Indirect Proofs

Outline for Today

- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
  - How do you show something is *not* true?
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

Logical Implication

Implications

- An ***implication*** is a statement of the form

If *P* is true, then *Q* is true.

- Some examples:
  - Math: If n is an even integer, then n^2 is an even integer.
  - Set Theory: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - Queen Bey: If you like it, then you should put a ring on it.

What Implications Mean

- Consider the simple statement

If I put fire near cotton, it will burn.

- Some questions to consider:
  - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? *(Scope)*
  - Does the fire cause the cotton to burn, or does the cotton burn for another reason? *(Causality)*
- These are significantly deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

Understanding Implications

“If there's a rainbow in the sky, then it's raining somewhere.”

- In mathematics, implication is *directional*.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about *causality*.
  - Rainbows do not cause rain. ☺

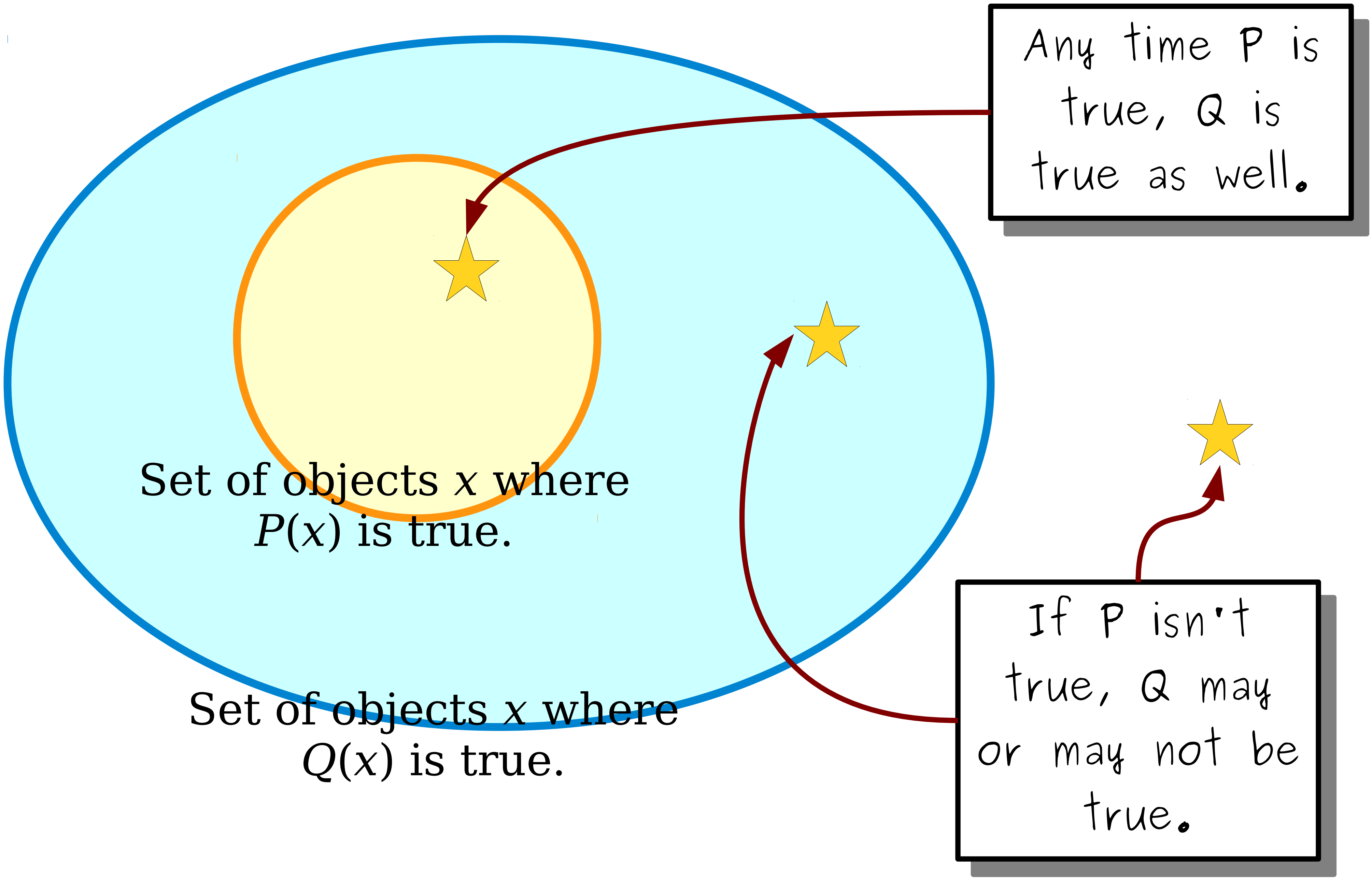
What Implications Mean

- In mathematics, a statement of the form

  For any x, if $P(x)$ is true, then $Q(x)$ is true

  means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



Negations

Negations

- A ***proposition*** is a statement that is either true or false.
  - Sentences that are questions or commands are not propositions.
- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - $\emptyset = \mathbb{R}$.
  - Moonlight is a good movie.
- The ***negation*** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the statement "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is *not* "it is sunny outside." ⚠

How do you find the negation of a statement?

The negation of the ***universal*** statement

Every *P* is a *Q*

is the *existential* statement

There is a *P* that is not a *Q*.

The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the *existential* statement

There exists an *x* where *P*(*x*) is false.

The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the *universal* statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

is the *universal* statement

For all *x*, *P*(*x*) is false.

Puppy Logic

- Consider the statement

I love all puppies.

Puppy Logic

- Consider the statement

I love all puppies.

What is the negation?

A. I don’t love any puppies.
B. I love some puppies and not others.
C. There is at least one puppy I don’t love.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

Puppy Logic

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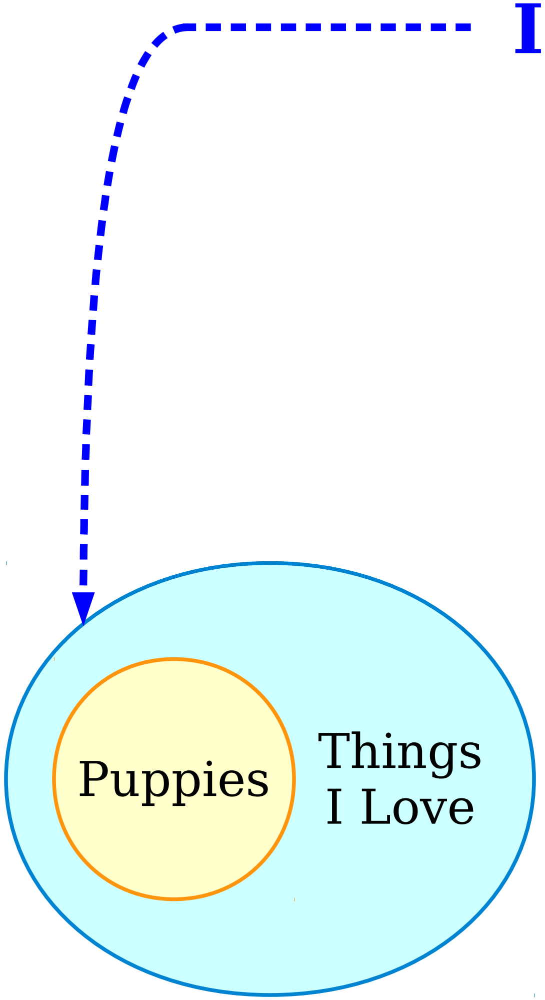


"I love all puppies."

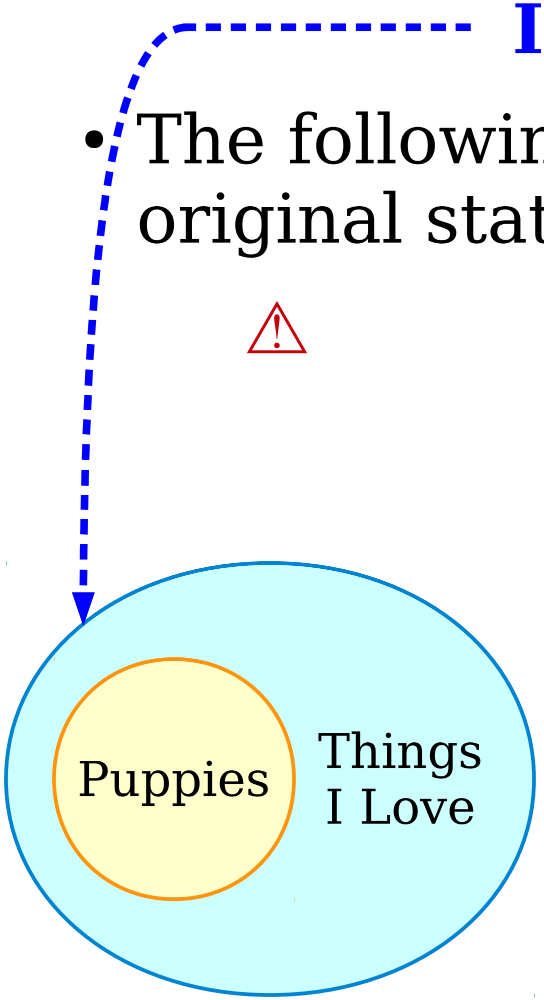
Puppy Logic

- Consider the statement

I love all puppies.

- The following statement is ***not*** the negation of the original statement:

⚠ **I don't love *any* puppies.** ⚠



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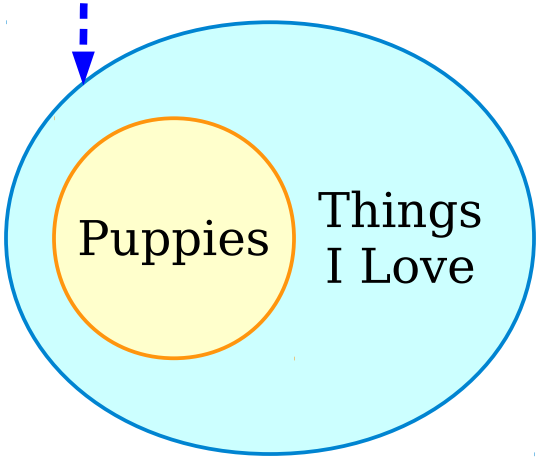

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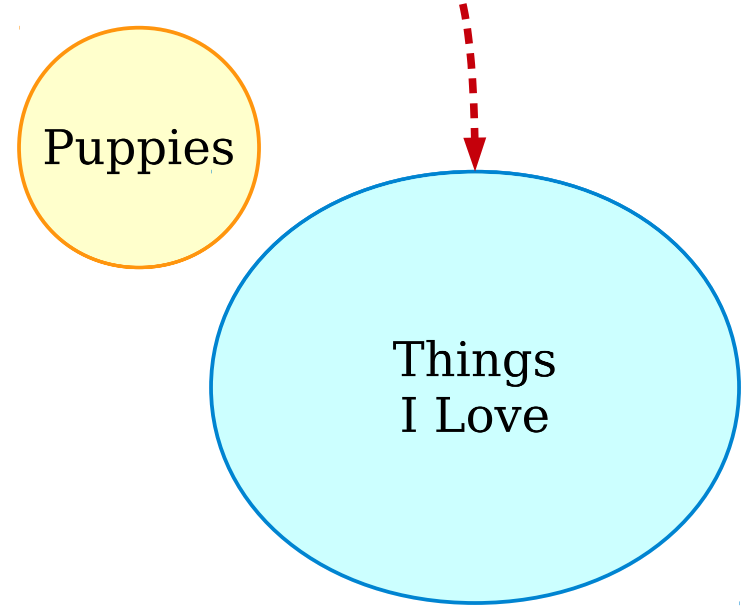

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Puppy Logic

- Consider the statement

I love all puppies.

- The following statement is ***not*** the negation of the original statement:

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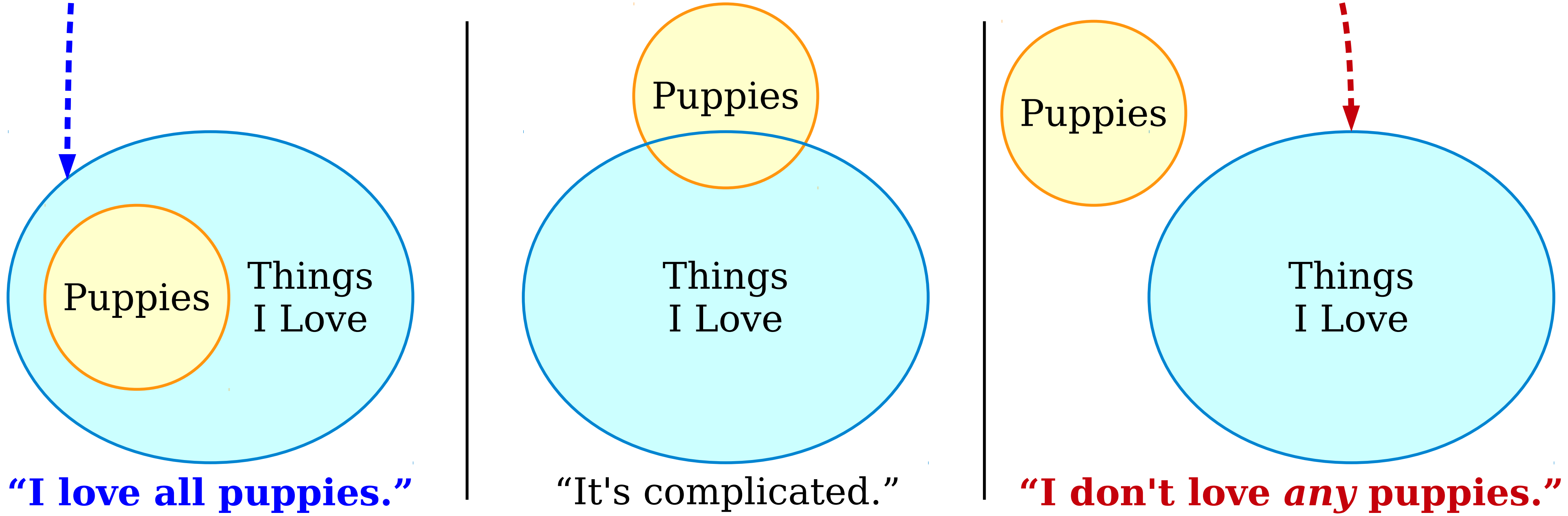


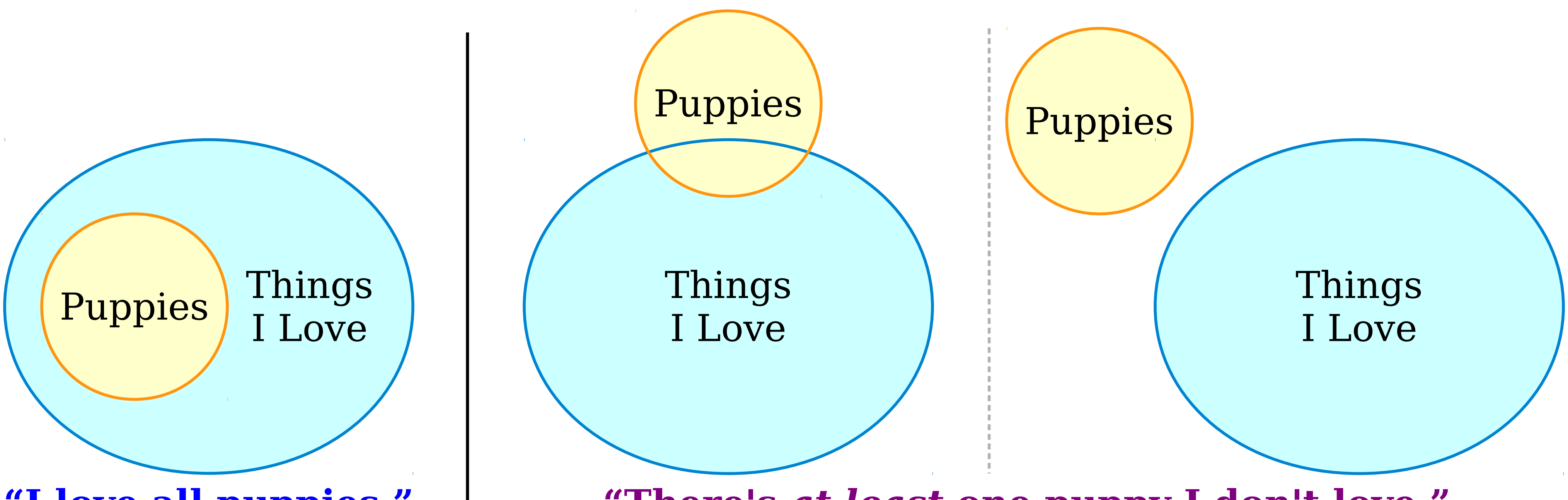
Puppy Logic

- Consider the statement

I love all puppies.

- Here's the proper negation of our initial statement about puppies:

There's *at least* one puppy I don't love.



"I love all puppies." **"There's *at least* one puppy I don't love."**

How do you negate an implication?

Let’s look at:

- Negation of an implication
- A close relative of negation: the Contrapositive

The negation of the statement

"If *P* is true,
then *Q* is true"

is the statement

"P is true,
and Q is false."

The negation of an implication
is not an implication!

“If your March Madness bracket is perfect, then you get an A in CS103.”

Which of the following is inconsistent with the above statement?

(A) Your bracket was terrible, and you got an A.
(B) Your bracket was terrible, and you got a B+.
(C) Your bracket was perfect, and you got a B+.
(D) Both (A) and (C)

The negation of the statement

"If your March Madness bracket is perfect, then you get an A in CS103."

is the statement

"Your March Madness bracket is perfect, and you still didn't get an A in CS103.

The negation of an implication is not an implication!

The negation of the statement

"For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

is the statement

"There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false."

The negation of an implication is not an implication!

The Contrapositive

- The ***contrapositive*** of the implication "If *P*, then *Q*" is "If not *Q*, then not *P*."
- For example:
  - "If your March Madness bracket is perfect, then you get an A in CS103."
  - Contrapositive: "If you didn't get an A in CS103 then your March Madness bracket wasn't perfect."
- Another example:
  - "If you like it, then you should put a ring on it."
  - Contrapositive: "If you shouldn't put a ring on it, then you don't like it."

Proof by Contrapositive

To prove the statement

“If *P* is true, then *Q* is true,”

you could choose to instead prove the equivalent statement

“If *Q* is false, then *P* is false.”

(if that seems easier).

This is called a ***proof by contrapositive***.

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We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

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Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

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$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if *n* is an integer:

If *n* is even, then n^2 is even.

If n^2 is even, then *n* is even.

- Therefore, if *n* is an integer:

***n* is even if and only if n^2 is even.**

- "If and only if" is often abbreviated ***iff***:

***n* is even iff n^2 is even.**

Proving Biconditionals

- To prove a theorem of the form ***P*** **iff** ***Q***, you need to prove that *P* implies *Q* and that *Q* implies *P*. (two separate proofs)
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that P is true, assume that P is *not* true.
  - Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
    - For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
  - This means that if P is false, something that cannot possibly happen, happens!
  - Therefore, P can't be false, so it must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We’ve seen sets of many different cardinalities:
  - $|\varnothing| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137\}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a “largest” set? That is, is there a set that’s bigger than every other set?

Theorem: There is no largest set.

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To prove this statement by contradiction, we're going to assume its negation.

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What is the negation of the statement "there is no largest set?"

Theorem: There is no largest set.

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What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

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What is the negation of the statement "there is no largest set?"

One option: **"there *is* a largest set."**

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Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

Theorem: There is no largest set.

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Now, consider the set $\wp(S)$.

Theorem: There is no largest set.

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Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S.

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Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

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We've reached a contradiction, so our assumption must have been wrong.

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Proving Implications

- To prove the implication

"If *P* is true, then *Q* is true."

- you can use these three techniques:
  - ***Direct Proof.***
    - Assume *P* and prove *Q*.
  - ***Proof by Contrapositive.***
    - Assume not *Q* and prove not *P*.
  - ***Proof by Contradiction.***
    - … what does this look like?

Theorem: If n is an integer and n^2 is even, then n is even.

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Proof: Assume for the sake of contradiction that ________

What is the assumption?

A. if n is odd, then n^2 is odd
B. n is an integer and n^2 is even, and n is odd
C. if n is an integer and n^2 is odd, then n is odd
D. n is an integer and n^2 is odd, and n is odd

Answer at **PollEv.com/cs103** or
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Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

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Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

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The three key pieces:

1. Say that the proof is by contradiction.
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In CS103, please include all these steps in your proofs!

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

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Theorem: If n is an integer and n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned} \tag{2}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Recap: Negating Implications

- To prove the statement

  "For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

  by contradiction, we do the following:
  - Assume this entire purple statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.
- What is the negation of the above purple statement?

  "There is an *x* where
  ***P*(*x*) is true and *Q*(*x*) is false"**

Recap: Contradictions and Implications

- To prove the statement

  “If *P* is true, then *Q* is true”

  using a proof by contradiction, do the following:
  - Assume that *P* is true and that *Q* is false.
  - Derive a contradiction.
  - Conclude that if *P* is true, *Q* must be as well.

Rational and Irrational Numbers

Rational and Irrational Numbers

- A number r is called a ***rational number*** if it can be written as

$$r = \frac{p}{q}$$

where p and q are integers and $q \neq 0$.

- A number that is not rational is called ***irrational***.

Simplest Forms

- *By definition*, if r is a rational number, then r can be written as p / q where p and q are integers and $q \neq 0$.
- ***Theorem:*** If r is a rational number, then r can be written as p / q where p and q are integers, $q \neq 0$, and p and q have no common factors other than 1 and -1.
  - That is, r can be written as a fraction in simplest form.
- We're just going to take this for granted for now, though with the techniques you'll see later in the quarter you'll be able to prove it!

Question: Are all real numbers rational?

Theorem: $\sqrt{2}$ is irrational.

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Multiplying both sides of equation (1) by q and squaring both sides

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (3) shows that q^2 is even. Our earlier theorem tells us that, because q^2 is even, q must also be even. But this is not possible – we know that p and q have no common factors other than 1 and -1, but we've shown that p and q must have two as a common factor.

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Vi Hart on Pythagoras and
the Square Root of Two:

http://www.youtube.com/watch?v=X1E7I7_r3Cw

What We Learned

- ***What's an implication?***
  - It's statement of the form "if *P*, then *Q*," and states that if *P* is true, then *Q* is true.
- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if *P*, then *Q*" is "if not *Q*, then not *P*.")
- ***What's a proof by contradiction?***
  - It's a proof of a statement *P* that works by showing that *P* cannot be false.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.

Handouts

- There are *six* (!) total handouts for today:
  - Handout 08: Guide to Proofs
  - Handout 09: Mathematical Vocabulary
  - Handout 10: Guide to Indirect Proofs
  - Handout 11: Ten Techniques to Get Unstuck
  - **Handout 12: Proofwriting Checklist**
  - **Handout 13: Problem Set One**
- Be sure to read handouts; there's a lot of really important information in there!

Announcements

- Problem Set 1 goes out today!
- ***Checkpoint*** due Monday, January 15 at 2:30PM.
    - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
    - We will get feedback back to you with comments on your proof technique and style.
    - The more effort you put in, the more you'll get out.
- ***Remaining problems*** due Friday, January 19 at 2:30PM.
    - Feel free to email staff list with questions, stop by office hours, or ask questions on Piazza!

Submitting Assignments

- All assignments should be submitted through GradeScope.
  - The programming portion of the assignment gets submitted separately from the written component.
  - The written component ***must*** be typed up; handwritten solutions don't scan well and get mangled in GradeScope.
- Summary of the late policy:
  - Everyone has *three* 24-hour late days.
  - Late days can't be used on checkpoints.
  - Nothing may be submitted more than ***two*** days past the due date.
- Because submission times are recorded automatically, we're strict about the submission deadlines.
- ***Very good idea:*** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
- ***Very bad idea:*** Wait until the last minute to submit.

Working in Pairs

- You can work on the problem sets individually or in pairs.
- Each person/pair should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handout 04 and Handout 05.

A Note on the Honor Code

Office hours have started!

Schedule is available
on the course website.

Appendix: Negating Statements

Scoping Implications

- Consider the following statements:

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

If n is even, then n^2 is even.

If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

- In the above statements, what are A, B, C, and n? Are they *specific* objects? Or do these claims hold for all objects?

Implications and Universals

- In discrete math, most implications involving unknown quantities are, implicitly, universal statements.*
- For example, the statement

  If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

  actually means

  For any sets A, B, and C,
  if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

** Your proofs should never use variables without officially introducing them though. This will become more clear next Wednesday.*

Negating Universal Statements

"For all *x*, P(*x*) is true"

becomes

"There is an *x* where P(*x*) is false."

Negating Existential Statements

"There exists an *x* where P(*x*) is true"

becomes

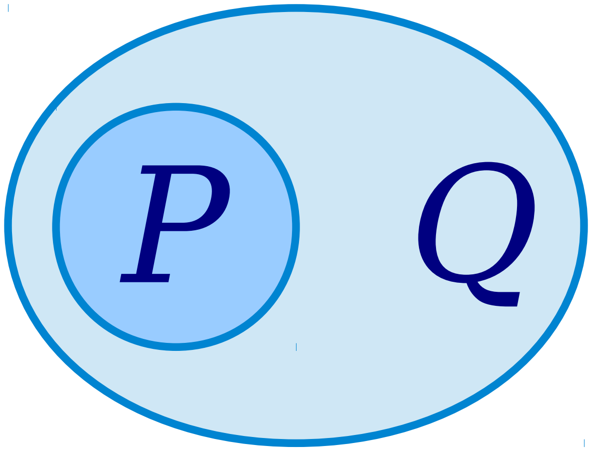
"For all *x*, P(*x*) is false."

Negating Implications

"For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

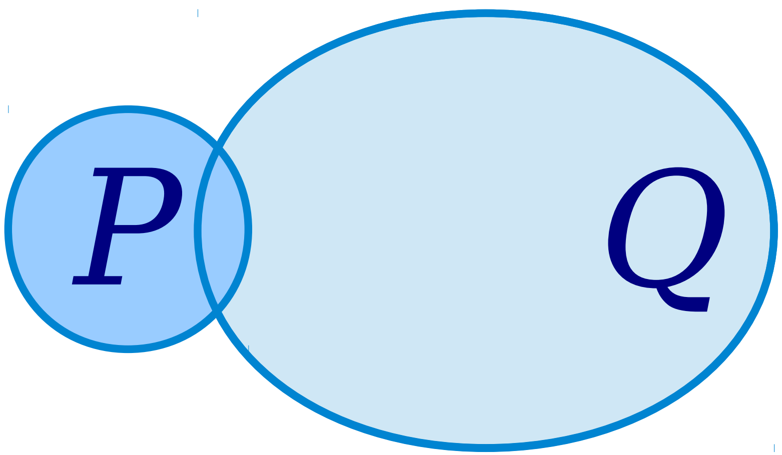
becomes

"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false"



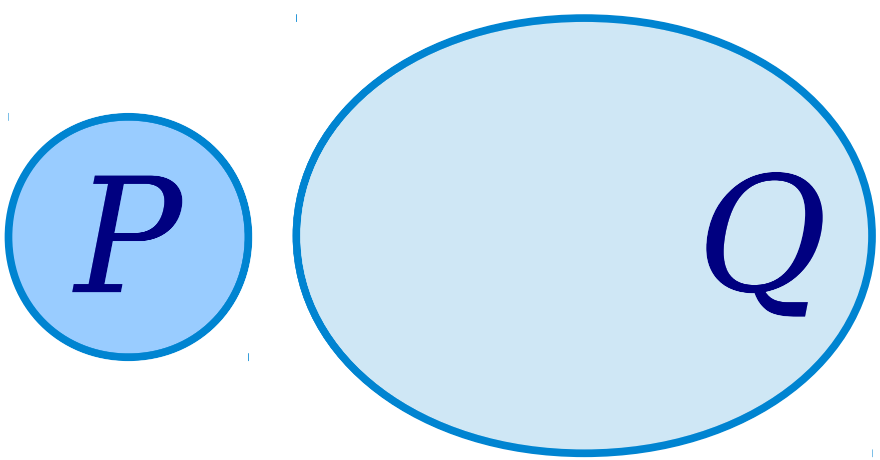
$P(x)$ implies $Q(x)$

"If $P(x)$ is true, then $Q(x)$ is true."



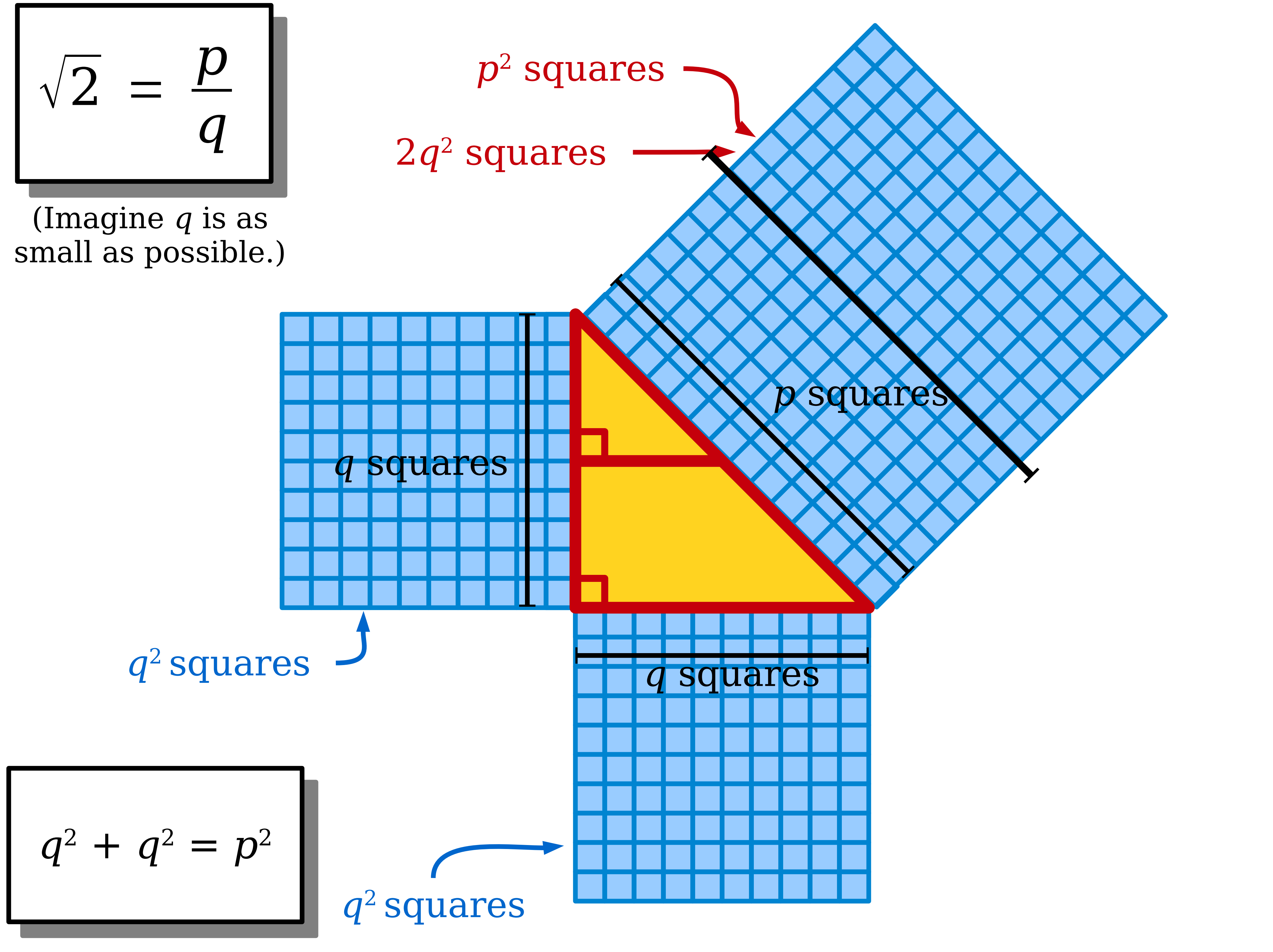
$P(x)$ does not imply $Q(x)$
-and-
$P(x)$ does not imply not $Q(x)$

"Sometimes $P(x)$ is true and $Q(x)$ is true,
-and-
sometimes $P(x)$ is true and $Q(x)$ is false."



$P(x)$ implies not $Q(x)$

If $P(x)$ is true, then $Q(x)$ is false



$\sqrt{2} = \frac{p}{q}$
(Imagine q is as small as possible.)
p^2 squares
$2q^2$ squares
p squares
q squares
q squares
q^2 squares
q^2 squares
$q^2 + q^2 = p^2$