Context-Free Grammars

Describing Languages

- We've seen two models for the regular languages:
  - ***Finite automata*** accept precisely the strings in the language.
  - ***Regular expressions*** describe precisely the strings in the language.
- Finite automata ***recognize*** strings in the language.
  - Perform a computation to determine whether a specific string is in the language.
- Regular expressions ***match*** strings in the language.
  - Describe the general shape of all strings in the language.

Context-Free Grammars

- A ***context-free grammar*** (or ***CFG***) is an entirely different formalism for defining a class of languages.
- ***Goal:*** Give a description of a language by recursively describing the structure of the strings in the language.
- CFGs are best explained by example...

Arithmetic Expressions

- Suppose we want to describe all legal arithmetic expressions using addition, subtraction, multiplication, and division.
- Here is one possible CFG:

$$
\begin{aligned}
\mathbf{E} &\rightarrow \texttt{int} \\
\mathbf{E} &\rightarrow \mathbf{E\ Op\ E} \\
\mathbf{E} &\rightarrow \texttt{(}\mathbf{E}\texttt{)} \\
\mathbf{Op} &\rightarrow \texttt{+} \\
\mathbf{Op} &\rightarrow \texttt{-} \\
\mathbf{Op} &\rightarrow \texttt{*} \\
\mathbf{Op} &\rightarrow \texttt{/}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{E} \\
\Rightarrow\ & \mathbf{E\ Op\ E} \\
\Rightarrow\ & \mathbf{E\ Op}\ \texttt{(}\mathbf{E}\texttt{)} \\
\Rightarrow\ & \mathbf{E\ Op}\ \texttt{(}\mathbf{E\ Op\ E}\texttt{)} \\
\Rightarrow\ & \mathbf{E}\ \texttt{*}\ \texttt{(}\mathbf{E\ Op\ E}\texttt{)} \\
\Rightarrow\ & \texttt{int * (}\mathbf{E\ Op\ E}\texttt{)} \\
\Rightarrow\ & \texttt{int * (int}\ \mathbf{Op\ E}\texttt{)} \\
\Rightarrow\ & \texttt{int * (int}\ \mathbf{Op}\ \texttt{int)} \\
\Rightarrow\ & \texttt{int * (int + int)}
\end{aligned}
$$

Arithmetic Expressions

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\mathbf{Op} &\rightarrow \texttt{+} \\
\mathbf{Op} &\rightarrow \texttt{-} \\
\mathbf{Op} &\rightarrow \texttt{*} \\
\mathbf{Op} &\rightarrow \texttt{/}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{E} \\
\Rightarrow\ & \mathbf{E\ Op\ E} \\
\Rightarrow\ & \mathbf{E\ Op}\ \texttt{int} \\
\Rightarrow\ & \texttt{int}\ \mathbf{Op}\ \texttt{int} \\
\Rightarrow\ & \texttt{int / int}
\end{aligned}
$$

Context-Free Grammars

- Formally, a context-free grammar is a collection of four items:
  - A set of ***nonterminal symbols*** (also called ***variables***),
  - A set of ***terminal symbols*** (the ***alphabet*** of the CFG)
  - A set of ***production rules*** saying how each nonterminal can be replaced by a string of terminals and nonterminals, and
  - A ***start symbol*** (which must be a nonterminal) that begins the derivation.

E → `int`

E → **E Op E**

E → `(`**E**`)`

Op → `+`

Op → `-`

Op → `*`

Op → `/`

Some CFG Notation

- In today's slides, capital letters in **Bold Red Uppercase** will represent nonterminals.
  - e.g. **A**, **B**, **C**, **D**
- Lowercase letters in `blue monospace` will represent terminals.
  - e.g. `t`, `u`, `v`, `w`
- Lowercase Greek letters in ***gray italics*** will represent arbitrary strings of terminals and nonterminals.
  - e.g. ***α***, ***γ***, ***ω***
- You don't need to use these conventions on your own; just make sure whatever you do is readable. ☺

A Notational Shorthand

E → `int` | **E Op E** | `(`**E**`)`

Op → `+` | `-` | `*` | `/`

Derivations

E → E Op E | `int` | (E)
Op → `+` | `*` | `-` | `/`

E
⇒ E Op E
⇒ E Op `(`E`)`
⇒ E Op `(`E Op E`)`
⇒ E `*` `(`E Op E`)`
⇒ `int * (`E Op E`)`
⇒ `int * (int` Op E`)`
⇒ `int * (int` Op `int)`
⇒ `int * (int + int)`

- A sequence of steps where nonterminals are replaced by the right-hand side of a production is called a ***derivation***.
- If string α derives string ω, we write $\alpha \Rightarrow^* \omega$.
- In the example on the left, we see E $\Rightarrow^*$ `int * (int + int)`.

The Language of a Grammar

- If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \boldsymbol{\omega} \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \boldsymbol{\omega} \}$$

- That is, $\mathscr{L}(G)$ is the set of strings of terminals derivable from the start symbol.

If G is a CFG with alphabet Σ and start symbol **S**,
then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \omega \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \omega \}$$

Consider the following CFG G over $\Sigma = \{$**a**, **b**, **c**, **d**$\}$:

$$\mathbf{S} \rightarrow \mathbf{Sa} \mid \mathbf{dT}$$
$$\mathbf{T} \rightarrow \mathbf{bTb} \mid \mathbf{c}$$

How many of the following strings are in $\mathscr{L}(G)$?

dca
cad
bcb
dTaa

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **a number**.

Context-Free Languages

- A language L is called a ***context-free language*** (or CFL) if there is a CFG G such that $L = \mathscr{L}(G)$.
- Questions:
  - What languages are context-free?
  - How are context-free and regular languages related?

From Regexes to CFGs

- CFGs consist purely of production rules of the form **A** → ***ω***. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

S → **`a*b`**

From Regexes to CFGs

- CFGs consist purely of production rules of the form **A** → ***ω***. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

$$
\begin{aligned}
\mathbf{S} &\rightarrow \mathbf{A}\mathtt{b} \\
\mathbf{A} &\rightarrow \mathbf{A}\mathtt{a} \mid \varepsilon
\end{aligned}
$$

From Regexes to CFGs

- CFGs consist purely of production rules of the form **A** → ***ω***. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

S → **a(b ∪ c*)**

From Regexes to CFGs

- CFGs consist purely of production rules of the form **A** → ***ω***. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{X}$$
$$\mathbf{X} \rightarrow \mathtt{b} \mid \mathbf{C}$$
$$\mathbf{C} \rightarrow \mathbf{C}\mathtt{c} \mid \varepsilon$$

Regular Languages and CFLs

- ***Theorem:*** Every regular language is context-free.
- ***Proof Idea:*** Use the construction from the previous slides to convert a regular expression for L into a CFG for L. ■
- ***Problem Set 8 Exercise:*** Instead, show how to convert a DFA/NFA into a CFG.

The Language of a Grammar

- Consider the following CFG G:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

a	a	a	a	b	b	b	b

$$\mathscr{L}(G) = \{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$$


Regular Languages
CFLs
All Languages

Why the Extra Power?

- Why do CFGs have more power than regular expressions?
- ***Intuition:*** Derivations of strings have unbounded "memory."

S → **aSb** | **ε**

a	a	a	a	b	b	b	b

Time-Out for Announcements!

Midterm Exam Logistics

- The next midterm is tonight from ***7:00PM – 10:00PM***. Locations are divvied up by last (family) name:
  - `A-I`: Go to ***Cubberley Auditorium***.
  - `J-Z`: Go to ***Cemex Auditorium***.
- The exam focuses on Lecture 06 – 13 (binary relations through induction) and PS3 – PS5. Finite automata onward is *not* tested.
  - Topics from earlier in the quarter (proofwriting, first-order logic, set theory, etc.) are also fair game, but that's primarily because the later material builds on this earlier material.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.

Our Advice

- ***Eat dinner tonight.*** You are not a brain in a jar. You are a rich, complex, beautiful biological system. Please take care of yourself.
- ***Read all the questions before diving into them.*** Tunnel vision can hurt you on an exam. There's evidence that spreading your time out leads to better outcomes.
- ***Reflect on how far you've come.*** How many of these questions would you have been able to *understand* two months ago? That's the mark that you're learning something!

Three Questions

- What is something you know now that, at the start of the quarter, you knew you didn’t know?
- What is something you know now that, at the start of the quarter, you *didn’t* know that you didn’t know?
- What is something you *don’t* know that, at the start of the quarter, you *didn’t* know that you didn’t know?

Back to CS103!

Designing CFGs

- Like designing DFAs, NFAs, and regular expressions, designing CFGs is a craft.
- When thinking about CFGs:
  - ***Think recursively:*** Build up bigger structures from smaller ones.
  - ***Have a construction plan:*** Know in what order you will build up the string.
  - ***Store information in nonterminals:*** Have each nonterminal correspond to some useful piece of information.

Designing CFGs

- Let $\Sigma = \{$**a**, **b**$\}$ and let $L = \{w \in \Sigma^* \mid w$ is a palindrome $\}$
- We can design a CFG for L by thinking inductively:
  - Base case: ε, **a**, and **b** are palindromes.
  - If ω is a palindrome, then **a**ω**a** and **b**ω**b** are palindromes.
  - No other strings are palindromes.

$$S \to \varepsilon \mid a \mid b \mid aSa \mid bSb$$

Designing CFGs

- Let Σ = {**(**, **)**} and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced parentheses }
- Some sample strings in L:

((()))

(())()

(()())(()())

((((()))(())))

ε

()()

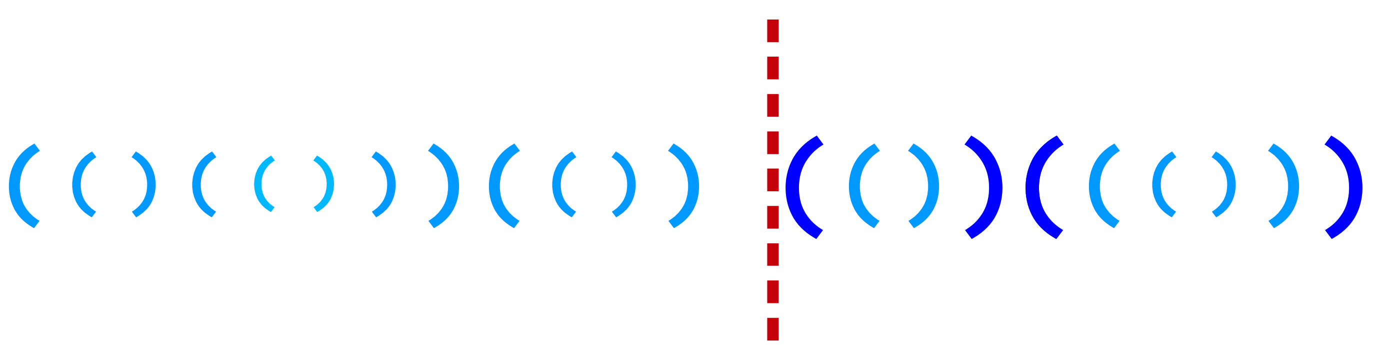
Designing CFGs

- Let $\Sigma = \{$(,)$\}$ and let $L = \{ w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis.

((()(()))(()))(())((()))

Designing CFGs

- Let Σ = {(,)} and let $L = \{ w \in \Sigma^* \mid w$ is a string of balanced parentheses }
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis.



Designing CFGs

- Let $\Sigma = \{(,)\}$ and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis. Removing the first parenthesis and the matching parenthesis forms two new strings of balanced parentheses.

$$\mathbf{S} \rightarrow \mathbf{(S)S} \mid \boldsymbol{\varepsilon}$$

Designing CFGs

- Let $\Sigma = \{$**a**, **b**$\}$ and let $L = \{ w \in \Sigma^* \mid w$ has the same number of **a**'s and **b**'s $\}$

How many of the following CFGs have language L?

S → aSb | bSa | ε

S → abS | baS | ε

S → abSba | baSab | ε

S → SbaS | SabS | ε

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **a number**.

Designing CFGs: A Caveat

- When designing a CFG for a language, make sure that it
  - generates all the strings in the language and
  - never generates a string outside the language.
- The first of these can be tricky – make sure to test your grammars!
- You'll design your own CFG for this language on Problem Set 8.

CFG Caveats II

- Is the following grammar a CFG for the language $\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$?

$$S \to \mathtt{a}S\mathtt{b}$$

- What strings in $\{\mathtt{a}, \mathtt{b}\}^*$ can you derive?
  - Answer: ***None!***
- What is the language of the grammar?
  - Answer: **Ø**
- When designing CFGs, make sure your recursion actually terminates!

Designing CFGs

- When designing CFGs, remember that each nonterminal can be expanded out independently of the others.
- Let $\Sigma = \{$a$,\ \stackrel{?}{=}\}$ and let $L = \{$a$^n \stackrel{?}{=}$ a$^n \mid n \in \mathbb{N}\}$.
- Is the following a CFG for L?

S → **X**≟**X**

X → a**X** | ε

S
⇒ **X**≟**X**
⇒ a**X**≟**X**
⇒ aa**X**≟**X**
⇒ aa≟**X**
⇒ aa≟a**X**
⇒ aa≟a

Finding a Build Order

- Let $\Sigma = \{$ **a**, **≟** $\}$ and let $L = \{$ $\mathtt{a}^n \stackrel{?}{=} \mathtt{a}^n \mid n \in \mathbb{N}$ $\}$.
- To build a CFG for L, we need to be more clever with how we construct the string.
  - If we build the strings of **a**'s independently of one another, then we can't enforce that they have the same length.
  - ***Idea:*** Build both strings of **a**'s at the same time.
- Here's one possible grammar based on that idea:

$$S \rightarrow \stackrel{?}{=} \mid aSa$$

$$
\begin{aligned}
& S \\
\Rightarrow\ & aSa \\
\Rightarrow\ & aaSaa \\
\Rightarrow\ & aaaSaaa \\
\Rightarrow\ & aaa\stackrel{?}{=}aaa
\end{aligned}
$$

Function Prototypes

- Let Σ = {`void`, `int`, `double`, `name`, `(`, `)`, `,`, `;`}.
- Let's write a CFG for C-style function prototypes!
- Examples:
  - `void name(int name, double name);`
  - `int name();`
  - `int name(double name);`
  - `int name(int, int name, int);`
  - `void name(void);`

Function Prototypes

- Here's one possible grammar:
  - **S** → **Ret** `name` `(`**Args**`);`
  - **Ret** → **Type** | `void`
  - **Type** → `int` | `double`
  - **Args** → `ε` | `void` | **ArgList**
  - **ArgList** → **OneArg** | **ArgList**, **OneArg**
  - **OneArg** → **Type** | **Type** `name`
- Fun question to think about: what changes would you need to make to support pointer types?

Summary of CFG Design Tips

- Look for recursive structures where they exist: they can help guide you toward a solution.
- Keep the build order in mind – often, you'll build two totally different parts of the string concurrently.
  - Usually, those parts are built in opposite directions: one's built left-to-right, the other right-to-left.
- Use different nonterminals to represent different structures.

Applications of Context-Free Grammars

CFGs for Programming Languages

BLOCK	→		STMT
		|	`{` STMTS `}`
STMTS	→		ε
		|	STMT STMTS
STMT	→		EXPR`;`
		|	`if (`EXPR`)` BLOCK
		|	`while (`EXPR`)` BLOCK
		|	`do` BLOCK `while (`EXPR`);`
		|	BLOCK
		|	…
EXPR	→		`identifier`
		|	`constant`
		|	EXPR `+` EXPR
		|	EXPR `-` EXPR
		|	EXPR `*` EXPR
		|	…

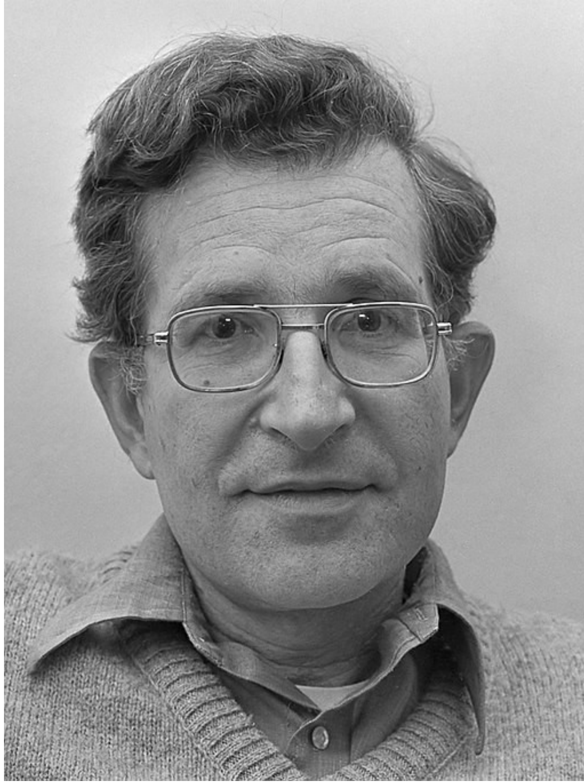
Grammars in Compilers

- One of the key steps in a compiler is figuring out what a program “means.”
- This is usually done by defining a grammar showing the high-level structure of a programming language.
- There are certain classes of grammars (LL(1) grammars, LR(1) grammars, LALR(1) grammars, etc.) for which it's easy to figure out how a particular string was derived.
- Tools like `yacc` or `bison` automatically generate parsers from these grammars.
- Curious to learn more? Take CS143!

Natural Language Processing

- By building context-free grammars for actual languages and applying statistical inference, it's possible for a computer to recover the likely meaning of a sentence.
  - In fact, CFGs were first called ***phrase-structure grammars*** and were introduced by Noam Chomsky in his seminal work *Syntactic Structures*.
  - They were then adapted for use in the context of programming languages, where they were called ***Backus-Naur forms***.
- Stanford's CoreNLP project is one place to look for an example of this.
- Want to learn more? Take CS124 or CS224N!

Biography Minute: Noam Chomsky



PC: Hans Peters / Anefo (via Wikimedia)

- Invented CFGs!
- Helped found fields of linguistics and cognitive science

- Today, perhaps more well known for political writing than linguistics
  - Made it onto President Nixon's "Enemies List"
  - Anti-capitalism, anti-imperialism, anti-war
  - Drawing on linguistics expertise, written extensively on state propaganda (*Manufacturing Consent*)

Next Time

- ***Turing Machines***
  - What does a computer with unbounded memory look like?
  - How would you program it?