

# Complexity Theory

## Part One

It may be that since one is customarily concerned with existence, [...] **decidability**, and so forth, one is not inclined to take seriously the question of the existence of a *better-than-decidable* algorithm.

- Jack Edmonds, "Paths, Trees, and Flowers"

# A Decidable Problem

- **Presburger arithmetic** is a logical system for reasoning about arithmetic.
  - $\forall x. x + 1 \neq 0$
  - $\forall x. \forall y. (x + 1 = y + 1 \rightarrow x = y)$
  - $\forall x. x + 0 = x$
  - $\forall x. \forall y. (x + y) + 1 = x + (y + 1)$
  - $(P(0) \wedge \forall y. (P(y) \rightarrow P(y + 1))) \rightarrow \forall x. P(x)$
- Given a statement, it is decidable whether that statement can be proven from the laws of Presburger arithmetic.
- Any Turing machine that decides whether a statement in Presburger arithmetic is true or false has to move its tape head at least  $2^{2^{cn}}$  times on some inputs of length  $n$  (for some fixed constant  $c$ ).

# For Reference

- Assume  $c = 1$ .

$$2^{2^0} = 2$$

$$2^{2^1} = 4$$

$$2^{2^2} = 16$$

$$2^{2^3} = 256$$

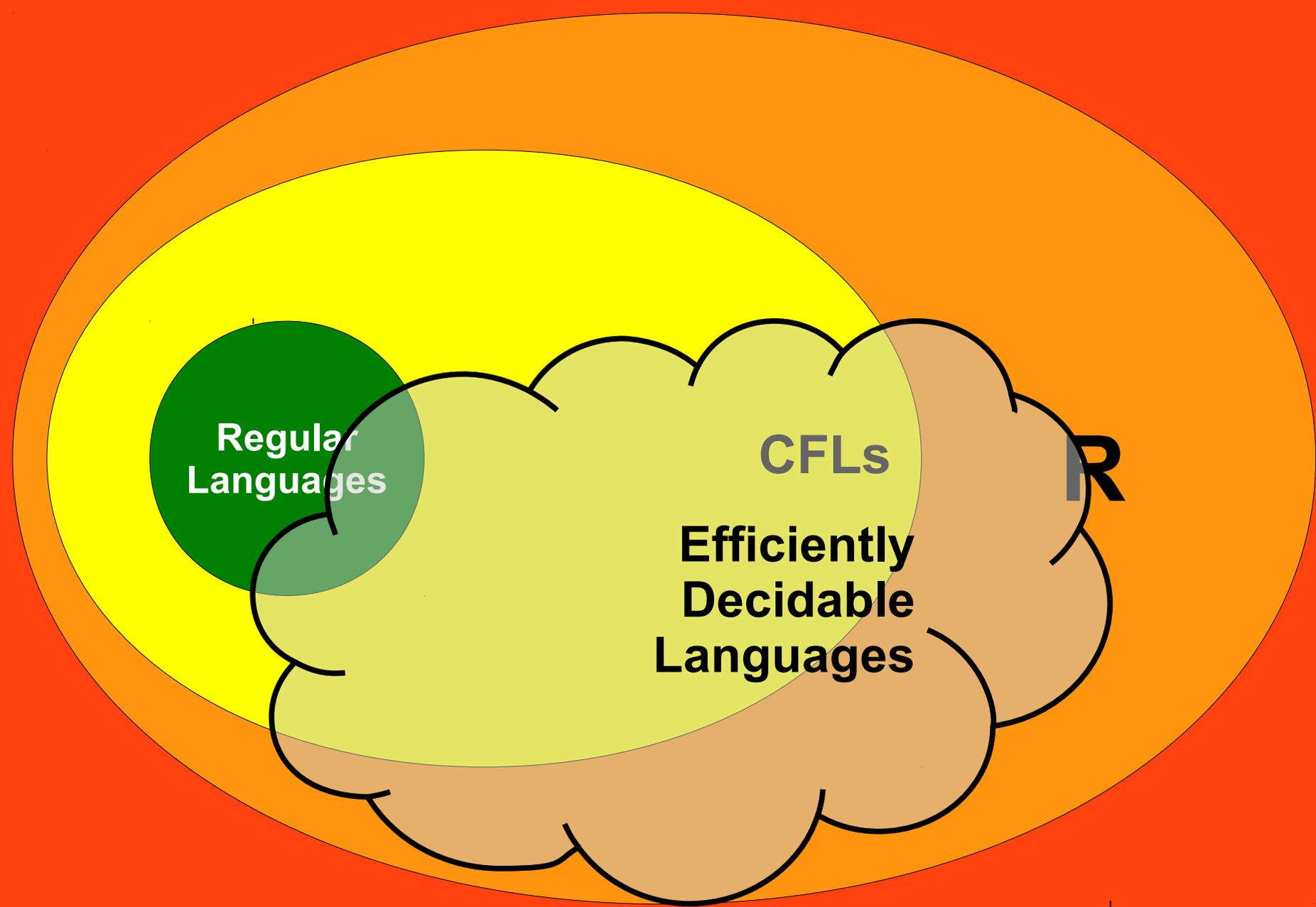
$$2^{2^4} = 65536$$

$$2^{2^5} = 18446744073709551616$$

$$2^{2^6} = 340282366920938463463374607431768211456$$

# The Limits of Decidability

- The fact that a problem is decidable does not mean that it is *feasibly* decidable.
- In ***computability theory***, we ask the question  
What problems can be solved by a computer?
- In ***complexity theory***, we ask the question  
What problems can be solved ***efficiently*** by a computer?
- In the remainder of this course, we will explore this question in more detail.



**Regular Languages**

**CFLs**

**Efficiently Decidable Languages**

**Undecidable Languages**

**R**

# The Setup

- In order to study computability, we needed to answer these questions:
  - What is “computation?”
  - What is a “problem?”
  - What does it mean to “solve” a problem?
- To study complexity, we need to answer these questions:
  - What does “complexity” even mean?
  - What is an “efficient” solution to a problem?

# Measuring Complexity

- Suppose that we have a decider  $D$  for some language  $L$ .
- How might we measure the complexity of  $D$ ?

Number of states.

Size of tape alphabet.

Size of input alphabet.

Amount of tape required.

- **Amount of time required.**

Number of times a given state is entered.

Number of times a given symbol is printed.

Number of times a given transition is taken.

(Plus a whole lot more...)

What is an efficient algorithm?

# Searching Finite Spaces

- Many decidable problems can be solved by searching over a large but finite space of possible options.
- Searching this space might take a staggeringly long time, but only finite time.
- From a decidability perspective, this is totally fine.
- From a complexity perspective, this may be totally unacceptable.

# A Sample Problem

4	3	11	9	7	13	5	6	1	12	2	8	0	10
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Goal: Find the length of the longest increasing subsequence of this sequence.

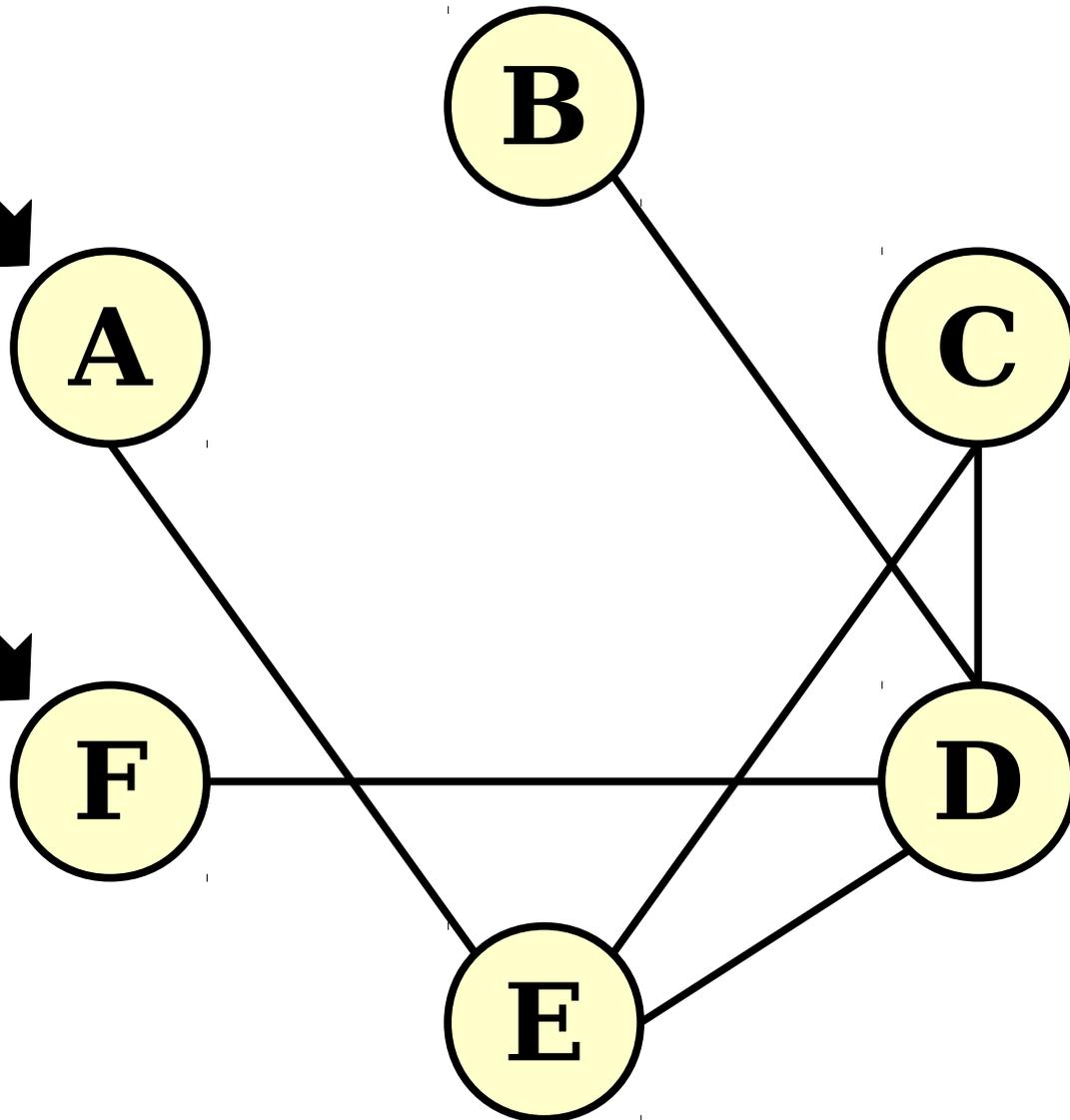
# Longest Increasing Subsequences

- ***One possible algorithm:*** try all subsequences, find the longest one that's increasing, and return that.
- There are  $2^n$  subsequences of an array of length  $n$ .
  - (Each subset of the elements gives back a subsequence.)
- Checking all of them to find the longest increasing subsequence will take time  $O(n \cdot 2^n)$ .
- Nifty fact: the age of the universe is about  $4.3 \times 10^{26}$  nanoseconds old. That's about  $2^{85}$  nanoseconds.
- Practically speaking, this algorithm doesn't terminate if you give it an input of size 100 or more.

# Longest Increasing Subsequences

- ***Theorem:*** There is an algorithm that can find the longest increasing subsequence of an array in time  $O(n \log n)$ .
- The algorithm is *beautiful* and surprisingly elegant. Look up ***patience sorting*** if you're curious.
- This algorithm works by exploiting particular aspects of how longest increasing subsequences are constructed. It's not immediately obvious that it works correctly.

# Another Problem



Goal: Determine the length of the shortest path from **A** to **F** in this graph.

# Shortest Paths

- It is possible to find the shortest path in a graph by listing off all sequences of nodes in the graph in ascending order of length and finding the first that's a path.
- This takes time  $O(n \cdot n!)$  in an  $n$ -node graph.
- For reference:  $29!$  nanoseconds is longer than the lifetime of the universe.

# Shortest Paths

- ***Theorem:*** It's possible to find the shortest path between two nodes in an  $n$ -node,  $m$ -edge graph in time  $O(m + n)$ .
- ***Proof idea:*** Use breadth-first search!
- The algorithm is a bit nuanced. It uses some specific properties of shortest paths and the proof of correctness is nontrivial.

# For Comparison

- ***Longest increasing subsequence:***
  - Naive:  $O(n \cdot 2^n)$
  - Fast:  $O(n^2)$
- ***Shortest path problem:***
  - Naive:  $O(n \cdot n!)$
  - Fast:  $O(n + m)$ .

# Defining Efficiency

- When dealing with problems that search for the “best” object of some sort, there are often at least exponentially many possible options.
- Brute-force solutions tend to take at least exponential time to complete.
- Clever algorithms often run in time  $O(n)$ , or  $O(n^2)$ , or  $O(n^3)$ , etc.

# Polynomials and Exponentials

- An algorithm runs in ***polynomial time*** if its runtime is some polynomial in  $n$ .
  - That is, time  $O(n^k)$  for some constant  $k$ .
- Polynomial functions “scale well.”
  - Small changes to the size of the input do not typically induce enormous changes to the overall runtime.
- Exponential functions scale terribly.
  - Small changes to the size of the input induce huge changes in the overall runtime.

# The Cobham-Edmonds Thesis

A language  $L$  can be ***decided efficiently*** if there is a TM that decides it in polynomial time.

Equivalently,  $L$  can be decided efficiently if it can be decided in time  $O(n^k)$  for some  $k \in \mathbb{N}$ .

Like the Church-Turing thesis, this is ***not*** a theorem!

It's an assumption about the nature of efficient computation, and it is somewhat controversial.

# The Cobham-Edmonds Thesis

- Efficient runtimes:
  - $4n + 13$
  - $n^3 - 2n^2 + 4n$
  - $n \log \log n$
- “Efficient” runtimes:
  - $n^{1,000,000,000,000}$
  - $10^{500}$
- Inefficient runtimes:
  - $2^n$
  - $n!$
  - $n^n$
- “Inefficient” runtimes:
  - $n^{0.0001 \log n}$
  - $1.0000000001^n$

# Why Polynomials?

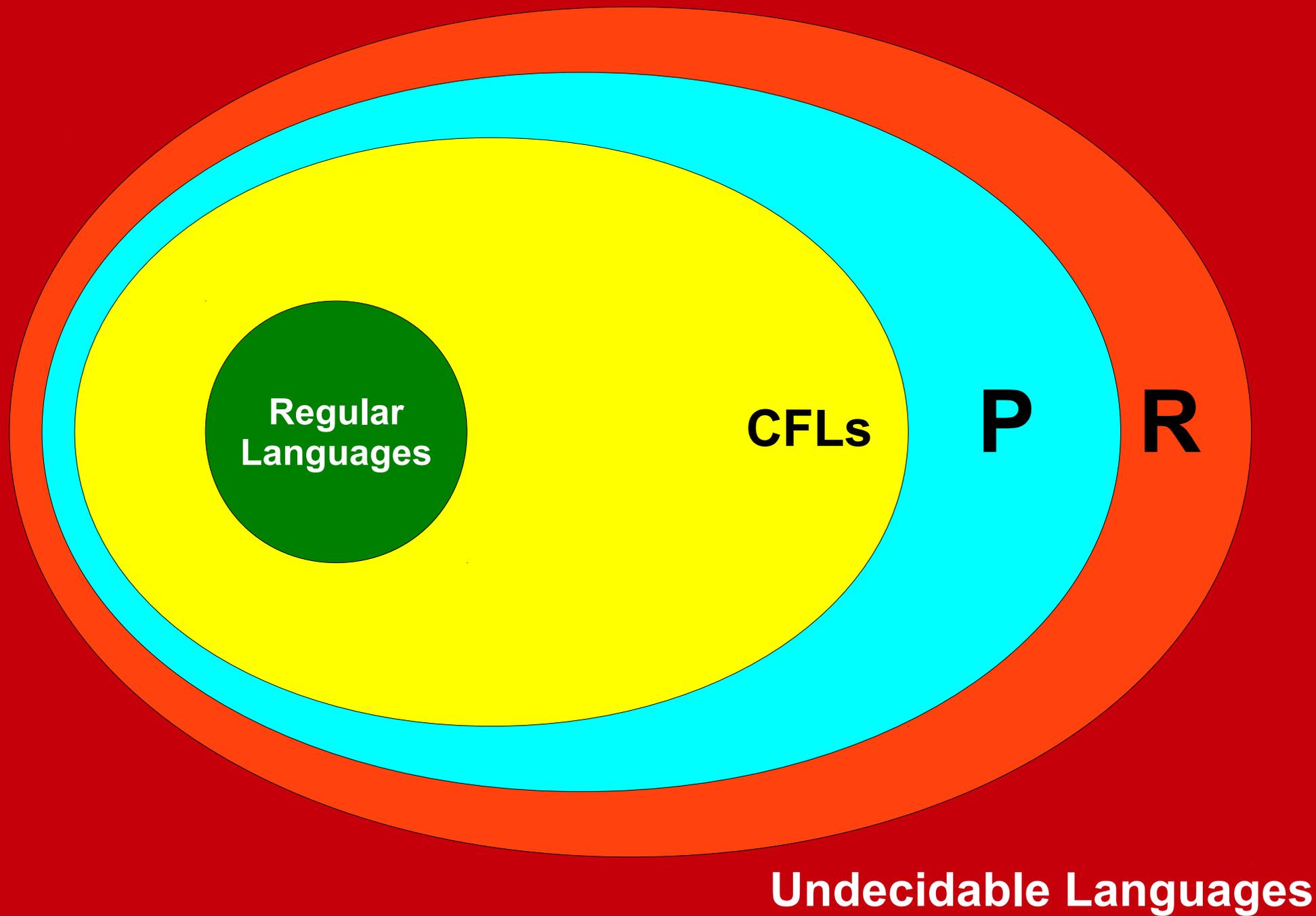
- Polynomial time *somewhat* captures efficient computation, but has a few edge cases.
- However, polynomials have very nice mathematical properties:
  - The sum of two polynomials is a polynomial. (Running one efficient algorithm after the other gives an efficient algorithm.)
  - The product of two polynomials is a polynomial. (Running one efficient algorithm a “reasonable” number of times gives an efficient algorithm.)
  - The *composition* of two polynomials is a polynomial. (Using the output of one efficient algorithm as the input to another efficient algorithm gives an efficient algorithm.)

# The Complexity Class **P**

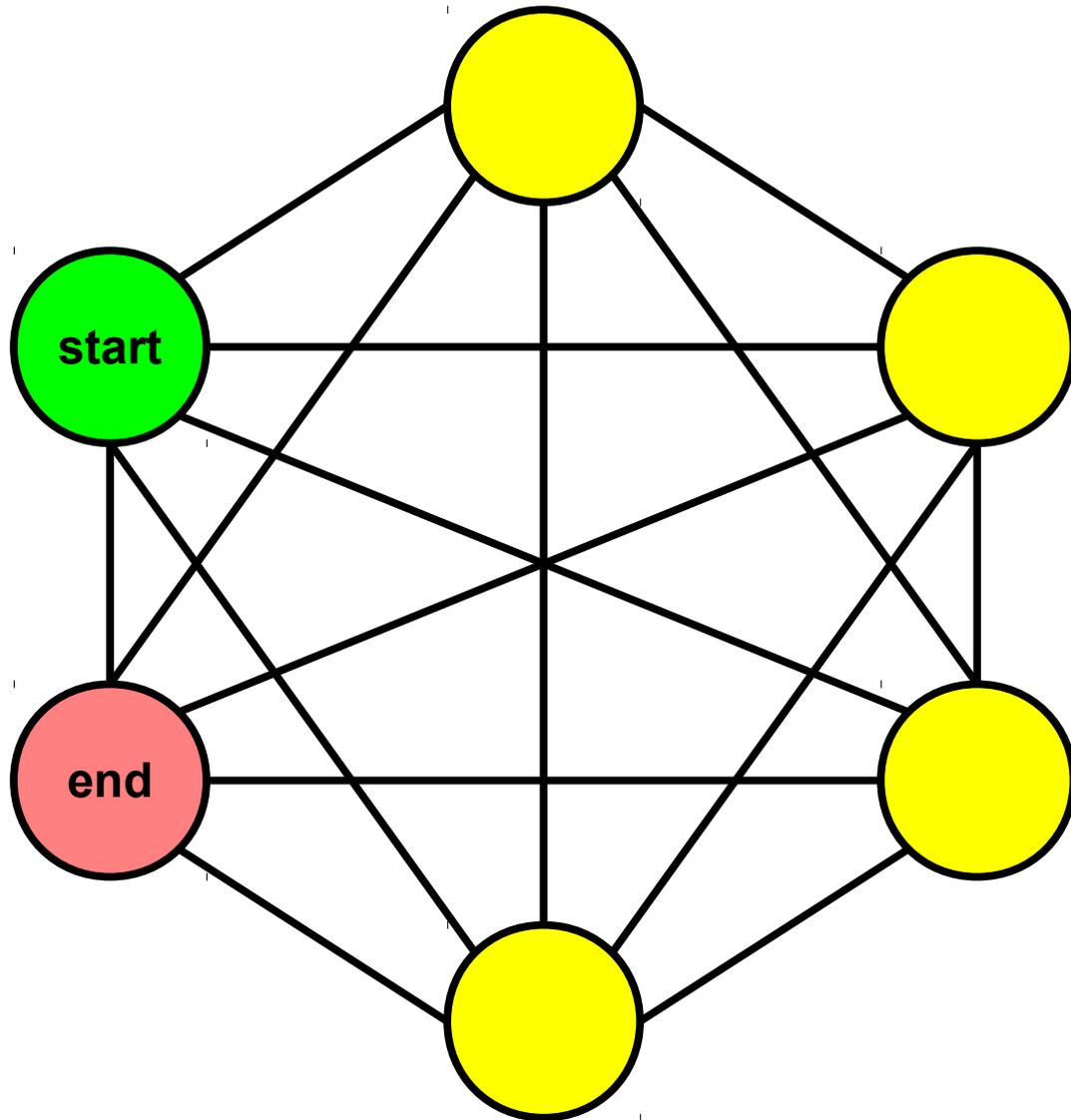
- The ***complexity class P*** (for *p*olynomial time) contains all problems that can be solved in polynomial time.
- Formally:  
$$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$$
- Assuming the Cobham-Edmonds thesis, a language is in **P** if it can be decided efficiently.

# Examples of Problems in **P**

- All regular languages are in **P**.
  - All have linear-time TMs.
- All CFLs are in **P**.
  - Requires a more nuanced argument (the *CYK algorithm* or *Earley's algorithm*.)
- And a *ton* of other problems are in **P** as well.
  - Curious? Take CS161!



What *can't* you do in polynomial time?



How many simple paths are there from the start node to the end node?



How many  
subsets of this  
set are there?

# An Interesting Observation

- There are (at least) exponentially many objects of each of the preceding types.
- However, each of those objects is not very large.
  - Each simple path has length no longer than the number of nodes in the graph.
  - Each subset of a set has no more elements than the original set.
- This brings us to our next topic...

What if you need to search a large space for a single object?

# Verifiers - Again

2	5	7	9	6	4	1	8	3
4	9	1	8	7	3	6	5	2
3	8	6	1	2	5	9	4	7
6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
1	6	3	2	5	7	4	9	8
5	7	8	4	9	6	2	3	1
9	2	4	3	8	1	7	6	5

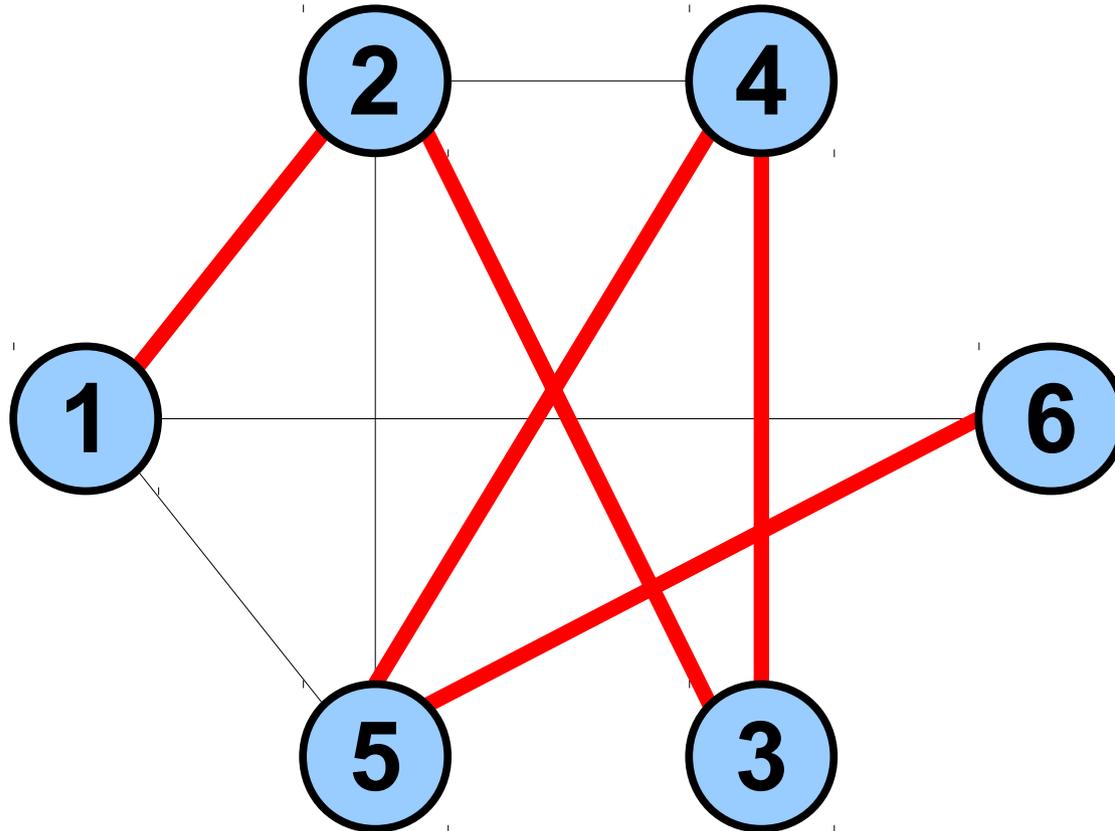
Does this Sudoku problem  
have a solution?

# Verifiers - Again

9	3	11	4	2	13	5	6	1	12	7	8	0	10
---	---	----	---	---	----	---	---	---	----	---	---	---	----

Is there an ascending subsequence of length at least 7?

# Verifiers - Again



Is there a simple path that goes through every node exactly once?

# Polynomial-Time Verifiers

- A ***polynomial-time verifier*** for  $L$  is a TM  $V$  such that
  - $V$  halts on all inputs.
  - $w \in L$  iff  $\exists c \in \Sigma^*. V$  accepts  $\langle w, c \rangle$ .
  - $V$ 's runtime is a polynomial in  $|w|$  (that is,  $V$ 's runtime is  $O(|w|^k)$  for some integer  $k$ )

# The Complexity Class **NP**

- The complexity class **NP** (*nondeterministic polynomial time*) contains all problems that can be verified in polynomial time.
- Formally:

$$\mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$

- The name **NP** comes from another way of characterizing **NP**. If you introduce *nondeterministic Turing machines* and appropriately define “polynomial time,” then **NP** is the set of problems that an NTM can solve in polynomial time.

And now...

The

***Most Important Question***

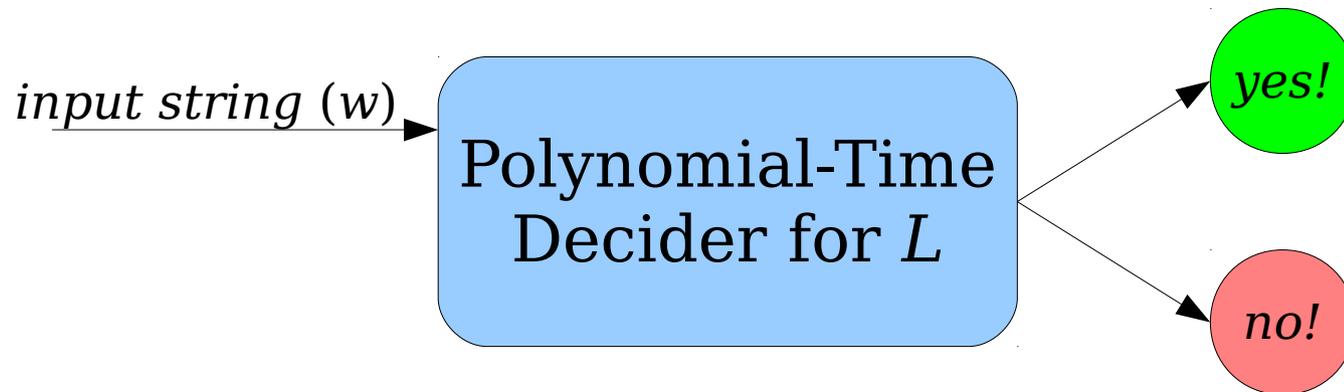
in

***Theoretical Computer Science***

What is the connection between **P** and **NP**?

**P** = {  $L$  | There is a polynomial-time decider for  $L$  }

**NP** = {  $L$  | There is a polynomial-time verifier for  $L$  }



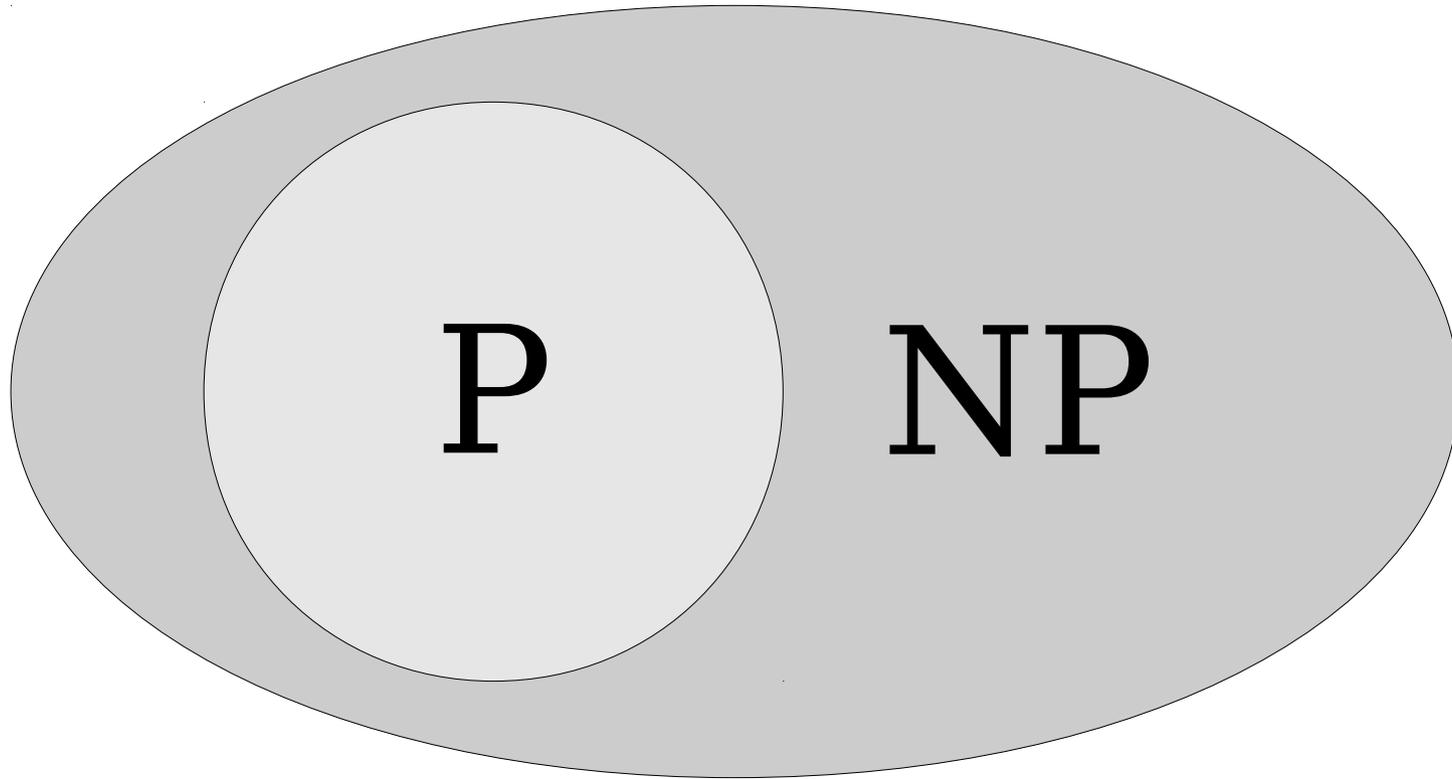
$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$

$\mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$

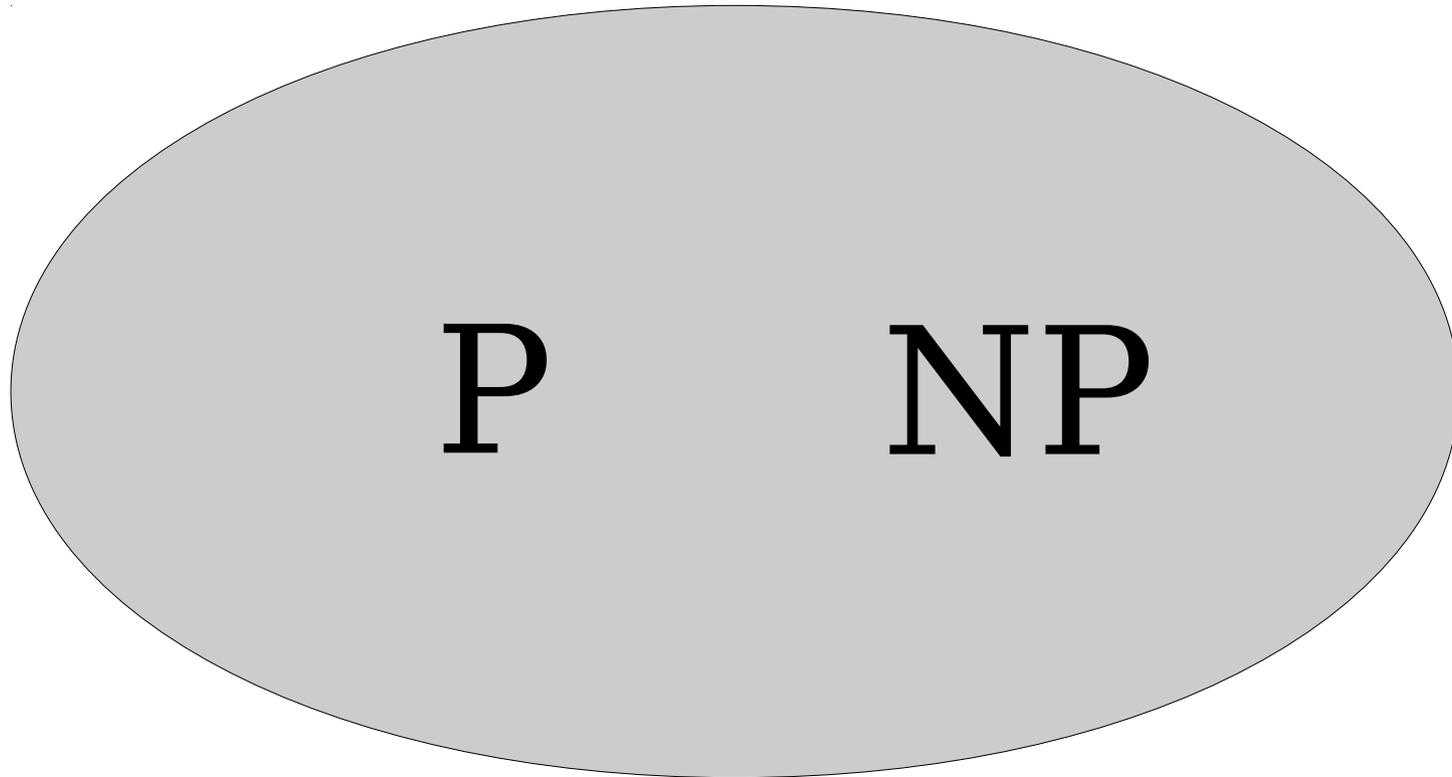


$\mathbf{P} \subseteq \mathbf{NP}$

# Which Picture is Correct?



# Which Picture is Correct?



Does **P** = **NP**?

# $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$

- The  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$  question is the most important question in theoretical computer science.
- With the verifier definition of  $\mathbf{NP}$ , one way of phrasing this question is

*If a solution to a problem can be **checked** efficiently, can that problem be **solved** efficiently?*
- An answer either way will give fundamental insights into the nature of computation.

# Why This Matters

- The following problems are known to be efficiently verifiable, but have no known efficient solutions:
  - Determining whether an electrical grid can be built to link up some number of houses for some price (Steiner tree problem).
  - Determining whether a simple DNA strand exists that multiple gene sequences could be a part of (shortest common supersequence).
  - Determining the best way to assign hardware resources in a compiler (optimal register allocation).
  - Determining the best way to distribute tasks to multiple workers to minimize completion time (job scheduling).
  - *And many more.*
- If  $P = NP$ , *all* of these problems have efficient solutions.
- If  $P \neq NP$ , *none* of these problems have efficient solutions.

# Why This Matters

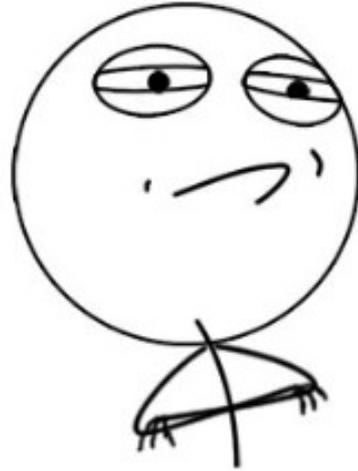
- If **P = NP**:
  - A huge number of seemingly difficult problems could be solved efficiently.
  - Our capacity to solve many problems will scale well with the size of the problems we want to solve.
- If **P  $\neq$  NP**:
  - Enormous computational power would be required to solve many seemingly easy tasks.
  - Our capacity to solve problems will fail to keep up with our curiosity.

# What We Know

- Resolving  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$  has proven *extremely difficult*.
- In the past 45 years:
  - Not a single correct proof either way has been found.
  - Many types of proofs have been shown to be insufficiently powerful to determine whether  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ .
  - A majority of computer scientists believe  $\mathbf{P} \neq \mathbf{NP}$ , but this isn't a large majority.
- Interesting read: Interviews with leading thinkers about  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ :
  - <http://web.eng.puc.cl/~jabaier/iic2212/poll-1.pdf>

# The Million-Dollar Question

**CHALLENGE ACCEPTED**



The Clay Mathematics Institute has offered a ***\$1,000,000 prize*** to anyone who proves or disproves  **$P = NP$** .

**Time-Out for Announcements!**

***Please evaluate this course in Axess.***  
Your comments really make a difference.

# Problem Set Nine

- Problem Set Nine is due this Friday at 2:30PM.
  - As a reminder, ***no late submissions will be accepted***. Please budget enough time to get your submission in!
  - ***Very smart idea***: submit at least three hours early.
- As always, feel free to ask questions in office hours or online via Piazza.

# Final Exam Logistics

- Our final exam is next Monday, December 11<sup>th</sup> from 3:30PM – 6:30PM. Locations are divvied up by last (family) name:
  - Abb – Ngu: Go to ***Cubberley Auditorium***.
  - Ogr – Zwa: Go to ***Cemex Auditorium***.
- The final exam is cumulative and covers topics from PS1 – PS9 and all lectures. Topics from this week are fair game but will be deemphasized.
- The exam is closed-computer, closed-book, and limited-note. You may bring one double-sided 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.
- Students with OAE accommodations: please contact us no later than tonight if you have not yet done so.

# Preparing for the Final

- On the course website you'll find
  - *five* practice final exams, which are all real exams with minor modifications, with solutions, and
  - a giant set of 45 practice problems (EPP3), with solutions.
- Our recommendation: Look back over the exams and problem sets and redo any problems that you didn't really get the first time around.
- Keep the TAs in the loop: stop by office hours to have them review your answers and offer feedback.

Your Questions

# “Motivation dying as finals approach? Any suggestions?”

For starters, hang in there! This is the home stretch!

It's hard to remain motivated about something that you're not inherently interested in. If there are any classes that you just feel totally burnt out in, stop by the prof's office hours and politely ask them what it is about the material that excites them or how it gets used downstream.

Alternatively - if you're physically worn out (low sleep, not eating well, not exercising, etc.), it is may be wise to take an evening to recharge and not focus on work. You'll make up for the time not spent working through the huge efficiency gains you'll get.

“Is it an **R** or a **R**? Example: “Prove that for every string  $w$ , there's (a OR an) **R** language containing  $w$  and an **RE** language containing  $w$ .””

I always thought the rule was to pronounce things out loud and see whether you get a vowel sound in the course of doing so. So I assume that it's probably “an R language” because it's read out loud as “an arrrrr language.” I did some Google searches on this one and the top hits seemed to suggest this was correct, but I'm not 100% sure!

“Can you give a pep talk to the women for why we should stick with this despite the upset partners who come onto us, having to prove ourselves constantly, etc.?”

I have a lot to say on this subject. I'll take this one in class.

Back to CS103!

What do we know about  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ ?

# Adapting our Techniques

# A Problem

- The **R** and **RE** languages correspond to problems that can be decided and verified, *period*, without any time bounds.
- To reason about what's in **R** and what's in **RE**, we used two key techniques:
  - **Universality**: TMs can run other TMs as subroutines.
  - **Self-Reference**: TMs can get their own source code.
- Why can't we just do that for **P** and **NP**?

***Theorem (Baker-Gill-Solovay):*** Any proof that purely relies on universality and self-reference cannot resolve  $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ .

***Proof:*** Take CS154!

So how *are* we going to  
reason about **P** and **NP**?

# Next Time

- ***Reducibility***
  - A technique for connecting problems to one another.
- ***NP-Completeness***
  - What are the hardest problems in **NP**?