

Practice Midterm Exam 4

This exam is closed-book and closed-computer. You may have a double-sided, 8.5" × 11" sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 32 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

You have three hours to complete this exam. There are 48 total points.

Question	Points	Graders
(1) Mathematical Logic	/ 12	
(2) Set Theory	/ 12	
(3) Proofwriting I	/ 12	
(4) Proofwriting II	/ 12	
	/ 48	

You can do this. Best of luck on the exam!

Problem One: Mathematical Logic**(12 Points)***(CS103 Midterm, Fall 2017)*

When we discussed first-order logic, we spent a decent amount of time talking about how to translate statements from English into first-order logic. You practiced this skill on Problem Set Two. In this problem, we'd like you to show us what you've learned about the art of first-order translation.

Leo Tolstoy's classic novel *Anna Karenina* begins with this famous line:

“All happy families are alike; each unhappy family is unhappy in its own way.”

This question explores the first-order structure of this sentence.

- i. **(3 Points)** Given the predicates

Happy(x), which states that x is happy;

Family(x), which states that x is a family; and

Alike(x, y), which states that x and y are alike,

translate the statement “all happy families are alike” into first-order logic. (You'll translate the second half of the opening line of *Anna Karenina* in the next part of this problem.)

- ii. **(5 Points)** Given the predicates

UnhappyFamily(x), which states that x is an unhappy family;

Reason(x), which states that x is a reason; and

UnhappyBecause(x, y), which states that x is unhappy because of y ,

translate the statement “each unhappy family is unhappy in its own way” into first-order logic. You should interpret the statement to mean “each unhappy family is unhappy for a reason, and no other unhappy families are unhappy for that same reason.”

We've talked a lot about negating and simplifying statements in first-order logic, a useful skill with applications to disproofs, proofs by contradiction, and proofs by contrapositive. You practiced this on Problem Set Two, and in this question we'd like you to demonstrate what you've learned.

iii. **(4 Points)** Consider the following statement in first-order logic:

$$(\forall x. (K(x) \leftrightarrow \forall z. \neg A(x, z))) \wedge (\forall x. (H(x) \rightarrow \exists y. (B(x, y) \wedge D(x, y))))$$

Give a statement in first-order logic that is the negation of this statement. As in Problem Set Two, your final formula must not have any negations in it, except for direct negations of predicates. Do not introduce any new predicates, functions, or constants.

While you're not required to show every step of your work, it does make it a bit easier for us to give you partial credit in the event that you accidentally make a mistake. If you do show each step, please clearly indicate what your final answer is.

As a courtesy to your TAs, please try to make the parenthesization of your final statement as clear as possible.

Problem Two: Set Theory**(12 Points)***(CS103 Midterm, Fall 2017)*

On Problem Set One, you wrote C++ code that took in a pair of objects S and T , then returned whether those objects had various properties. For example, you wrote code to test whether $S \in T$, whether $S \subseteq \wp(\wp(T))$, whether $S = \{T\}$, etc.

In this problem, we'd like you run this process in reverse. Below are five C++ functions that take as input some number of objects, then determine whether some property holds of them. For each of those functions, write a statement using set theory notation indicating what that function checks for. No justification is necessary.

To receive full credit, express your answers purely using set theory notation (\in , \subseteq , \cup , $=$, \wp , etc.) and without using any quantifiers or propositional connectives. For example, if a piece of code checked whether S was a subset of T , you should write $S \subseteq T$ rather than $\forall x. (x \in S \rightarrow x \in T)$. Answers using quantifiers or propositional connectives will only receive partial credit.

A syntax refresher:

- The function `isSet(S)` returns whether S is a set.
- The function `asSet(S)` returns a view of object S as a set.
- The notation `S.count(x)` returns whether $x \in S$.
- The notation `S.size()` returns $|S|$.
- The notation `for (Object x: S) { ... }` iterates over all elements of S , one at a time.

```
bool mysteryPredicateOne(Object S) { // 1 Point
    return isSet(S) && asSet(S).size() == 0;
}
```

What this code tests for: _____

```
bool mysteryPredicateTwo(Object S, Object T) { // 2 Points
    if (!isSet(S) || !isSet(T)) return false;

    for (Object x: asSet(S)) {
        if (asSet(T).count(x)) return true;
    }

    for (Object x: asSet(T)) {
        if (asSet(S).count(x)) return true;
    }

    return false;
}
```

What this code tests for: _____

(Continued on the next page)

```

bool mysteryPredicateThree(Object S, Object T) { // 3 Points
    if (!isSet(T)) return false;

    for (Object X: asSet(T)) {
        if (isSet(X) && asSet(X).size() == 1) {
            if (asSet(X).count(S)) return true;
        }
    }

    return false;
}

```

What this code tests for: _____

```

bool mysteryPredicateFour(Object S, Object T, Object U) { // 3 Points
    if (!isSet(S) || !isSet(T) || !isSet(U)) return false;

    for (Object x: asSet(S)) {
        if (!asSet(T).count(x)) return false; // Note the ! sign here.
        if ( asSet(U).count(x)) return false; // There is no ! sign here.
    }

    return true;
}

```

What this code tests for: _____

```

bool mysteryPredicateFive(Object S) { // 3 Points
    if (!isSet(S)) return false;

    for (Object T: asSet(S)) {
        if (!isSet(T)) return false;
        for (Object x: asSet(T)) {
            if (!asSet(S).count(x)) return false;
        }
    }

    return true;
}

```

What this code tests for: _____

Problem Three: Proofwriting I**(12 Points)***(CS103 Midterm, Fall 2017)*

In our first week of class, we proved that $\sqrt{2}$ is irrational, and on Problem Set One, you proved that $\sqrt{3}$ is irrational. This question will give you a chance to demonstrate what you've learned about indirect proof techniques and how to reason about irrational numbers.

This problem builds up to a proof that *infinitely many* natural numbers have irrational square roots, greatly expanding the scope of numbers that we've shown are irrational.

- i. **(4 Points)** In part (ii) of this problem, you'll need to make use of the following lemma:

Lemma: If m and n are integers where mn is even, then at least one of m and n is even.

Prove this lemma ***using a proof by contrapositive***. In the course of doing so, please call back to the formal definition of odd and even numbers as appropriate.

As a refresher, here's the lemma you proved on the previous page:

Lemma: If m and n are integers where mn is even, then at least one of m and n is even.

You'll now use this lemma to prove the following theorem:

Theorem: If n is an odd natural number, then $\sqrt{2n}$ is irrational.

ii. **(8 Points)** Prove this theorem *using a proof by contradiction*. Some hints:

- You may want to structure this proof along the lines of the proof that $\sqrt{2}$ is irrational.
- Remember that if r is a rational number, it's always possible to write r as p/q where p and q are integers, where $q \neq 0$, and where p and q have no common factors other than +1 and -1.

Feel free to use this page for scratch work. There's extra space on the next page to write your final answer.

(Extra space for your answer to Problem Three, if you need it.)

Problem Four: Proofwriting II**(12 Points)***(CS103 Midterm, Fall 2017)*

You now have quite a bit of experience writing proofs about sets. On Problem Set One, you practiced writing proofs and disproofs about properties of sets. On Problem Set Two, you explored hereditary sets and proved some results about them. This problem is designed to let you show us what you've learned about writing proofs pertaining to sets.

This particular problem concerns a special kind of a set called a *transitive set*. Formally speaking, we'll say that a set S is *transitive* if $S \subseteq \wp(S)$. These sets have all sorts of fun properties and show up in the study of the mathematical nature of infinity.

- i. **(4 Points)** Below are a list of sets. For each set, determine whether or not it's a transitive set. No justification is required, and there's no penalty for an incorrect guess.

$$\mathbb{N}$$
 Transitive

 Not Transitive

$$\emptyset$$
 Transitive

 Not Transitive

$$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$$
 Transitive

 Not Transitive

$$\{\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}$$
 Transitive

 Not Transitive

As a refresher from the previous page, we'll say that a set S is a *transitive set* if $S \subseteq \wp(S)$.

ii. **(8 Points)** Prove the following result using a *direct proof*:

If S is a transitive set, then $\wp(S)$ is also a transitive set.

As you write up your proof of this result, please make sure to call back to the formal definitions of the subset relation and the power set. Setting the proof up properly – clearly articulating your start and end points, introducing new variables as appropriate, etc. – will make things easier.

Feel free to use this page as scratch space if you'd like. There's more space to write your answer on the next page.

(Extra space for your answer to Problem Four, if you need it.)