Mathematical Induction

Part One

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

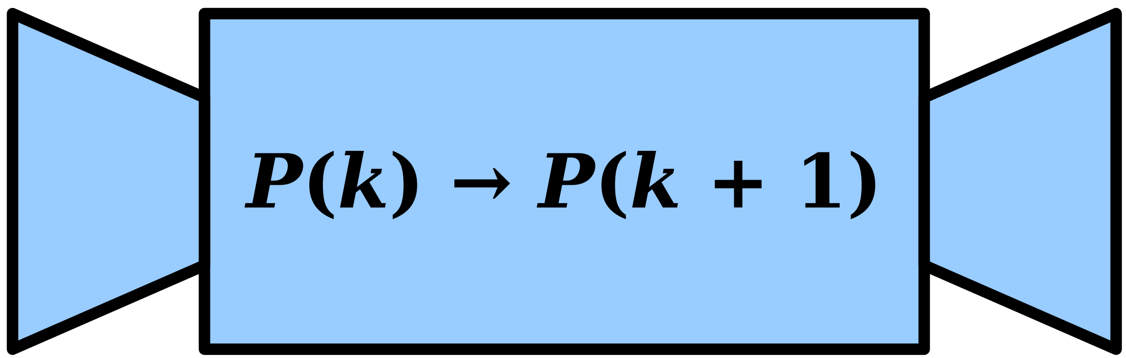
Induction, Intuitively

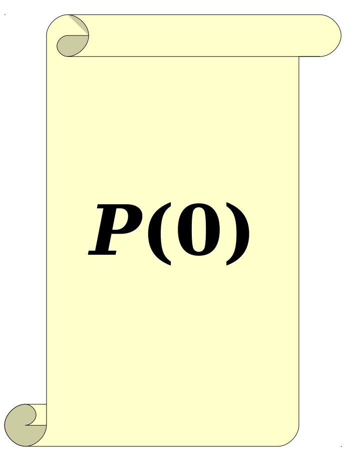
$$\boldsymbol{P(0)}$$

$$\boldsymbol{\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))}$$

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

Why Induction Works





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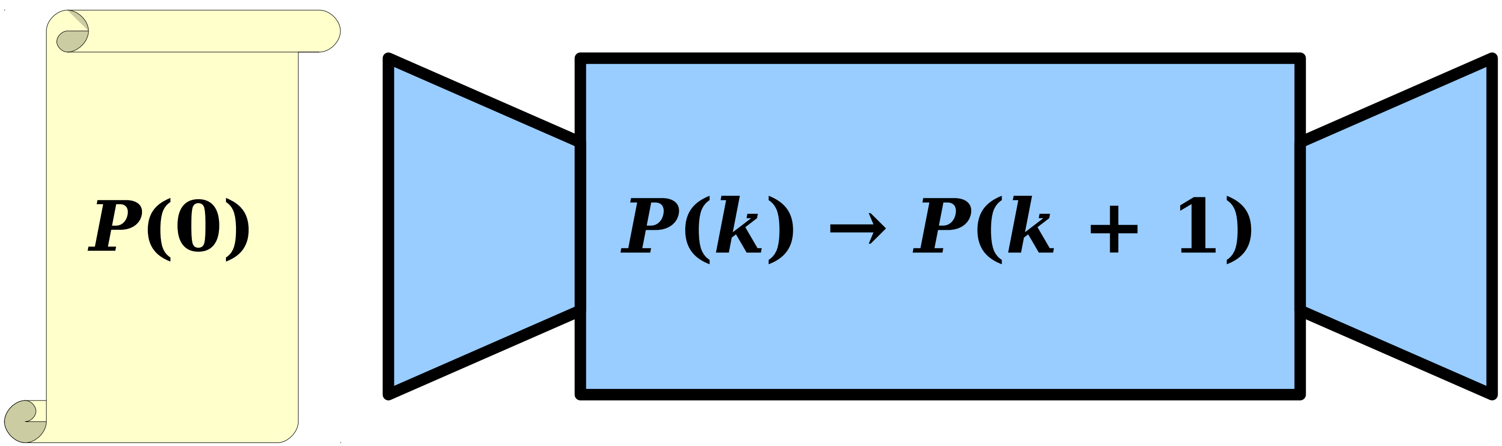

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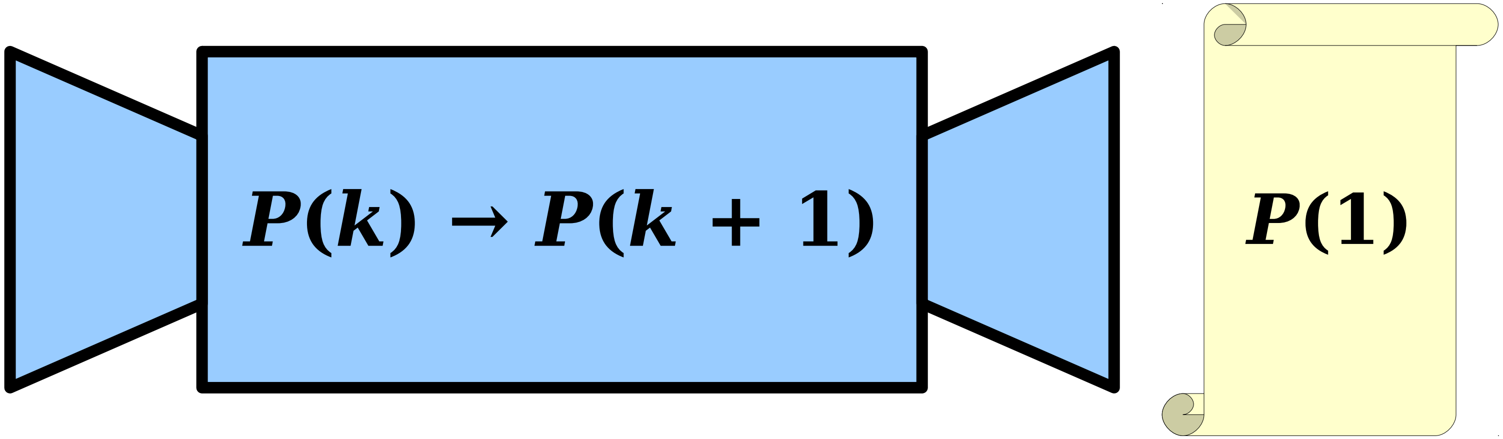

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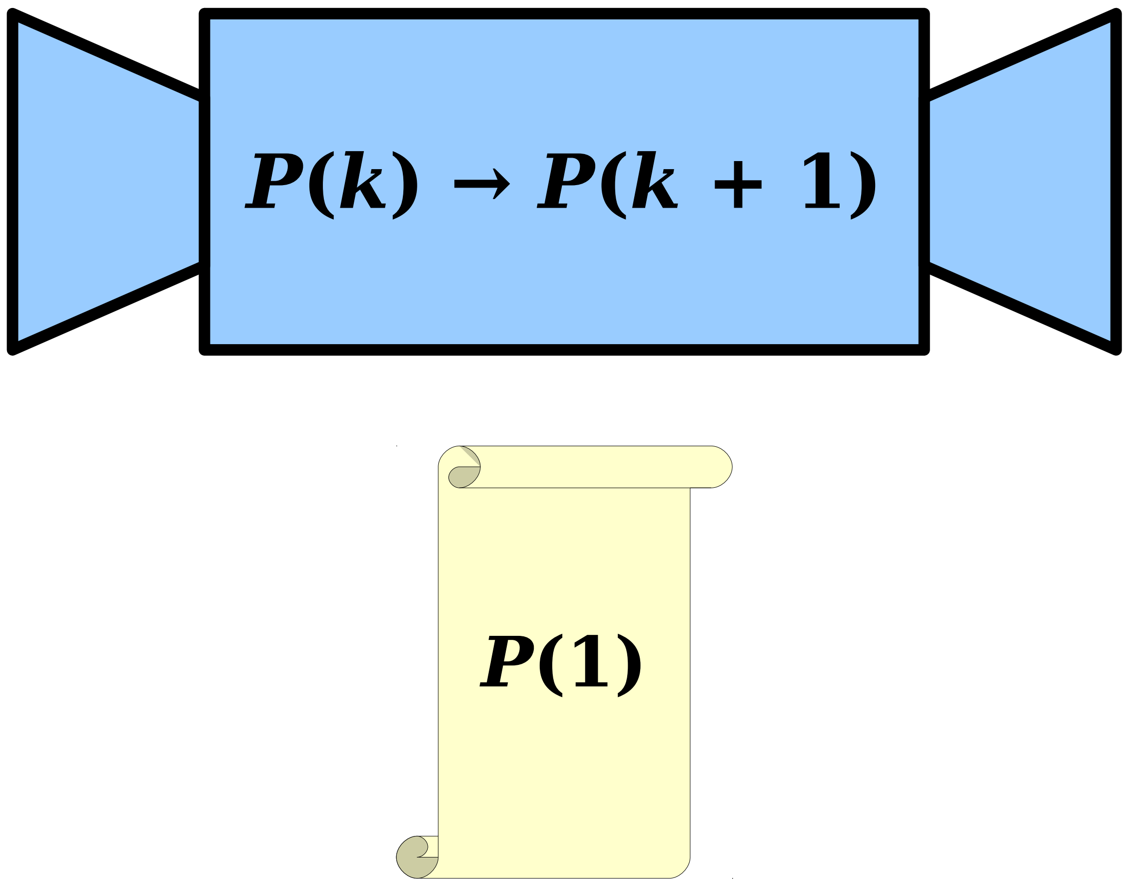

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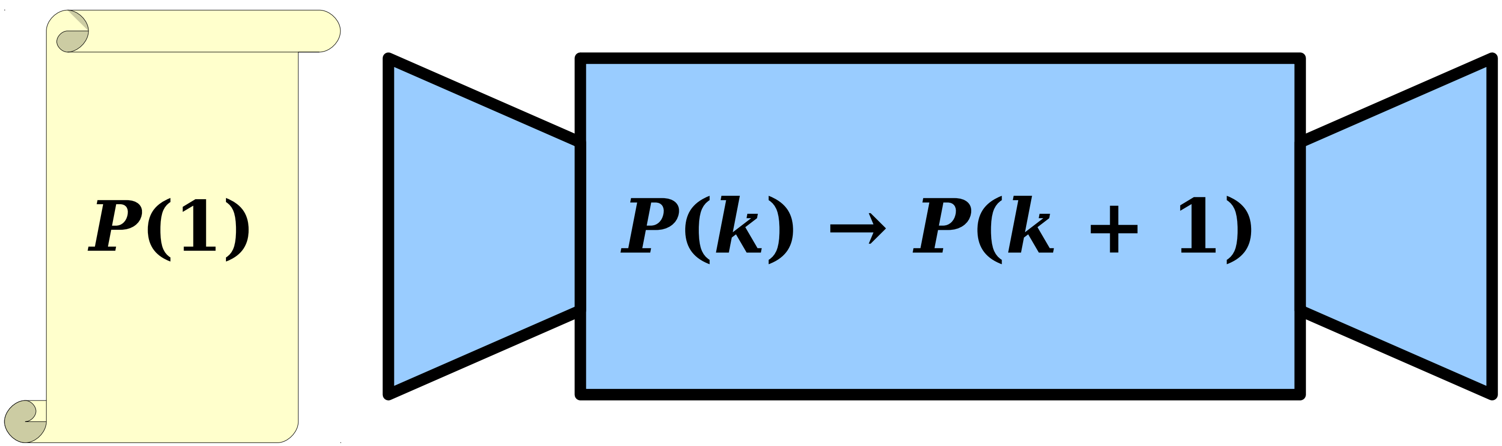

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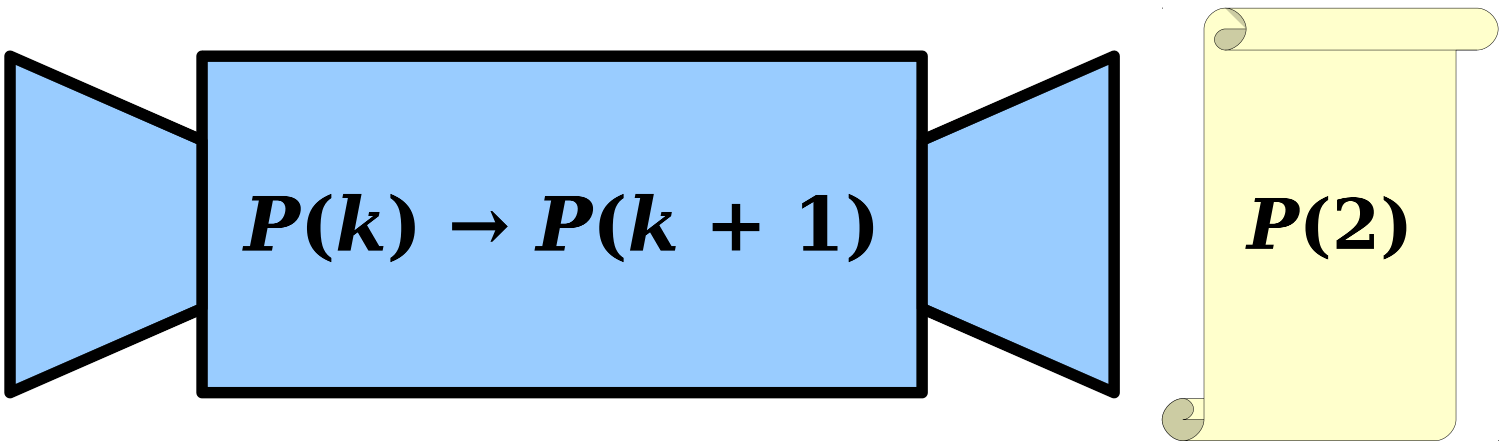

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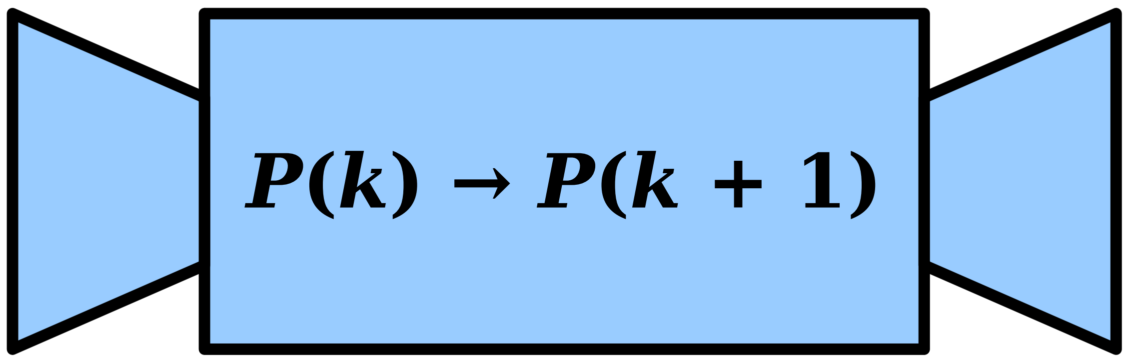

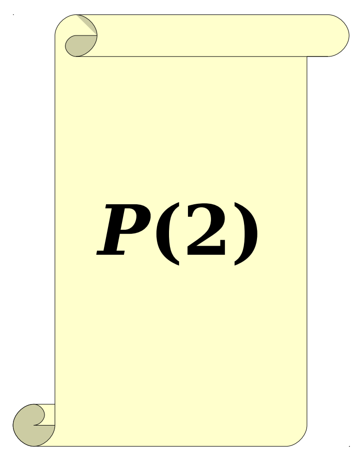

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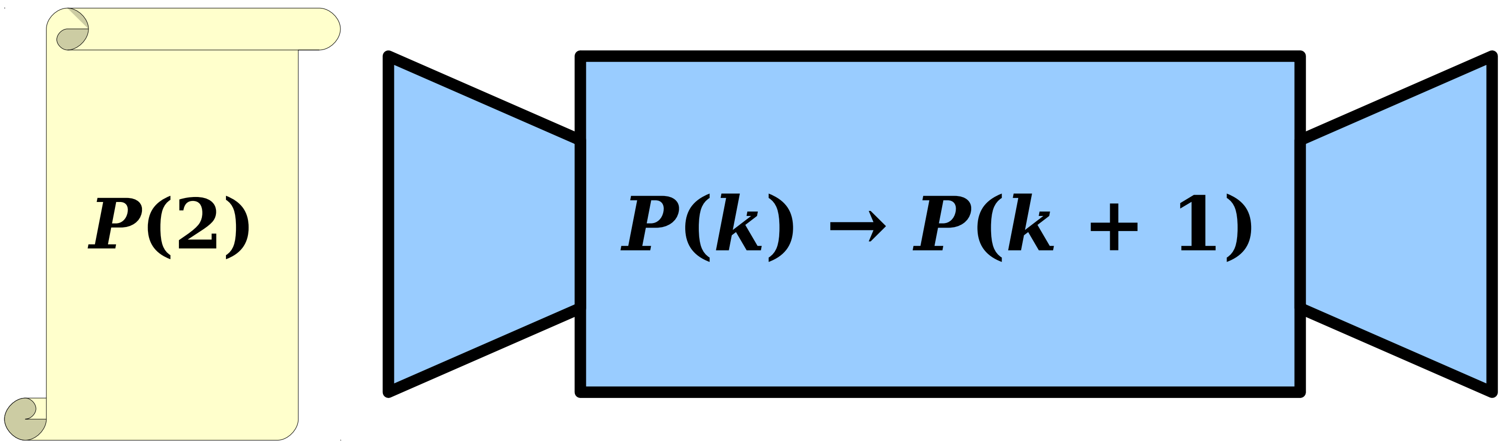

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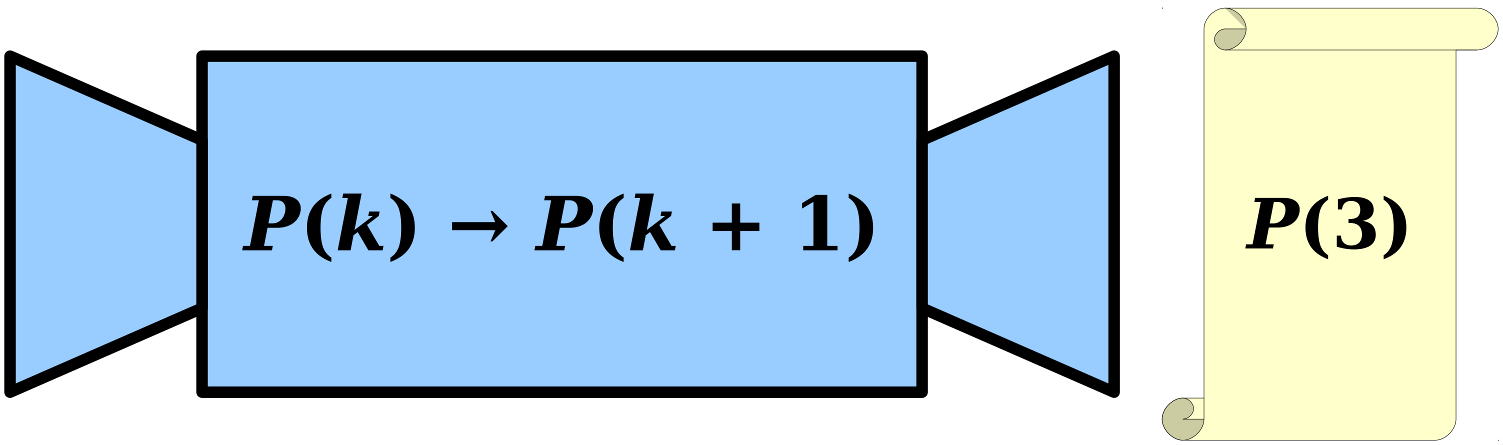

Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers n.
- In a proof by induction, there are three steps:
  - Prove that $P(0)$ is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if $P(k)$ is true, then $P(k+1)$ is true.
    - This is called the ***inductive step***.
    - The assumption that $P(k)$ is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Some Sums

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 \quad = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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At the start of the proof, we tell the reader what property we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly stating $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to prove it.

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Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to simplify a complex expression. This is a common theme in inductive proofs.

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For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

A Quick Aside

- This result helps explain the range of numbers that can be stored in an `int`.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. If we have time, we'll see one later today.

Structuring a Proof by Induction

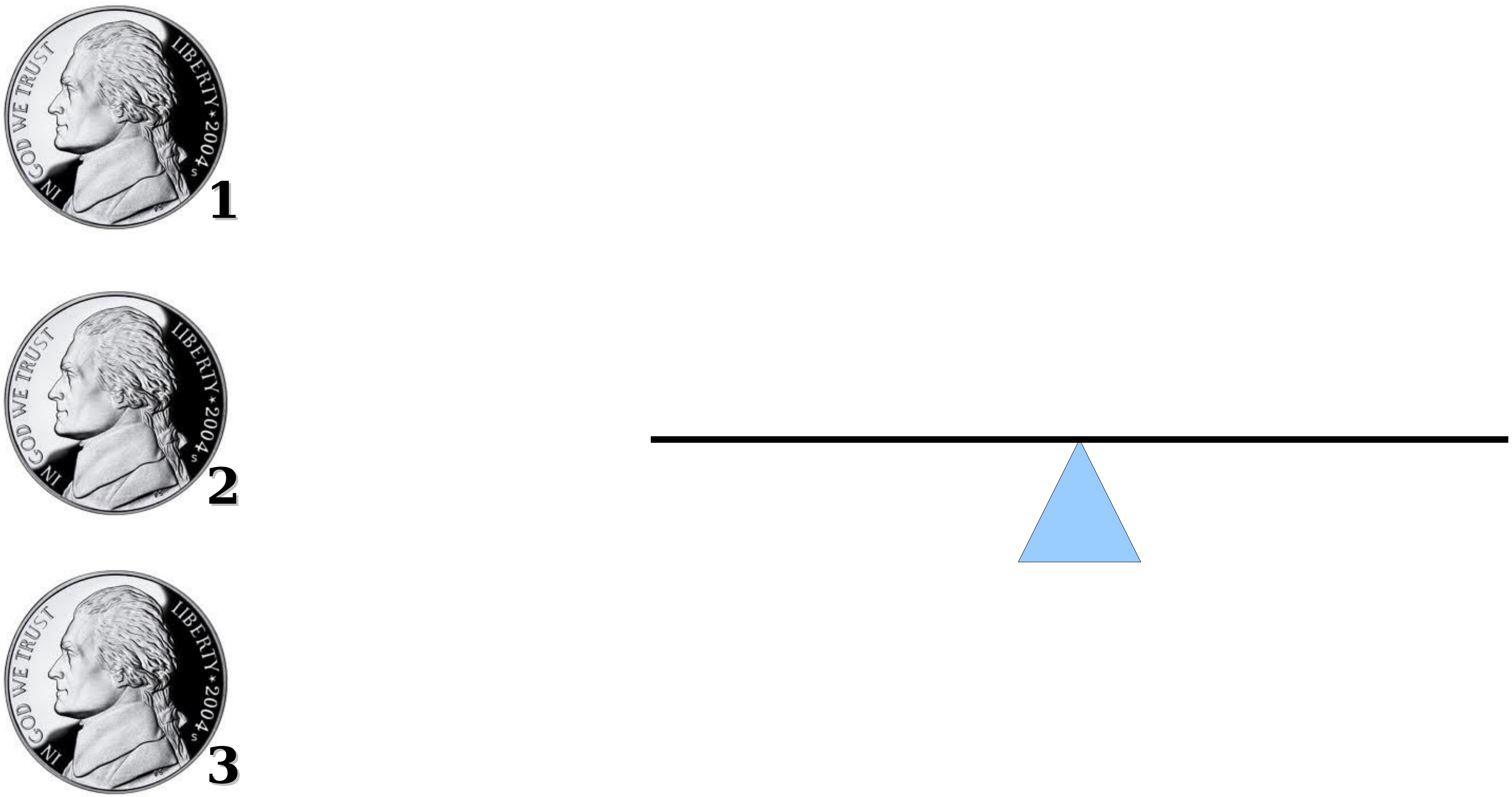
- Define some predicate P that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that $P(0)$ is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming $P(k)$ for some arbitrary natural number k, then write out exactly what that means.
  - Say that you're going to prove $P(k+1)$, then write out exactly what that means.
  - Prove that $P(k+1)$ using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

The Counterfeit Coin Problem

Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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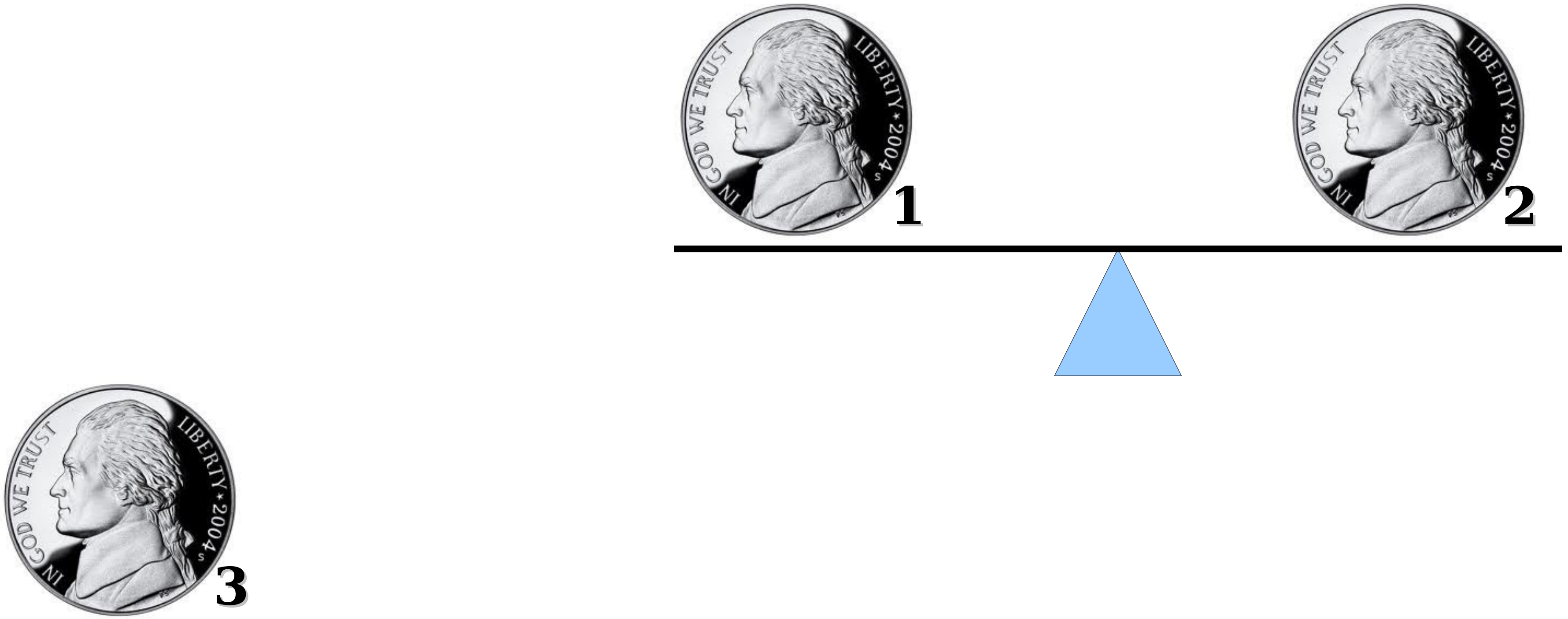

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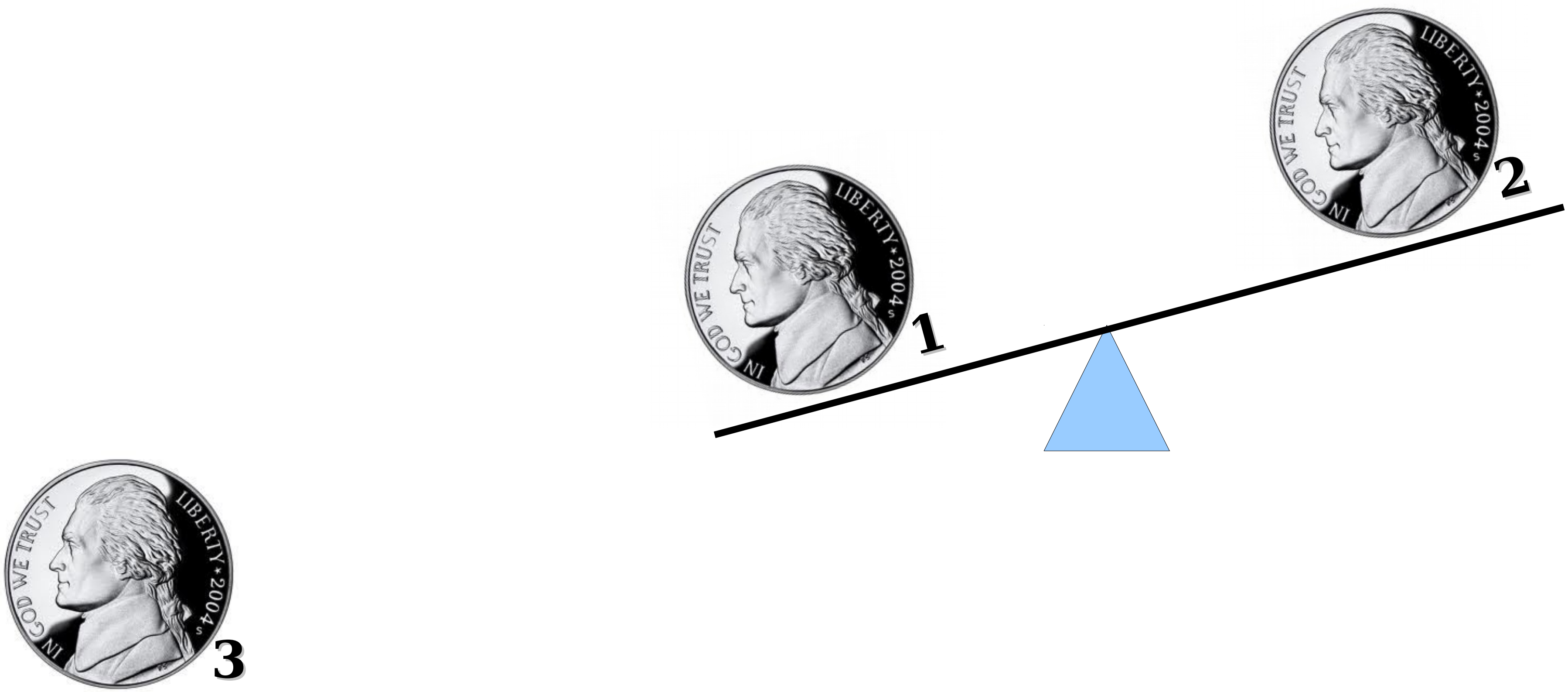

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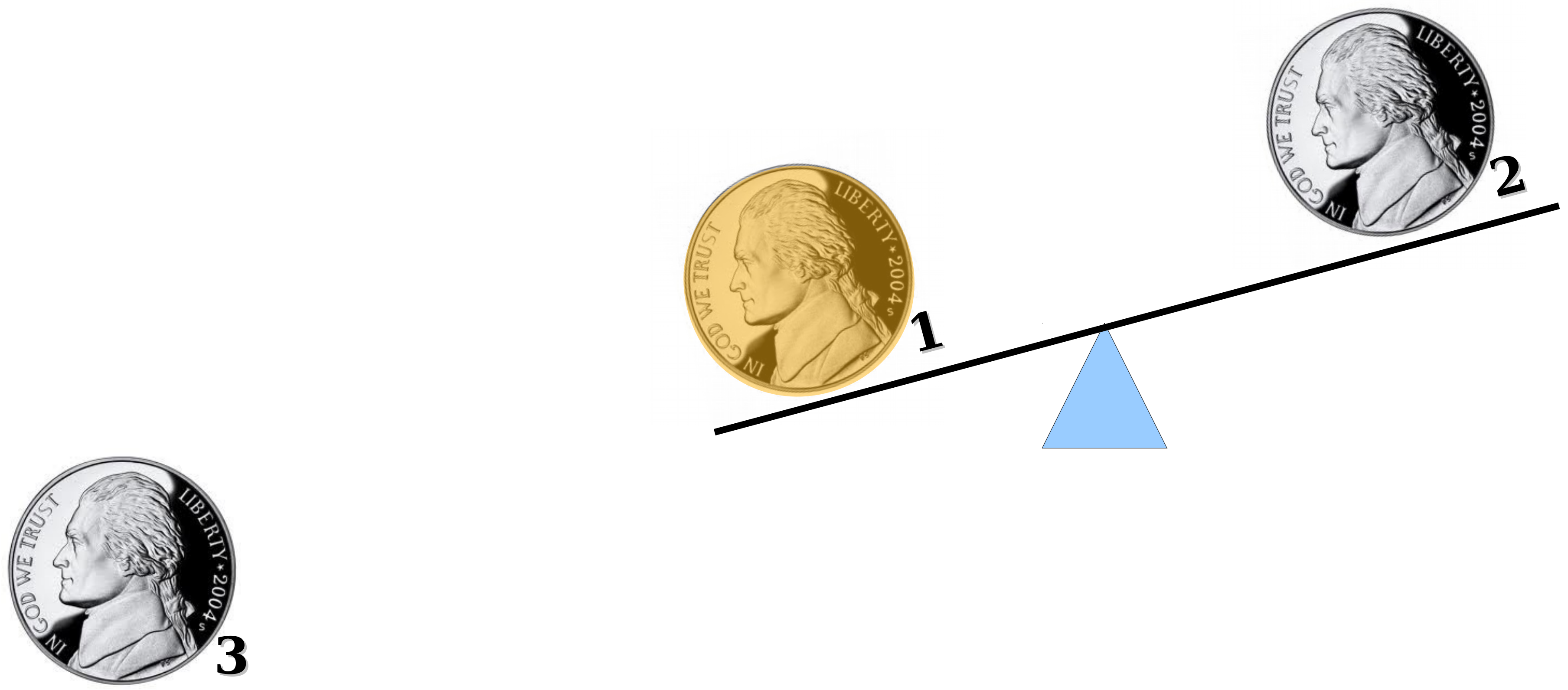

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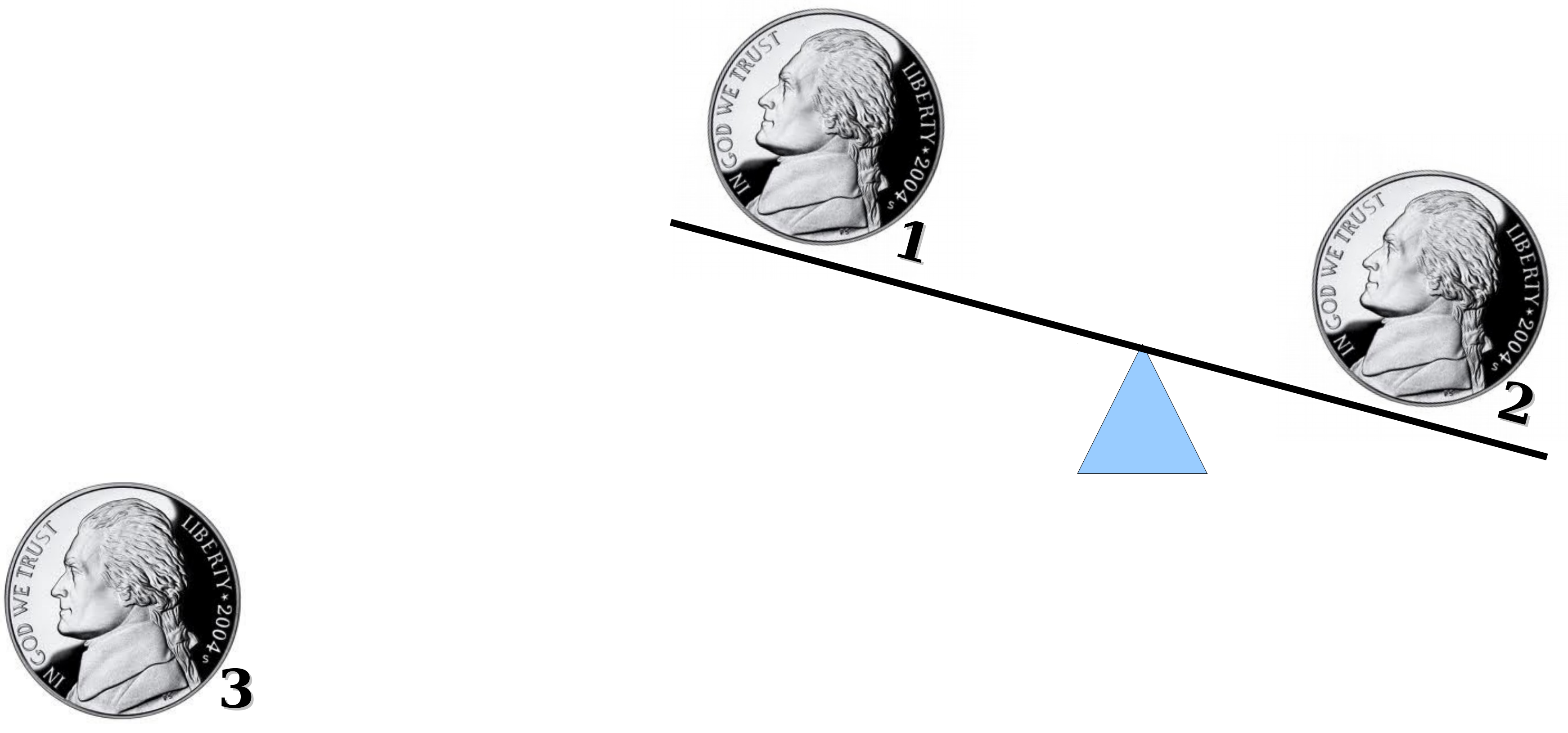

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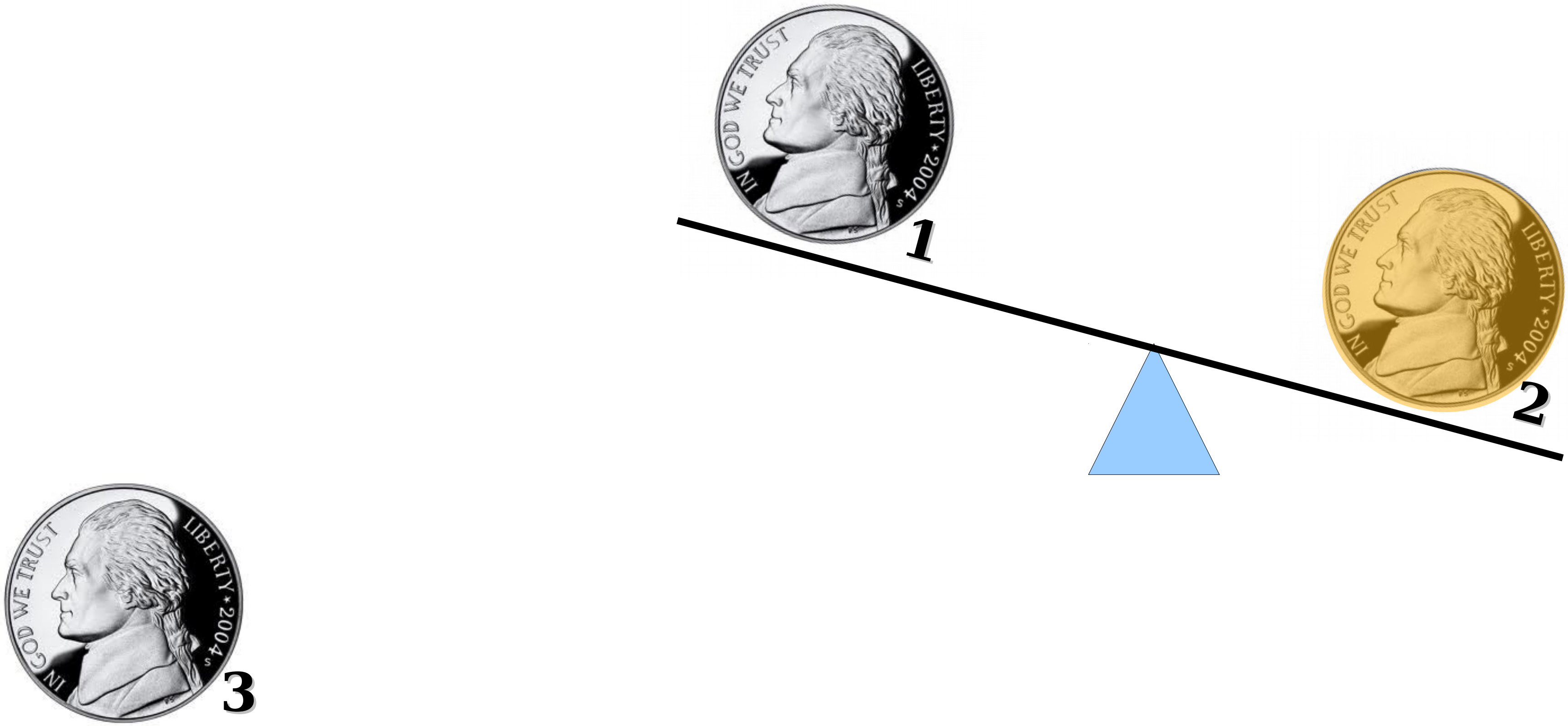

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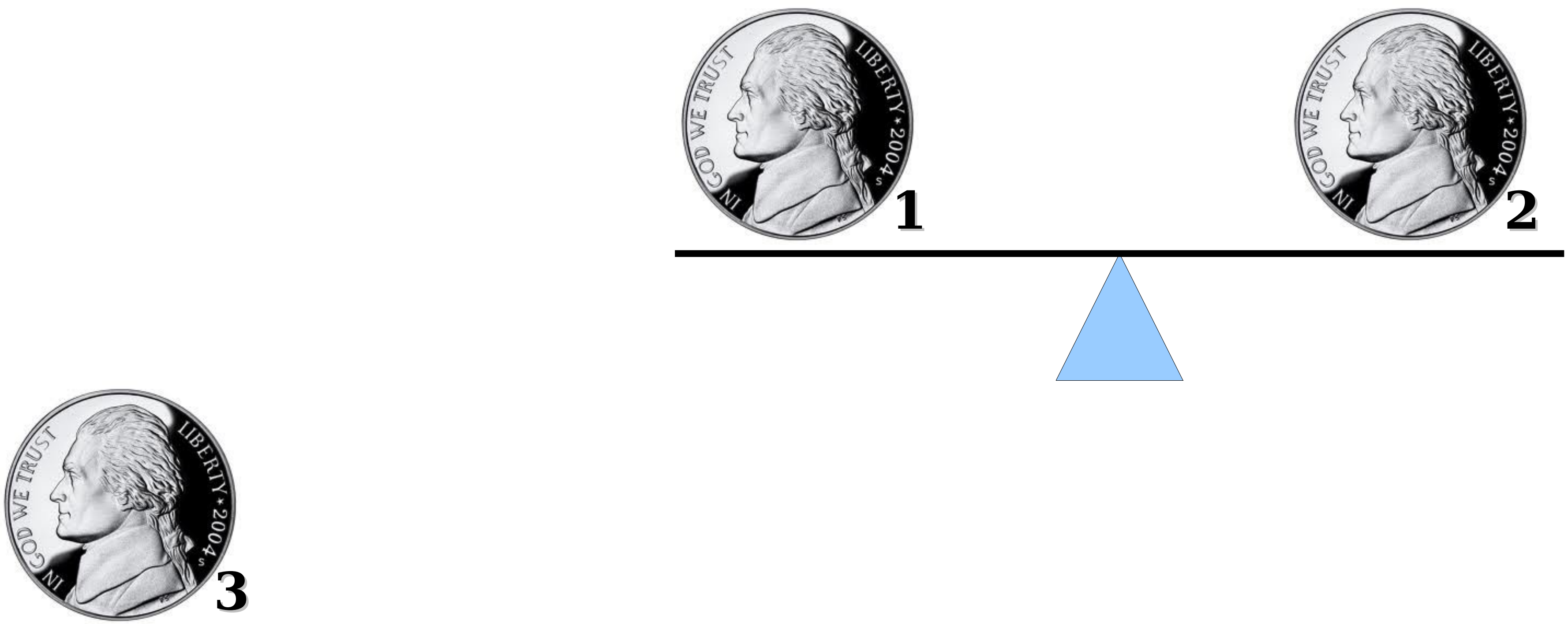

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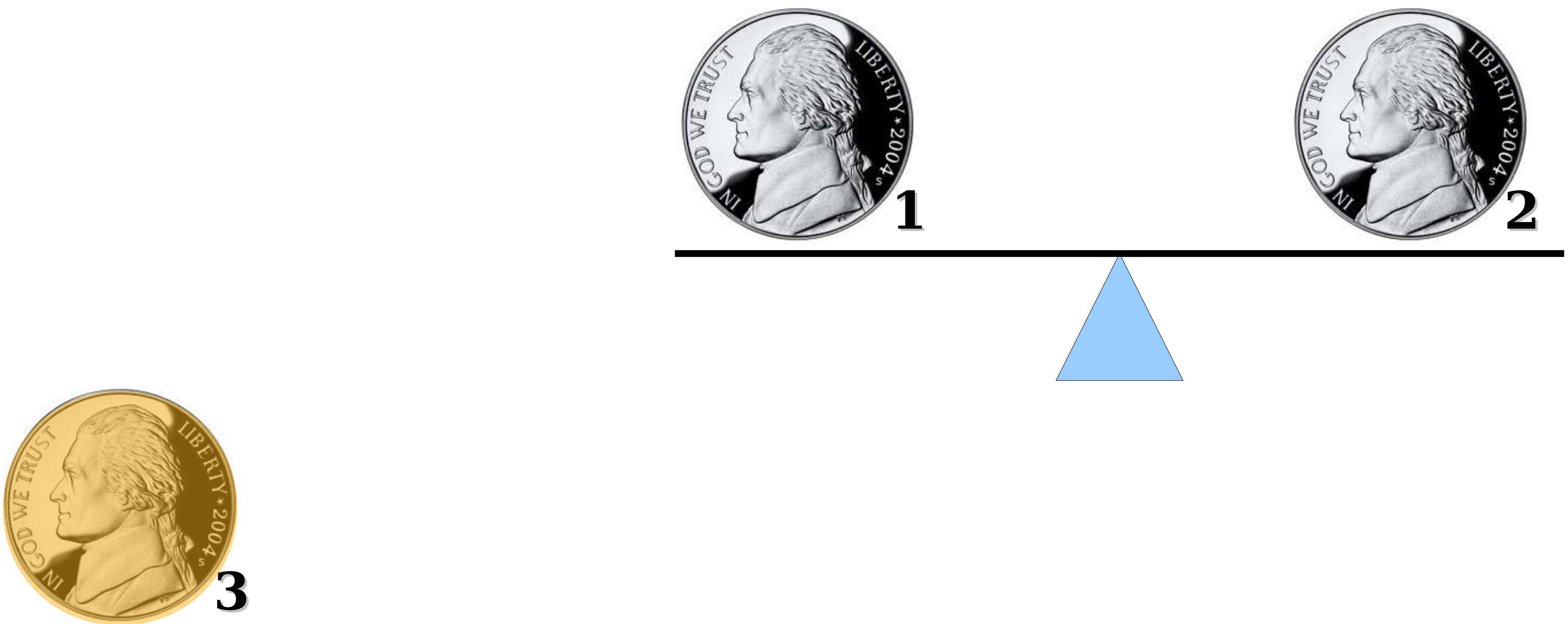
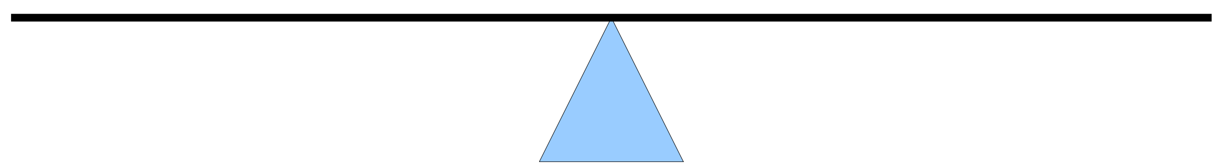


A Harder Problem

- You are given a set of ***nine*** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only ***two*** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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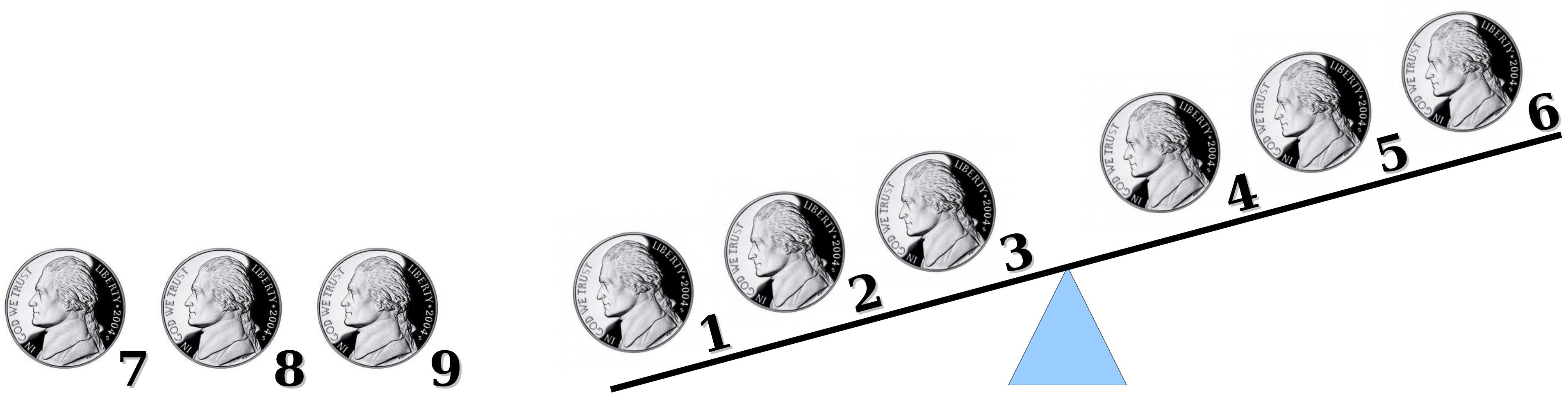

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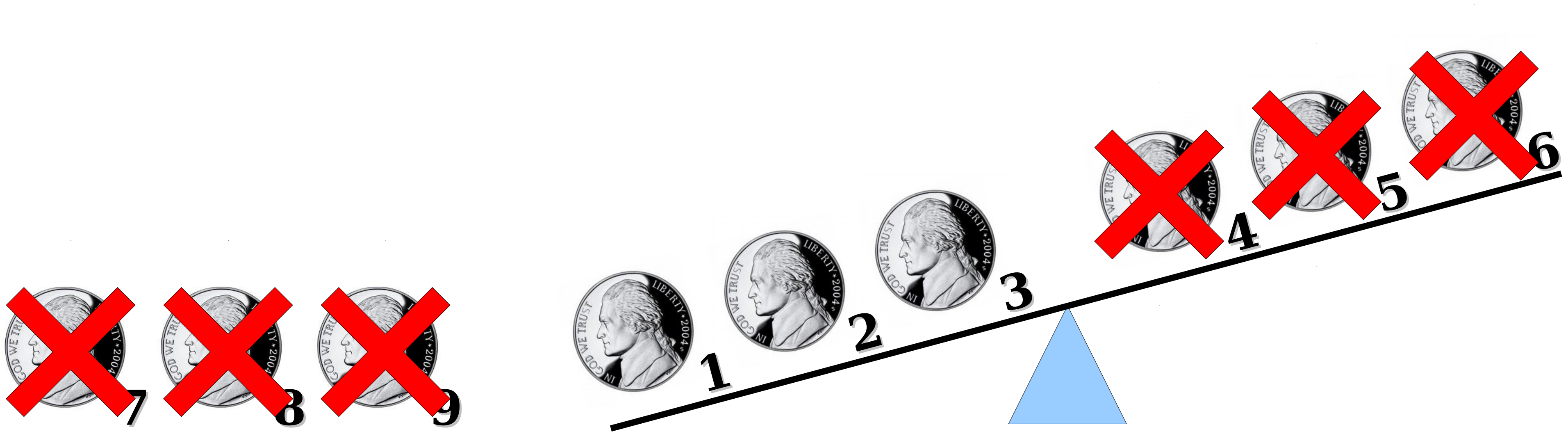

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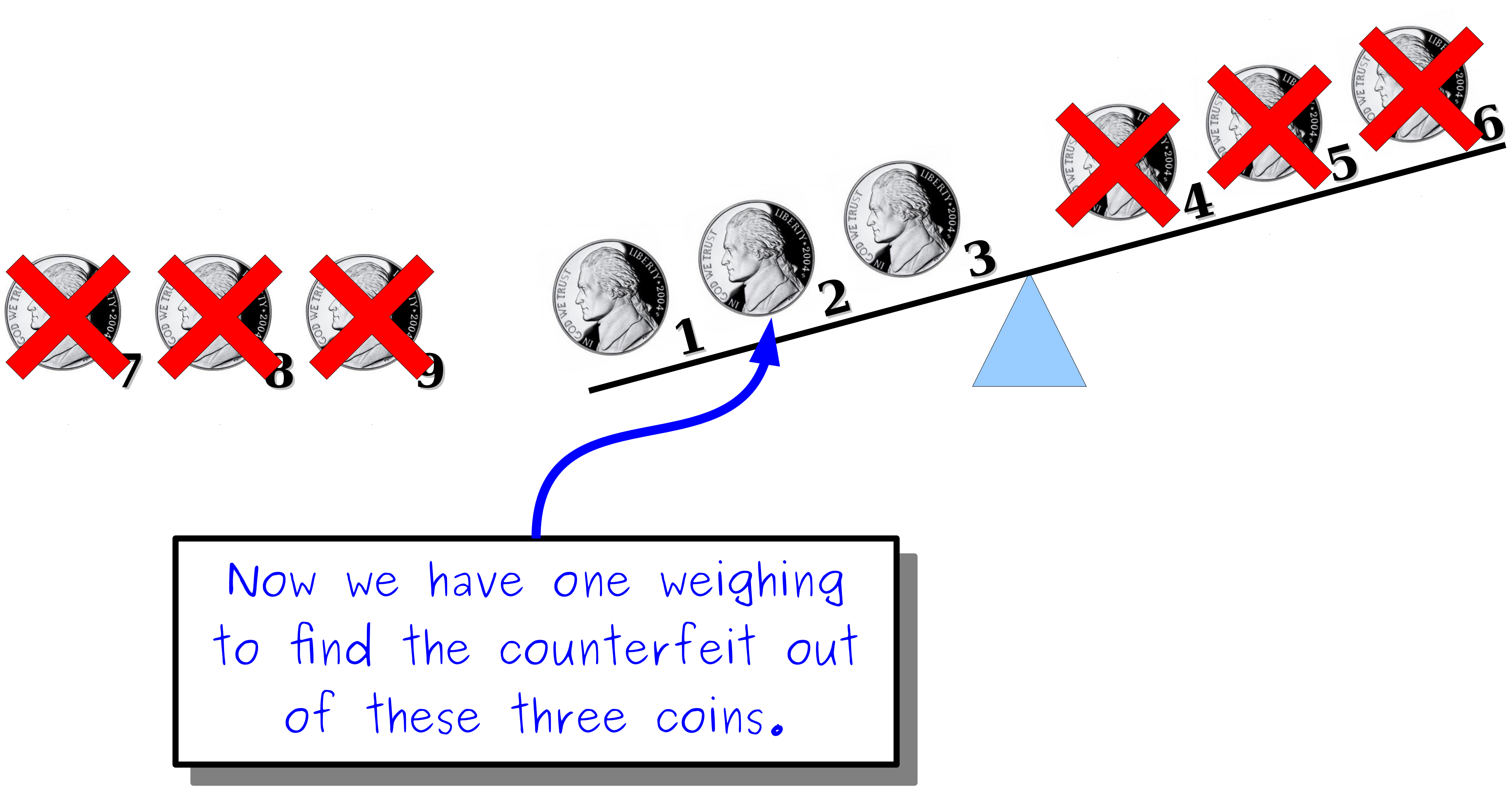

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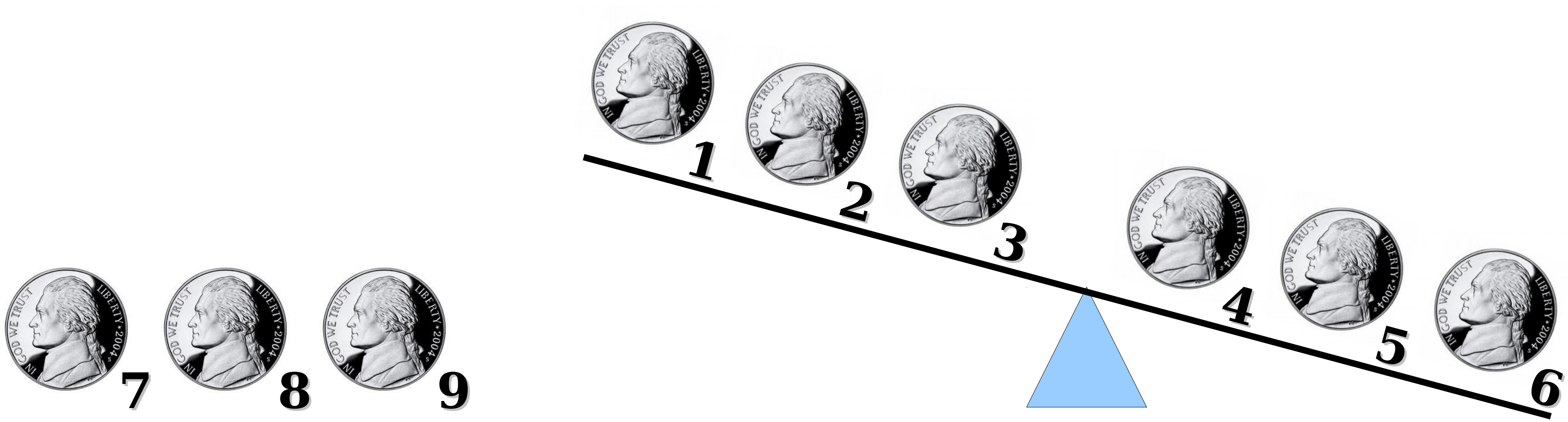

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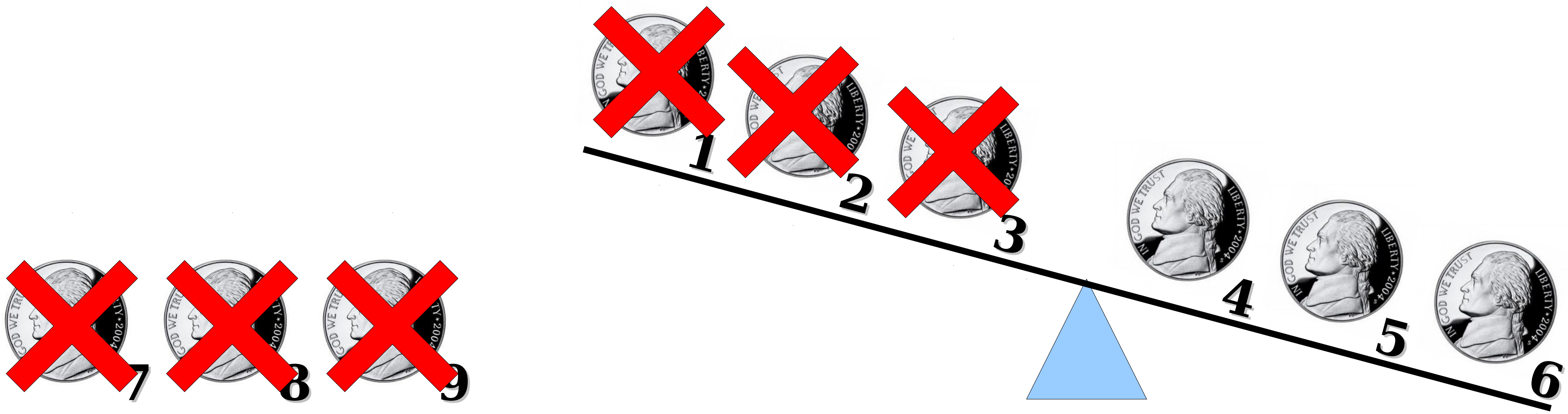

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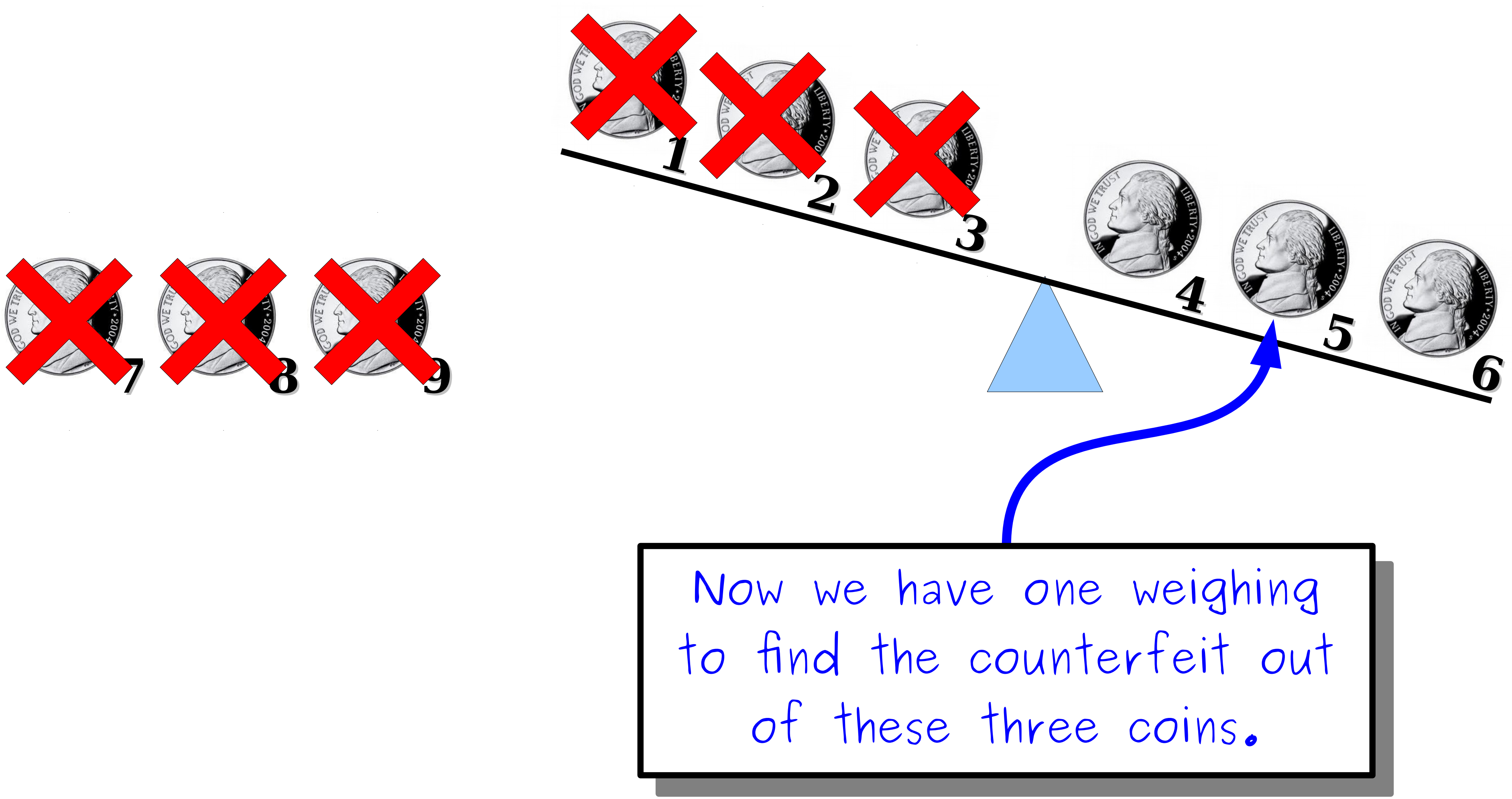

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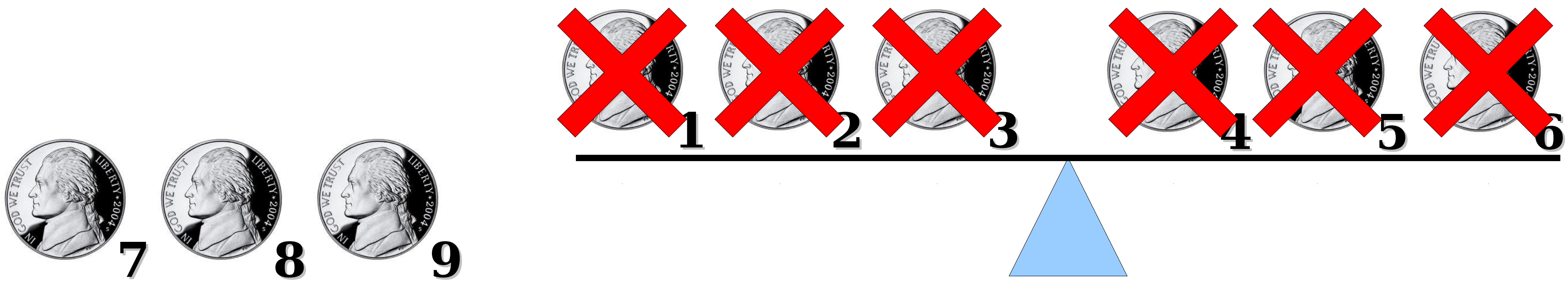

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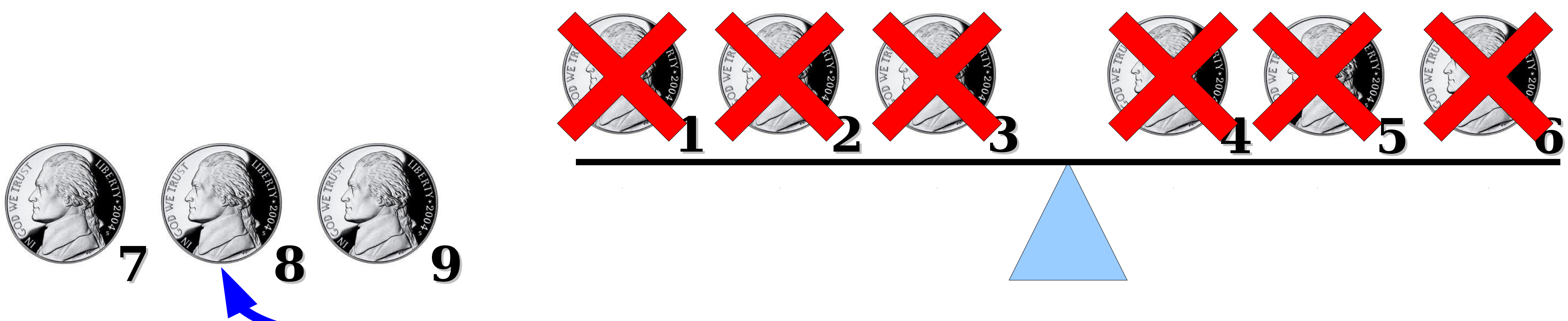

Now we have one weighing to find the counterfeit out of these three coins.

Can we generalize this?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - ***One*** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of ***three*** coins.
- If we have two weighings, we can find the counterfeit out of ***nine*** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

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At the start of the proof, we tell the reader what property we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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In a proof by induction, we need to prove that

- ☐ $P(0)$ is true
- ☐ If $P(k)$ is true, then $P(k+1)$ is true.

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As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0=1$ coins with one coin heavier than the rest, we can find that coin with zero weighings.

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Here, we state what P(0) actually says. Now, can go prove this using any proof techniques we'd like!

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For the inductive step, suppose $P(k)$ is true for some arbitrary $k \in \mathbb{N}$, so we can find the heavier of 3^k coins in k weighings.

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin.

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> Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

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Some Fun Problems

- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get k weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have k weighings?
  - Can you find the counterfeit out of a group of more than 3^k coins with k weighings?
  - Can you find the counterfeit out of any group of at most 3^k coins with k weighings?

Time-Out for Announcements!

CS+SOCIAL GOOD

SPRING MIXER

WEDNESDAY, MAY 3 | 6:00-7:30 PM

MEYER GREEN

Come relax, eat food, and talk about social good.

***There will be music, macaroons,
Pinkberry, Sprinkles, and a photo booth!***

Limited seats;
apply here!

Problem Set Two: A Common Mistake

Theorem: If S is a hereditary set, then $\wp(S)$ is also a hereditary set.

Proof: Let S be a hereditary set. We need to prove that $\wp(S)$ is also a hereditary set.

$\wp(S)$ is the set of all subsets of S, or $\wp(S) = \{ T \mid T \subseteq S \}$. Some set A is a subset of a set B if every element of A is an element of B.

Pick any subset of S. All elements of that subset are elements of S because of the definition of a subset. All elements of S are hereditary sets because S is hereditary. This means that every element of every subset of S is a hereditary set, so every subset of S is a hereditary set, so $\wp(S)$ is hereditary. ■

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1. Don't ***state*** definitions; ***use*** them instead.

2. Give names to quantities you manipulate.

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Midterm Exam

- The first midterm exam is ***tomorrow, May 2nd***, from ***7:00PM – 10:00PM***. Locations are divvied up by last (family) name:
  - `Abb – Niu`: Go to Hewlett 200.
  - `Nor – Vas`: Go to Hewlett 201.
  - `Vil – Yim`: Go to Hewlett 102.
  - `You – Zuc`: Go to Hewlett 103.
- You're responsible for Lectures 00 – 05 and topics covered in PS1 – PS2. Later lectures and problem sets won't be tested.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.

A Recommended Pre-Midterm Video:

https://youtu.be/Xs9aGVUZ3YA

Three Questions

- What is something you know right now that, at the start of the quarter, you knew you didn’t know?
- What is something you know right now that, at the start of the quarter, you *didn’t* know you didn’t know?
- What is something you *don’t* know right now that, at the start of the quarter, you *didn’t* know you didn’t know?

Your Questions

“How will this class help us to get jobs or internships. I feel as if it is too theoretical and will not have real world applications if research isnt your thing”

I have a lot I'd like to say on this topic. Let me start with an email I got in Fall quarter…

Hey Keith!

I hope you're doing well! I just wanted to let you know something cool I was thinking about. I was in an interview with Salesforce yesterday, and they wanted me to write a program to identify all the words in a file that rhymed.

I started thinking about it and realized that rhyming was just an equivalence relation - so I could basically just build a dictionary where each key was one element of an equivalence class and the values were members of that class. It was really awesome to see this in real life!

Thanks again,
[signature]

The best engineers I know are the ones who are able to find the right model, or the right abstraction, or the right way of thinking about the problem they're trying to solve.

Back to CS103!

How Not To Induct

Something's Wrong...

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Theorem: The sum of the first n powers of two is 2^n.

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make sure to prove the base case!

Otherwise, your argument is invalid!

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$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) \\ &= 2^{k+1}. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

Theorem: The sum of the first n powers of two is 2^n.

Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is 2^n." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

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$$2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k$$

$$2^k + 2^k \qquad \text{(via (1))}$$

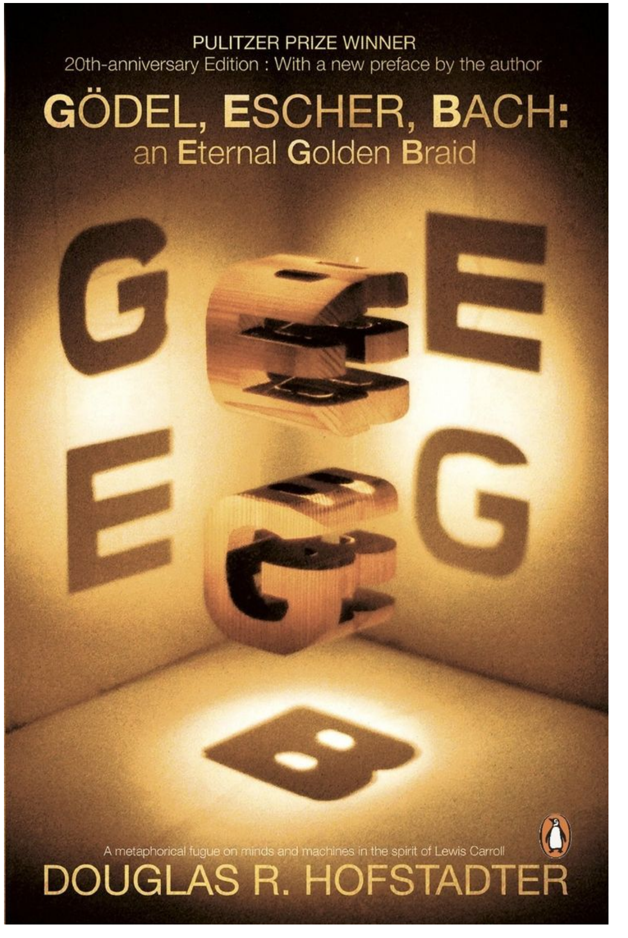
Therefore, $P(k + 1)$

You can prove **anything** from a faulty assumption. This is called the **principle of explosion.**

The `MU` Puzzle

Gödel, Escher Bach: An Eternal Golden Braid

- Pulitzer-Prize winning book exploring recursion, computability, and consciousness.
- Written by Douglas Hofstadter, cognitive scientist at Indiana University.
- A great (but dense!) read.



The `MU` Puzzle

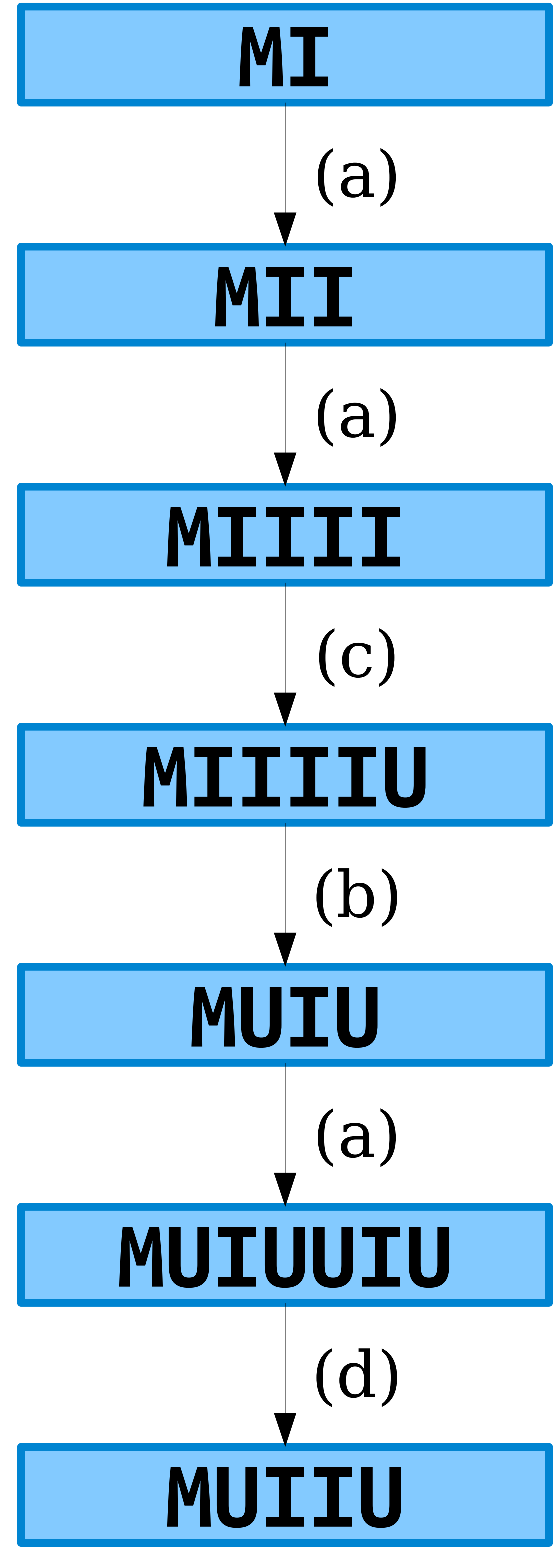
- Begin with the string `MI`.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the `M`: for example, `MIIU` becomes `MIIUIIU`, or `MI` becomes `MII`.
  - Replace `III` with `U`: `MIIII` becomes `MUI` or `MIU`.
  - Append `U` to the string if it ends in `I`: `MI` becomes `MIU`.
  - Remove any `UU`: `MUUU` becomes `MU`.
- ***Question***: How do you transform `MI` to `MU`?

(a) Double the string after an `M`.

(b) Replace `III` with `U`.

(c) Append `U`, if the string ends in `I`.

(d) Delete `UU` from the string.



Try It!

Starting with `MI`, apply these operations to make `MU`:

(a) Double the string after an `M`.

(b) Replace `III` with `U`.

(c) Append `U`, if the string ends in `I`.

(d) Delete `UU` from the string.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting **I**'s


MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

The Key Insight

- Initially, the number of **`I`**'s is *not* a multiple of three.
- To make **`MU`**, the number of **`I`**'s must end up as a multiple of three.
- Can we *ever* make the number of **`I`**'s a multiple of three?

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

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Proof: By contrapositive; we'll prove that if $n – 3$ is a multiple of three, then n is also a multiple of three. Because $n – 3$ is a multiple of three, we can write $n – 3 = 3k$ for some integer k. Then $n = 3(k+1)$, so n is also a multiple of three, as required. ■

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Proof: Let n be a number that isn't a multiple of three. If n is congruent to one modulo three, then $n = 3k + 1$ for some integer k. This means $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$, so $2n$ is not a multiple of three. Otherwise, n must be congruent to two modulo three, so $n = 3k + 2$ for some integer k. Then $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$, and so $2n$ is not a multiple of three. ■

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Theorem: The `MU` puzzle has no solution.

Proof: Assume for the sake of contradiction that the `MU` puzzle has a solution and that we can convert `MI` to `MU`. This would mean that at the very end, the number of `I`'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of `I`'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the `MU` puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a ***loop invariant***.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Next Time

- ***Variations on Induction***
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.