# Context-Free Grammars

# Describing Languages

- We've seen two models for the regular languages:
  - ***Finite automata*** accept precisely the strings in the language.
  - ***Regular expressions*** describe precisely the strings in the language.
- Finite automata ***recognize*** strings in the language.
  - Perform a computation to determine whether a specific string is in the language.
- Regular expressions ***match*** strings in the language.
  - Describe the general shape of all strings in the language.

# Context-Free Grammars

- A ***context-free grammar*** (or ***CFG***) is an entirely different formalism for defining a class of languages.
- Goal: Give a procedure for listing off all strings in the language.
- CFGs are best explained by example...

# Arithmetic Expressions

- Suppose we want to describe all legal arithmetic expressions using addition, subtraction, multiplication, and division.
- Here is one possible CFG:

$$
\begin{aligned}
\mathbf{E} &\rightarrow \texttt{int} \\
\mathbf{E} &\rightarrow \mathbf{E}\ \mathbf{Op}\ \mathbf{E} \\
\mathbf{E} &\rightarrow \texttt{(}\mathbf{E}\texttt{)} \\
\mathbf{Op} &\rightarrow \texttt{+} \\
\mathbf{Op} &\rightarrow \texttt{-} \\
\mathbf{Op} &\rightarrow \texttt{*} \\
\mathbf{Op} &\rightarrow \texttt{/}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{E} \\
\Rightarrow\ & \mathbf{E}\ \mathbf{Op}\ \mathbf{E} \\
\Rightarrow\ & \mathbf{E}\ \mathbf{Op}\ \texttt{(}\mathbf{E}\texttt{)} \\
\Rightarrow\ & \mathbf{E}\ \mathbf{Op}\ \texttt{(}\mathbf{E}\ \mathbf{Op}\ \mathbf{E}\texttt{)} \\
\Rightarrow\ & \mathbf{E}\ \texttt{*}\ \texttt{(}\mathbf{E}\ \mathbf{Op}\ \mathbf{E}\texttt{)} \\
\Rightarrow\ & \texttt{int}\ \texttt{*}\ \texttt{(}\mathbf{E}\ \mathbf{Op}\ \mathbf{E}\texttt{)} \\
\Rightarrow\ & \texttt{int}\ \texttt{*}\ \texttt{(int}\ \mathbf{Op}\ \mathbf{E}\texttt{)} \\
\Rightarrow\ & \texttt{int}\ \texttt{*}\ \texttt{(int}\ \mathbf{Op}\ \texttt{int)} \\
\Rightarrow\ & \texttt{int * (int + int)}
\end{aligned}
$$

# Arithmetic Expressions

- Suppose we want to describe all legal arithmetic expressions using addition, subtraction, multiplication, and division.
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$$
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\mathbf{E} &\to \mathbf{E}\ \mathbf{Op}\ \mathbf{E} \\
\mathbf{E} &\to \texttt{(}\mathbf{E}\texttt{)} \\
\mathbf{Op} &\to \texttt{+} \\
\mathbf{Op} &\to \texttt{-} \\
\mathbf{Op} &\to \texttt{*} \\
\mathbf{Op} &\to \texttt{/}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{E} \\
\Rightarrow\ & \mathbf{E}\ \mathbf{Op}\ \mathbf{E} \\
\Rightarrow\ & \mathbf{E}\ \mathbf{Op}\ \texttt{int} \\
\Rightarrow\ & \texttt{int}\ \mathbf{Op}\ \texttt{int} \\
\Rightarrow\ & \texttt{int / int}
\end{aligned}
$$

# Context-Free Grammars

- Formally, a context-free grammar is a collection of four objects:
  - A set of ***nonterminal symbols*** (also called ***variables***),
  - A set of ***terminal symbols*** (the ***alphabet*** of the CFG)
  - A set of ***production rules*** saying how each nonterminal can be replaced by a string of terminals and nonterminals, and
  - A ***start symbol*** (which must be a nonterminal) that begins the derivation.

**E** → `int`

**E** → **E Op E**

**E** → `(`**E**`)`

**Op** → `+`

**Op** → `-`

**Op** → `*`

**Op** → `/`

# Some CFG Notation

- Capital letters in **Bold Red Uppercase** will represent nonterminals.
  - i.e. **A**, **B**, **C**, **D**
- Lowercase letters in **`blue monospace`** will represent terminals.
  - i.e. **`t`**, **`u`**, **`v`**, **`w`**
- Lowercase Greek letters in ***gray italics*** will represent arbitrary strings of terminals and nonterminals.
  - i.e. $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\omega}$

# A Notational Shorthand

**E** → `int`

**E** → **E Op E**

**E** → `(`**E**`)`

**Op** → `+`

**Op** → `-`

**Op** → `*`

**Op** → `/`

# A Notational Shorthand

$\mathbf{E} \rightarrow$ `int` | $\mathbf{E}$ $\mathbf{Op}$ $\mathbf{E}$ | `(`$\mathbf{E}$`)`

$\mathbf{Op} \rightarrow$ `+` | `-` | `*` | `/`

# Derivations

$$\mathbf{E} \rightarrow \mathbf{E}\ \mathbf{Op}\ \mathbf{E} \mid \texttt{int} \mid (\mathbf{E})$$
$$\mathbf{Op} \rightarrow \texttt{+} \mid \texttt{*} \mid \texttt{-} \mid \texttt{/}$$

$\mathbf{E}$

$\Rightarrow \mathbf{E}\ \mathbf{Op}\ \mathbf{E}$

$\Rightarrow \mathbf{E}\ \mathbf{Op}\ \texttt{(}\mathbf{E}\texttt{)}$

$\Rightarrow \mathbf{E}\ \mathbf{Op}\ \texttt{(}\mathbf{E}\ \mathbf{Op}\ \mathbf{E}\texttt{)}$

$\Rightarrow \mathbf{E}\ \texttt{*}\ \texttt{(}\mathbf{E}\ \mathbf{Op}\ \mathbf{E}\texttt{)}$

$\Rightarrow \texttt{int * (}\mathbf{E}\ \mathbf{Op}\ \mathbf{E}\texttt{)}$

$\Rightarrow \texttt{int * (int}\ \mathbf{Op}\ \mathbf{E}\texttt{)}$

$\Rightarrow \texttt{int * (int}\ \mathbf{Op}\ \texttt{int)}$

$\Rightarrow \texttt{int * (int + int)}$

- A sequence of steps where nonterminals are replaced by the right-hand side of a production is called a ***derivation***.
- If string $\boldsymbol{\alpha}$ derives string $\boldsymbol{\omega}$, we write $\boldsymbol{\alpha} \Rightarrow^* \boldsymbol{\omega}$.
- In the example on the left, we see $\mathbf{E} \Rightarrow^* \texttt{int * (int + int)}$.

# The Language of a Grammar

- If $G$ is a CFG with alphabet $\Sigma$ and start symbol $\mathbf{S}$, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \ \boldsymbol{\omega} \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \boldsymbol{\omega} \ \}$$

- That is, $\mathscr{L}(G)$ is the set of strings derivable from the start symbol.
- Note: $\boldsymbol{\omega}$ must be in $\Sigma^*$, the set of strings made from terminals. Strings involving nonterminals aren't in the language.

# Context-Free Languages

- A language $L$ is called a ***context-free language*** (or CFL) if there is a CFG $G$ such that $L = \mathscr{L}(G)$.
- Questions:
  - What languages are context-free?
  - How are context-free and regular languages related?

# From Regexes to CFGs

- CFGs consist purely of production rules of the form $\mathbf{A} \to \boldsymbol{\omega}$. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

$$\mathbf{S} \to \texttt{a*b}$$

# From Regexes to CFGs

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- However, we can convert regular expressions to CFGs as follows:

$$\mathbf{S} \to \mathbf{A}\mathtt{b}$$
$$\mathbf{A} \to \mathbf{A}\mathtt{a} \mid \varepsilon$$

# From Regexes to CFGs

- CFGs consist purely of production rules of the form **A** → ***ω***. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

$$\mathbf{S} \rightarrow \mathtt{a(b \cup c^*)}$$

# From Regexes to CFGs

- CFGs consist purely of production rules of the form $\mathbf{A} \to \boldsymbol{\omega}$. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

$$\mathbf{S} \to \mathtt{a}\mathbf{X}$$
$$\mathbf{X} \to \mathtt{b} \mid \mathbf{C}$$
$$\mathbf{C} \to \mathbf{C}\mathtt{c} \mid \varepsilon$$

# Regular Languages and CFLs

- ***Theorem:*** Every regular language is context-free.
- ***Proof Idea:*** Use the construction from the previous slides to convert a regular expression for $L$ into a CFG for $L$. ■
- ***Problem Set Exercise:*** Instead, show how to convert a DFA/NFA into a CFG.

# The Language of a Grammar

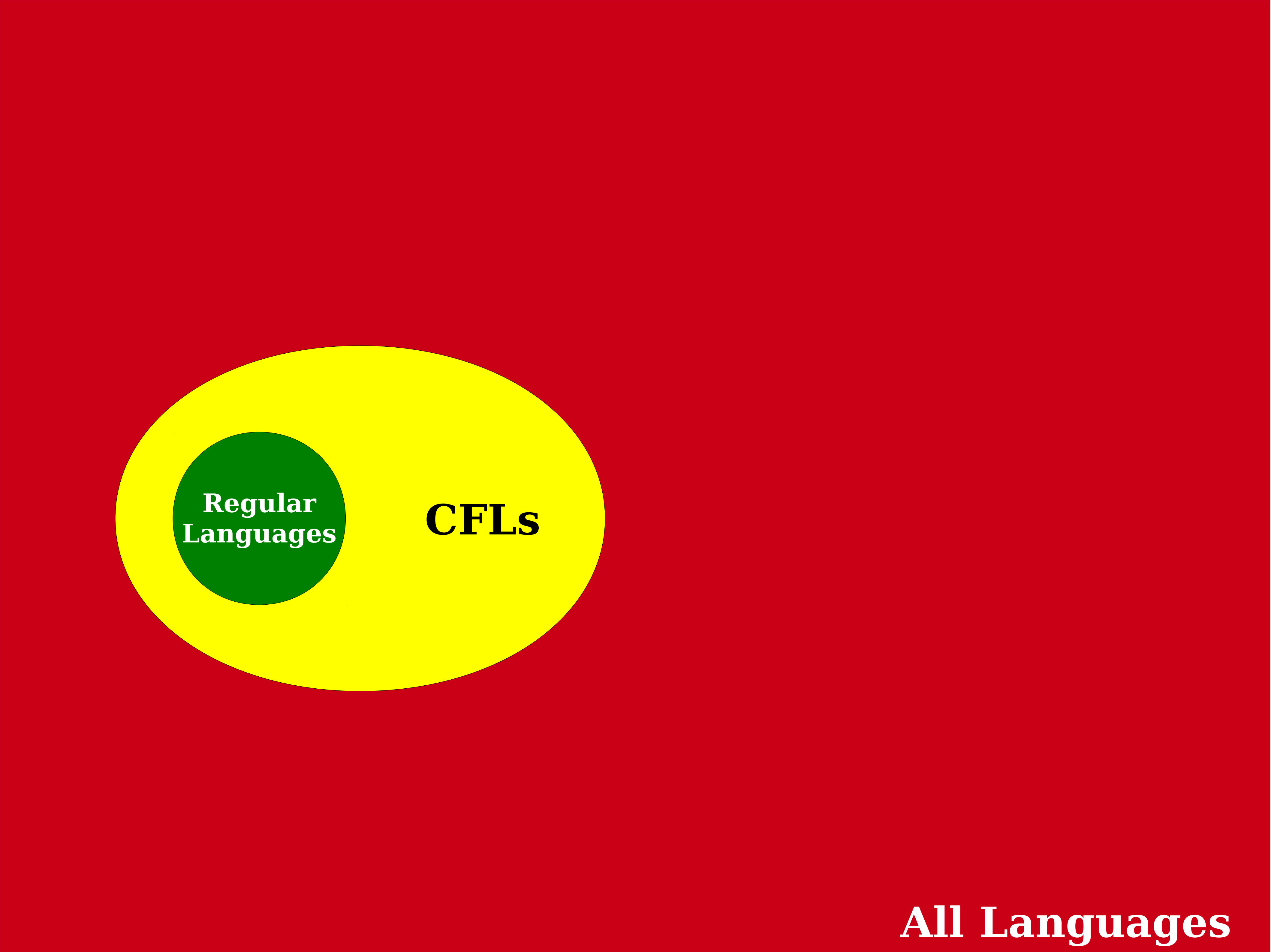
- Consider the following CFG $G$:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a | a | a | a | b | b | b | b |

$$\mathscr{L}(G) = \{\ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\ \}$$


Regular Languages
CFLs
All Languages

# Why the Extra Power?

- Why do CFGs have more power than regular expressions?
- ***Intuition:*** Derivations of strings have unbounded "memory."

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a | a | a | a | b | b | b | b |

Time-Out for Announcements!

# Problem Set Seven

- Problem Set Six was due at the start of today's lecture.
  - Want to use late days? Submit up to Monday at 3:00PM.
- Problem Set Seven goes out now. It's due next Friday.
  - Play around with the Myhill-Nerode theorem and the limits of regular languages!
  - Play around with your very own CFGs!

# Midterms Graded

- Midterms have been graded. They're available for pickup in the Gates building.
  - SCPD students: we've sent the exams back to the SCPD office. You should hear back from them soon.
- Solutions and stats are available in the Gates building in the normal handout filing cabinet.

# Midterm Regrades

- If you believe that we made a grading error on the exam, you can submit it for a regrade. To do so, fill out the form online, staple it to your exam, and hand it to Keith by next Friday.
- Please only submit regrades if you
  - believe that we actually graded your exam incorrectly, and
  - you've talked about the exam with the course staff and they agree with you.
- ***Your score can go down if you ask for a regrade***. Please be sure you really want to ask for it before you submit a regrade request.

Back to CS103!

# Designing CFGs

- Like designing DFAs, NFAs, and regular expressions, designing CFGs is a craft.
- When thinking about CFGs:
  - ***Think recursively:*** Build up bigger structures from smaller ones.
  - ***Have a construction plan:*** Know in what order you will build up the string.
  - ***Store information in nonterminals:*** Have each nonterminal correspond to some useful piece of information.

# Designing CFGs

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{w \in \Sigma^* \mid w$ is a palindrome $\}$
- We can design a CFG for $L$ by thinking inductively:
  - Base case: ε, **a**, and **b** are palindromes.
  - If $\omega$ is a palindrome, then $\mathtt{a}\omega\mathtt{a}$ and $\mathtt{b}\omega\mathtt{b}$ are palindromes.

$$\mathbf{S} \to \varepsilon \mid \mathtt{a} \mid \mathtt{b} \mid \mathtt{a}\mathbf{S}\mathtt{a} \mid \mathtt{b}\mathbf{S}\mathtt{b}$$

# Designing CFGs

- Let $\Sigma$ = { **(**, **)** } and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Some sample strings in $L$:

**((()))**

**(())()**

**(()())(()())**

**((((()))(())))**

**ε**

**()()**

# Designing CFGs

- Let $\Sigma = \{ (, ) \}$ and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis.

((()(()))(()))(())((()))

# Designing CFGs

- Let Σ = { (, ) } and let $L$ = {$w$ ∈ Σ* | $w$ is a string of balanced parentheses }
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis.

((()(()))(())) (())((()))

# Designing CFGs

- Let $\Sigma = \{ (, ) \}$ and let $L = \{ w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
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( ( ) ( ( ) ) ) ( ( ) ) ( ( ) ) ( ( ( ) ) )

# Designing CFGs

- Let $\Sigma = \{$ **(**, **)** $\}$ and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis. Removing the first parenthesis and the matching parenthesis forms two new strings of balanced parentheses.

$$\mathbf{S} \rightarrow \mathbf{(S)S} \mid \boldsymbol{\varepsilon}$$

# Designing CFGs: A Caveat

- Let $\Sigma = \{$**a**, **b**$\}$ and let $L = \{w \in \Sigma^* \mid w$ has the same number of **a**'s and **b**'s $\}$
- Is this a CFG for $L$?

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \mathtt{b}\mathbf{S}\mathtt{a} \mid \varepsilon$$

- Can you derive the string **abba**?

# Designing CFGs: A Caveat

- When designing a CFG for a language, make sure that it
  - generates all the strings in the language and
  - never generates a string outside the language.
- The first of these can be tricky – make sure to test your grammars!
- You'll design your own CFG for this language on the next problem set.

# CFG Caveats II

- Is the following grammar a CFG for the language $\{\ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\ \}$?

$$\mathbf{S} \to \mathtt{a}\mathbf{S}\mathtt{b}$$

- What strings can you derive?
  - Answer: **None!**
- What is the language of the grammar?
  - Answer: $\boldsymbol{\emptyset}$
- When designing CFGs, make sure your recursion actually terminates!

# CFG Caveats III

- When designing CFGs, remember that each nonterminal can be expanded out independently of the others.
- Let $\Sigma = \{\mathtt{a}, \stackrel{?}{=}\}$ and let $L = \{\mathtt{a}^n\stackrel{?}{=}\mathtt{a}^n \mid n \in \mathbb{N}\}$.
- Is the following a CFG for $L$?

$$\mathbf{S} \to \mathbf{X}\stackrel{?}{=}\mathbf{X}$$

$$\mathbf{X} \to \mathtt{a}\mathbf{X} \mid \varepsilon$$

$$\begin{aligned}
&\mathbf{S} \\
\Rightarrow\ &\mathbf{X}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathtt{a}\mathbf{X}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathtt{aa}\mathbf{X}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathtt{aa}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathtt{aa}\stackrel{?}{=}\mathtt{a}\mathbf{X} \\
\Rightarrow\ &\mathtt{aa}\stackrel{?}{=}\mathtt{a}
\end{aligned}$$

# Finding a Build Order

- Let $\Sigma = \{\mathtt{a}, \overset{?}{=}\}$ and let $L = \{\mathtt{a}^n\overset{?}{=}\mathtt{a}^n \mid n \in \mathbb{N}\}$.
- To build a CFG for $L$, we need to be more clever with how we construct the string.
  - If we build the strings of a's independently of one another, then we can't enforce that they have the same length.
  - ***Idea:*** Build both strings of a's at the same time.
- Here's one possible grammar based on that idea:

$$\mathbf{S} \to \overset{?}{=} \mid \mathtt{a}\mathbf{S}\mathtt{a}$$

$$\begin{aligned} & \mathbf{S} \\ \Rightarrow\ & \mathtt{a}\mathbf{S}\mathtt{a} \\ \Rightarrow\ & \mathtt{aa}\mathbf{S}\mathtt{aa} \\ \Rightarrow\ & \mathtt{aaa}\mathbf{S}\mathtt{aaa} \\ \Rightarrow\ & \mathtt{aaa}\overset{?}{=}\mathtt{aaa} \end{aligned}$$

# Function Prototypes

- Let $\Sigma$ = {`void`, `int`, `double`, `name`, `(`, `)`, `,`, `;`}.
- Let's write a CFG for C-style function prototypes!
- Examples:
  - `void name(int name, double name);`
  - `int name();`
  - `int name(double name);`
  - `int name(int, int name, int);`
  - `void name(void);`

# Function Prototypes

- Here's one possible grammar:
  - **S** → **Ret** `name` `(`**Args**`)` `;`
  - **Ret** → **Type** | `void`
  - **Type** → `int` | `double`
  - **Args** → ε | `void` | **ArgList**
  - **ArgList** → **OneArg** | **ArgList**, **OneArg**
  - **OneArg** → **Type** | **Type** `name`
- Fun question to think about: what changes would you need to make to support pointer types?

# Summary of CFG Design Tips

- Look for recursive structures where they exist: they can help guide you toward a solution.
- Keep the build order in mind – often, you'll build two totally different parts of the string concurrently.
  - Usually, those parts are built in opposite directions: one's built left-to-right, the other right-to-left.
- Use different nonterminals to represent different structures.

# Applications of Context-Free Grammars

# CFGs for Programming Languages

```
BLOCK  →  STMT
       |  { STMTS }

STMTS  →  ε
       |  STMT STMTS

STMT   →  EXPR;
       |  if (EXPR) BLOCK
       |  while (EXPR) BLOCK
       |  do BLOCK while (EXPR);
       |  BLOCK
       |  …

EXPR   →  identifier
       |  constant
       |  EXPR + EXPR
       |  EXPR – EXPR
       |  EXPR * EXPR
       |  …
```

# Grammars in Compilers

- One of the key steps in a compiler is figuring out what a program “means.”
- This is usually done by defining a grammar showing the high-level structure of a programming language.
- There are certain classes of grammars (LL(1) grammars, LR(1) grammars, LALR(1) grammars, etc.) for which it's easy to figure out how a particular string was derived.
- Tools like `yacc` or `bison` automatically generate parsers from these grammars.
- Curious to learn more? Take CS143!

# Natural Language Processing

- By building context-free grammars for actual languages and applying statistical inference, it's possible for a computer to recover the likely meaning of a sentence.
  - In fact, CFGs were first called ***phrase-structure grammars*** and were introduced by Noam Chomsky in his seminal work *Syntactic Structures*.
  - They were then adapted for use in the context of programming languages, where they were called ***Backus-Naur forms***.
- Stanford's CoreNLP project is one place to look for an example of this.
- Want to learn more? Take CS124 or CS224N!

# Next Time

- **Turing Machines**
  - What does a computer with unbounded memory look like?
  - How do you program them?