Binary Relations

Part One

Outline for Today

- ***Binary Relations***
  - Reasoning about connections between objects.
- ***Equivalence Relations***
  - Reasoning about clusters.
- ***A Fundamental Theorem***
  - How do we know we have the “right” definition for something?

Relationships

- In CS103, you've seen examples of relationships
  - between sets:

    $$A \subseteq B$$

  - between numbers:

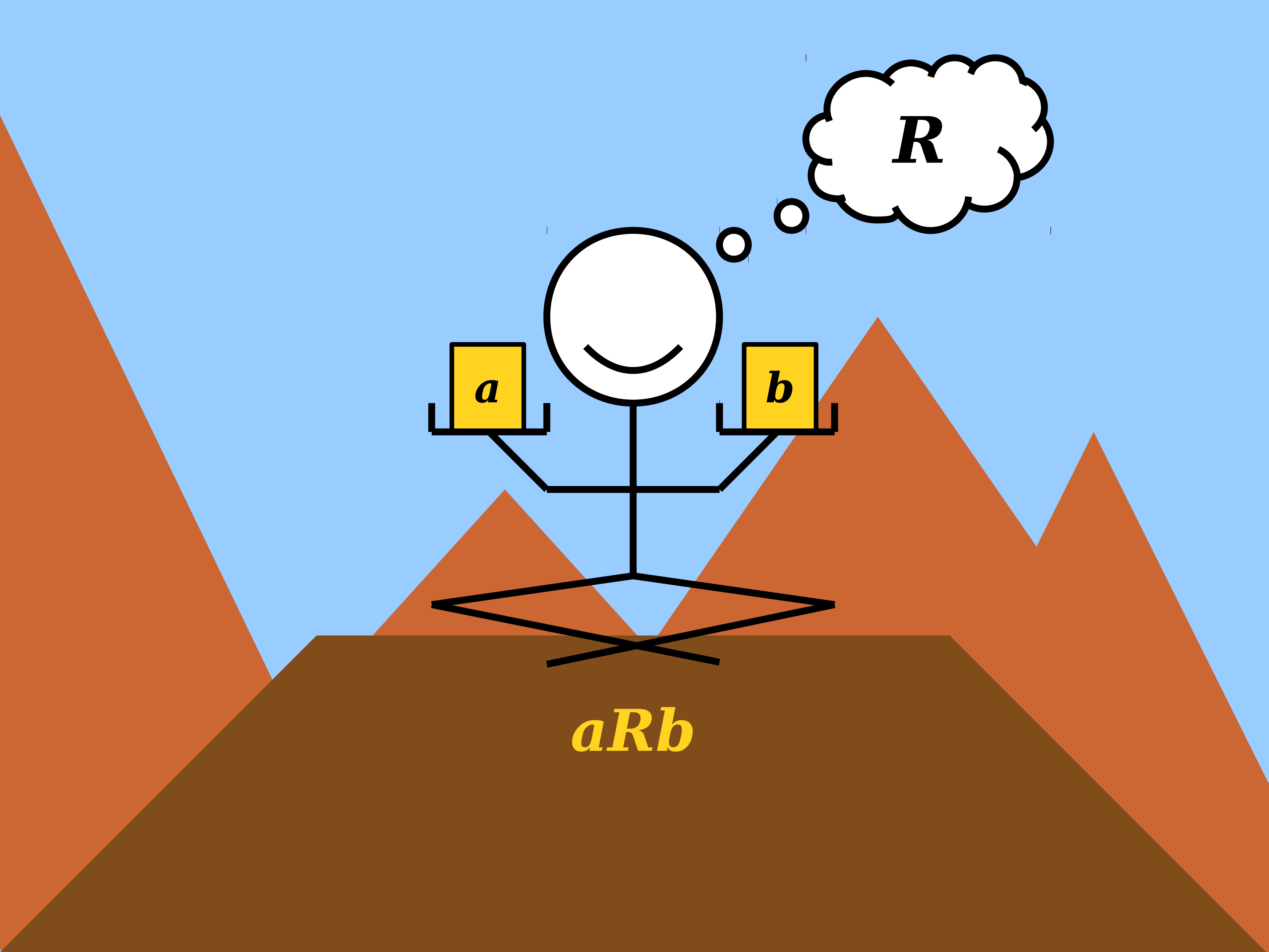
    $$x < y \qquad x \equiv_k y \qquad x \leq y$$

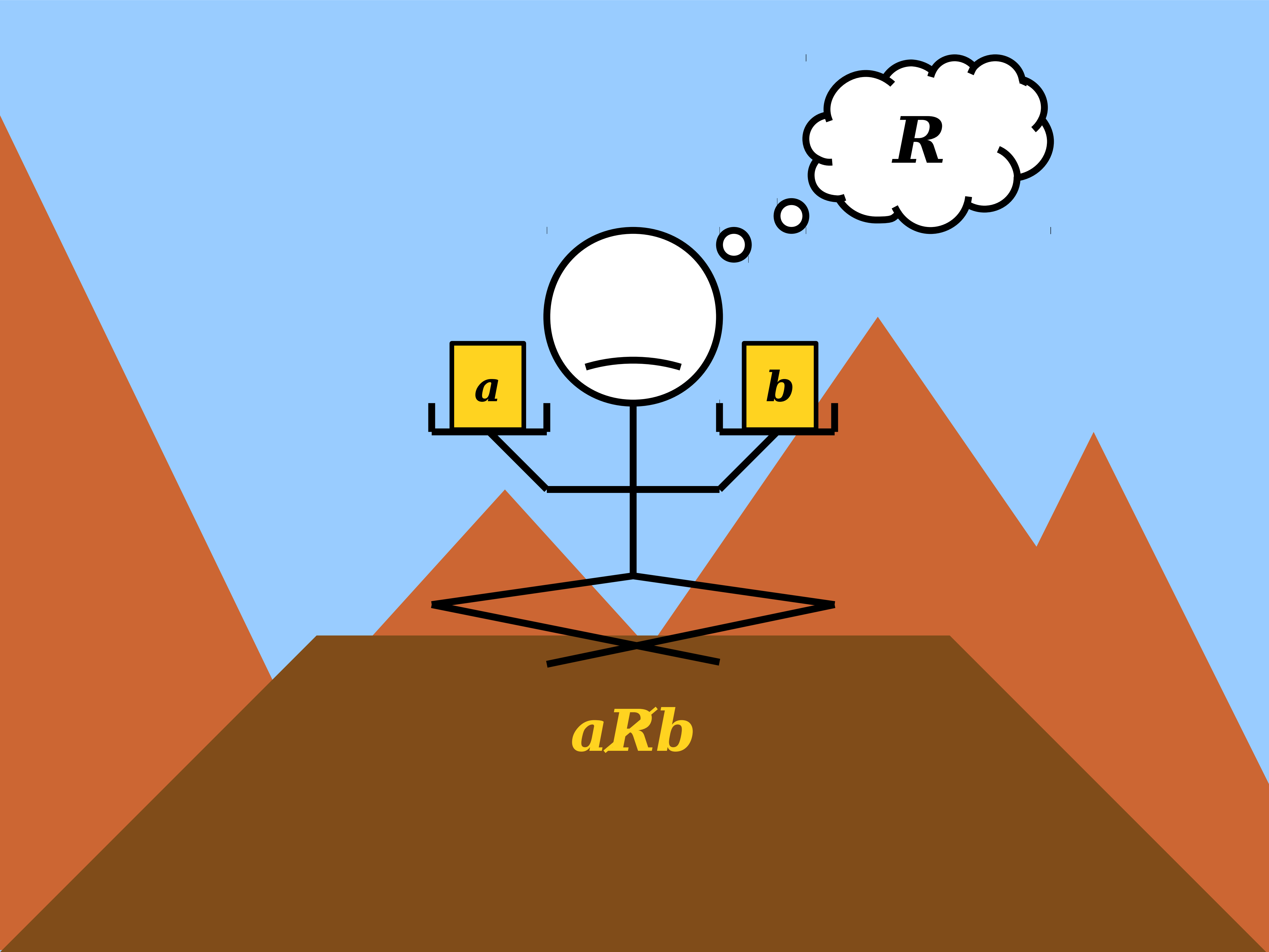
  - between people:

    $$p \text{ loves } q$$

- Since these relations focus on connections between two objects, they are called ***binary relations***.
  - The “binary” here means “pertaining to two things,” not “made of zeros and ones.”

What exactly is a binary relation?


R
a
b
aRb


R
a
b
$a \not R b$

Binary Relations

- A ***binary relation over a set A*** is a predicate R that can be applied to pairs of elements drawn from A.
- If R is a binary relation over A and it holds for the pair (a, b), we write $\boldsymbol{aRb}$.

$3 = 3$ $\quad\quad$ $5 < 7$ $\quad\quad$ $\emptyset \subseteq \mathbb{N}$

- If R is a binary relation over A and it does not hold for the pair (a, b), we write $\boldsymbol{a \not{R} b}$.

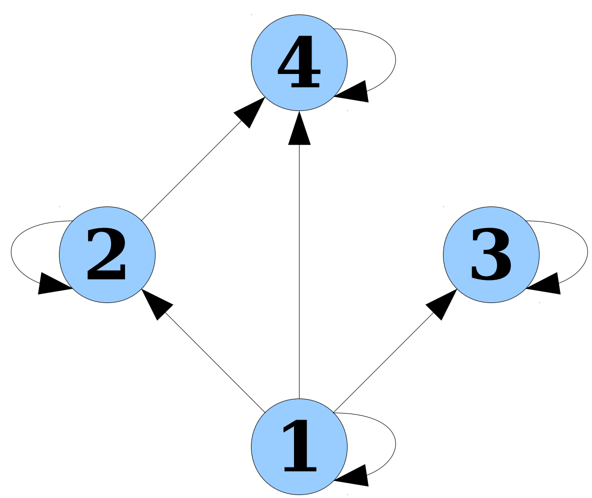
$4 \neq 3$ $\quad\quad$ $4 \not< 3$ $\quad\quad$ $\mathbb{N} \not\subseteq \emptyset$

Properties of Relations

- Generally speaking, if R is a binary relation over a set A, the order of the operands is significant.
  - For example, $3 < 5$, but $5 \not< 3$.
  - In some relations order is irrelevant; more on that later.
- Relations are always defined relative to some underlying set.
  - It's not meaningful to ask whether ☺ $\subseteq$ 15, for example, since $\subseteq$ is defined over sets, not arbitrary objects.

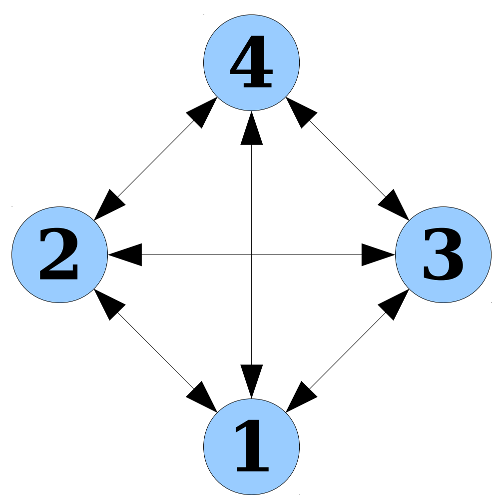
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a \mid b$ (meaning "a divides b") over the set {1, 2, 3, 4} looks like this:



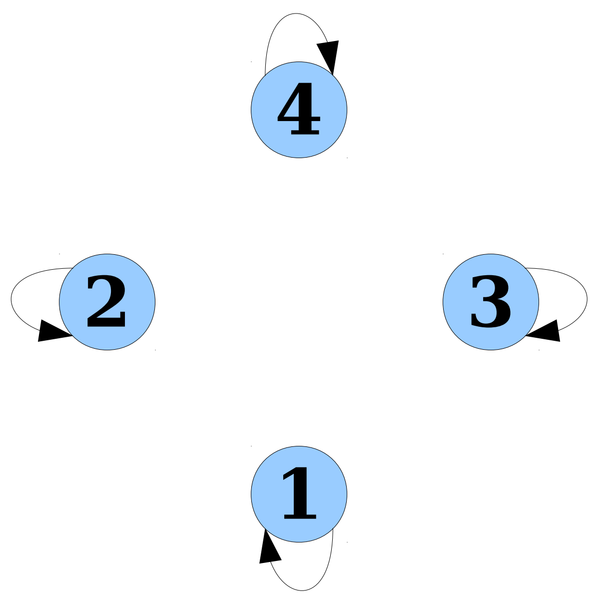
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a \neq b$ over the set $\{1, 2, 3, 4\}$ looks like this:



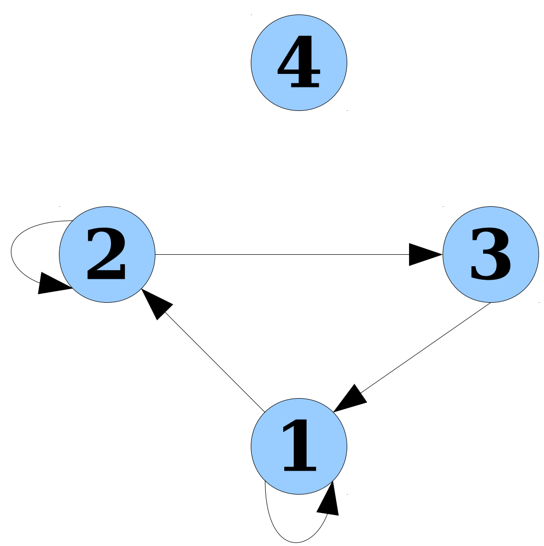
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a = b$ over the set $\{1, 2, 3, 4\}$ looks like this:



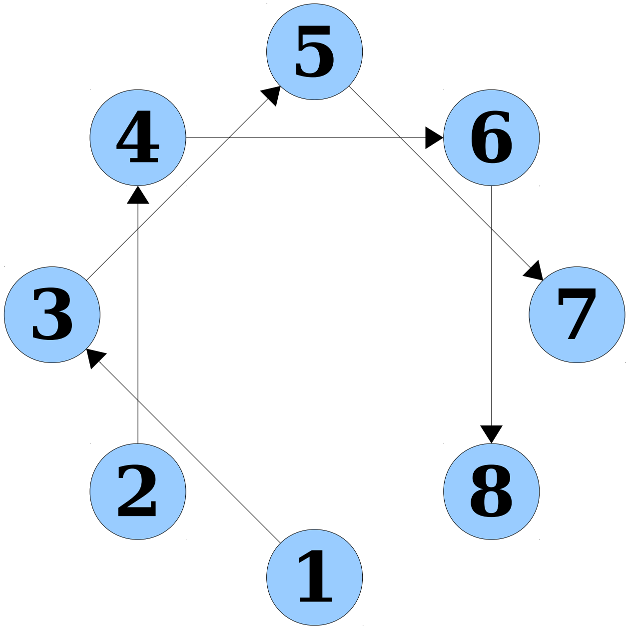
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: below is some relation over {1, 2, 3, 4} that's a totally valid relation even though there doesn't appear to be a simple unifying rule.



Below is a picture of a binary relation R over the set {1, 2, ..., 8}. Which of the following is a correct definition of the relation R?

A. xRy if $x = 3$ and $y = 5$
B. xRy if $y = x + 2$
C. yRx if $y = x + 2$
D. $R = +2$
E. None of these
F. More than one of these



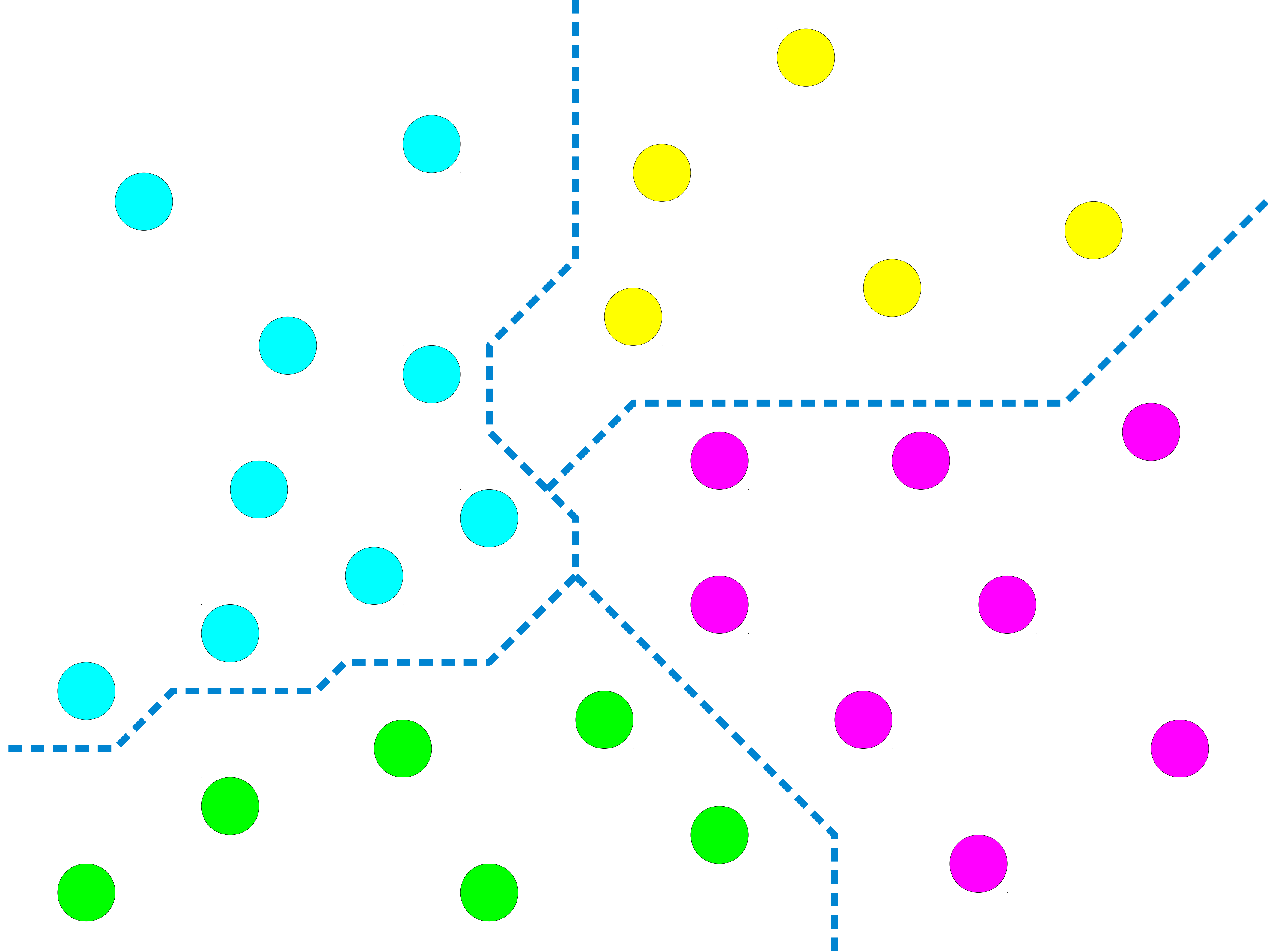
Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

Capturing Structure

Capturing Structure

- Binary relations are an excellent way for capturing certain structures that appear in computer science.
- Today, we'll look at one of them (***partitions***), and next time we'll see another (***prerequisites***).
- Along the way, we'll explore how to write proofs about definitions given in first-order logic.

Partitions



Partitions

- A ***partition of a set*** is a way of splitting the set into disjoint, nonempty subsets so that every element belongs to exactly one subset.
  - Two sets are ***disjoint*** if their intersection is the empty set; formally, sets S and T are disjoint if $S \cap T = \emptyset$.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.
- There doesn't have to be any rhyme or reason to what those pieces are, though often there is one.

Partitions and Clustering

- If you have a set of data, you can often learn something from the data by finding a "good" partition of that data and inspecting the partitions.
  - Usually, the term ***clustering*** is used in data analysis rather than *partitioning*.
- Interested to learn more? Take CS161 or CS246!

What's the connection between partitions and binary relations?

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

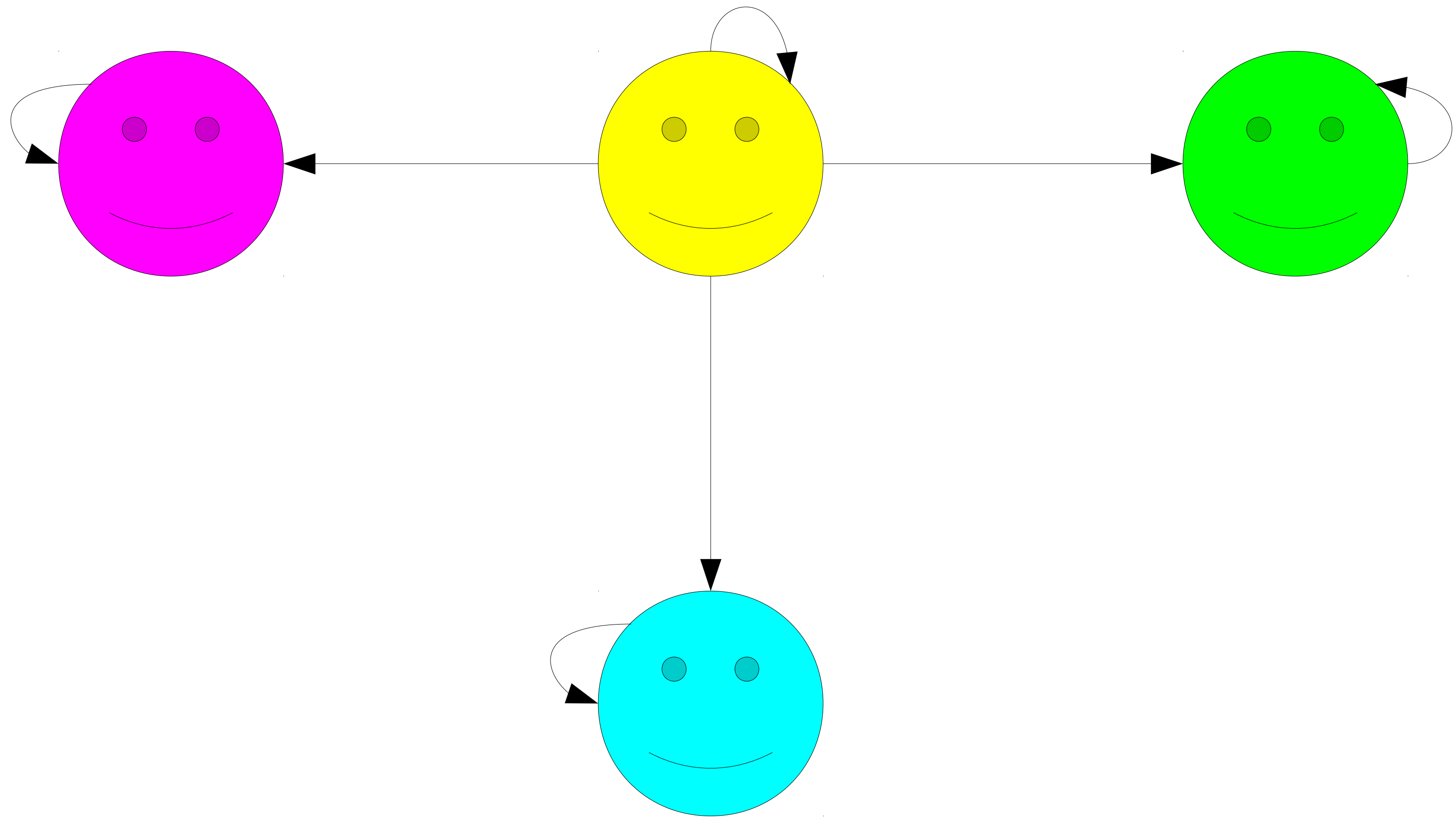
Reflexivity

- Some relations always hold from any element to itself.
- Examples:
  - $x = x$ for any x.
  - $A \subseteq A$ for any set A.
  - $x \equiv_k x$ for any x.
- Relations of this sort are called ***reflexive***.
- Formally speaking, a binary relation R over a set A is reflexive if the following first-order statement is true:

$$\forall a \in A.\ aRa$$

(*"Every element is related to itself."*)

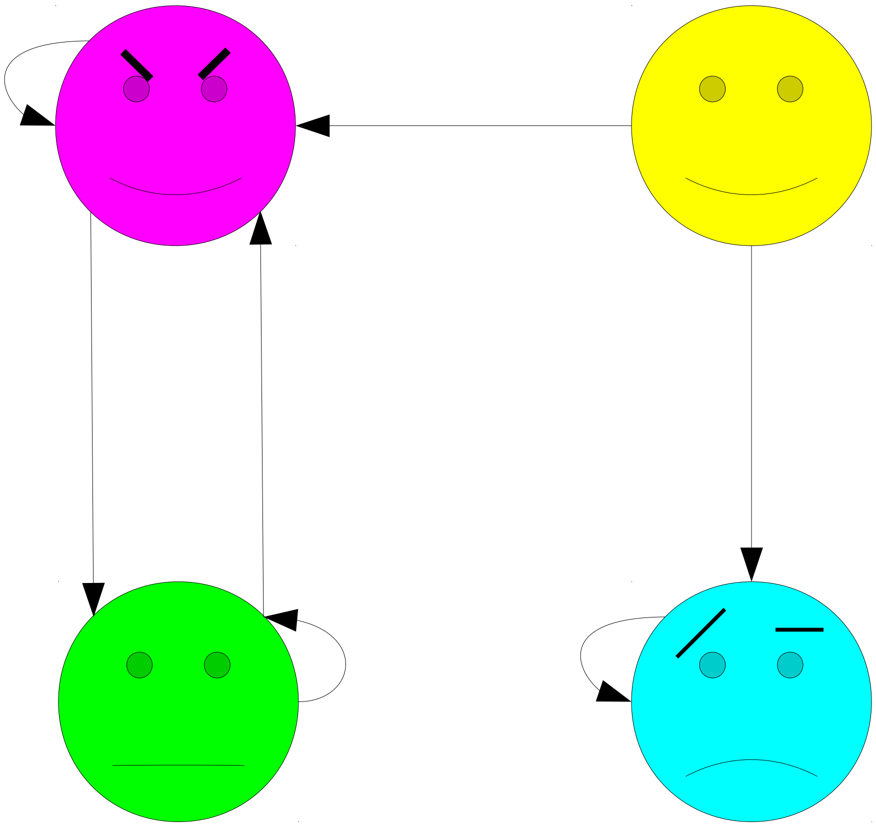
Reflexivity Visualized



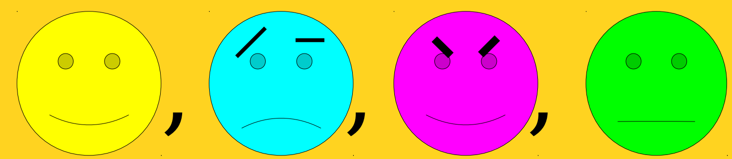
$\forall a \in A.\ aRa$

(*"Every element is related to itself."*)

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, **4**, or **5**.

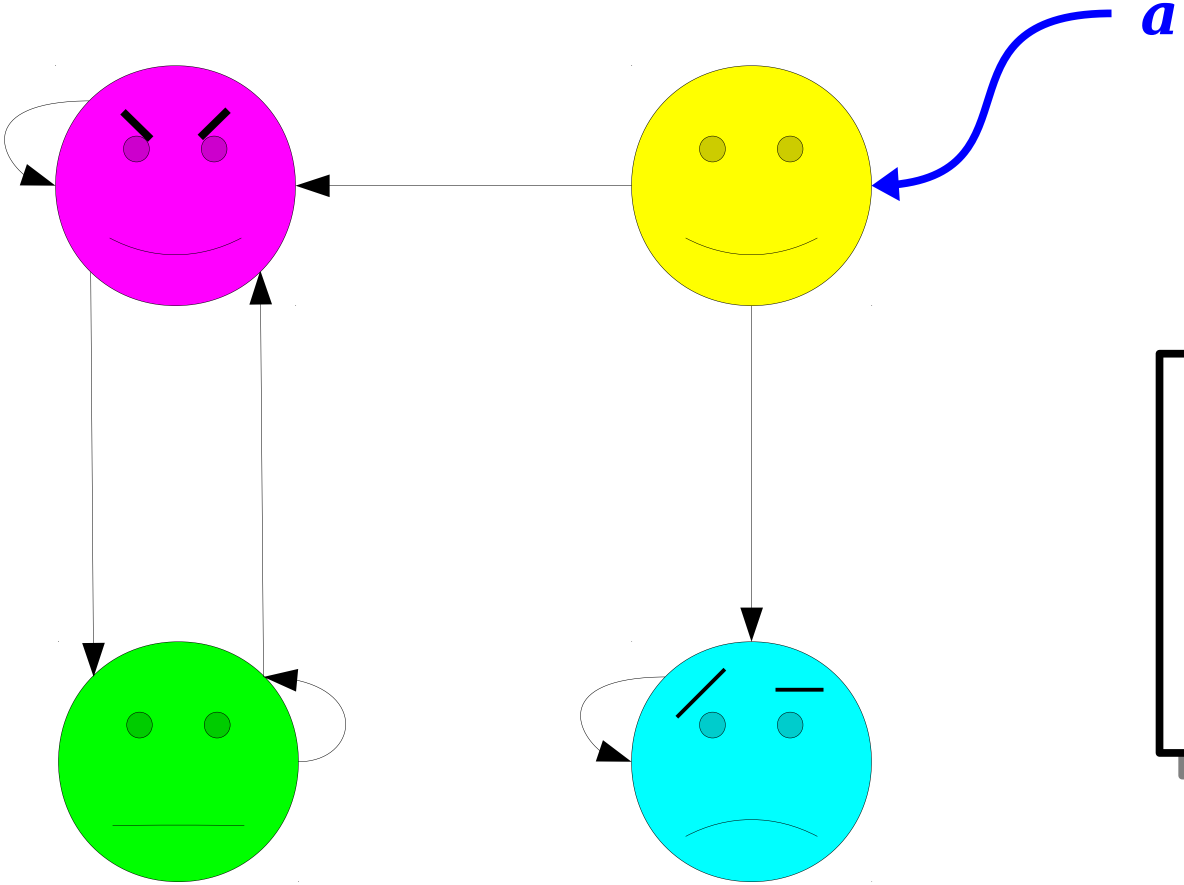


Let R be the binary relation given by the drawing to the left. How many of the following objects are reflexive?

R, 

$$\forall a \in A.\ aRa$$

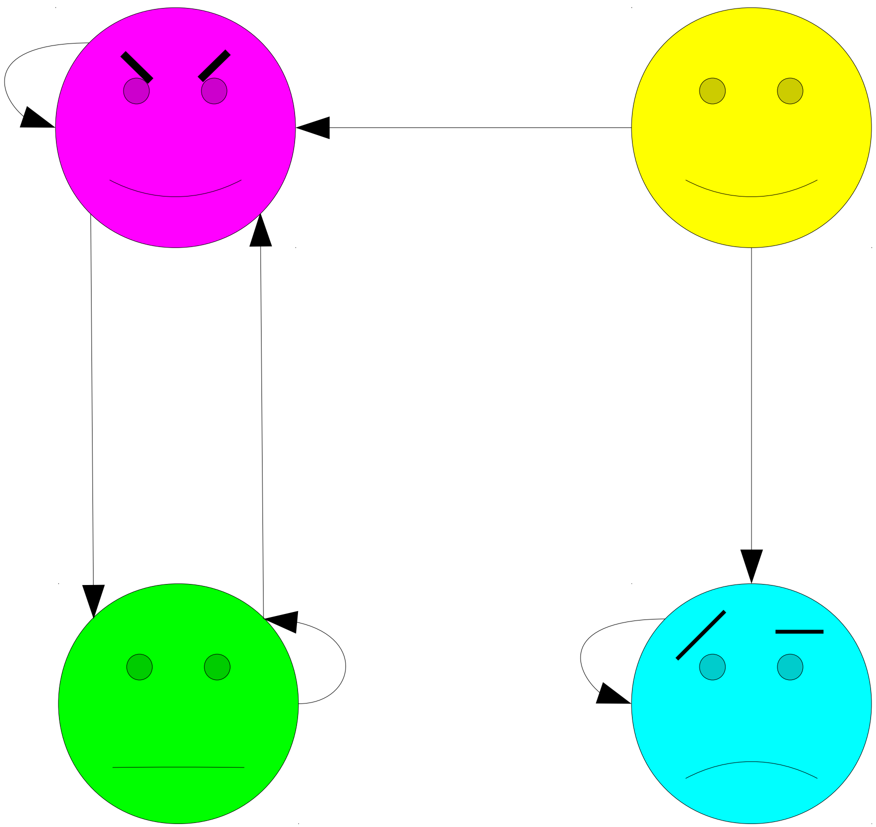
(“Every element is related to itself.”)

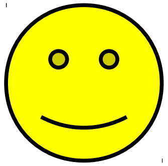


This means that R is not reflexive, since the first-order logic statement given below is not true.

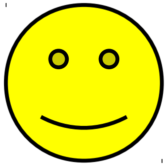
$$\forall a \in A.\ aRa$$

(*"Every element is related to itself."*)



Is  reflexive?

Reflexivity is a property of ***relations***, not ***individual objects***.

∀*a* ∈ ??. ***a***  ***a***

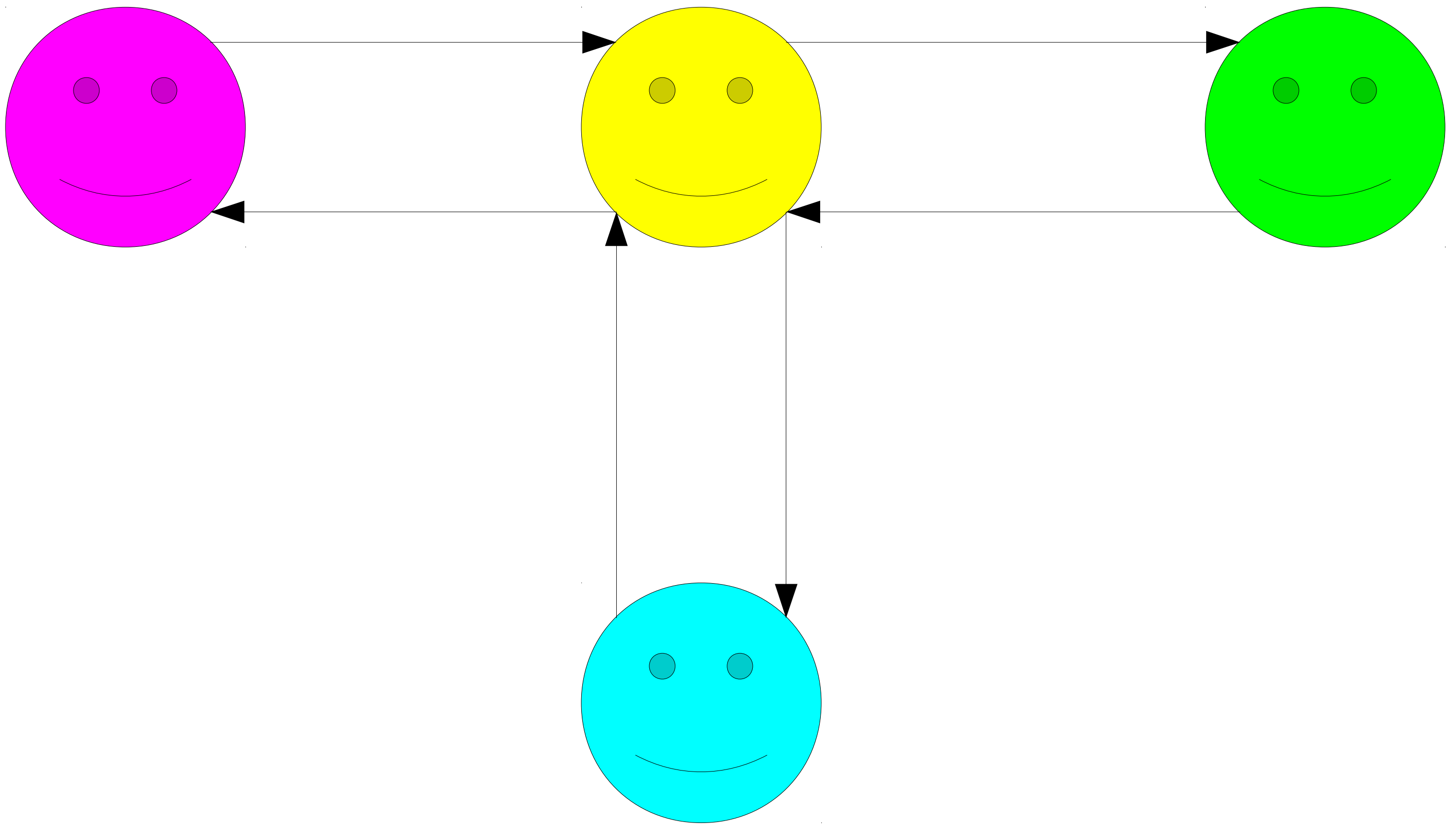
Symmetry

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $x = y$, then $y = x$.
  - If $x \equiv_k y$, then $y \equiv_k x$.
- These relations are called ***symmetric***.
- Formally: a binary relation R over a set A is called *symmetric* if the following first-order statement is true about R:

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

(*"If a is related to b, then b is related to a."*)

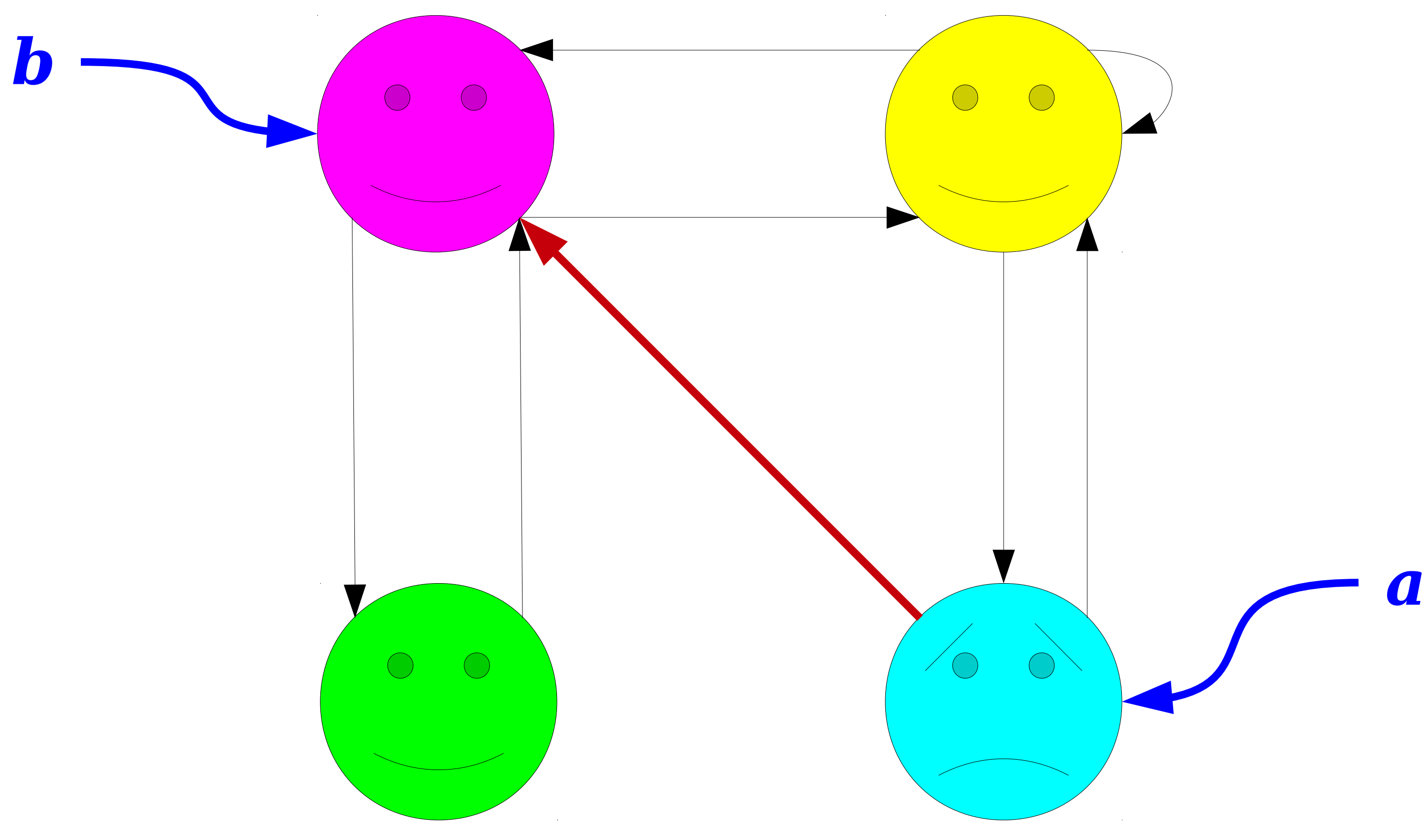
Symmetry Visualized



$$\forall a \in A. \forall b \in A. (aRb \rightarrow bRa)$$

("If a is related to b, then b is related to a.")

Is This Relation Symmetric?

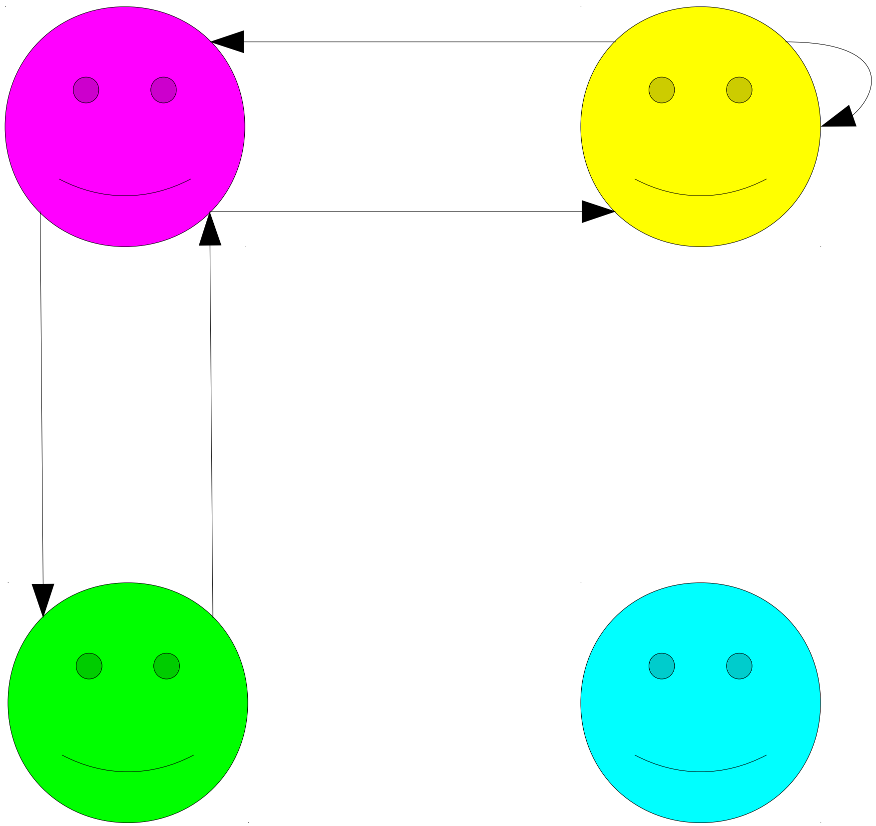


$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

(*"If a is related to b, then b is related to a."*)

Is this relation symmetric?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **Y** or **N**.



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

(*“If a is related to b, then b is related to a.”*)

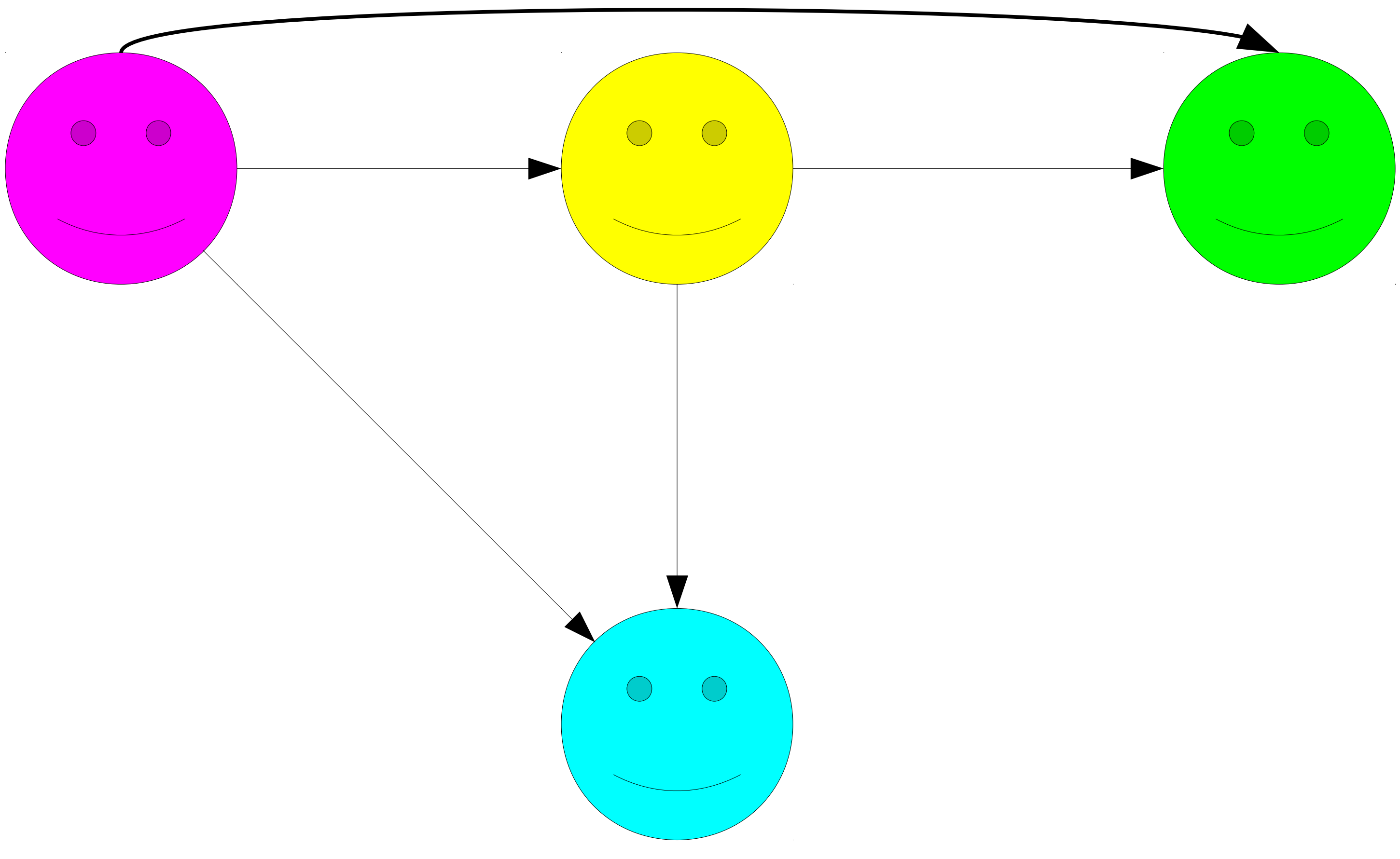
Transitivity

- Many relations can be chained together.
- Examples:
  - If $x = y$ and $y = z$, then $x = z$.
  - If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
- These relations are called ***transitive***.
- A binary relation R over a set A is called *transitive* if the following first-order statement is true about R:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

(*"Whenever a is related to b and b is related to c, we know a is related to c."*)

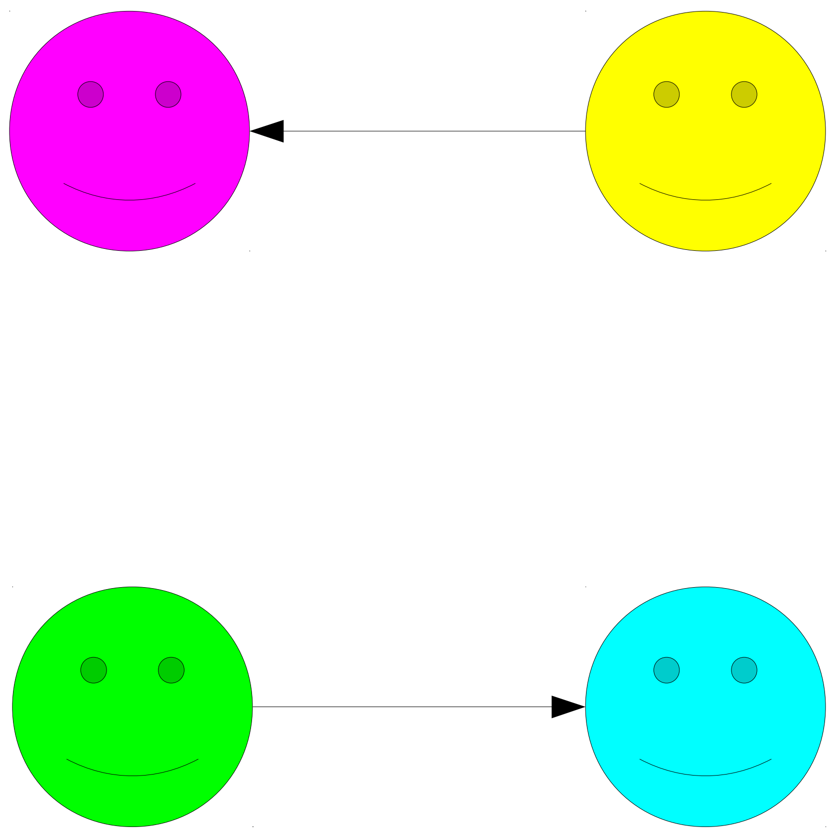
Transitivity Visualized



$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

(*"Whenever a is related to b and b is related to c, we know a is related to c.*)

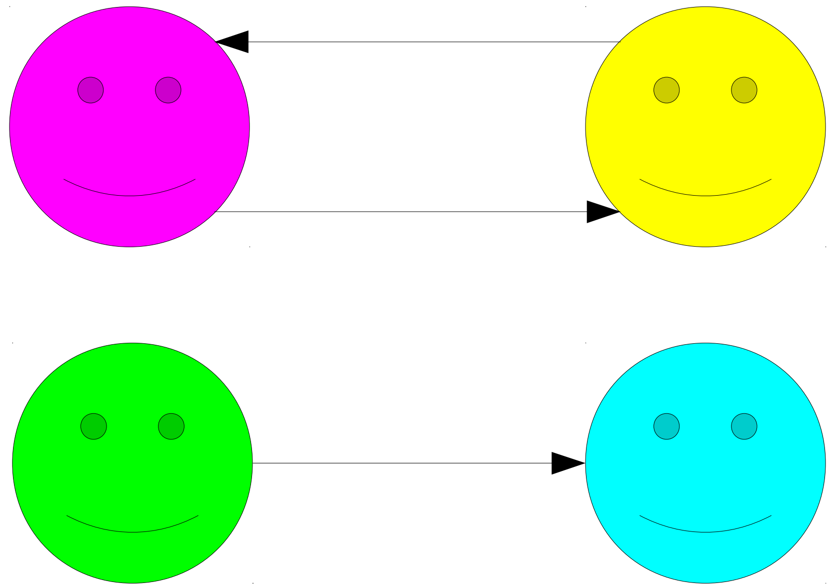
Is This Relation Transitive?



$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow aRc)$$

(*"Whenever a is related to b and b is related to c, we know a is related to c."*)

Is this relation transitive?



Answer at **PollEv.com/cs103** or
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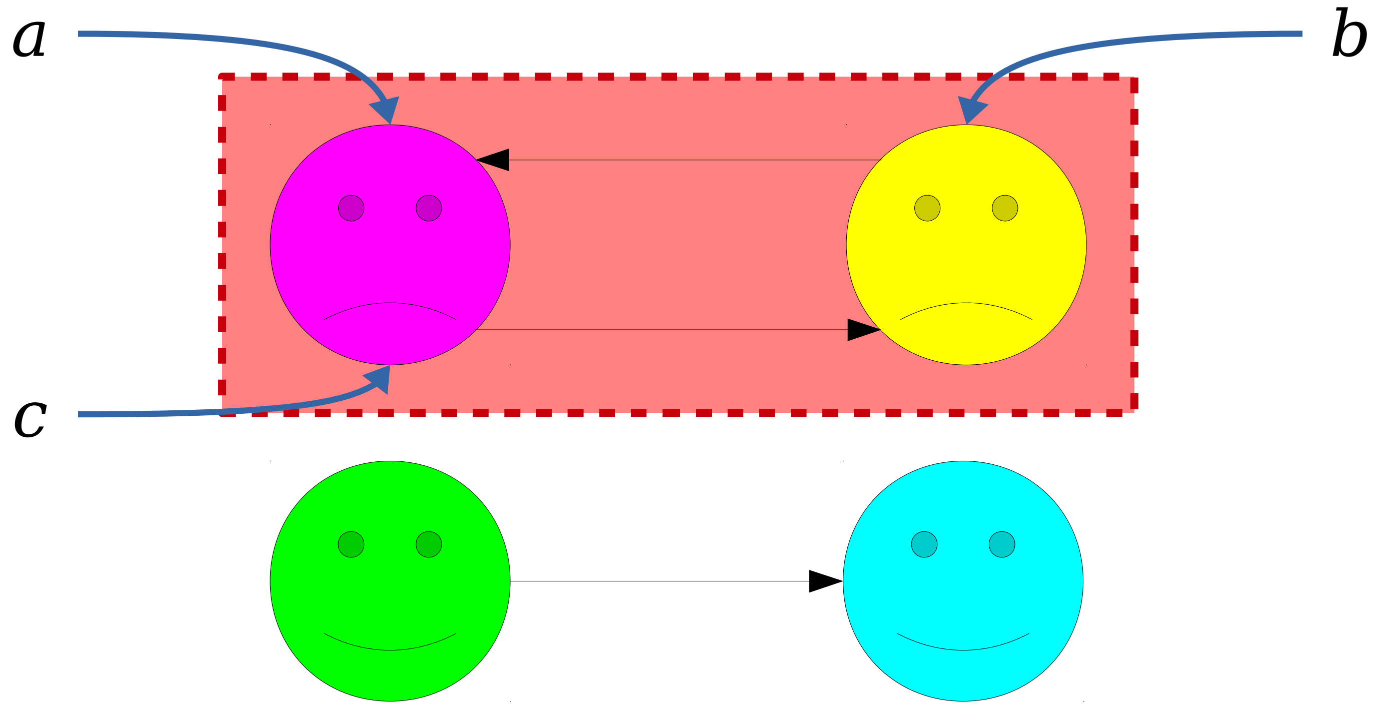
$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

(*"Whenever a is related to b and b is related to c, we know a is related to c."*)

Is This Relation Transitive?



$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$

(*"Whenever a is related to b and b is related to c, we know a is related to c.*)

Equivalence Relations

- An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.
- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - x has the same color as y
  - x has the same shape as y.

Binary relations give us a ***common language*** to describe ***common structures***.

Equivalence Relations

- Most modern programming languages include some sort of hash table data structure.
  - Java: `HashMap`
  - C++: `std::unordered_map`
  - Python: `dict`
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways…

Equivalence Relations

"The `equals` method implements an equivalence relation on non-null object references:

- It is *reflexive*: for any non-null reference value `x`, `x.equals(x)` should return true.
- It is *symmetric*: for any non-null reference values `x` and `y`, `x.equals(y)` should return true if and only if `y.equals(x)` returns true.
- It is *transitive*: for any non-null reference values `x`, `y`, and `z`, if `x.equals(y)` returns `true` and `y.equals(z)` returns true, then `x.equals(z)` should return `true`."

Java 8 Documentation

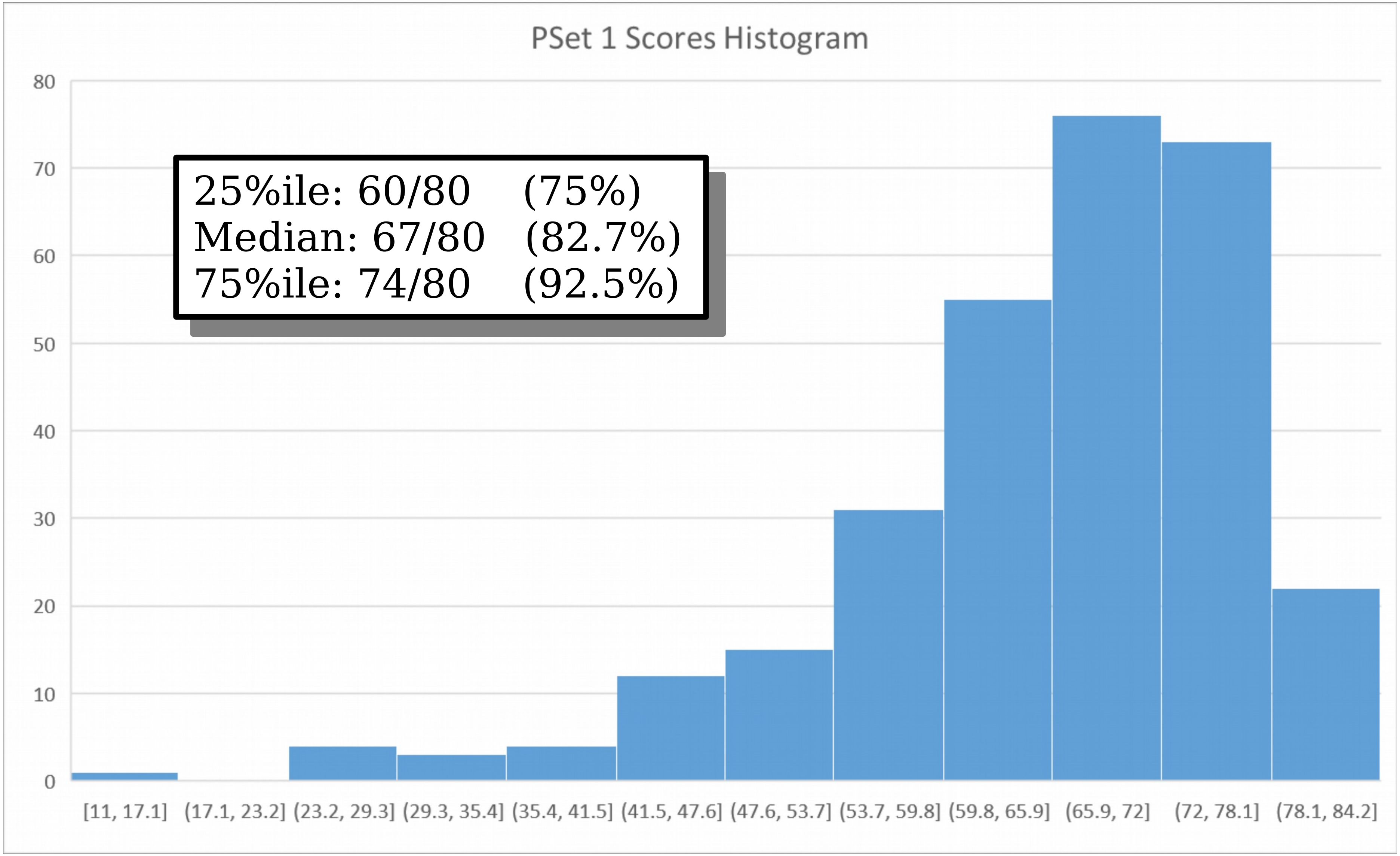
Equivalence Relations

"Each unordered associative container is parameterized by `Key`, by a function object type `Hash` that meets the Hash requirements (17.6.3.4) and acts as a hash function for argument values of type `Key`, and by a binary predicate `Pred` that induces an equivalence relation on values of type `Key`. Additionally, `unordered_map` and `unordered_multimap` associate an arbitrary mapped type `T` with the `Key`."

C++14 ISO Spec, §23.2.5/3

Time-Out for Announcements!

Interpreting your Pset 1 Grade



25%ile: 60/80 (75%)
Median: 67/80 (82.7%)
75%ile: 74/80 (92.5%)

Research Info Session

- CURIS (Undergraduate Research Institute "in" CS—har har har) is a summer research experience in our dept
- Unbelievable cutting-edge projects
- See if grad school might be of interest
- Learn more:

  Tuesday, 1/30 at 5:30pm in Gates 219

Back to CS103!

Equivalence Relation Proofs

- Let's suppose you've found a binary relation R over a set A and want to prove that it's an equivalence relation.
- How exactly would you go about doing this?

An Example Relation

- Consider the binary relation ~ defined over the set ℤ:

a~b **if** ***a+b*** **is even**

- Some examples:

0~4 1~9 2~6 5~5

- Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

a~b **if** ***some property of a and b holds***

This is the general template for defining a relation. Although we're using "if" rather than "iff" here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with "if" rather than "iff." Check the "Mathematical Vocabulary" handout for details.

What properties must ~ have to be an equivalence relation?

Reflexivity
Symmetry
Transitivity

Let's prove each property independently.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation ~ is reflexive.

Proof:

What is the formal definition of reflexivity?

$$\forall a \in \mathbb{Z}.\ a \sim a$$

Therefore, we'll choose an arbitrary integer a, then go prove that $a \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation ~ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the ~ relation, this means that we need to prove that $a+a$ is even.

To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even. Therefore, $a \sim a$ holds, as required. ■

***a*~*b* if *a*+*b* is even**

Lemma 2: The binary relation ~ is symmetric.

Which of the following works best as the opening of this proof?

A. Consider any integers a and b. We will prove $a \sim b$ and $b \sim a$.
B. Pick $\forall a \in \mathbb{Z}$ and $\forall b \in \mathbb{Z}$. We will prove $a \sim b \rightarrow b \sim a$.
C. Consider any integers a and b where $a \sim b$ and $b \sim a$.
D. Consider any integer a where $a \sim a$.
E. The relation ~ is symmetric if for any $a, b \in \mathbb{Z}$, we have $a \sim b \rightarrow b \sim a$.
F. Consider any integers a and b where $a \sim b$. We will prove $b \sim a$.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

$a \sim b$ if $a + b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof:

What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. (a \sim b \to b \sim a)$$

Therefore, we'll choose arbitrary integers a and b where $a \sim b$, then prove that $b \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required. ■

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation ~ is transitive.

Proof:

What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \to a \sim c)$$

Therefore, we'll choose arbitrary integers a, b, and c where $a \sim b$ and $b \sim c$, then prove that $a \sim c$.

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required. ■

An Observation

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the $\sim$ relation, this means that we need to prove that $a+a$ is even.

To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even. Therefore, $a \sim a$ holds, as required. ■

The formal definition of reflexivity is given in first-order logic, but this proof does not contain any first-order logic symbols!

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required. ■

The formal definition of symmetry is given in first-order logic, but this proof does not contain any first-order logic symbols!

$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

so

$$a+c = 2k +$$

So there is an integer r,
$a+c = 2r$. Thus $a+c$ is ev

The formal definition of transitivity is given in first-order logic, but this proof does not contain any first-order logic symbols!

First-Order Logic and Proofs

- First-order logic is an excellent tool for giving formal definitions to key terms.
- While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.
- Follow the example of these proofs:
  - Use the FOL definitions to determine what to assume and what to prove.
  - Write the proof in plain English using the conventions we set up in the first week of the class.
- ***Please, please, please, please, please internalize the contents of this slide!***