

Indirect Proofs

Outline for Today

- ***What is an Implication?***
 - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
 - How do you show something is *not* true?
- ***Proof by Contrapositive***
 - What's a contrapositive?
 - And some applications!
- ***Proof by Contradiction***
 - The basic method.
 - And some applications!

Logical Implication

Implications

- An ***implication*** is a statement of the form

If P is true, then Q is true.

- Some examples:
 - Math: If n is an even integer, then n^2 is an even integer.
 - Set Theory: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
 - Queen Bey: If you like it, then you should put a ring on it.

What Implications Mean

- Consider the simple statement

If I put fire near cotton, it will burn.
- Some questions to consider:
 - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? (*Scope*)
 - Does the fire cause the cotton to burn, or does the cotton burn for another reason? (*Causality*)
- These are significantly deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

Understanding Implications

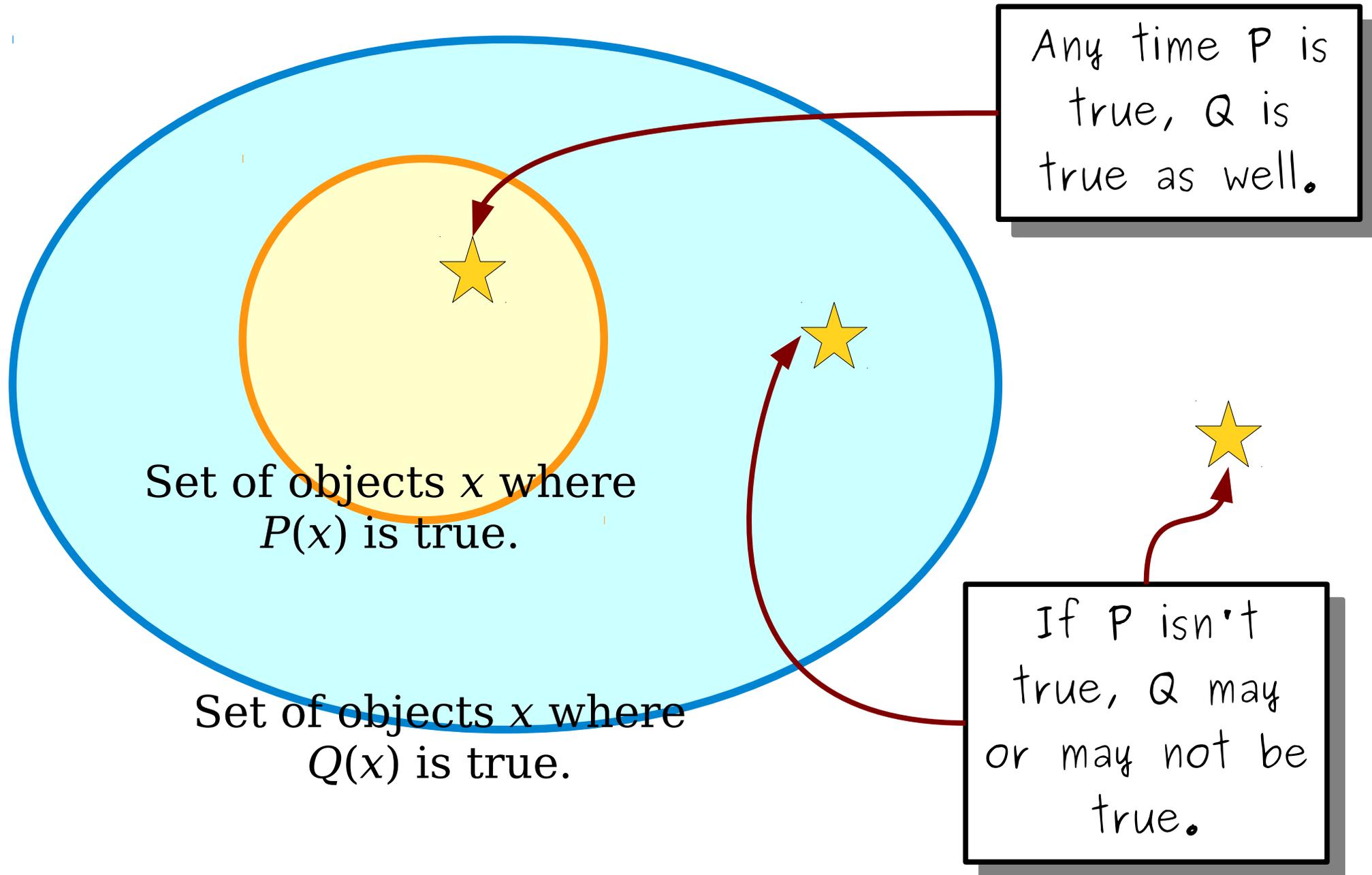
**“If there's a rainbow in the sky,
then it's raining somewhere.”**

- In mathematics, implication is *directional*.
 - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
 - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about *causality*.
 - Rainbows do not cause rain. ☺

What Implications Mean

- In mathematics, a statement of the form **For any x , if $P(x)$ is true, then $Q(x)$ is true** means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



Negations

Negations

- A **proposition** is a statement that is either true or false.
 - Sentences that are questions or commands are not propositions.
- Some examples:
 - If n is an even integer, then n^2 is an even integer.
 - $\emptyset = \mathbb{R}$.
 - Moonlight is a good movie.
- The **negation** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the statement “it is snowing outside.”
 - Its negation is “it is not snowing outside.”
 - Its negation is *not* “it is sunny outside.” ⚠

How do you find the negation
of a statement?

The negation of the *universal* statement

Every P is a Q

is the *existential* statement

There is a P that is not a Q .

The negation of the *universal* statement

For all x , $P(x)$ is true.

is the *existential* statement

There exists an x where $P(x)$ is false.

The negation of the *existential* statement

There exists a P that is a Q

is the *universal* statement

Every P is not a Q .

The negation of the *existential* statement

There exists an x where $P(x)$ is true

is the *universal* statement

For all x , $P(x)$ is false.

Puppy Logic

- Consider the statement

I love all puppies.

What is the negation?

- A. I don't love any puppies.
- B. I love some puppies and not others.
- C. There is at least one puppy I don't love.

Answer at [Pollevo.com/cs103](https://www.pollevo.com/cs103) or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

Puppy Logic

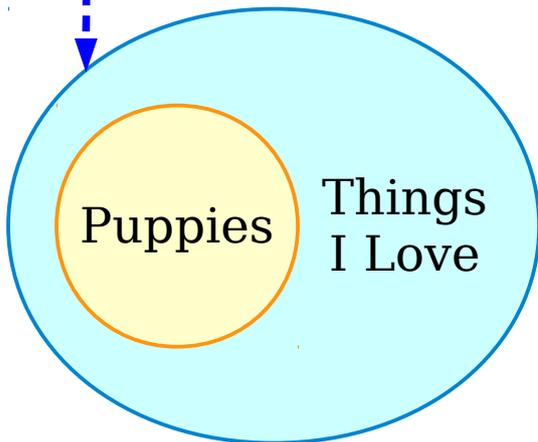
- Consider the statement

I love all puppies.

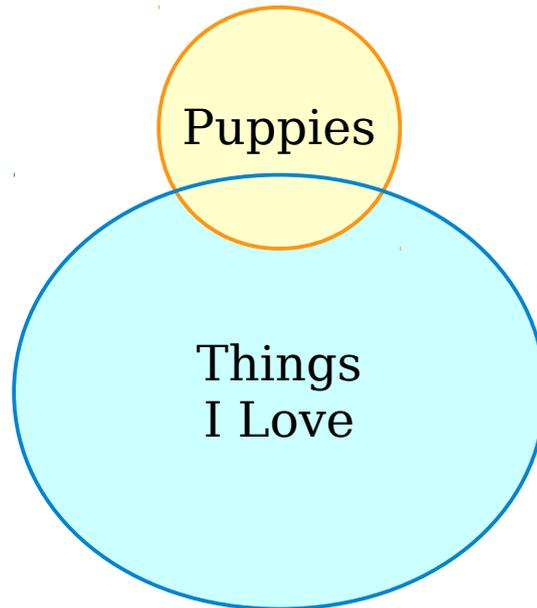
- The following statement is **not** the negation of the original statement:



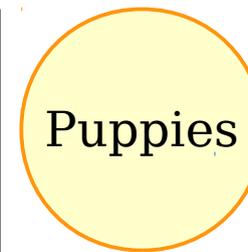
I don't love *any* puppies.



"I love all puppies."



"It's complicated."



"I don't love *any* puppies."

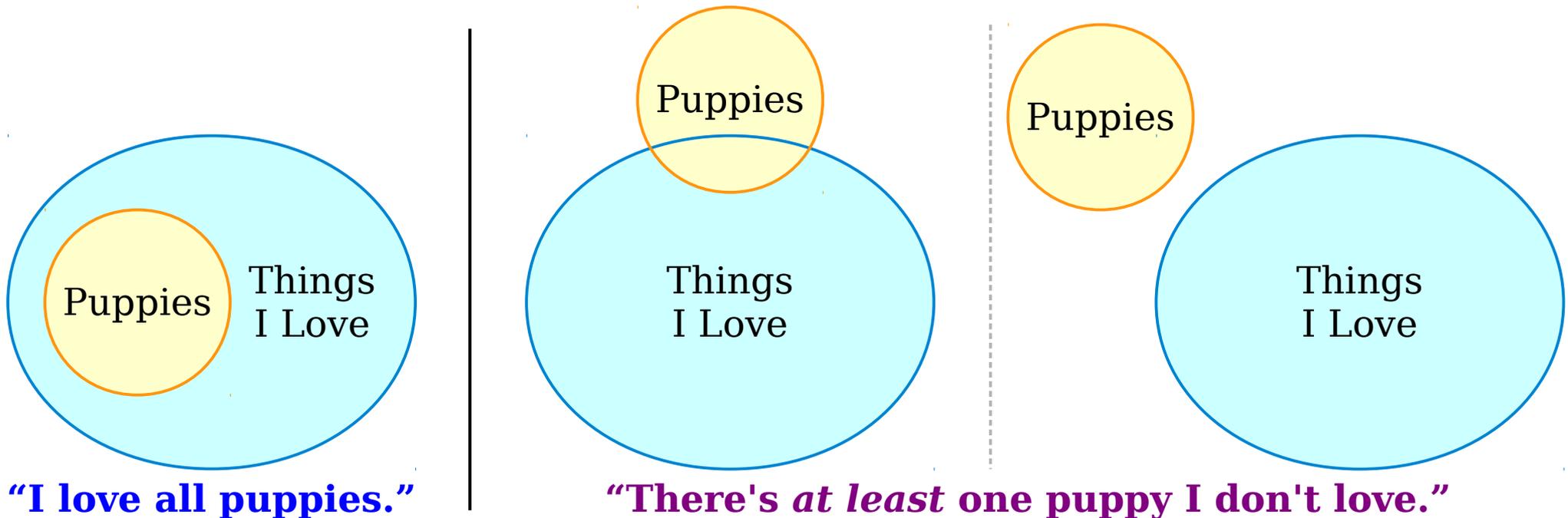
Puppy Logic

- Consider the statement

I love all puppies.

- Here's the proper negation of our initial statement about puppies:

There's at least one puppy I don't love.



How do you negate an implication?

Let's look at:

- Negation of an implication
- A close relative of negation: the Contrapositive

The negation of the statement

**“If P is true,
then Q is true”**

is the statement

**“ P is true,
and Q is false.”**

***The negation of an implication
is not an implication!***

**“If your March Madness bracket is perfect,
then you get an A in CS103.”**

**Which of the following is inconsistent
with the above statement?**

- (A) Your bracket was terrible, and you got an A.
- (B) Your bracket was terrible, and you got a B+.
- (C) Your bracket was perfect, and you got a B+.
- (D) Both (A) and (C)

The negation of the statement

**“If your March Madness bracket is perfect,
then you get an A in CS103.”**

is the statement

**“Your March Madness bracket is perfect,
and you still didn’t get an A in CS103.**

***The negation of an implication
is not an implication!***

The negation of the statement

**“For any x , if $P(x)$ is true,
then $Q(x)$ is true”**

is the statement

**“There is at least one x where
 $P(x)$ is true **and** $Q(x)$ is false.”**

***The negation of an implication
is not an implication!***

The Contrapositive

- The ***contrapositive*** of the implication “If P , then Q ” is “If not Q , then not P .”
- For example:
 - “If your March Madness bracket is perfect, then you get an A in CS103.”
 - Contrapositive: “If you didn’t get an A in CS103 then your March Madness bracket wasn’t perfect.”
- Another example:
 - “If you like it, then you should put a ring on it.”
 - Contrapositive: “If you shouldn’t put a ring on it, then you don’t like it.”

Proof by Contrapositive

To prove the statement

“If P is true, then Q is true,”

you could choose to instead prove the
equivalent statement

“If Q is false, then P is false.”

(if that seems easier).

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive;

We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$.

Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned}n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1.\end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

Therefore, n^2 is odd. ■

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Let n be a
there is some
Squaring
simplifying

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.
2. Explicitly state the contrapositive of what we want to prove.
3. Go prove the contrapositive.

From this
(namely, 2
Therefore

Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if n is an integer:

If n is even, then n^2 is even.

If n^2 is even, then n is even.

- Therefore, if n is an integer:

n is even if and only if n^2 is even.

- “If and only if” is often abbreviated *iff*:

n is even iff n^2 is even.

Proving Biconditionals

- To prove a theorem of the form **P iff Q** , you need to prove that P implies Q and that Q implies P . (two separate proofs)
- You can use any proof techniques you'd like to show each of these statements.
 - In our case, we used a direct proof for one and a proof by contrapositive for the other.

Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle, *The Adventure of the Blanched Soldier*

Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
 - To prove that P is true, assume that P is *not* true.
 - Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
 - For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
 - This means that if P is false, something that cannot possibly happen, happens!
 - Therefore, P can't be false, so it must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
 - $|\emptyset| = 0$
 - $|\{1, 2, 3\}| = 3$
 - $|\{n \in \mathbb{N} \mid n < 137\}| = 137$
 - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- **Question:** Is there a “largest” set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there is a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember - proofs are meant to be read by other people!

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S . This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Proving Implications

- To prove the implication

“If P is true, then Q is true.”
- you can use these three techniques:
 - ***Direct Proof.***
 - Assume P and prove Q .
 - ***Proof by Contrapositive.***
 - Assume not Q and prove not P .
 - ***Proof by Contradiction.***
 - ... what does this look like?

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that _____

What is the assumption?

- A. if n is odd, then n^2 is odd
- B. n is an integer and n^2 is even, and n is odd
- C. if n is an integer and n^2 is odd, then n is odd
- D. n is an integer and n^2 is odd, and n is odd

Answer at [PollEv.com/cs103](https://pollev.com/cs103) or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \quad (1)$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned} \quad (2)$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

Since n is odd we know that there is an integer k such

The three key pieces:

1. Say that the proof is by contradiction.
2. Say what the negation of the original statement is.
3. Say you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Recap: Negating Implications

- To prove the statement

“For any x , if $P(x)$ is true, then $Q(x)$ is true”

by contradiction, we do the following:

- Assume this entire purple statement is false.
 - Derive a contradiction.
 - Conclude that the statement is true.
- What is the negation of the above purple statement?

**“There is an x where
 $P(x)$ is true and $Q(x)$ is false”**

Recap: Contradictions and Implications

- To prove the statement

“If P is true, then Q is true”

using a proof by contradiction, do the following:

- Assume that P is true and that Q is false.
- Derive a contradiction.
- Conclude that if P is true, Q must be as well.

Rational and Irrational Numbers

Rational and Irrational Numbers

- A number r is called a **rational number** if it can be written as

$$r = \frac{p}{q}$$

where p and q are integers and $q \neq 0$.

- A number that is not rational is called **irrational**.

Simplest Forms

- *By definition*, if r is a rational number, then r can be written as p / q where p and q are integers and $q \neq 0$.
- ***Theorem:*** If r is a rational number, then r can be written as p / q where p and q are integers, $q \neq 0$, and p and q have no common factors other than 1 and -1.
 - That is, r can be written as a fraction in simplest form.
- We're just going to take this for granted for now, though with the techniques you'll see later in the quarter you'll be able to prove it!

Question: Are all real numbers rational?

Theorem: $\sqrt{2}$ is irrational.

Proof: Assume for the sake of contradiction that $\sqrt{2}$ is rational. This means that there must be integers p and q where $q \neq 0$, where p and q have no common divisors other than 1 and -1, and where

$$p / q = \sqrt{2}. \quad (1)$$

Multiplying both sides of equation (1) by q and squaring both sides shows us that

$$p^2 = 2q^2. \quad (2)$$

From equation (2), we see that p^2 is even. Earlier, we proved that if p^2 is even, then p must also be even. Therefore, we know that there is some integer k such that $p = 2k$. Substituting this into equation (2) and simplifying gives us the following:

$$\begin{aligned} p^2 &= 2q^2 \\ (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2 \end{aligned} \quad (3)$$

Equation (3) shows that q^2 is even. Our earlier theorem tells us that, because q^2 is even, q must also be even. But this is not possible - we know that p and q have no common factors other than 1 and -1, but we've shown that p and q must have two as a common factor.

We have reached a contradiction, so our original assumption must have been wrong. Therefore, $\sqrt{2}$ is irrational. ■

Theorem: $\sqrt{2}$ is irrational.

Proof: Assume for the sake of contradiction that $\sqrt{2}$ is rational. This means that there must be integers p and q where $q \neq 0$, where p and q have no common divisors other than 1 and -1, and where

$$p / q = \sqrt{2}. \quad (1)$$

Multiplying both sides of equation (1) by q and squaring both sides

The three key pieces:

1. State that the proof is by contradiction.
2. State what you are assuming is the negation of the statement to prove.
3. State you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (3) shows that q^2 is even. Our earlier theorem tells us that, because q^2 is even, q must also be even. But this is not possible - we know that p and q have no common factors other than 1 and -1, but we've shown that p and q must have two as a common factor.

We have reached a contradiction, so our original assumption must have been wrong. Therefore, $\sqrt{2}$ is irrational. ■

Vi Hart on Pythagoras and
the Square Root of Two:

http://www.youtube.com/watch?v=X1E7I7_r3Cw

What We Learned

- ***What's an implication?***

- It's statement of the form “if P , then Q ,” and states that if P is true, then Q is true.

- ***How do you negate formulas?***

- It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.

- ***What is a proof by contrapositive?***

- It's a proof of an implication that instead proves its contrapositive.
- (The contrapositive of “if P , then Q ” is “if not Q , then not P .”)

- ***What's a proof by contradiction?***

- It's a proof of a statement P that works by showing that P cannot be false.

Next Time

- ***Mathematical Logic***
 - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
 - Reasoning about simple statements.
- ***Propositional Equivalences***
 - Simplifying complex statements.

Handouts

- There are *six* (!) total handouts for today:
 - Handout 08: Guide to Proofs
 - Handout 09: Mathematical Vocabulary
 - Handout 10: Guide to Indirect Proofs
 - Handout 11: Ten Techniques to Get Unstuck
 - **Handout 12: Proofwriting Checklist**
 - **Handout 13: Problem Set One**
- Be sure to read handouts; there's a lot of really important information in there!

Announcements

- Problem Set 1 goes out today!
- **Checkpoint** due Monday, January 15 at 2:30PM.
 - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
 - We will get feedback back to you with comments on your proof technique and style.
 - The more effort you put in, the more you'll get out.
- **Remaining problems** due Friday, January 19 at 2:30PM.
 - Feel free to email staff list with questions, stop by office hours, or ask questions on Piazza!

Submitting Assignments

- All assignments should be submitted through GradeScope.
 - The programming portion of the assignment gets submitted separately from the written component.
 - The written component **must** be typed up; handwritten solutions don't scan well and get mangled in GradeScope.
- Summary of the late policy:
 - Everyone has *three* 24-hour late days.
 - Late days can't be used on checkpoints.
 - Nothing may be submitted more than **two** days past the due date.
- Because submission times are recorded automatically, we're strict about the submission deadlines.
- **Very good idea:** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
- **Very bad idea:** Wait until the last minute to submit.

Working in Pairs

- You can work on the problem sets individually or in pairs.
- Each person/pair should only submit a single problem set, and should submit it only once.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handout 04 and Handout 05.

A Note on the Honor Code

Office hours have started!

Schedule is available
on the course website.

Appendix: Negating Statements

Scoping Implications

- Consider the following statements:
 - If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.**
 - If n is even, then n^2 is even.**
 - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.**
- In the above statements, what are A , B , C , and n ? Are they *specific* objects? Or do these claims hold for all objects?

Implications and Universals

- In discrete math, most implications involving unknown quantities are, implicitly, universal statements.*
- For example, the statement

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

actually means

**For any sets A , B , and C ,
if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.**

** Your proofs should never use variables without officially introducing them though. This will become more clear next Wednesday.*

Negating Universal Statements

“For all x , $P(x)$ is true”

becomes

“There is an x where $P(x)$ is false.”

Negating Existential Statements

“There exists an x where $P(x)$ is true”

becomes

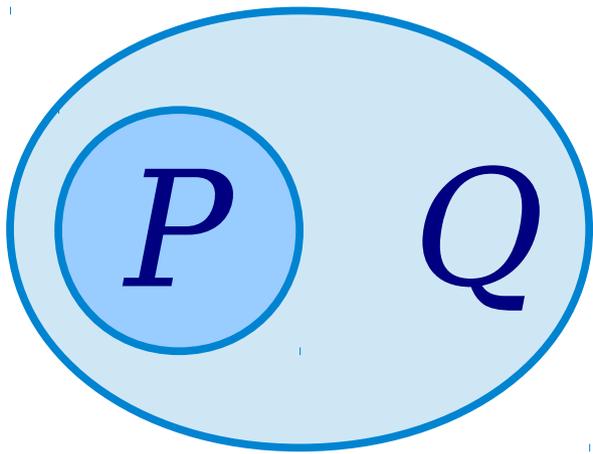
“For all x , $P(x)$ is false.”

Negating Implications

“For every x , if $P(x)$ is true, then $Q(x)$ is true”

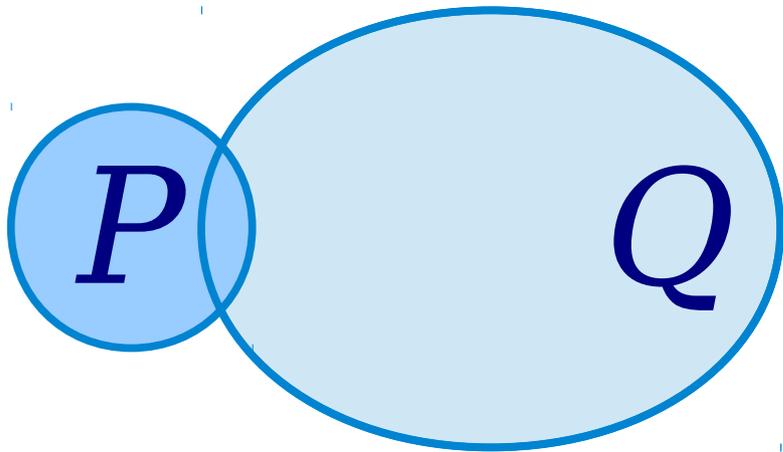
becomes

“There is an x where $P(x)$ is true and $Q(x)$ is false”



$P(x)$ implies $Q(x)$

“If $P(x)$ is true, then $Q(x)$ is true.”



$P(x)$ does not imply $Q(x)$

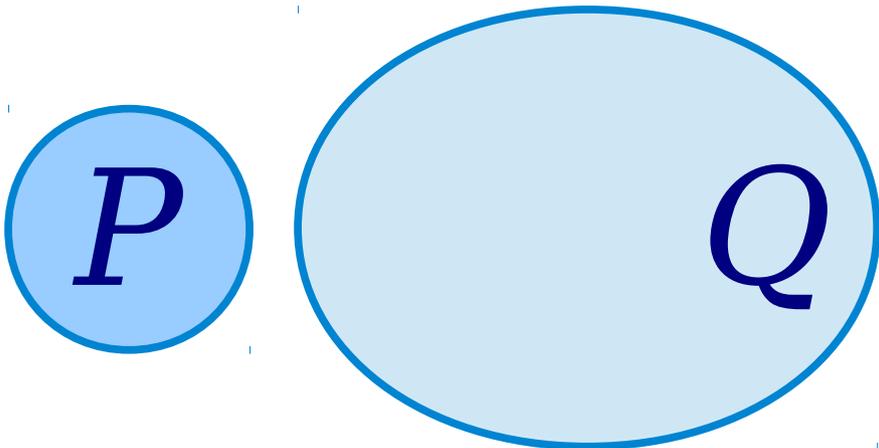
-and-

$P(x)$ does not imply not $Q(x)$

“Sometimes $P(x)$ is true and $Q(x)$ is true,

-and-

sometimes $P(x)$ is true and $Q(x)$ is false.”

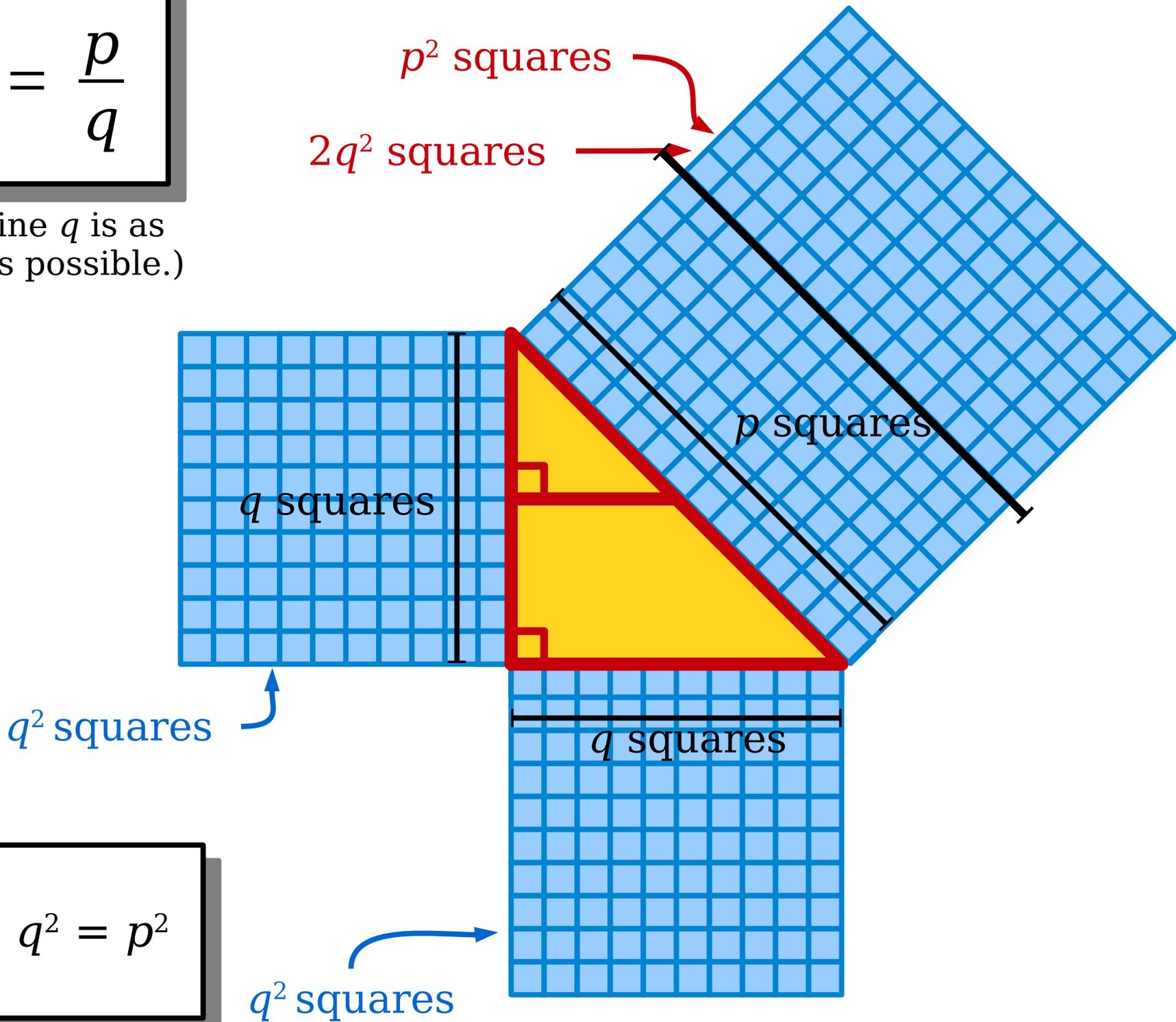


$P(x)$ implies not $Q(x)$

If $P(x)$ is true, then $Q(x)$ is false

$$\sqrt{2} = \frac{p}{q}$$

(Imagine q is as small as possible.)



$$q^2 + q^2 = p^2$$