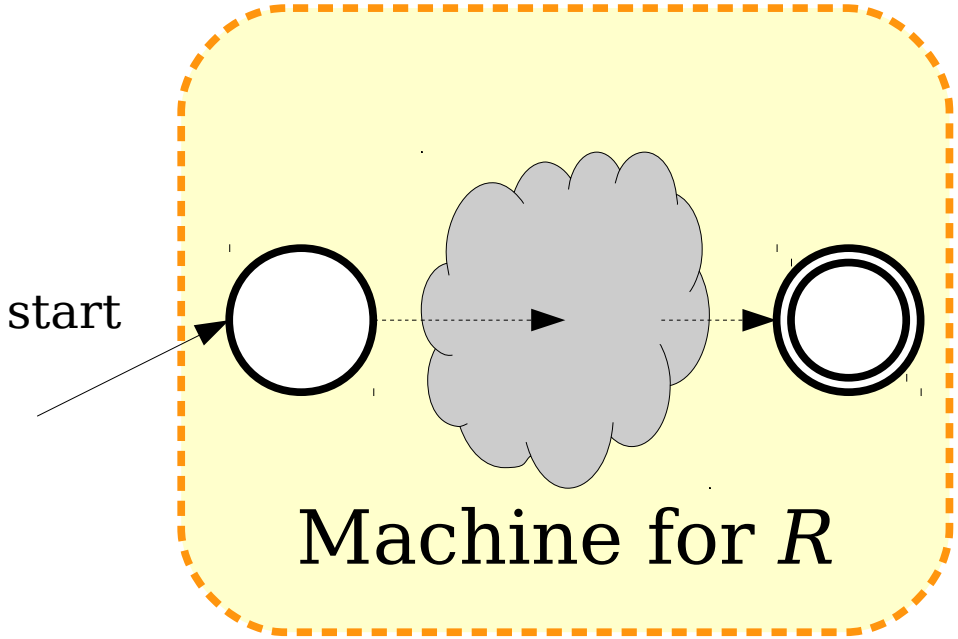

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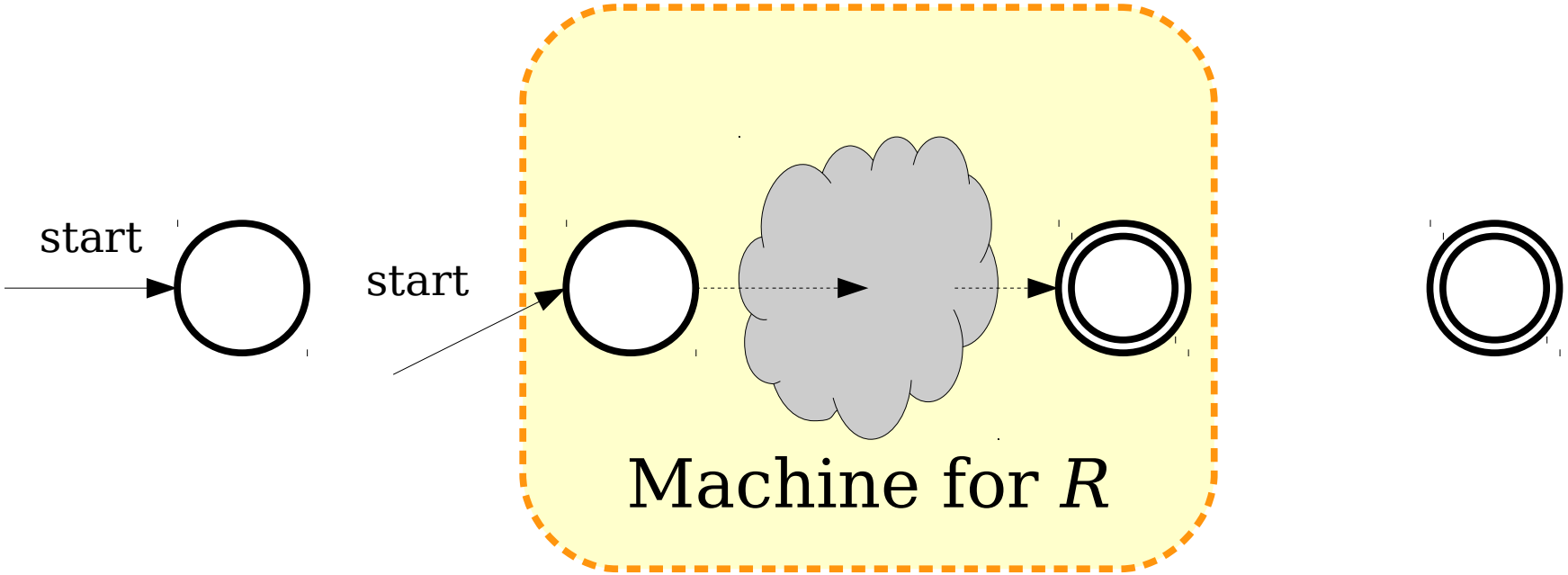

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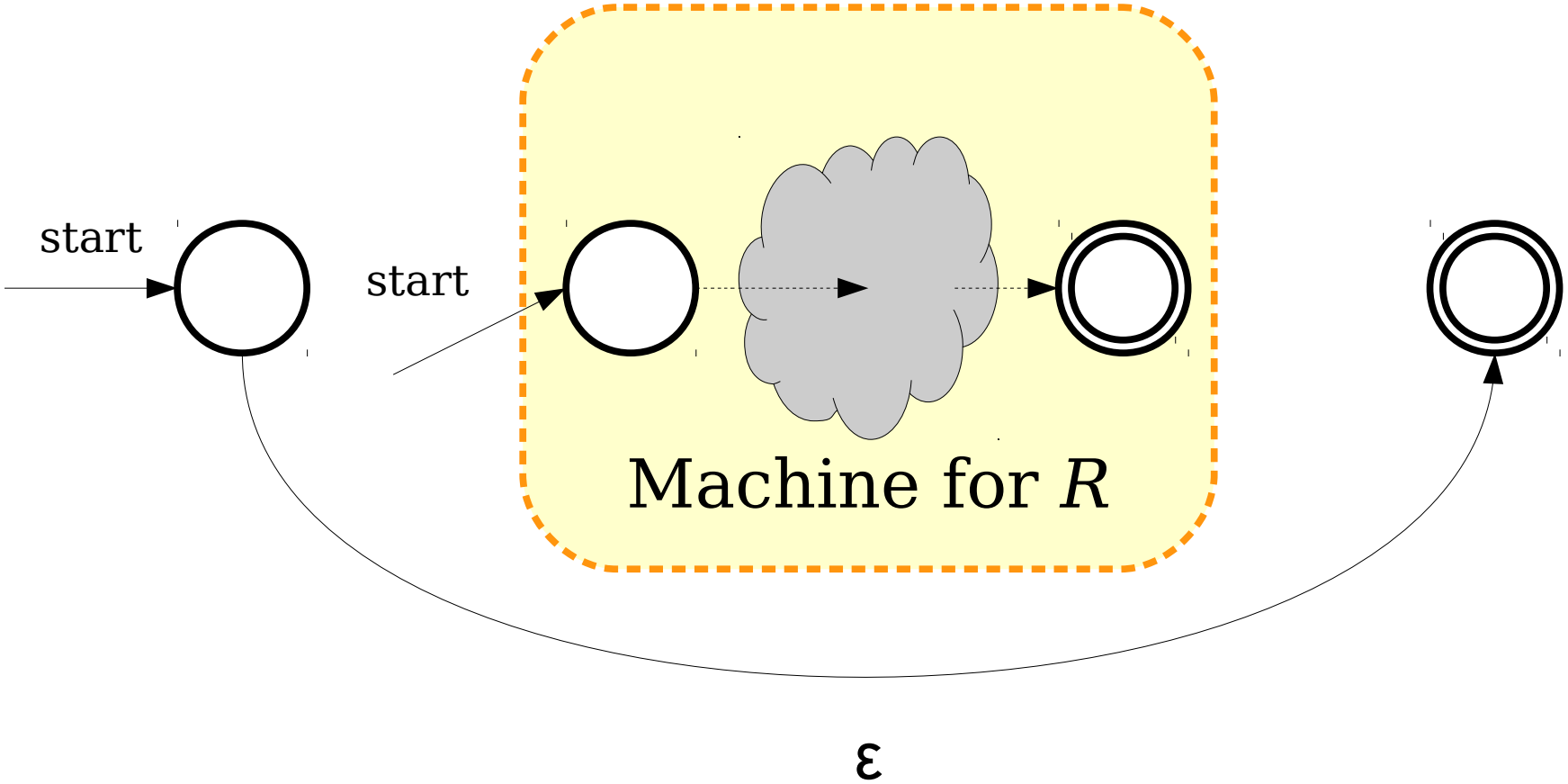
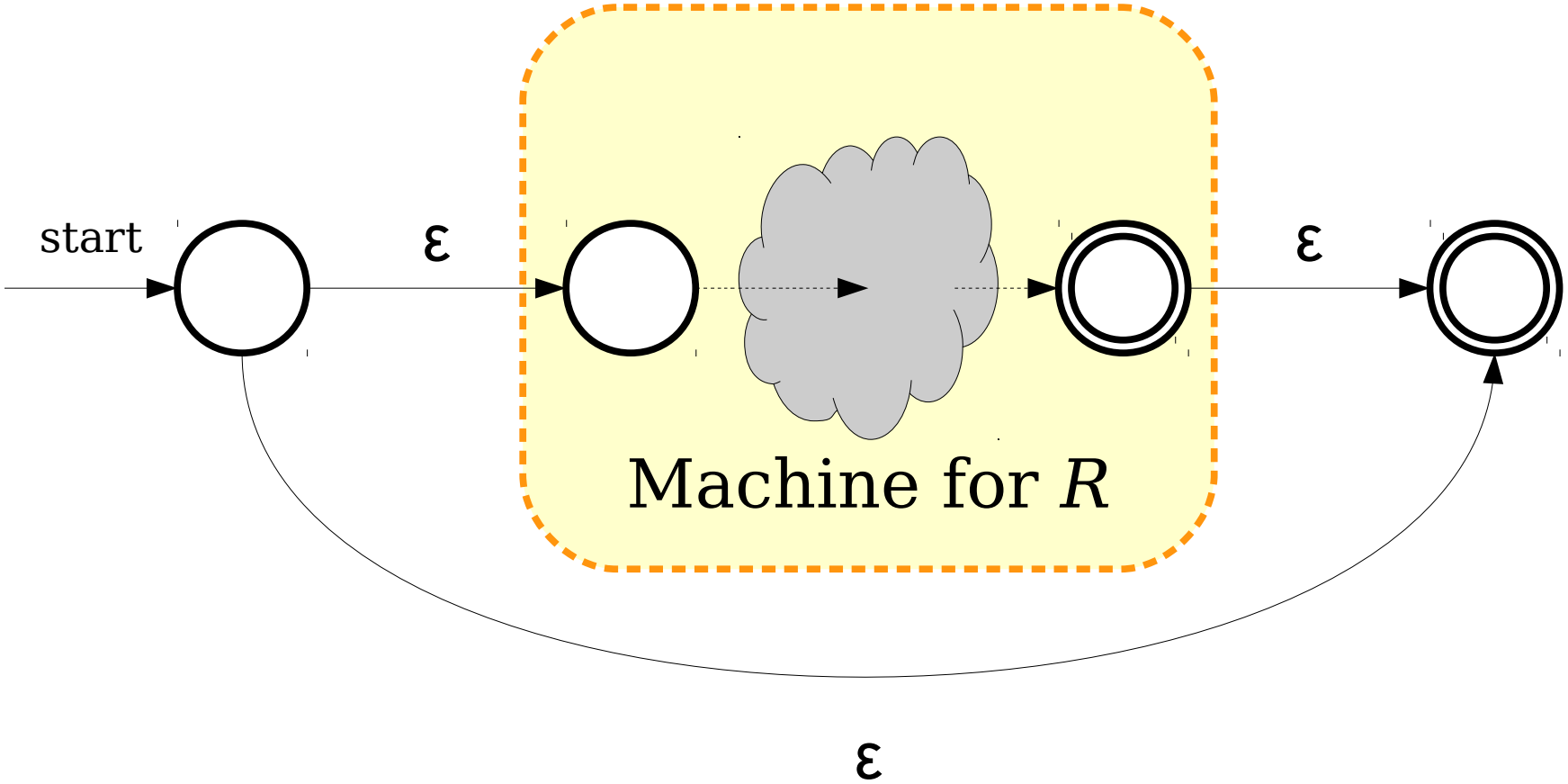
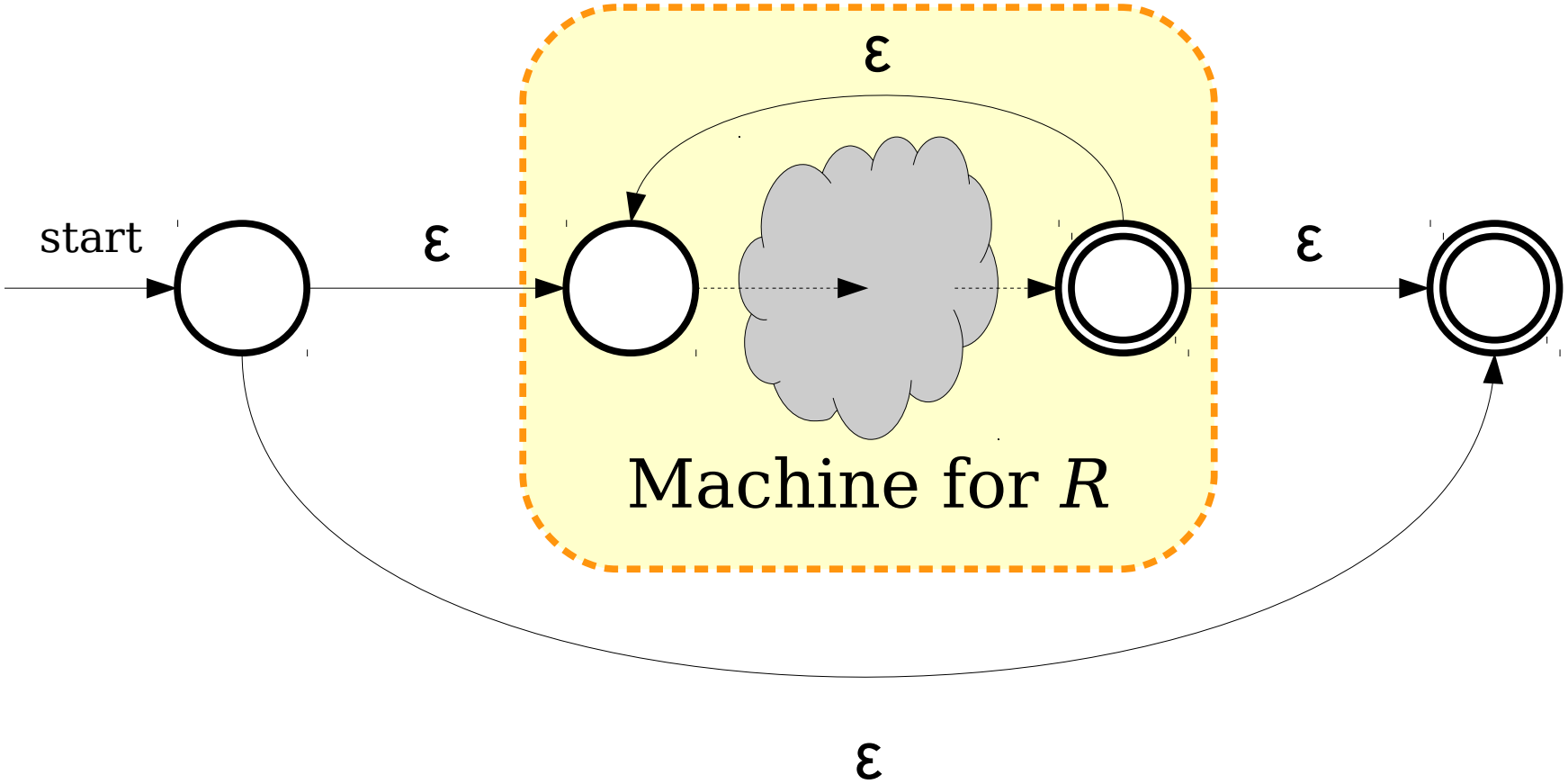


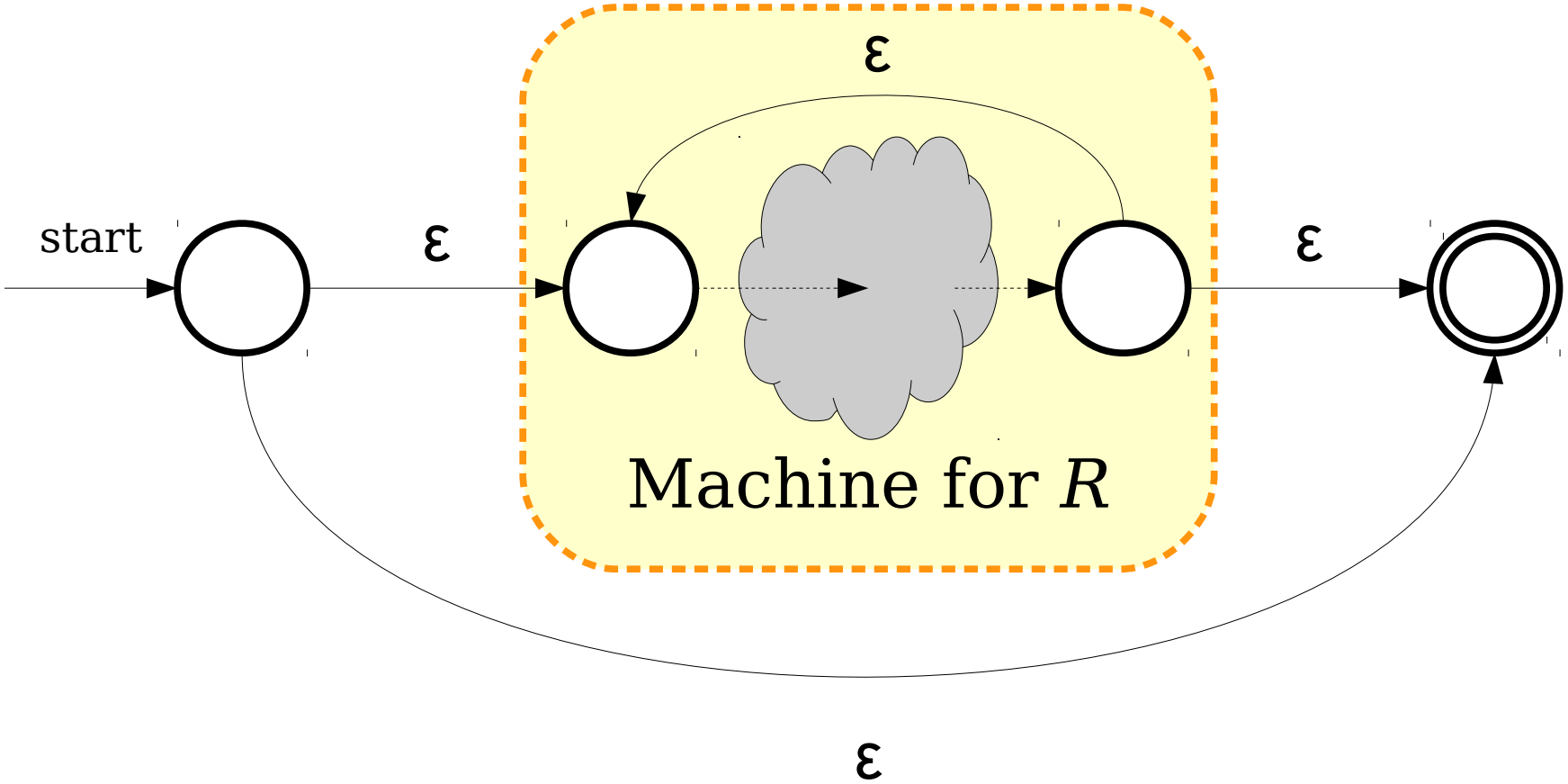
# Construction for $R^*$



# Construction for $R^*$



# Construction for $R^*$



# Why This Matters

- Many software tools work by matching regular expressions against text.
- One possible algorithm for doing so:
  - Convert the regular expression to an NFA.
  - (Optionally) Convert the NFA to a DFA using the subset construction.
  - Run the text through the finite automaton and look for matches.
- This is actually used in practice! The compiled matching automata run extremely quickly.

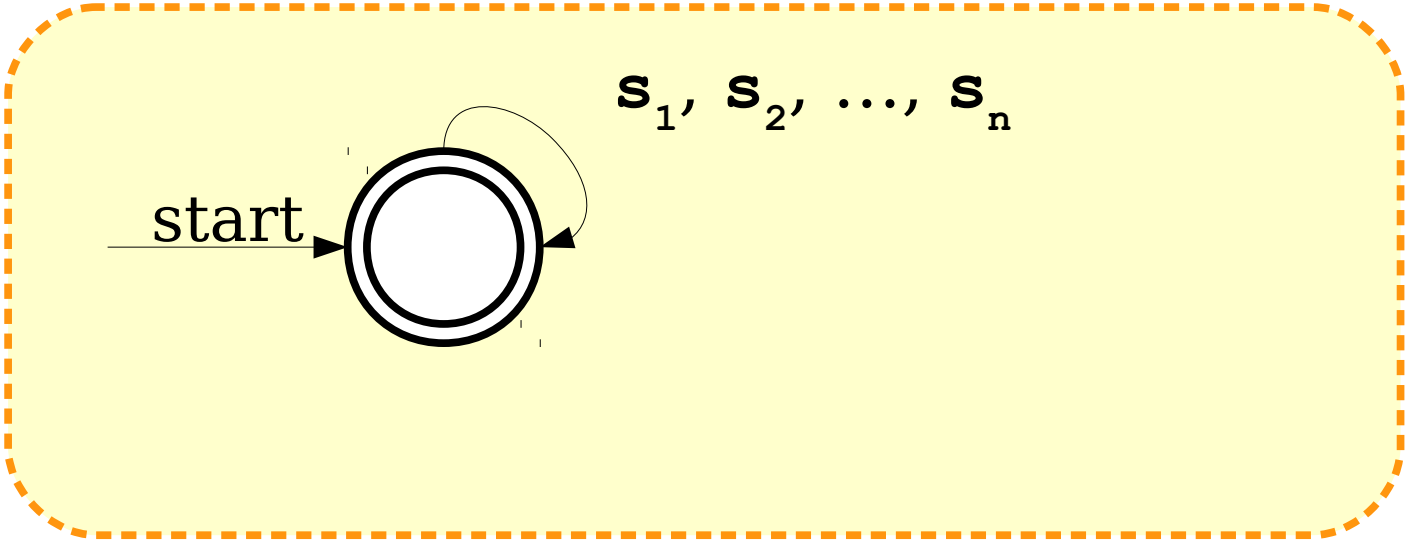
# The Power of Regular Expressions

***Theorem:*** If $L$ is a regular language, then there is a regular expression for $L$.

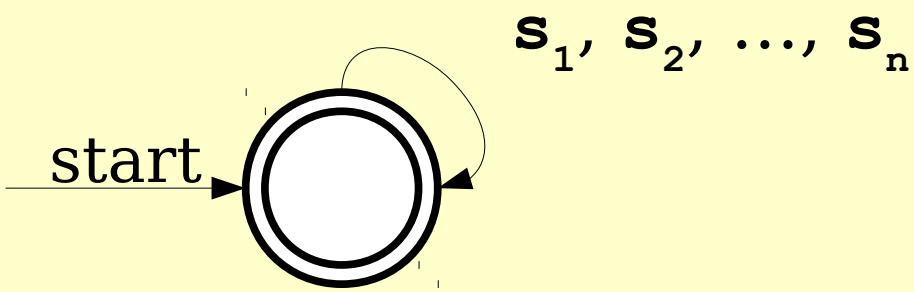
***This is not obvious!***

***Proof idea:*** Show how to convert an arbitrary NFA into a regular expression.

# From NFAs to Regular Expressions



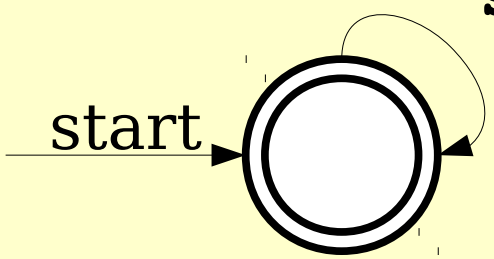
# From NFAs to Regular Expressions




Regular expression: $(s_1 \mid s_2 \mid \ldots \mid s_n)^*$

# From NFAs to Regular Expressions

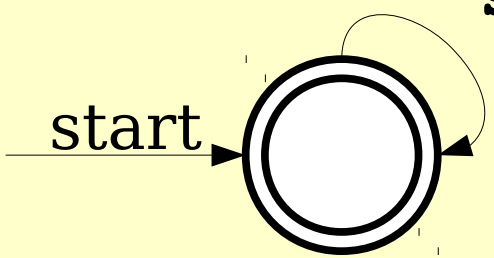
$s_1 \mid s_2 \mid \ldots \mid s_n$




Regular expression: $(s_1 \mid s_2 \mid \ldots \mid s_n)^*$

# From NFAs to Regular Expressions

$s_1 \mid s_2 \mid \ldots \mid s_n$

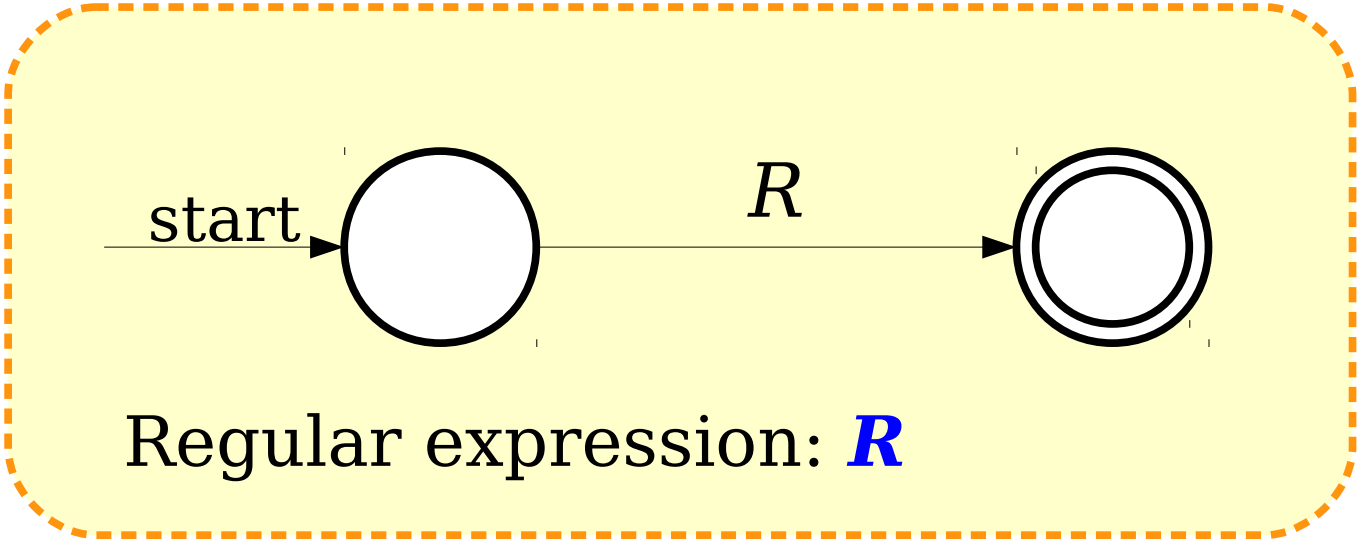



Regular expression: $(s_1 \mid s_2 \mid \ldots \mid s_n)^*$

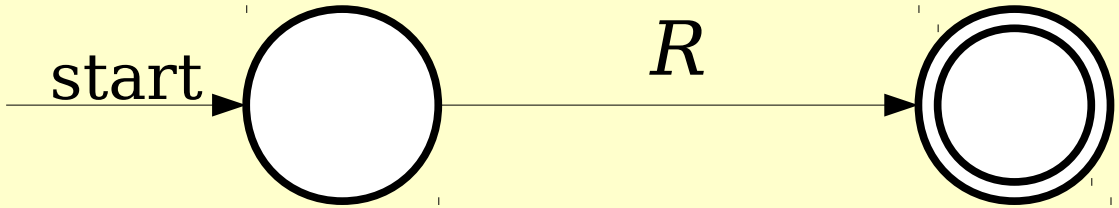
Key idea: Label transitions with arbitrary regular expressions.

# From NFAs to Regular Expressions

# From NFAs to Regular Expressions



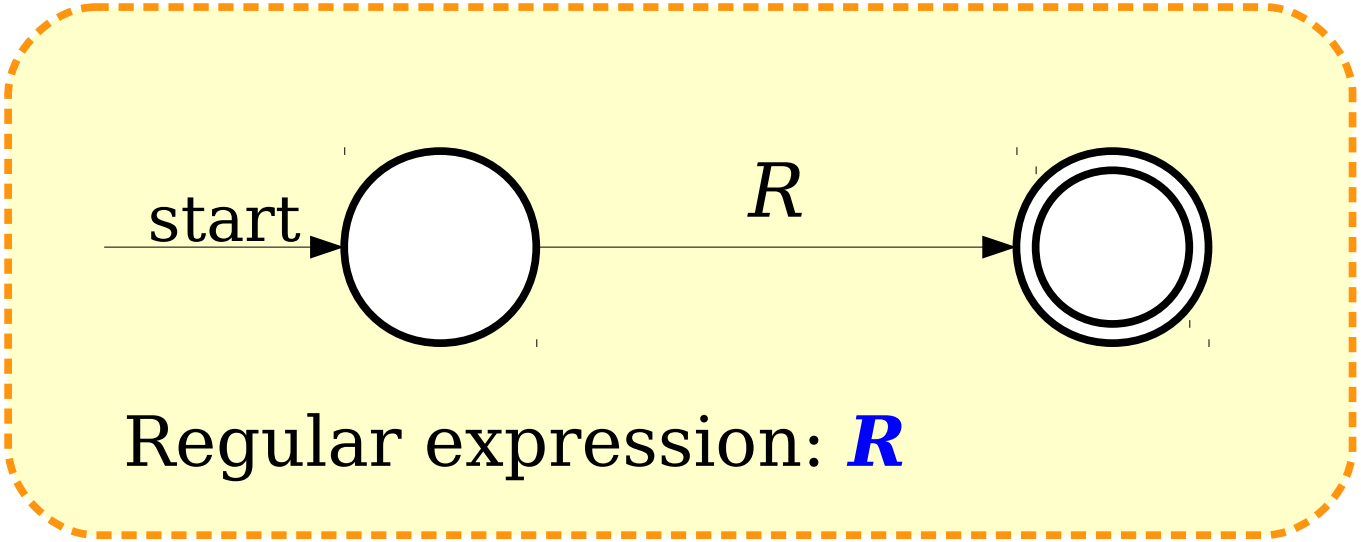
# From NFAs to Regular Expressions

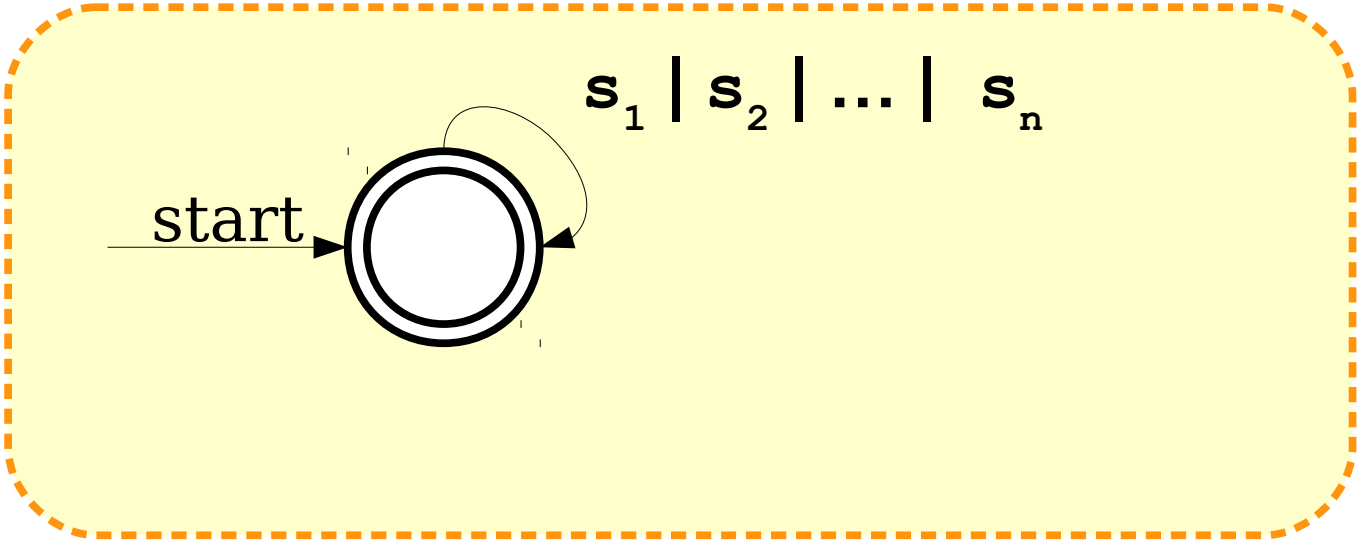



Regular expression: ***R***

Key idea: If we can convert any NFA into something that looks like this, we can easily read off the regular expression.

# From NFAs to Regular Expressions



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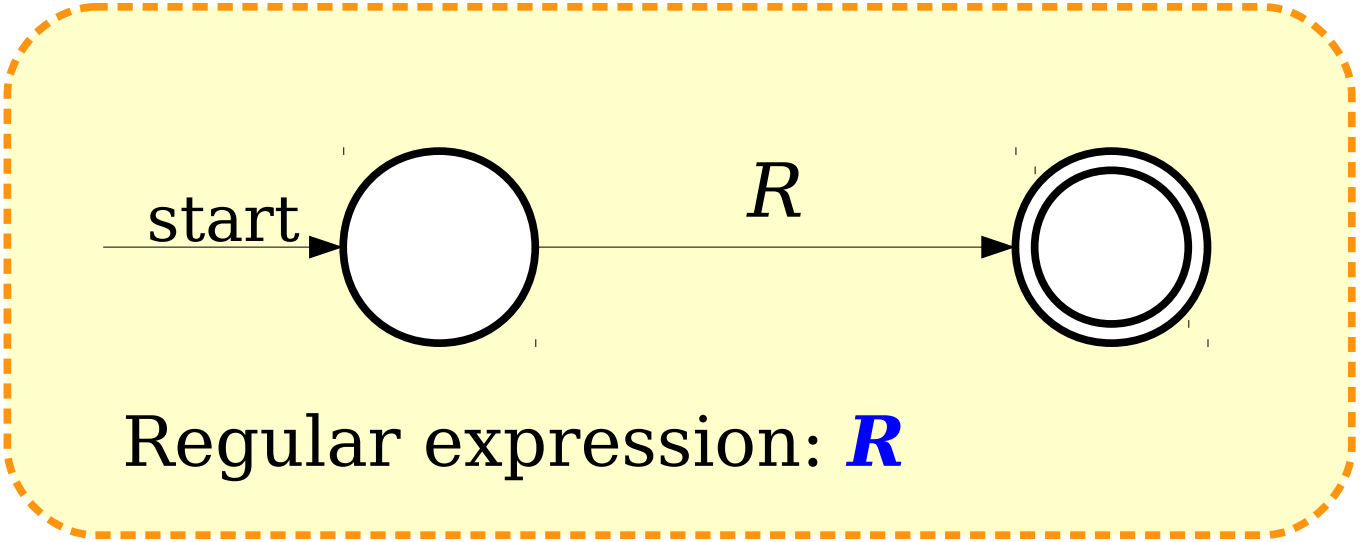


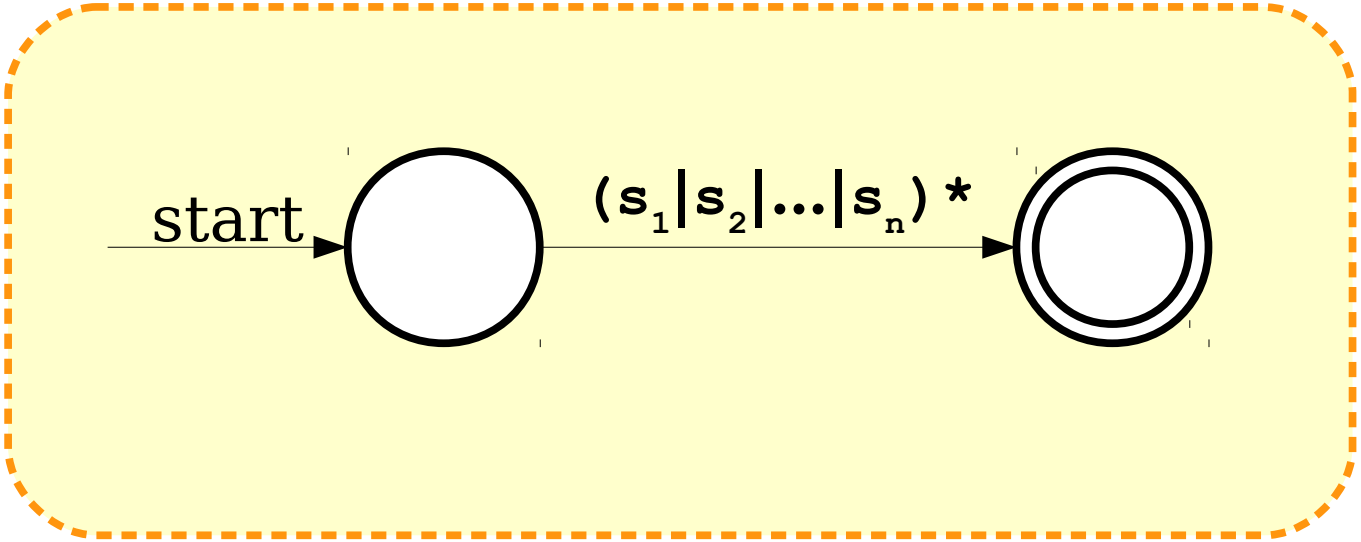

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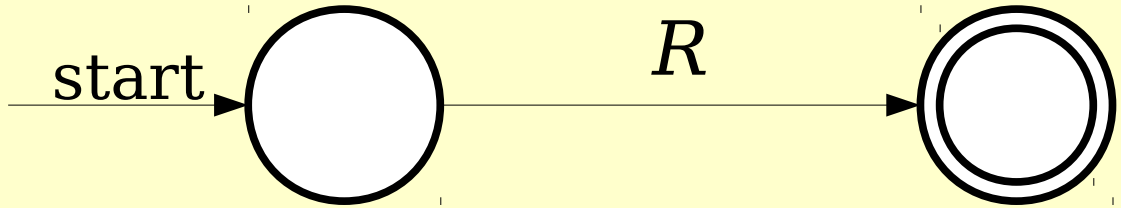
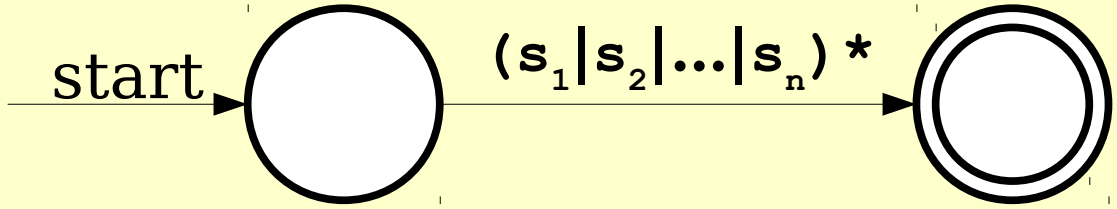
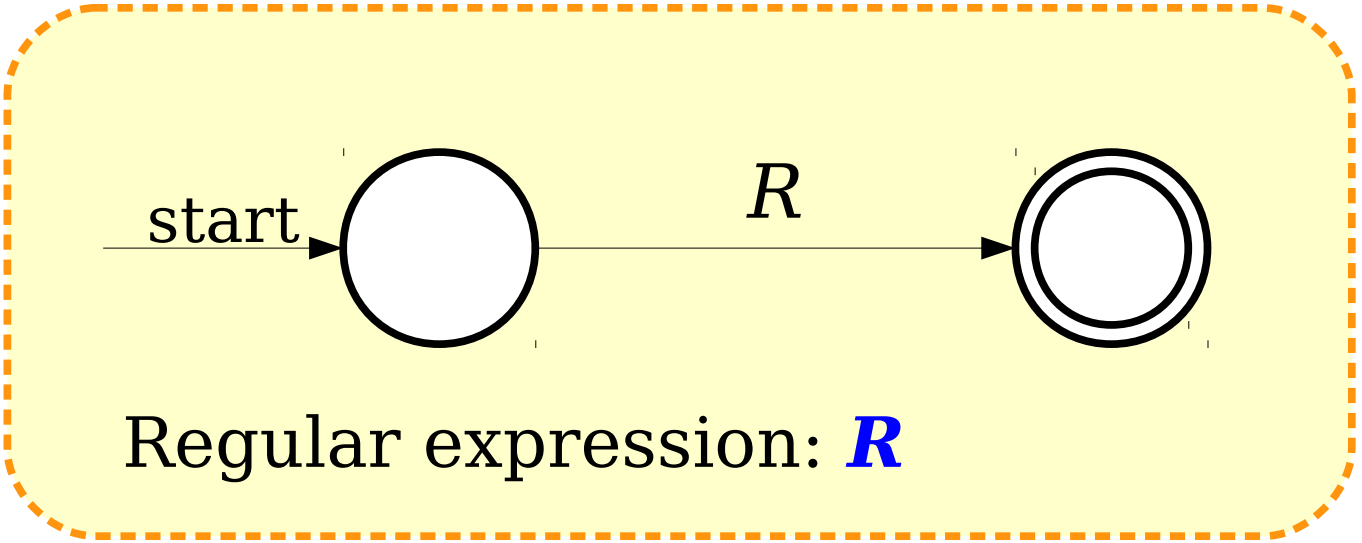



Regular expression: ***R***

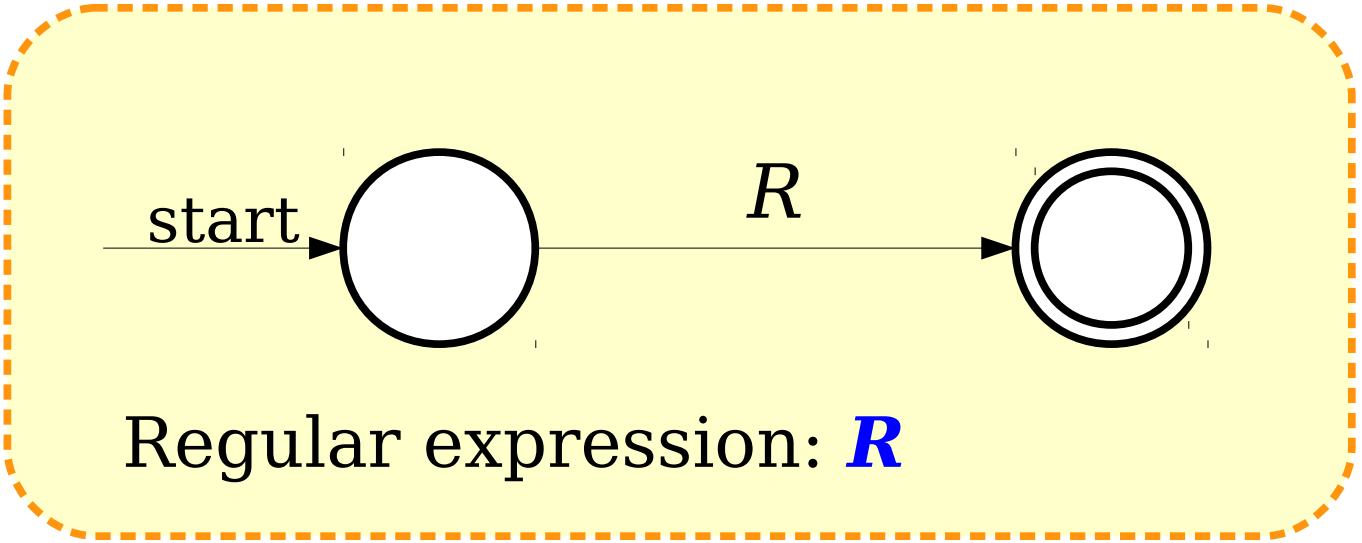


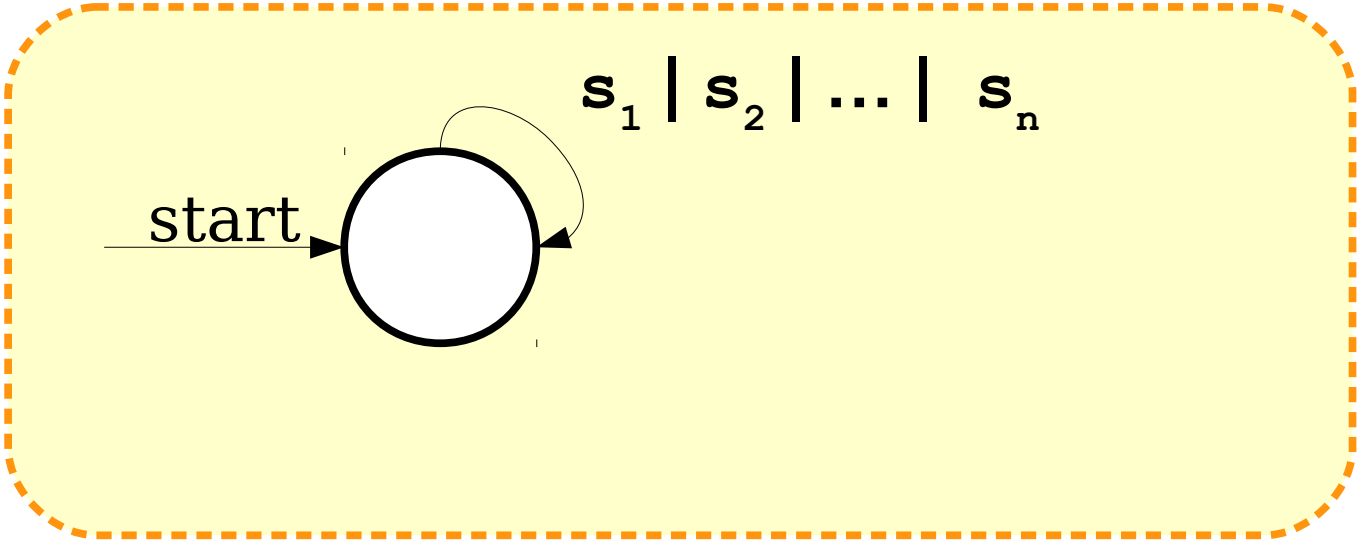

Regular expression: $(s_1|s_2| \ldots |s_n)*$

# From NFAs to Regular Expressions



# From NFAs to Regular Expressions



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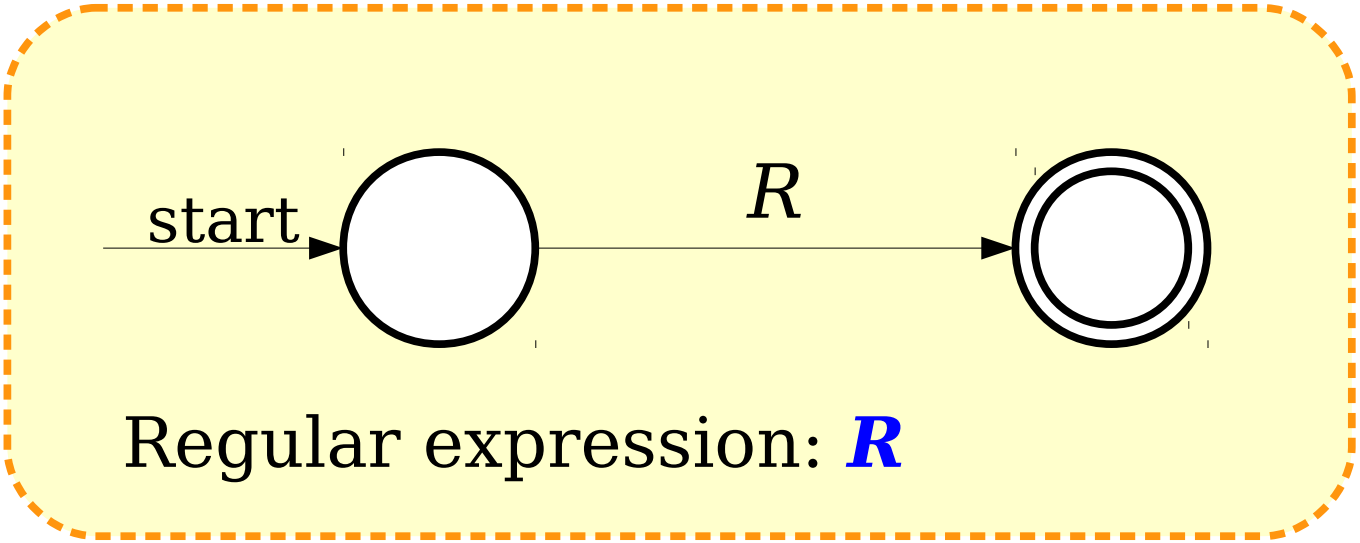


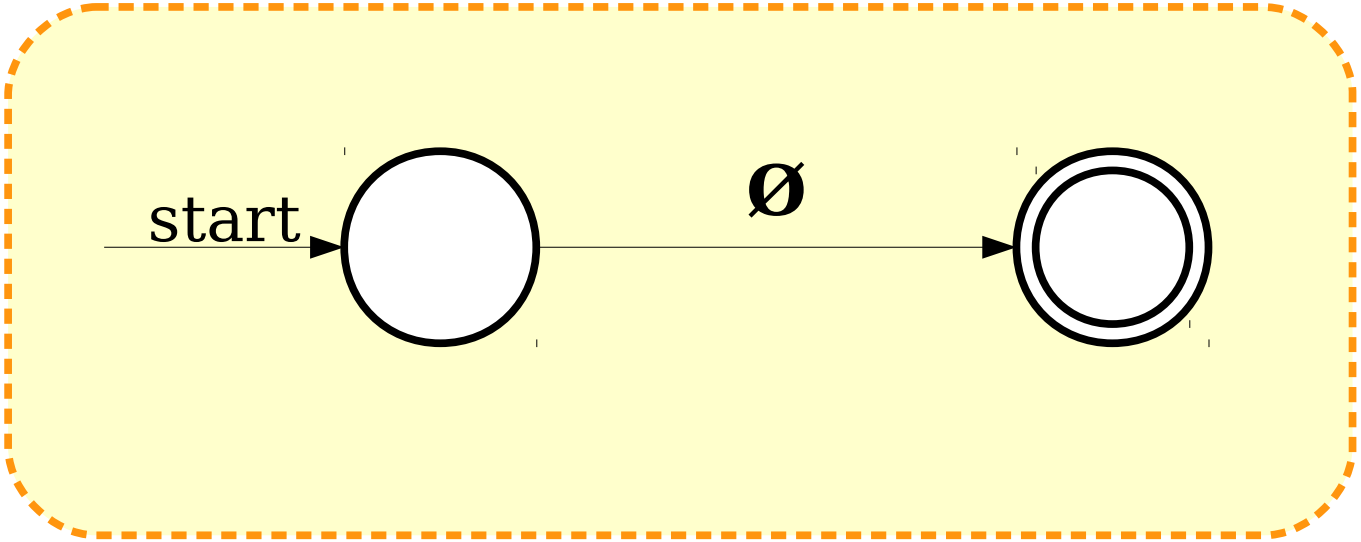

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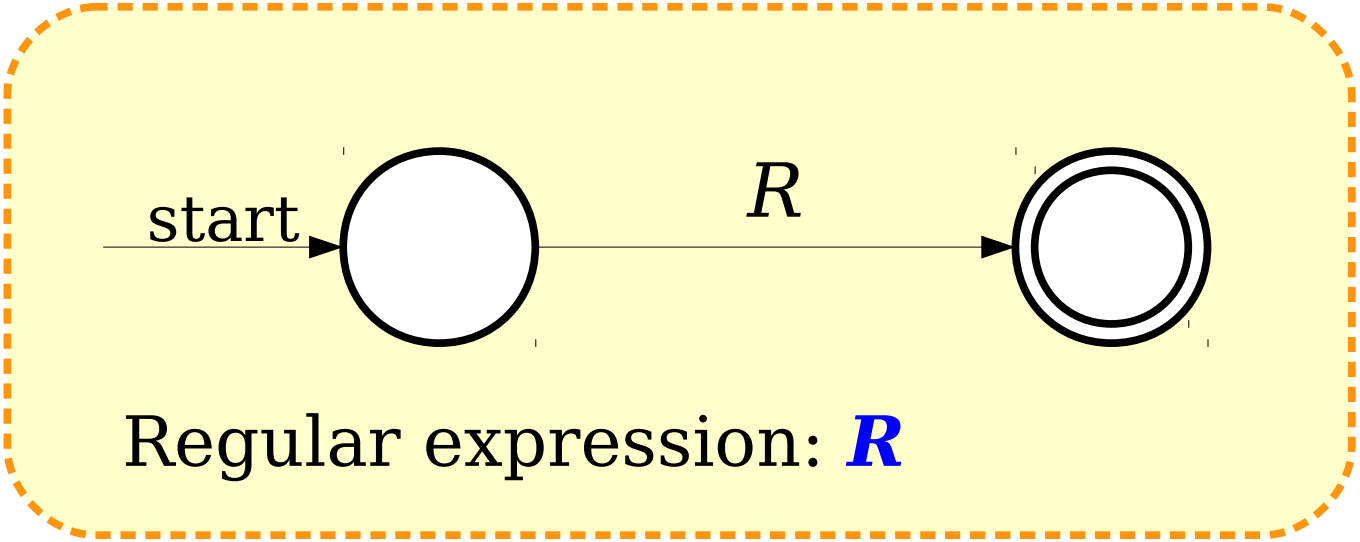


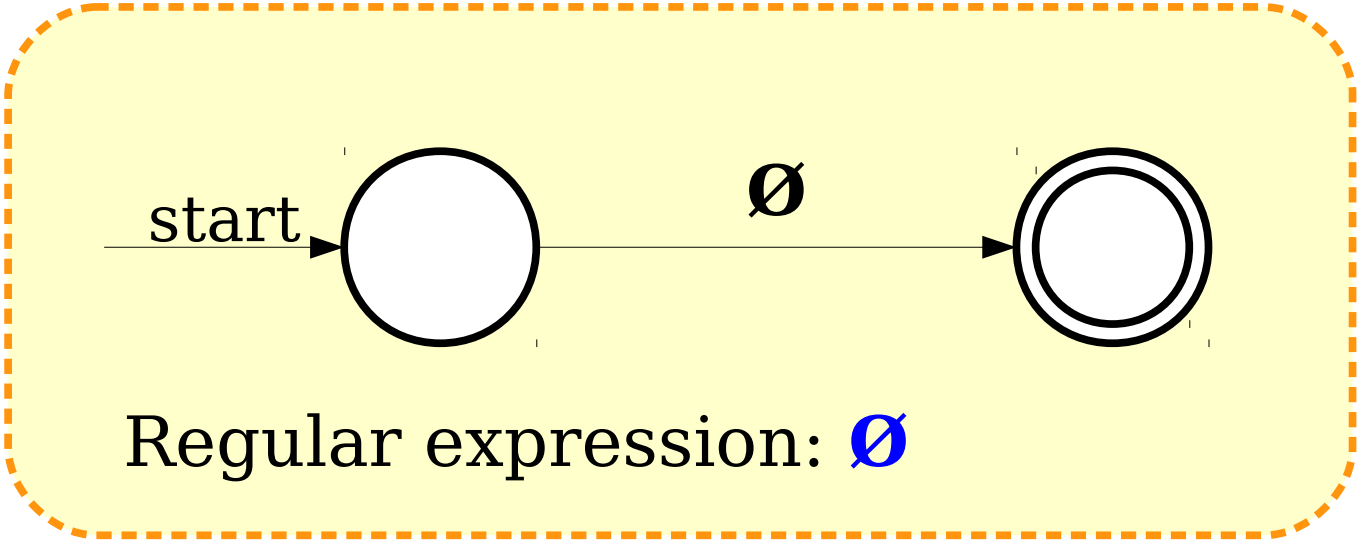

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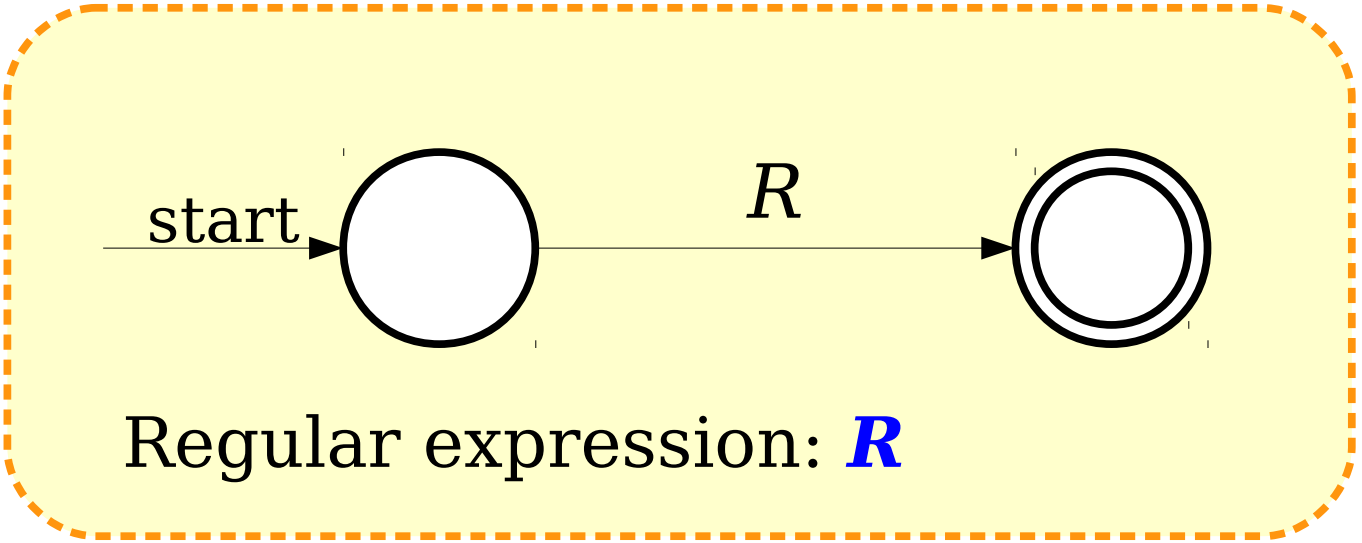

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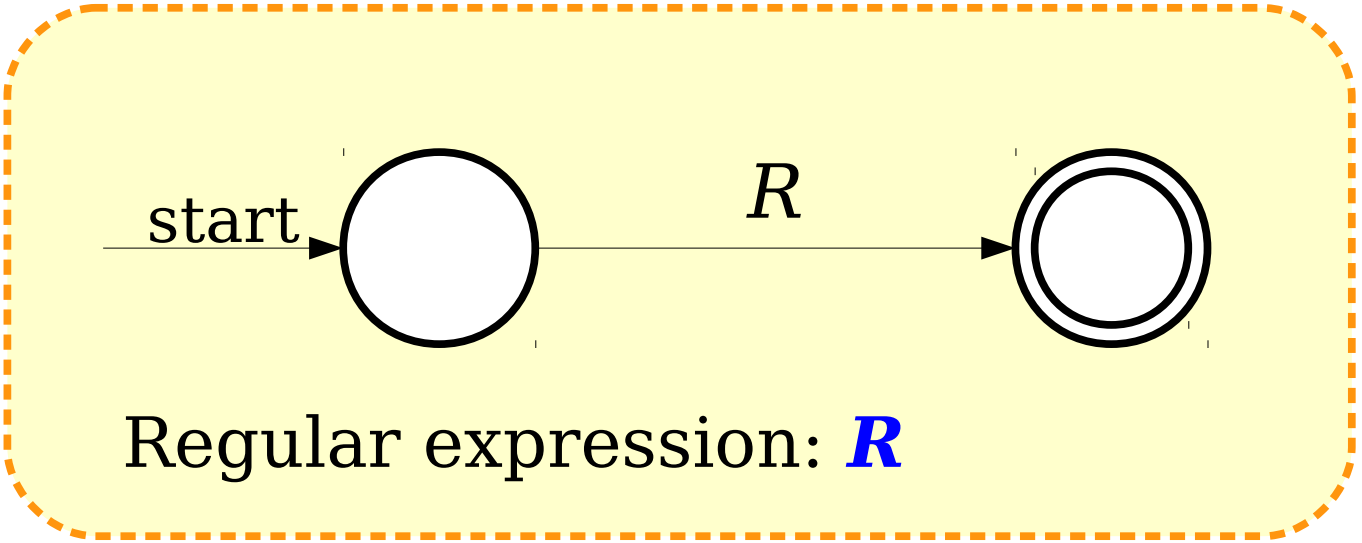


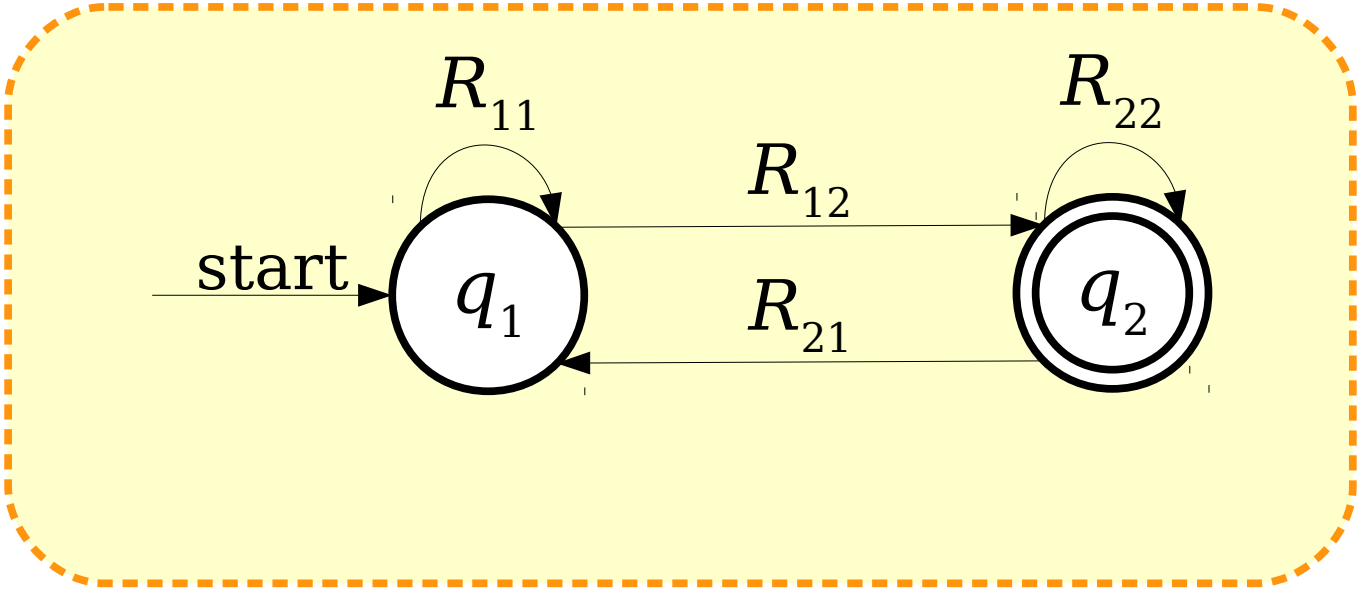

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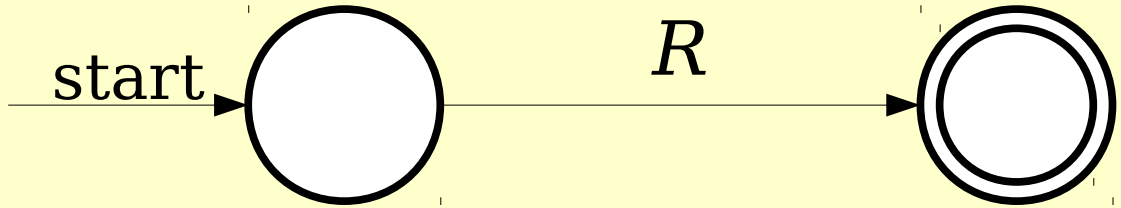
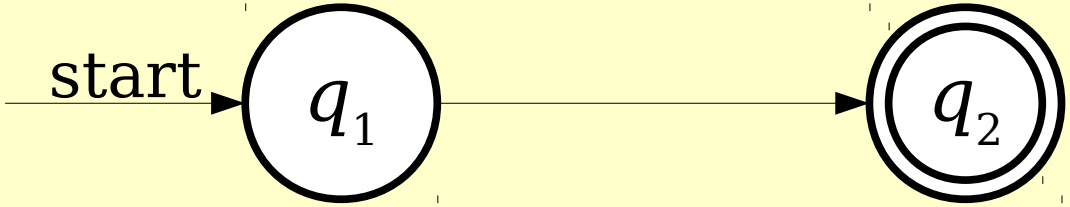



Regular expression: $\boldsymbol{R}$

$$\boldsymbol{R_{11}{}^* R_{12} (R_{22} \mid R_{21} R_{11}{}^* R_{12})^*}$$



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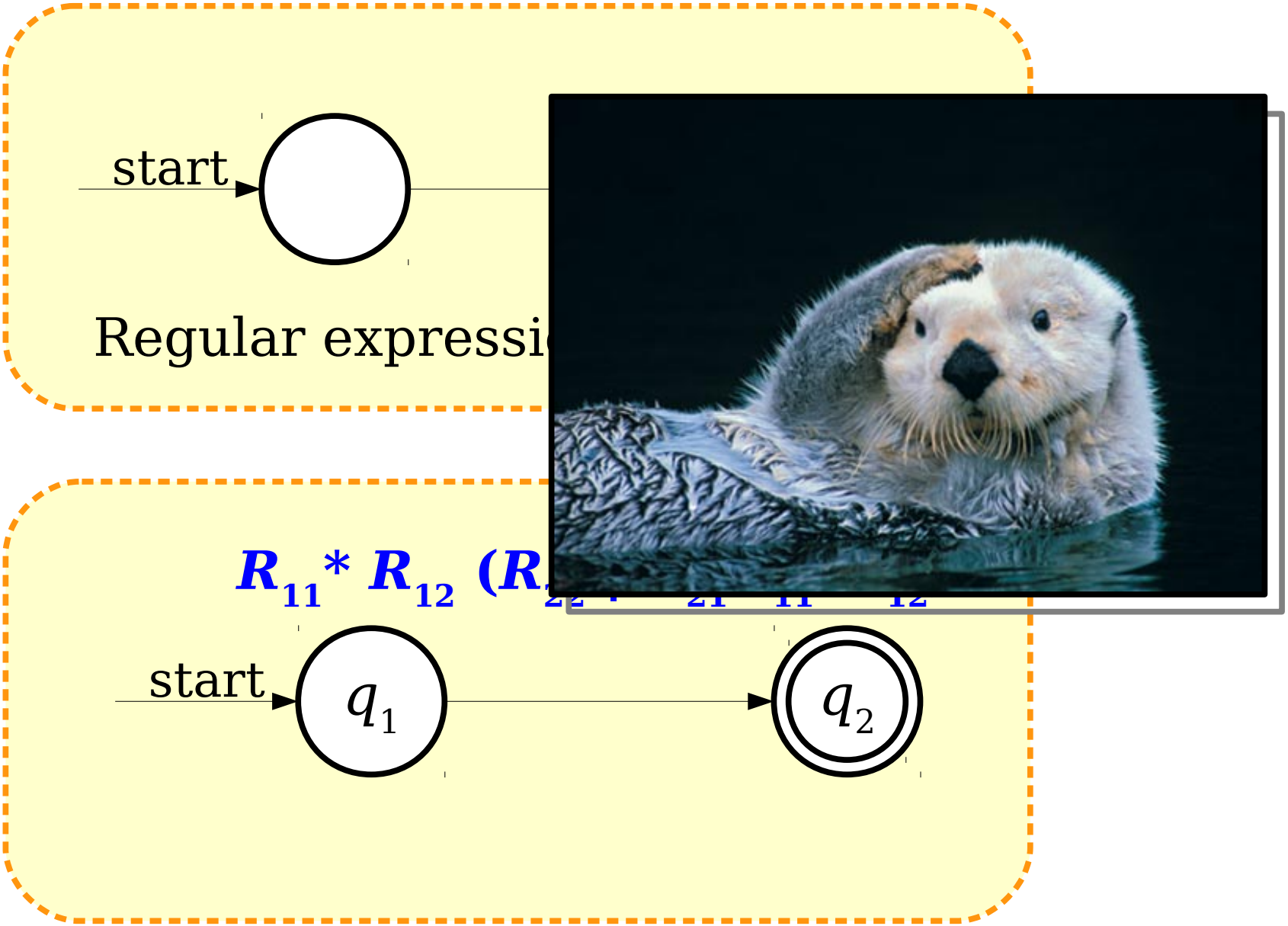

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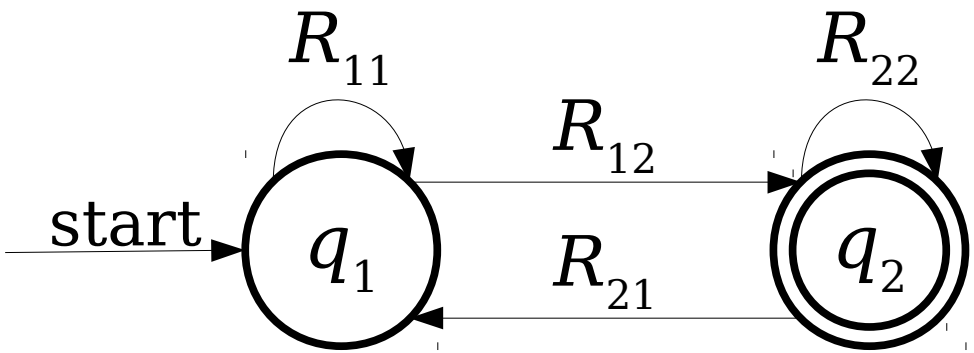

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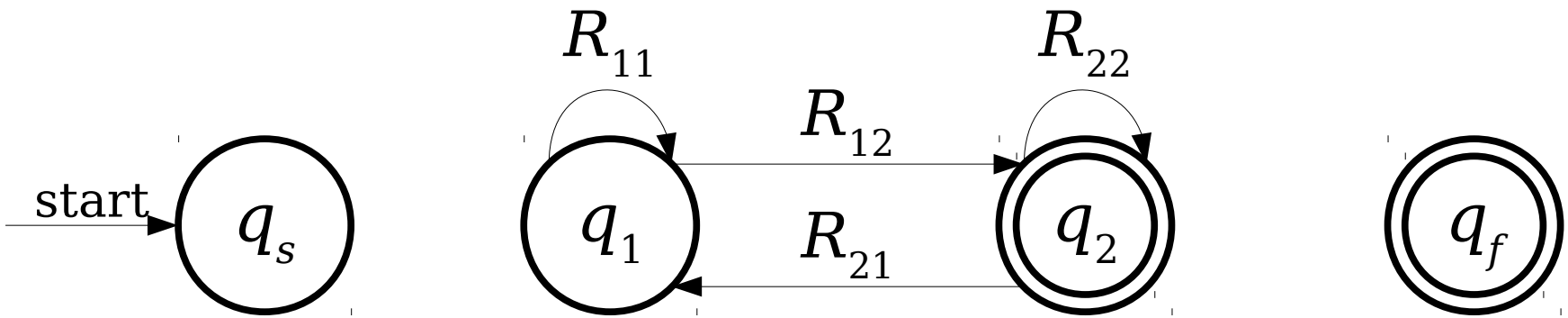

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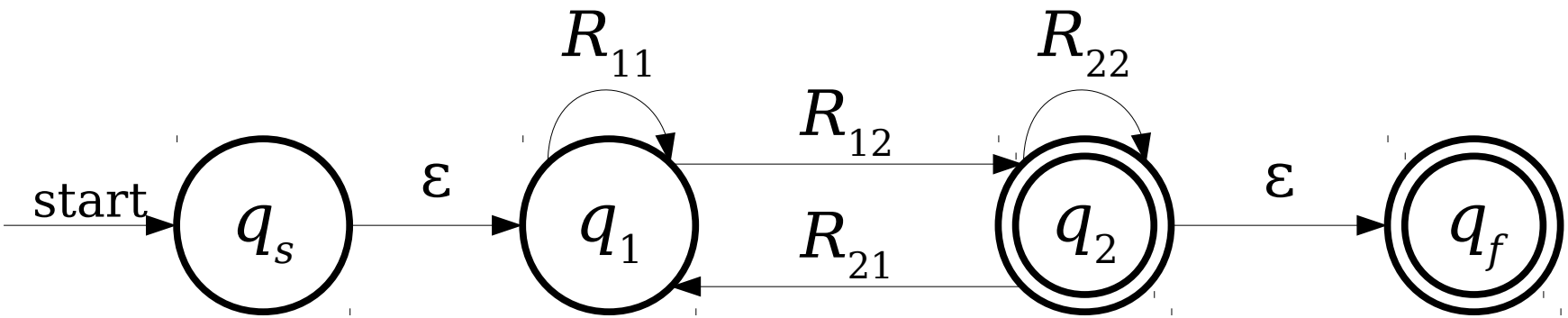

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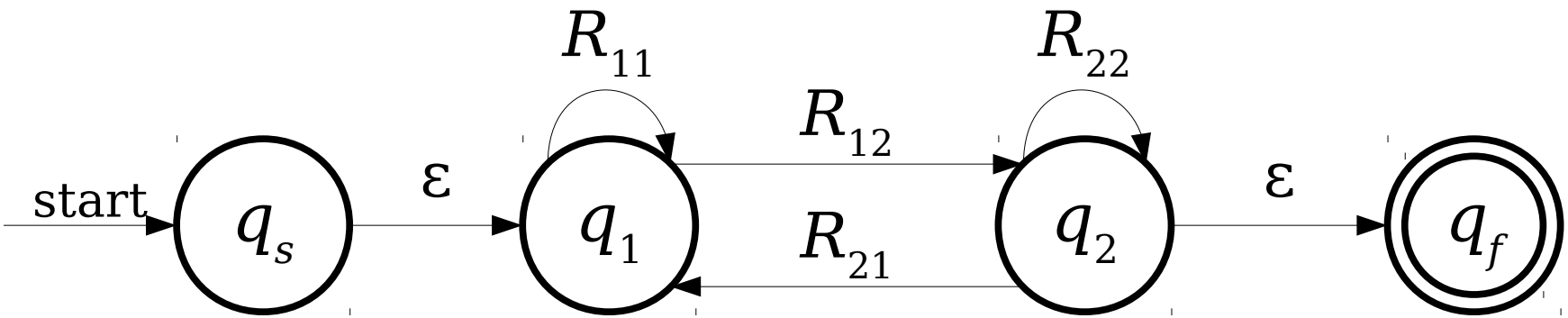

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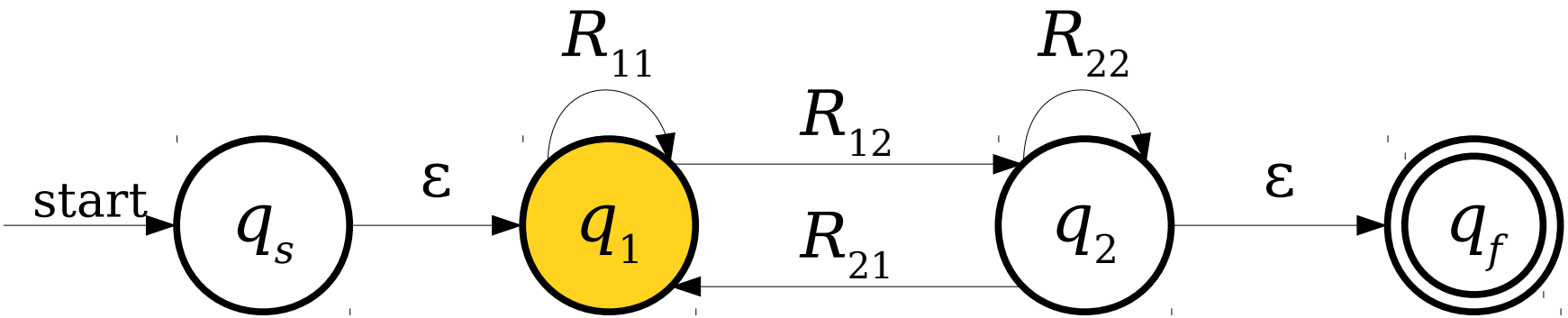
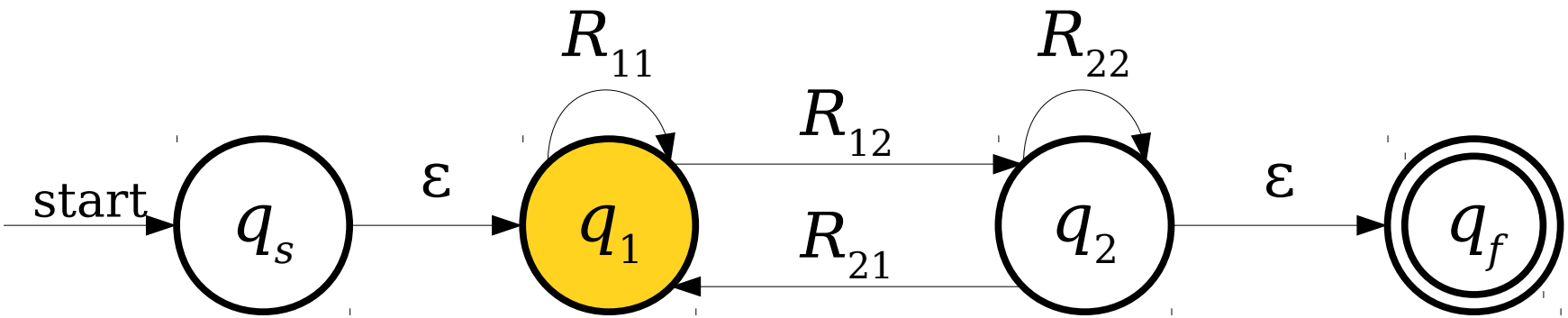
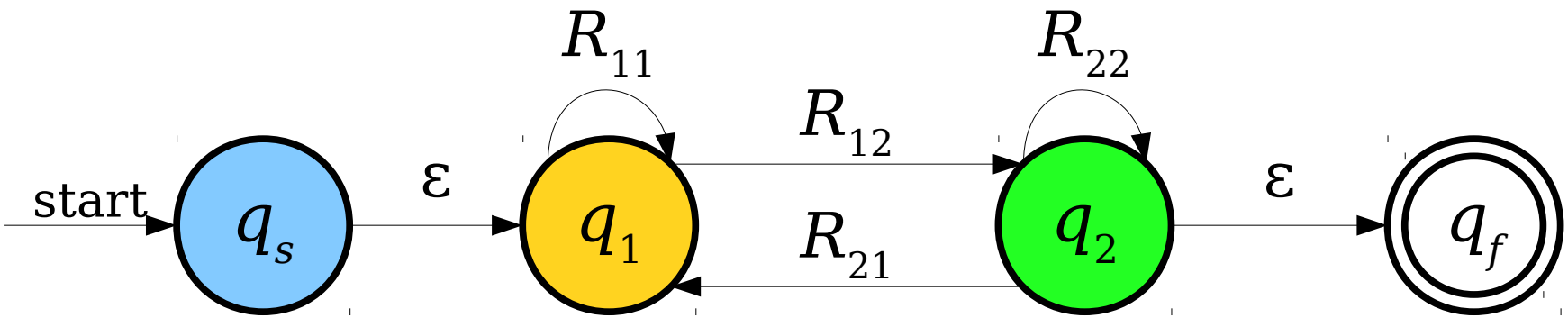


# From NFAs to Regular Expressions




Could we eliminate this state from the NFA?

# From NFAs to Regular Expressions



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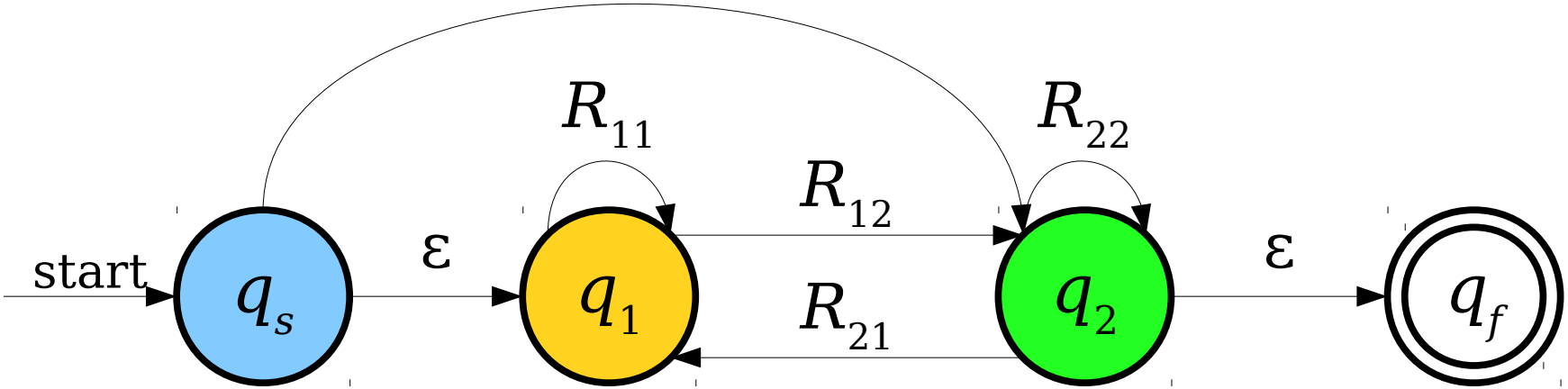

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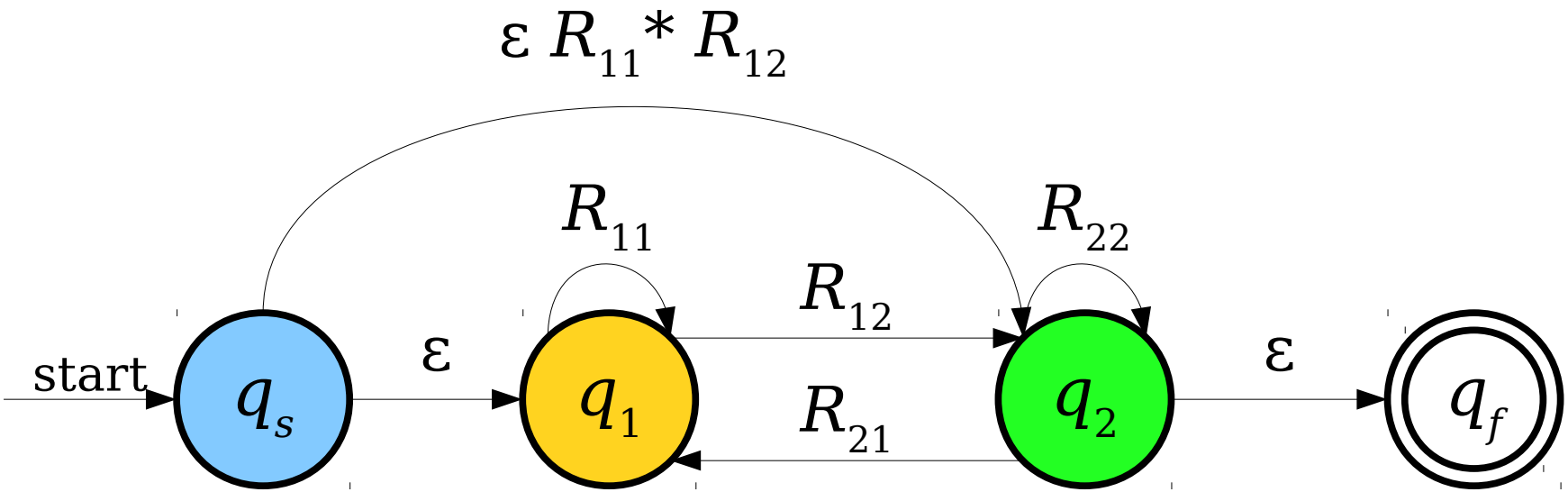
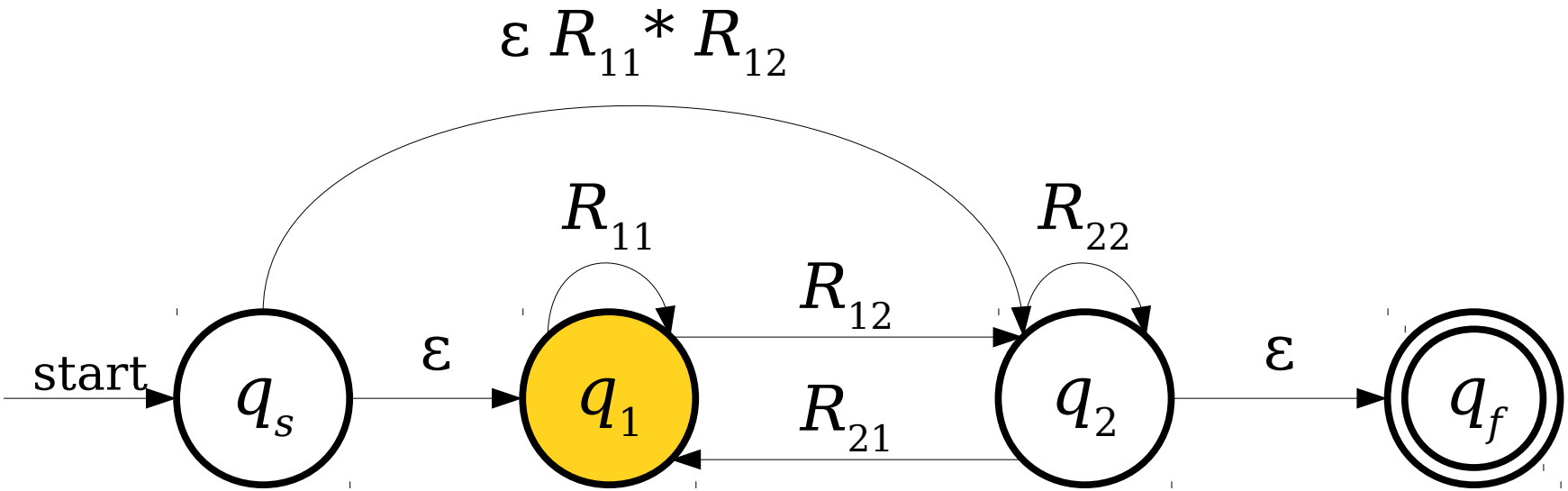



Note: We're using concatenation and Kleene closure in order to skip this state.

# From NFAs to Regular Expressions



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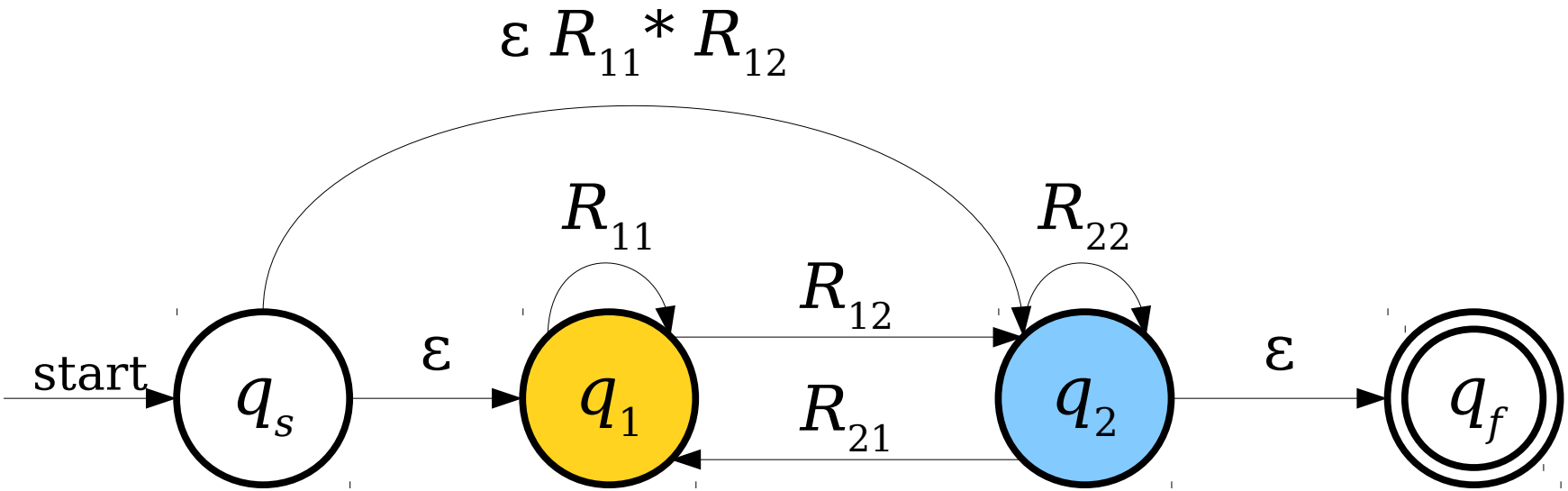

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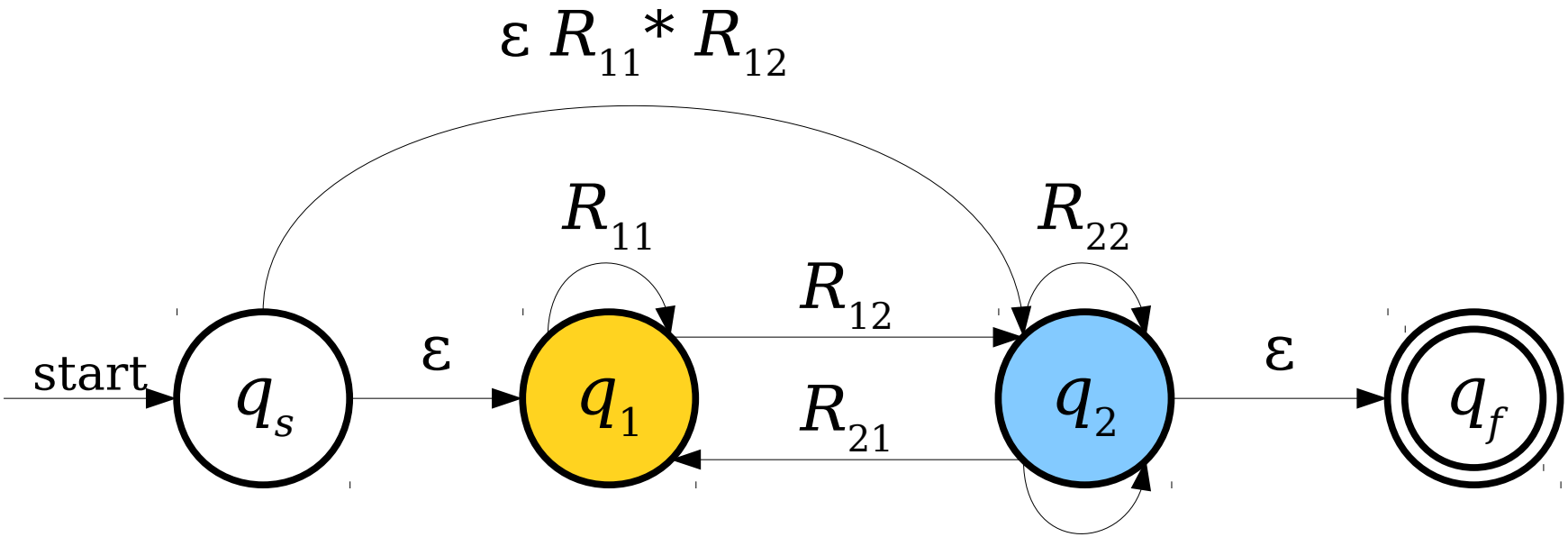

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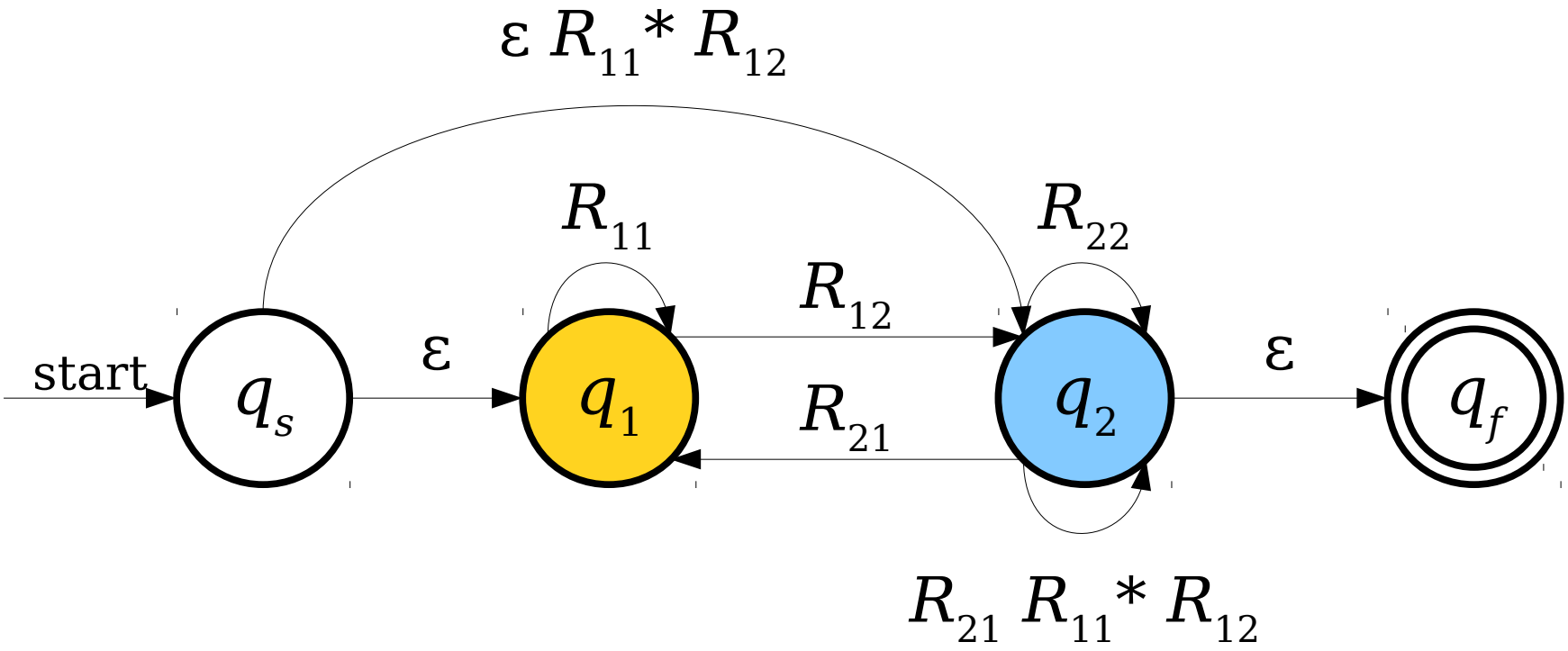
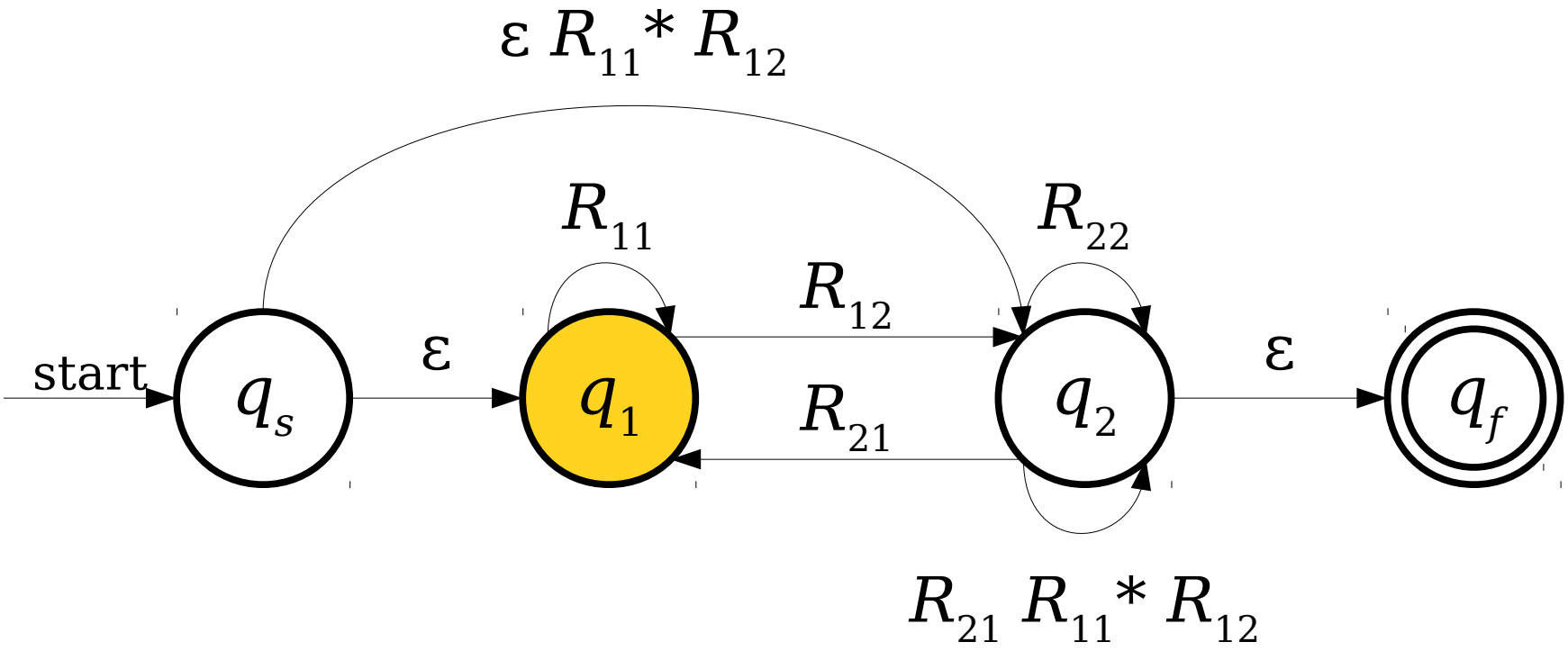


# From NFAs to Regular Expressions



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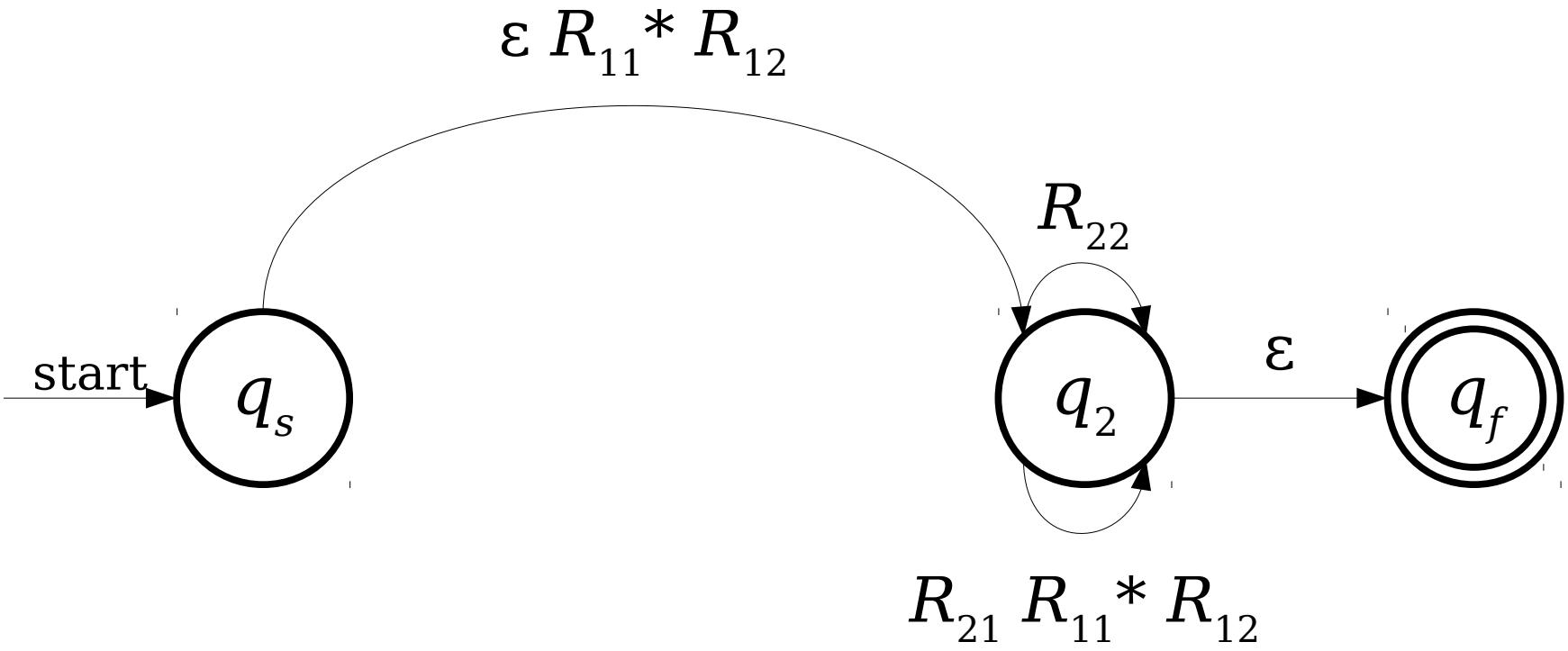

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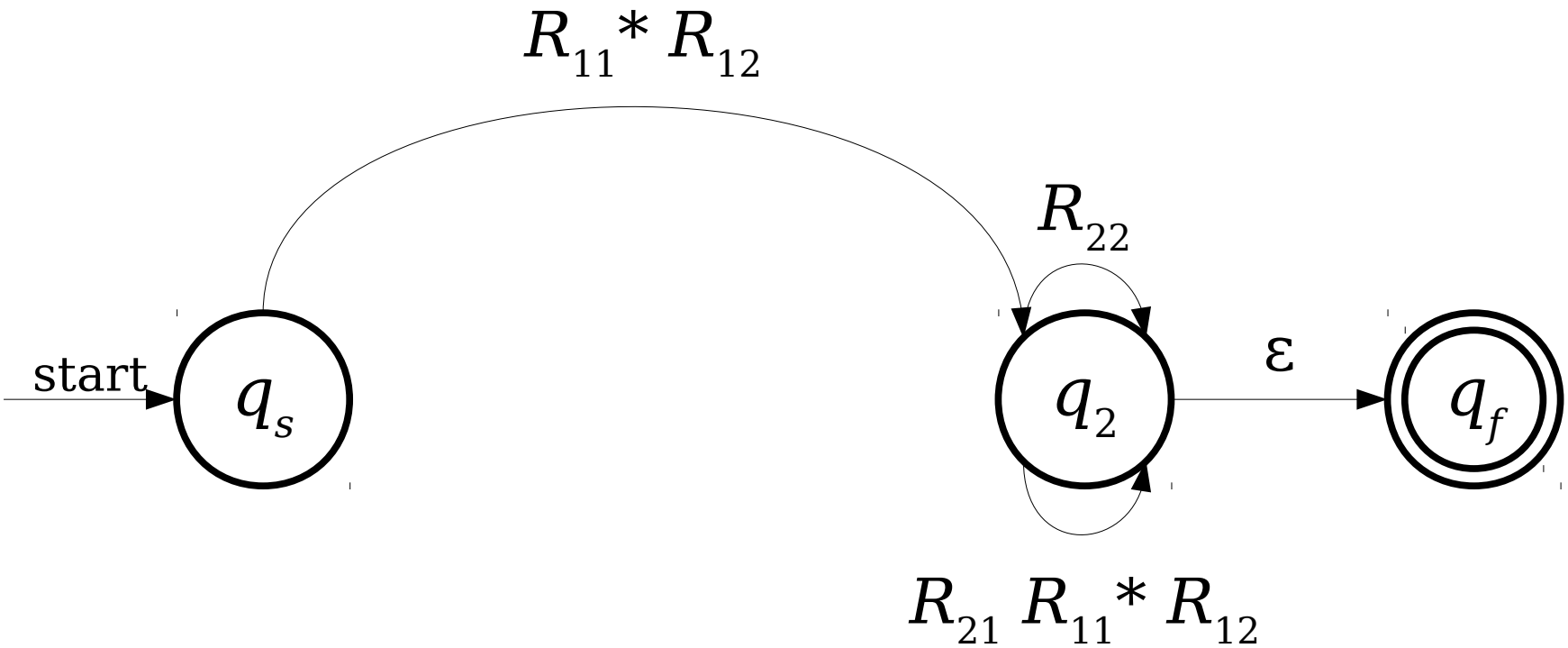
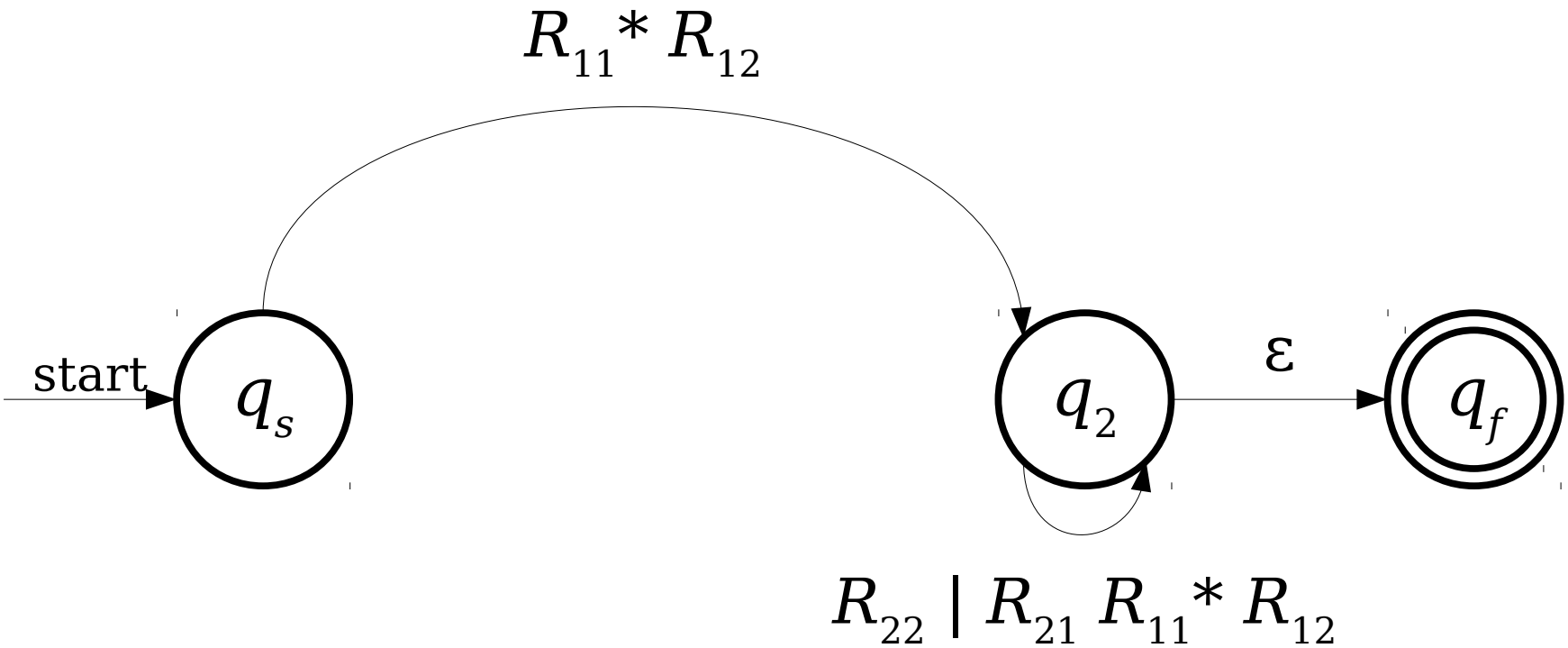
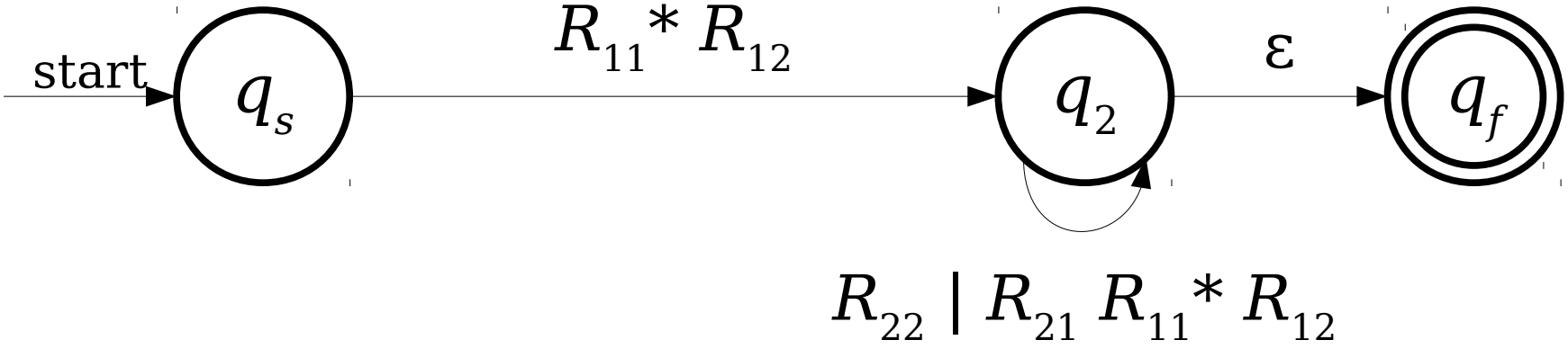


# From NFAs to Regular Expressions

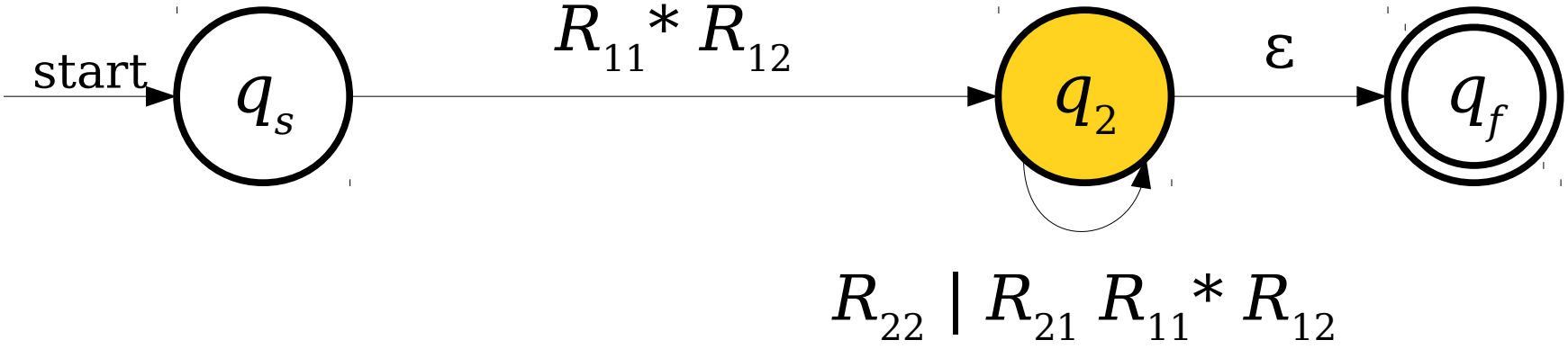



Note: We're using union to combine these transitions together.

# From NFAs to Regular Expressions



# From NFAs to Regular Expressions



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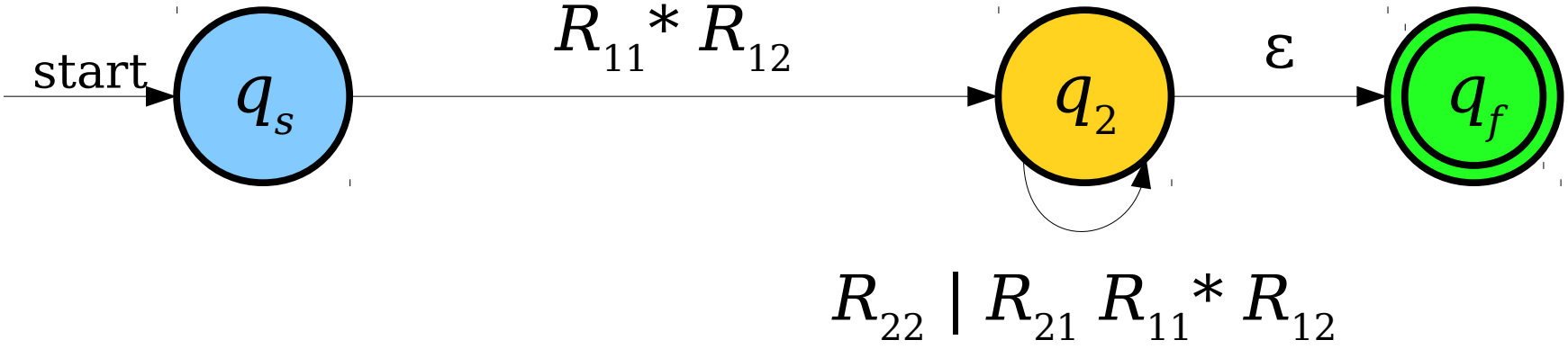
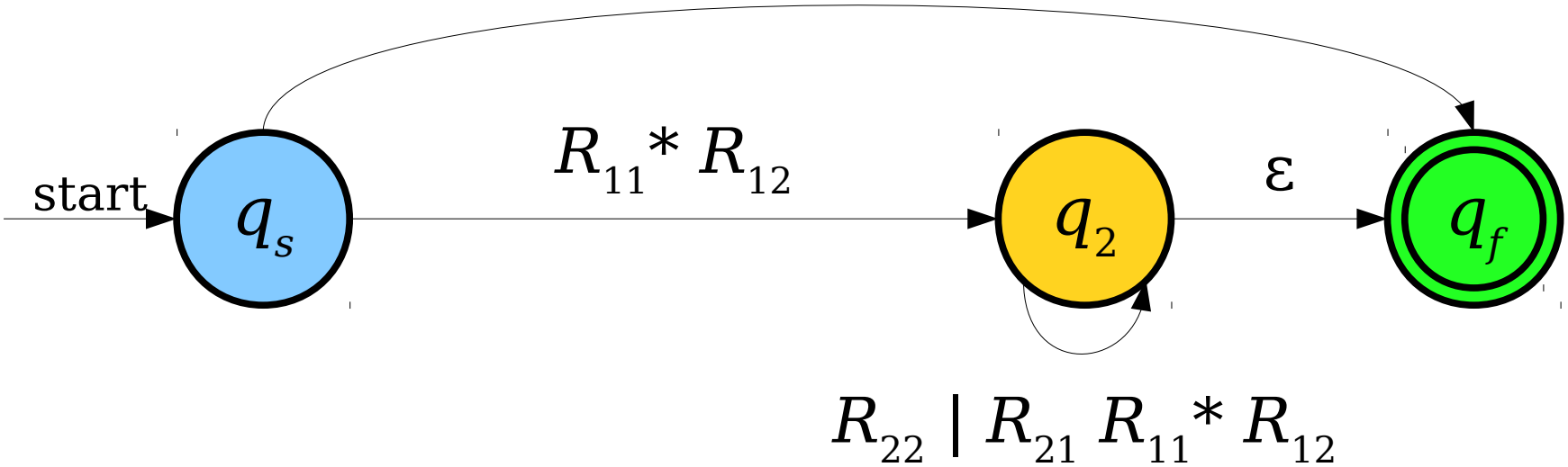
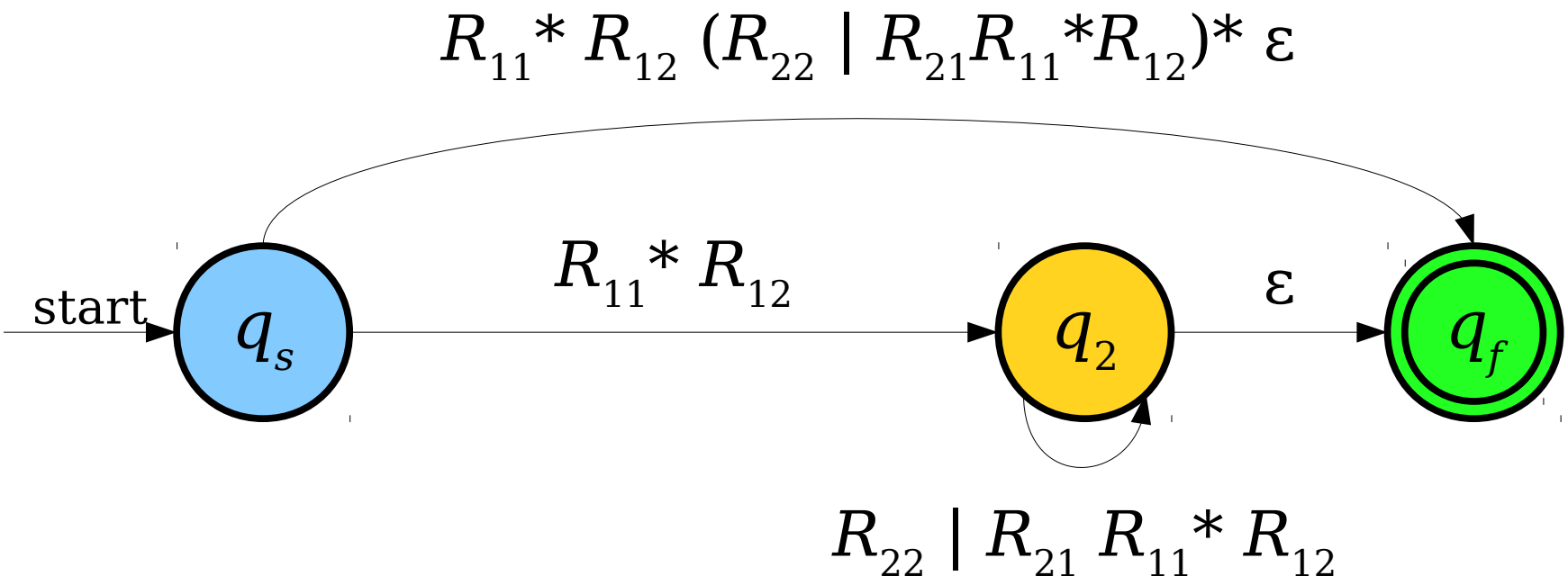


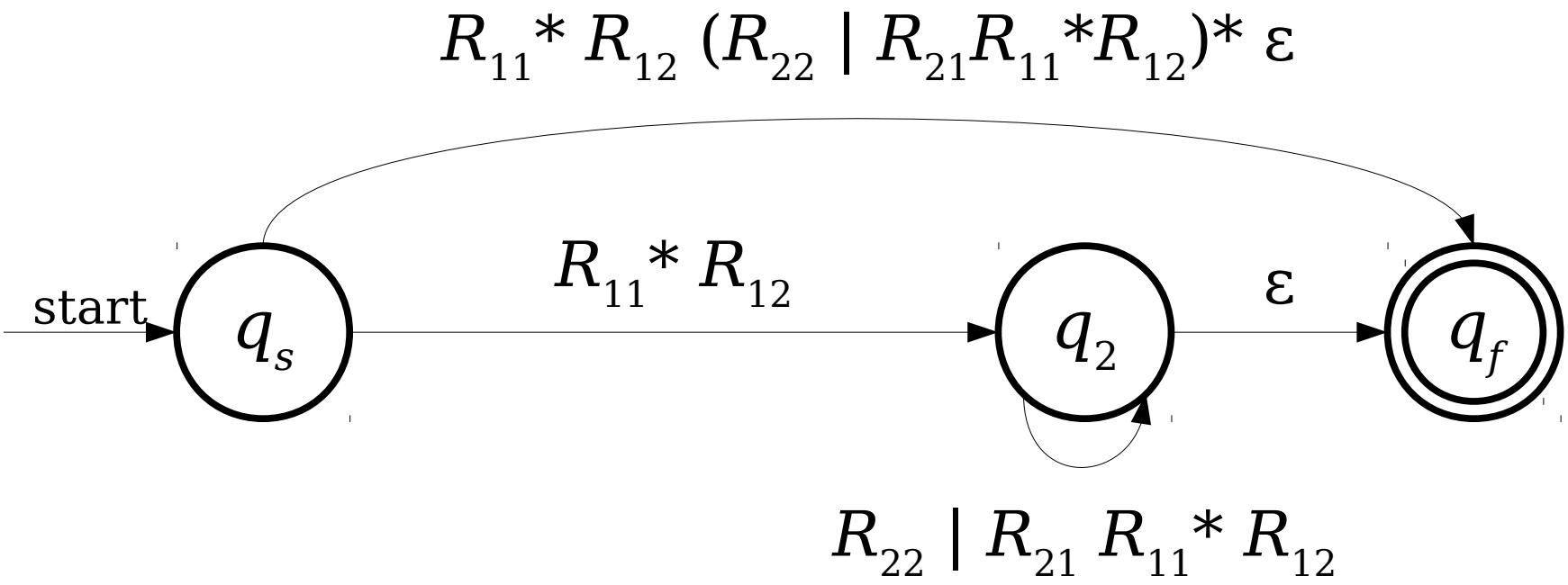
# From NFAs to Regular Expressions



# From NFAs to Regular Expressions

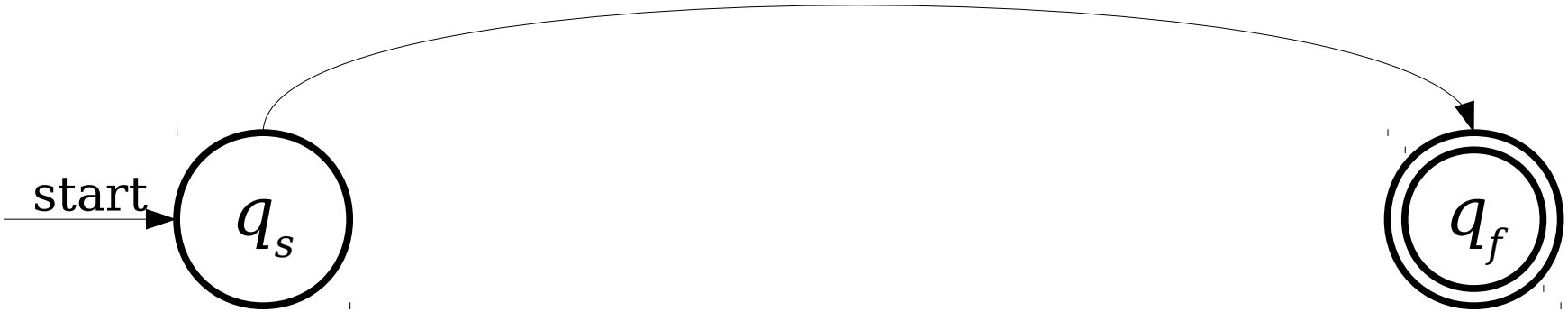


# From NFAs to Regular Expressions

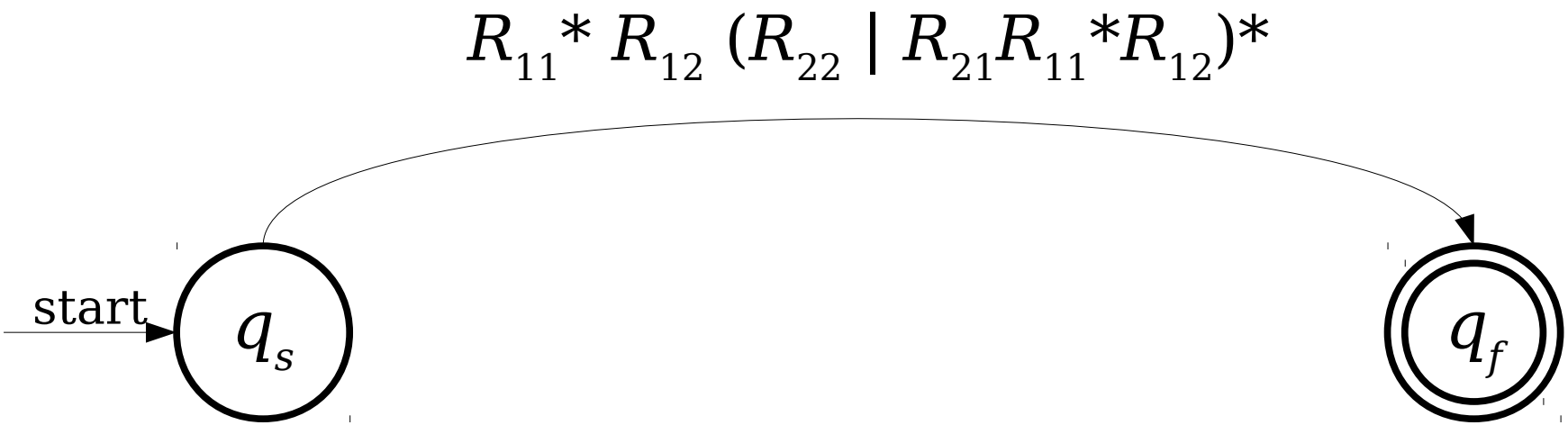


# From NFAs to Regular Expressions

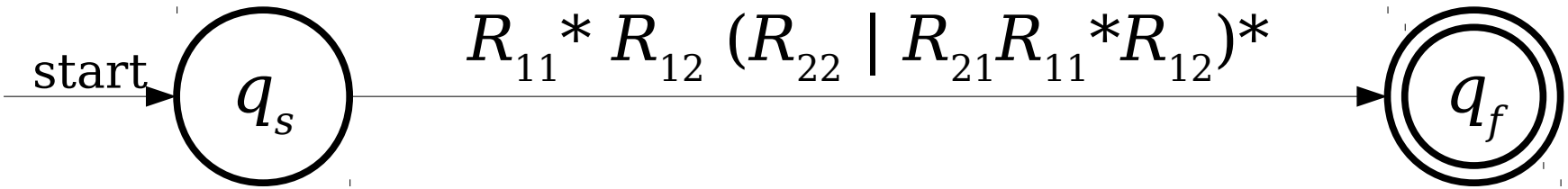
$$R_{11}{}^* \, R_{12} \, (R_{22} \mid R_{21} R_{11}{}^* R_{12})^* \, \varepsilon$$



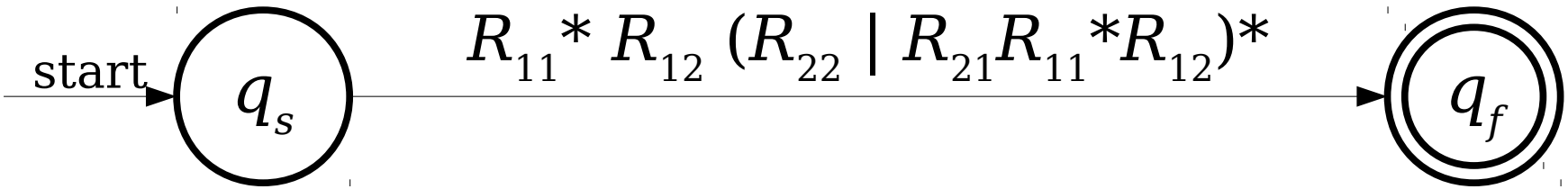
# From NFAs to Regular Expressions

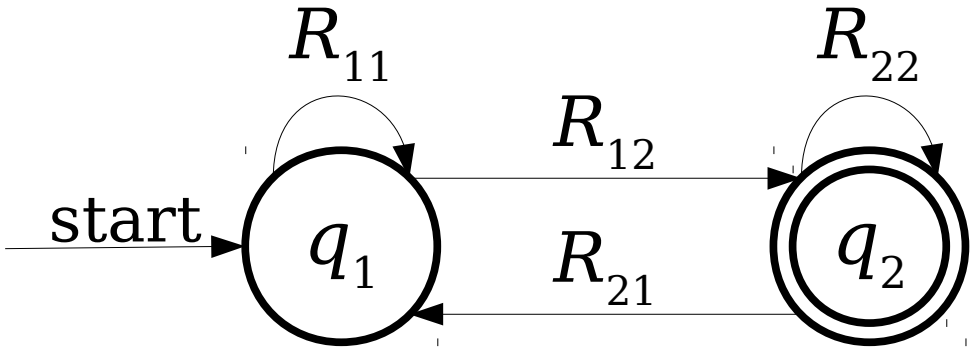


# From NFAs to Regular Expressions



# From NFAs to Regular Expressions



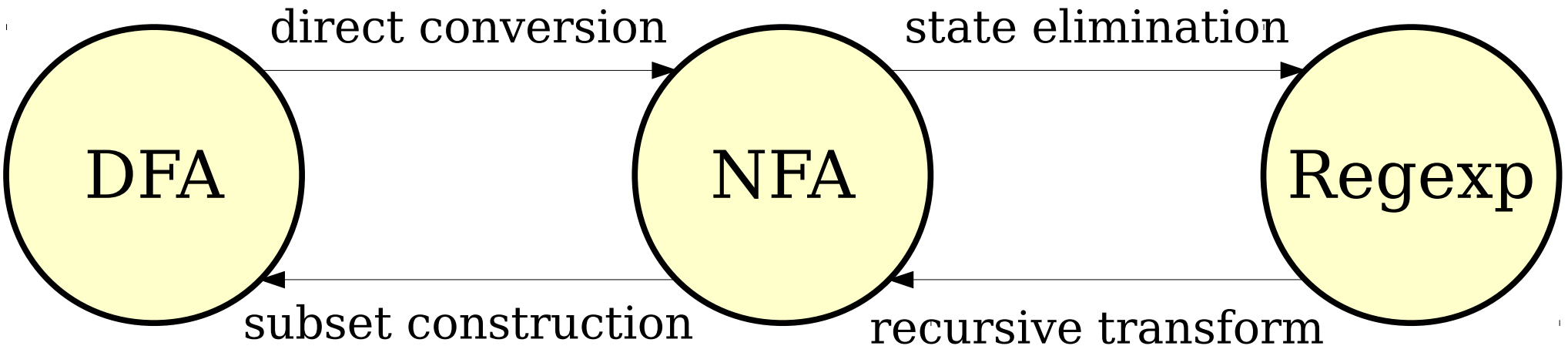
# The Construction at a Glance

- Start with an NFA for the language $L$.
- Add a new start state $q_s$ and accept state $q_f$ to the NFA.
  - Add ε-transitions from each original accepting state to $q_f$, then mark them as not accepting.
- Repeatedly remove states other than $q_s$ and $q_f$ from the NFA by "shortcutting" them until only two states remain: $q_s$ and $q_f$.
- The transition from $q_s$ to $q_f$ is then a regular expression for the NFA.

# Eliminating a State

- To eliminate a state $q$ from the automaton, do the following for each pair of states $q_0$ and $q_1$, where there's a transition from $q_0$ into $q$ and a transition from $q$ into $q_1$:
  - Let $R_{in}$ be the regex on the transition from $q_0$ to $q$.
  - Let $R_{out}$ be the regex on the transition from $q$ to $q_1$.
  - If there is a regular expression $R_{stay}$ on a transition from $q$ to itself, add a new transition from $q_0$ to $q_1$ labeled $R_{in}(R_{stay})^*R_{out}$.
  - If there isn't, add a new transition from $q_0$ to $q_1$ labeled $R_{in}R_{out}$.
- If a pair of states has multiple transitions between them labeled $R_1$, $R_2$, …, $R_k$, replace them with a single transition labeled $R_1 \mid R_2 \mid \ldots \mid R_k$.

# Our Transformations



***Theorem:*** The following are all equivalent:

- $L$ is a regular language.
- There is a DFA $D$ such that $\mathscr{L}(D) = L$.
- There is an NFA $N$ such that $\mathscr{L}(N) = L$.
- There is a regular expression $R$ such that $\mathscr{L}(R) = L$.

# Why This Matters

- The equivalence of regular expressions and finite automata has practical relevance.
  - Tools like `grep` and `flex` that use regular expressions capture all the power available via DFAs and NFAs.
- This also is hugely theoretically significant: the regular languages can be assembled "from scratch" using a small number of operations!

# Next Time

- **Applications of Regular Languages**
  - Answering “so what?”
- **Intuiting Regular Languages**
  - What makes a language regular?
- **The Myhill-Nerode Theorem**
  - The limits of regular languages.