

## Practice Second Midterm Exam III

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This exam is closed-book and closed-computer. You may have a double-sided, 8.5" × 11" sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 36 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question	Points	Grader
(1) Functions and Cardinality	/ 8	
(2) Graphs and Binary Relations	/ 10	
(3) Graphs and the Pigeonhole Principle	/ 8	
(4) Induction and Functions	/ 10	
	<b>/ 36</b>	

***You can do this. Best of luck on the exam!***

**Problem One: Functions and Cardinality****(8 Points)***(Midterm exam, Spring 2017)*

On Problem Set Four, you explored how set cardinality relates back to bijective functions between sets. This problem is designed to let you show us what you've learned in the course of working through those problems.

Consider the following two sets:

$$\mathbb{E} = \{ n \in \mathbb{Z} \mid n \text{ is even} \}$$

$$\mathbb{O} = \{ n \in \mathbb{Z} \mid n \text{ is odd} \}$$

Notice that these are sets of *integers*, not *natural numbers*.

In this problem, we're going to ask you to prove that  $|\mathbb{E}| = |\mathbb{O}|$ .

i. **(1 Point)** Draw a picture showing a way to pair off the elements of  $\mathbb{E}$  with the elements of  $\mathbb{O}$ .

ii. **(1 Point)** Based on the picture you came up with in part (i), define a bijection  $f : \mathbb{E} \rightarrow \mathbb{O}$ . (You'll prove that your function is a bijection in part (iii). We're just expecting the definition of  $f$  here.)

- iii. **(6 Points)** Prove that  $|\mathbb{E}| = |\mathbb{O}|$  by proving the function you came up with in part (ii) is a bijection. In the course of writing up your proof, please briefly prove that the function you've chosen is well-defined (that is, every input in the domain produces an output in the codomain.)

**Problem Two: Graphs and Binary Relations****(10 Points)***(Midterm exam, Spring 2017)*

On Problem Set Three you explored strict orders, and on Problem Set Four you explored different properties of graphs. In this problem, you'll explore directed graphs through the lens of binary relations. We hope this gives you a chance to demonstrate what you've learned about working with different types of binary relations and with concepts like paths and connectivity.

Let  $G = (V, E)$  be a *directed* graph. A **directed path** is a sequence  $P$  of one or more nodes  $v_1, v_2, \dots, v_k$  where, for each consecutive pair of nodes  $v_i, v_{i+1}$  in the sequence, the edge  $(v_i, v_{i+1})$  exists in  $E$ .

Consider the following binary relation  $R$  over the set  $V$ :

$$xRy \quad \text{if} \quad \text{there is a directed path from } x \text{ to } y \text{ in } G.$$

(As a reminder, the “if” in the above context means “is defined to mean” and is not an implication.)

- i. **(2 Points)** Prove that  $R$  is transitive.

Let  $G = (V, E)$  be a directed graph. On the previous page, we defined the relation  $R$  over  $V$  as follows:

$$xRy \quad \text{if} \quad \text{there is a directed path from } x \text{ to } y \text{ in } G.$$

Now, consider the following relation  $S$  over the set  $V$ :

$$xSy \quad \text{if} \quad xRy \text{ and } y \not R x$$

(Again, the “if” in the above context means “is defined to mean” and is not an implication.) Intuitively,  $xSy$  means that there’s a directed path from  $x$  to  $y$ , but there’s no directed path from  $y$  back to  $x$ .

- ii. **(8 Points)** Prove that  $S$  is a strict order over  $V$ . Feel free to use the result from part (i) of this problem in the course of writing up your answer.

*(Extra space for your answer to Problem Two, part (ii), if you need it.)*

**Problem Three: Graphs and the Pigeonhole Principle****(8 Points)***(Midterm exam, Spring 2017)*

On Problem Set Four, you explored bipartite graphs and independent sets. This problem explores a generalization of bipartite graphs and its connection to independent sets.

A graph  $G = (V, E)$  is called a ***k-partite graph*** if there are  $k$  sets  $V_1, V_2, \dots, V_k$  such that

- every node  $v \in V$  belongs to exactly one set  $V_i$ , and
- every edge  $e \in E$  has its endpoints in two different sets  $V_i$  and  $V_j$ .

These sets are called the ***k-partite classes*** of  $G$ .

Now, a quick refresher. An ***independent set*** in a set  $G = (V, E)$  is a set  $I \subseteq V$  where

$$\forall u \in I. \forall v \in I. \{u, v\} \notin E.$$

Let  $G = (V, E)$  be a  $k$ -partite graph for some natural number  $k \geq 2$ . Prove that if  $G$  has exactly  $n$  nodes, then  $G$  has an independent set of size at least  $\lceil n/k \rceil$ .

*(Extra space for your answer to Problem Three, if you need it.)*

**Problem Four: Induction and Functions****(10 Points)***(Midterm exam, Spring 2017)*

Let  $f : A \rightarrow A$  be an arbitrary function from a set  $A$  to itself. We can inductively define a class of functions  $f^0, f^1, f^2$ , etc. called the **powers of  $f$**  as follows:

$$\begin{aligned}f^0(x) &= x \\f^{n+1}(x) &= (f \circ f^n)(x)\end{aligned}$$

This question explores properties of powers of functions.

- i. **(2 Points)** Let  $f : A \rightarrow A$  be an arbitrary function. Prove that  $f^1(x) = f(x)$  for all  $x \in A$ .

As a refresher, if  $f : A \rightarrow A$ , then we can inductively define a class of functions  $f^0, f^1, f^2$ , etc. called the **powers of  $f$**  as follows:

$$\begin{aligned}f^0(x) &= x \\f^{n+1}(x) &= (f \circ f^n)(x)\end{aligned}$$

Now, a new definition. A **fixed point** of a function  $f : A \rightarrow A$  is an element  $a \in A$  such that  $f(a) = a$ .

ii. **(8 Points)** Let  $f : A \rightarrow A$  be an arbitrary function. Your task is to prove the following statement:

For any natural number  $n$ , and for any element  $a \in A$ ,  
if  $a$  is a fixed point of  $f$ , then  $a$  is a fixed point of  $f^n$ .

To do so, we'd like you to use induction. Specifically, use induction to prove that the statement  $P(n)$  defined below is true for all natural numbers  $n$ :

$P(n)$  is the statement "for any  $a \in A$ , if  $a$  is a fixed point of  $f$ , then  $a$  is a fixed point of  $f^n$ ."

*(Extra space for your answer to Problem Four, part (ii), if you need it.)*