Mathematical Logic

Part Three

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

"For any natural number n,
n is even iff n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of n, the following is true."

“All *A*'s are *B*'s”

translates as

$$\forall x.\ (A(x) \rightarrow B(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (A(x) \to B(x))$$

If x is a counterexample, it *must* have property A but not have property B.

“Some *A* is a *B*”

translates as

$$\exists x.\ (A(x) \wedge B(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (A(x) \land B(x))$$

If x is an example, it *must* have property A on top of property B.

The Aristotelian Forms

"All *A*s are *B*s"

$$\forall x.\ (A(x) \to B(x))$$

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

"No *A*s are *B*s"

$$\forall x.\ (A(x) \to \neg B(x))$$

"Some *A*s aren't *B*s"

$$\exists x.\ (A(x) \land \neg B(x))$$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- $Person(p)$, which states that p is a person, and
- $Loves(x, y)$, which states that x loves y,

write a sentence in first-order logic that means “everybody loves someone else.”

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

How many of the following first-order logic statements are correct translations of "everyone loves someone else?"

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \to Loves(p, q)))$$

$$\exists p.\ (Person(p) \to \forall q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “Everyone loves someone else.”

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “There is someone everyone else loves.”

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

For Comparison

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

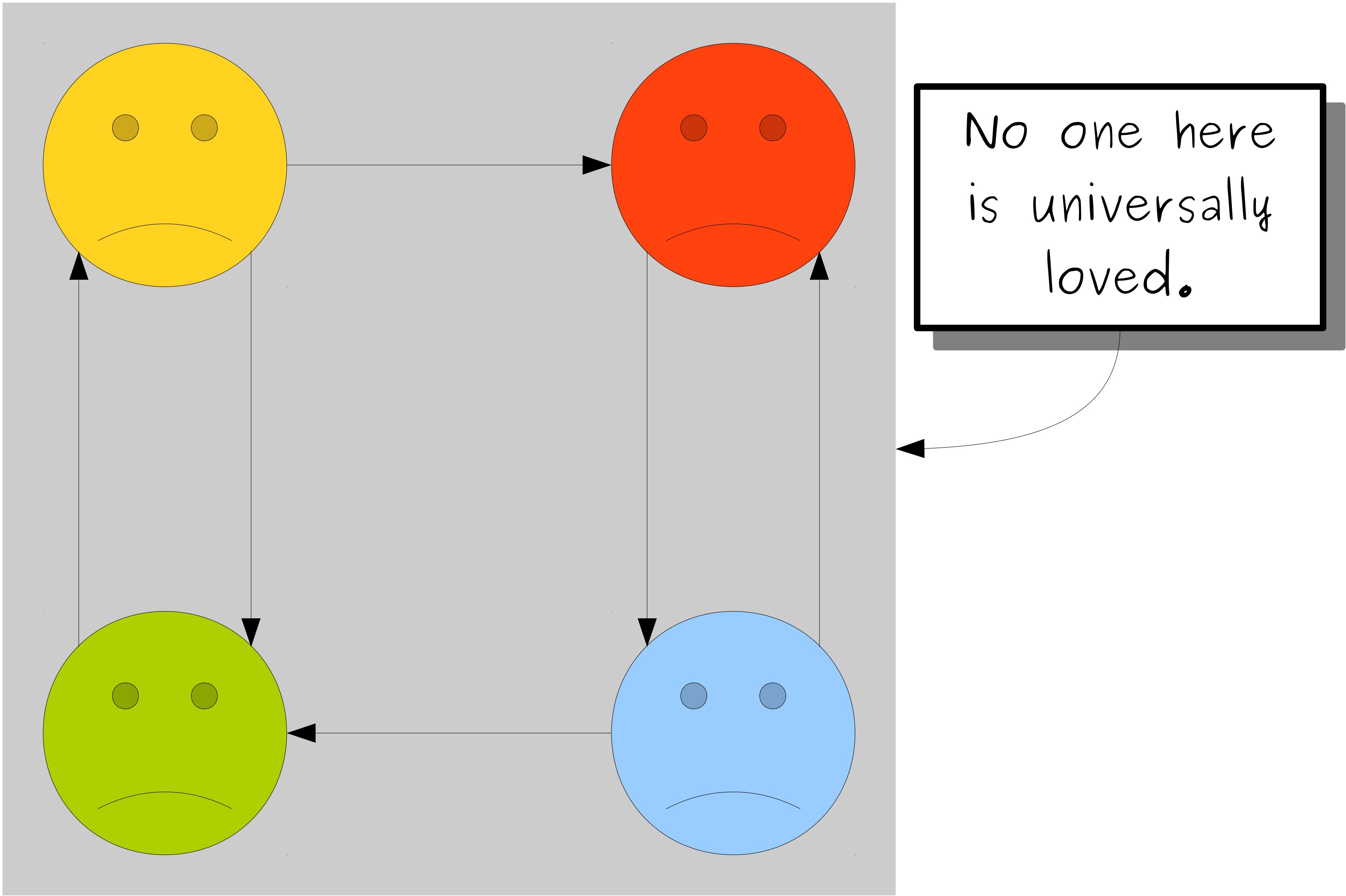
There is some person

who everyone

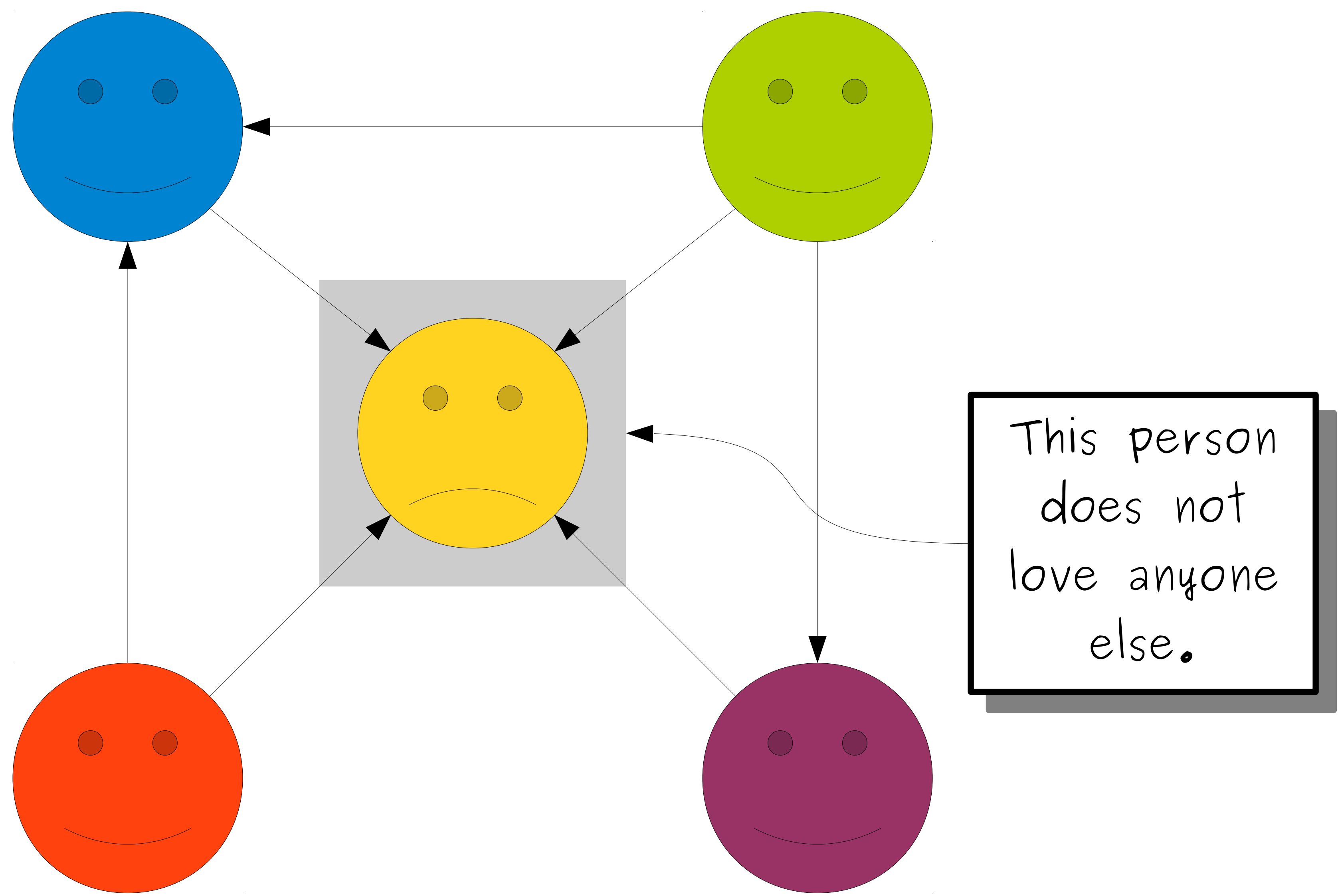
who isn't them

loves.

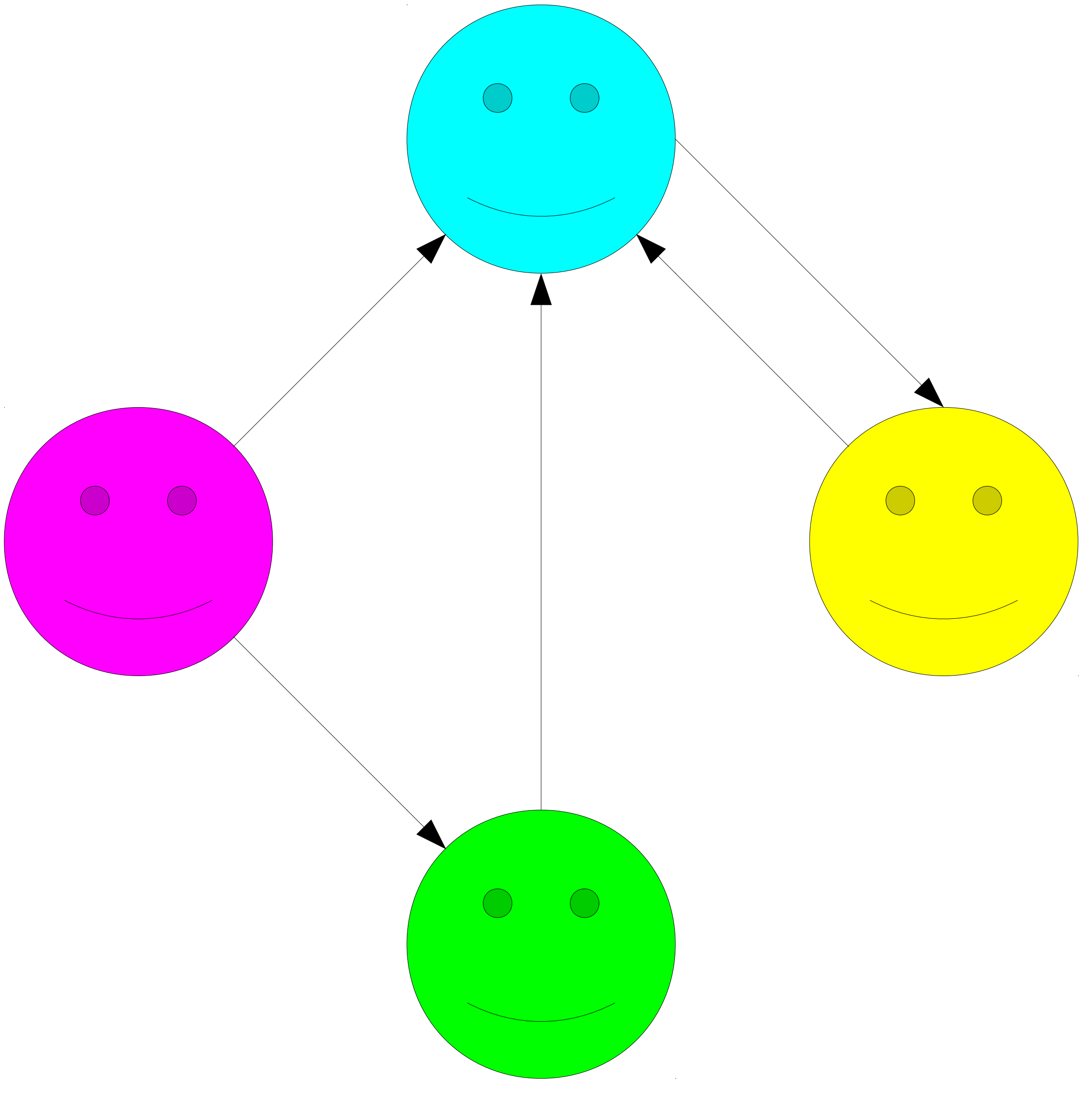
Everyone Loves Someone Else



There is Someone Everyone Else Loves



Everyone Loves Someone Else ***and*** There is Someone Everyone Else Loves



$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\land$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \rightarrow Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice of x, there's some choice of y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

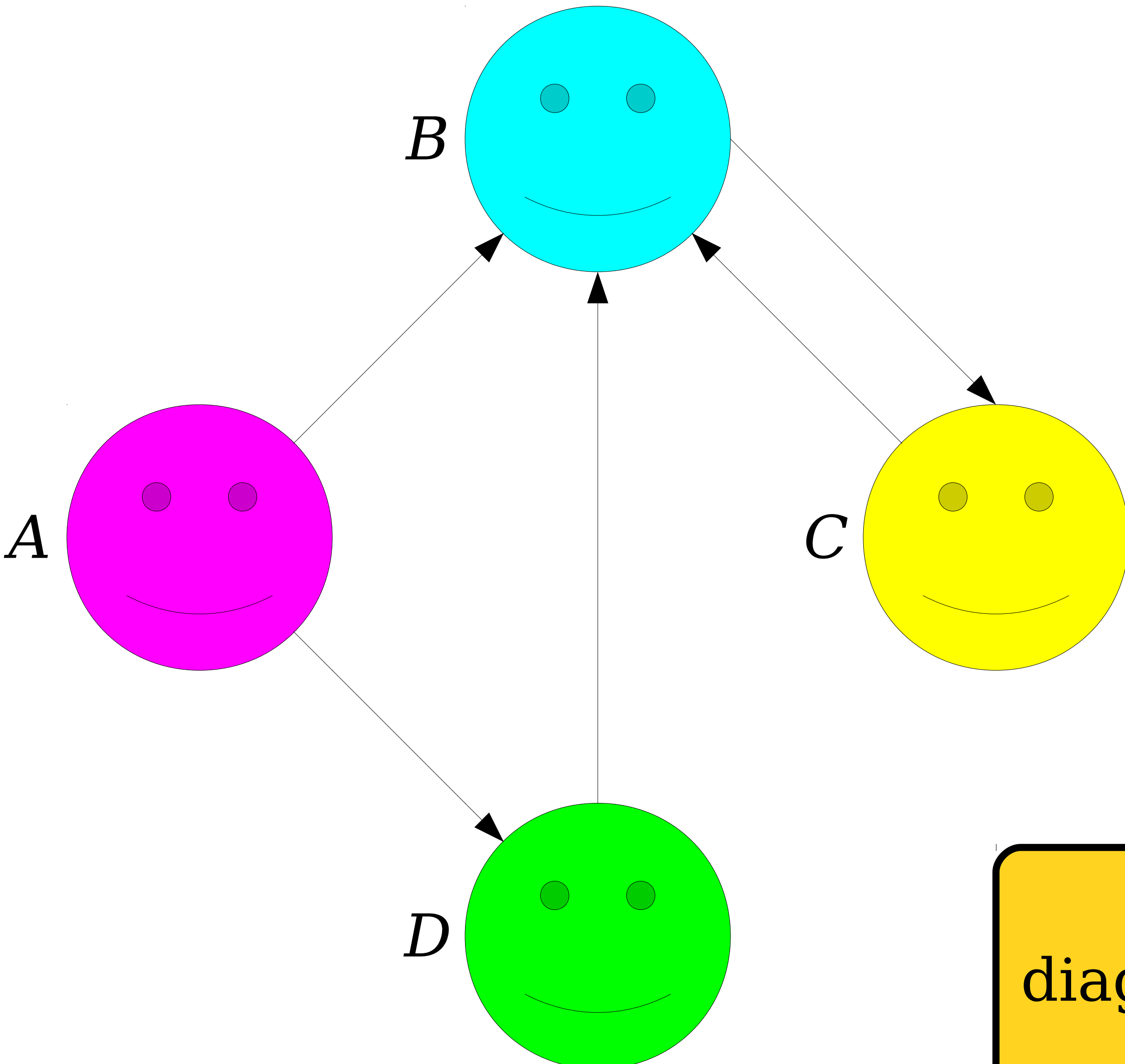
Quantifier Ordering

- The statement

  $$\exists x.\ \forall y.\ P(x, y)$$

  means “there is some x where for any choice of y, we get that $P(x, y)$ is true.”
- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!



Which person in this diagram do you most aspire to be?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, or **D**.

Time-Out for Announcements!

Problem Set Two

- Problem Set Two is due this Friday at 2:30PM.
  - Once we're done with this lecture, you'll know everything you need to complete it!
  - Have questions? Feel free to stop by office hours or to ask on Piazza.
- Hopefully you've taken a few minutes to read over all the problems by now. If not, we'd strongly recommend doing so.
- ***Good idea:*** Aim to complete Q1 – Q5 by the end of the evening.

Problem Set One Solutions

- Problem Set One solutions are now available.
- ***Please take the time to read over these solutions***.
  - For non-proof questions, make sure that you understand the intuition behind the answers. If they match yours, great! If not, that would be a great question to ask us.
  - For proofs, look over the style and formatting. Compare them against yours. How do they compare?
  - Each question has a “Why We Asked This Question” section at the end. Make sure you read over it – it would be a shame if you did a problem and didn’t hit the key insight we wanted you to have.

Apply to Section Lead!

- Want to teach a CS106A/B/X section? Already completed CS106B or CS106X? Apply to section lead at

  https://cs198.stanford.edu

- Application is due ***Thursday, February 1st***.
- There's a second round of hiring later this quarter for folks currently in CS106B/X – stay tuned!
- ***This is an amazing program. Highly recommended!***

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “two sets are equal if and only if they contain the same elements.”

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall S.\ (Set(S) \to \\
&\quad \forall T.\ (Set(T) \to \\
&\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad) \\
&)
\end{aligned}
$$

You sometimes see the universal quantifier pair with the ↔ connective. This is especially common when talking about sets because two sets are equal when they have precisely the same elements.

Mechanics: Negating Statements

Which of the following is the negation of the statement
$\forall x.\ \exists y.\ Loves(x, y)$?

A. $\forall x.\ \forall y.\ \neg Loves(x, y)$
B. $\forall x.\ \exists y.\ \neg Loves(x, y)$
C. $\exists x.\ \forall y.\ \neg Loves(x, y)$
D. $\exists x.\ \exists y.\ \neg Loves(x, y)$
E. None of these.
F. Two or more of these.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg\forall x.\ A \quad \equiv \quad \exists x.\ \neg A$$

$$\neg\exists x.\ A \quad \equiv \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$

$$\exists x.\ \neg\exists y.\ Loves(x, y)$$

$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \equiv \quad p \to \neg q$$

$$\neg(p \to q) \quad \equiv \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\exists x.\ (Puppy(x) \land Cute(x))$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set with no elements.")

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

("Every set contains at least one element.")

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*"There is a set that doesn't contain anything"*)

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

(*"Everything is a set that contains something"*)

Remember: ∀ usually goes with →, not ∧

Restricted Quantifiers

Quantifying Over Sets

- The notation

  $$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

  $$\exists x \in S.\ P(x)$$

  means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ \varphi$$

$$\exists x \in S.\ \varphi$$

  is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x)$. $Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y)$. $R(y)$. ⚠

⚠ $\exists P(x)$. $Q(x)$ ⚠

Expressing Uniqueness

Using the predicate

- $Level(l)$, which states that l is a level,

write a sentence in first-order logic that means “there is only one level.”

A fun diversion:

http://www.onemorelevel.com/game/there_is_only_one_level

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \rightarrow \neg Level(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \wedge \\
&\quad \forall x.\ (Level(x) \rightarrow x = l) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular $\forall$ and $\exists$ quantifiers.

Next Time

- ***Binary Relations***
  - How do we model connections between objects?
- ***Equivalence Relations***
  - How do we model the idea that objects can be grouped into clusters?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?