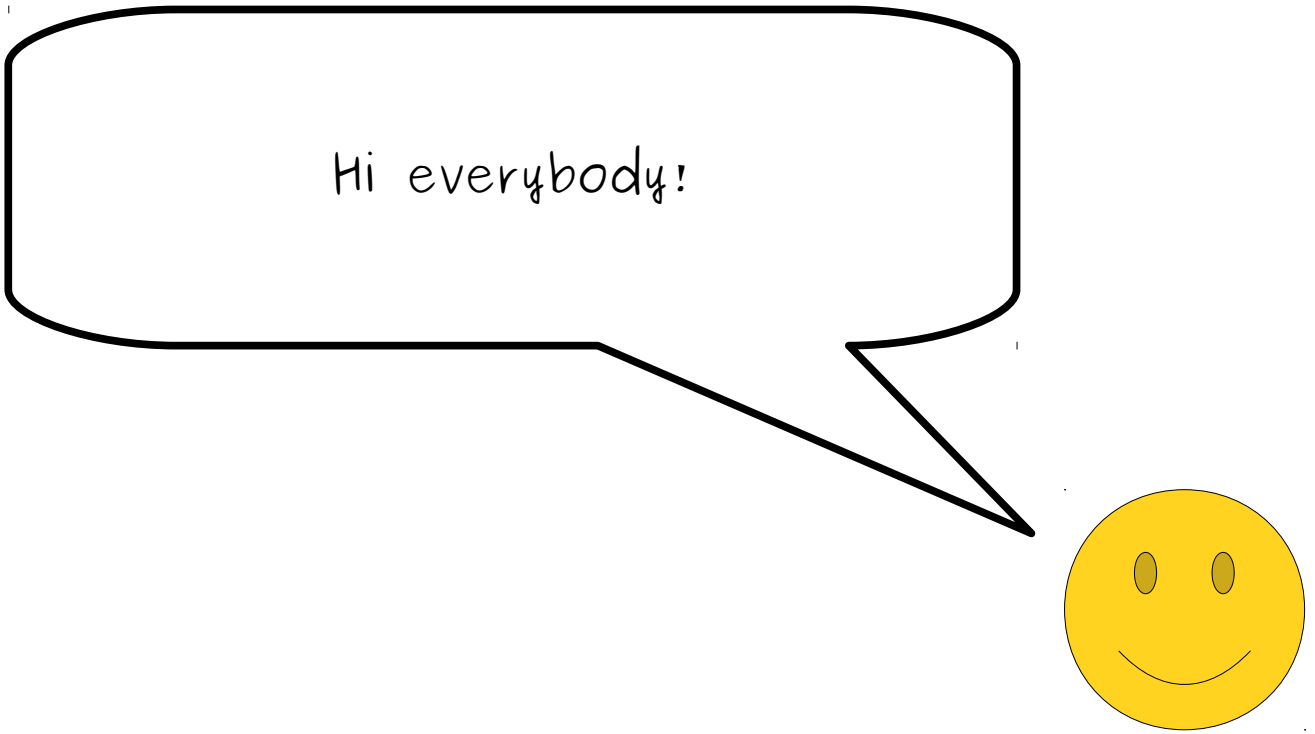
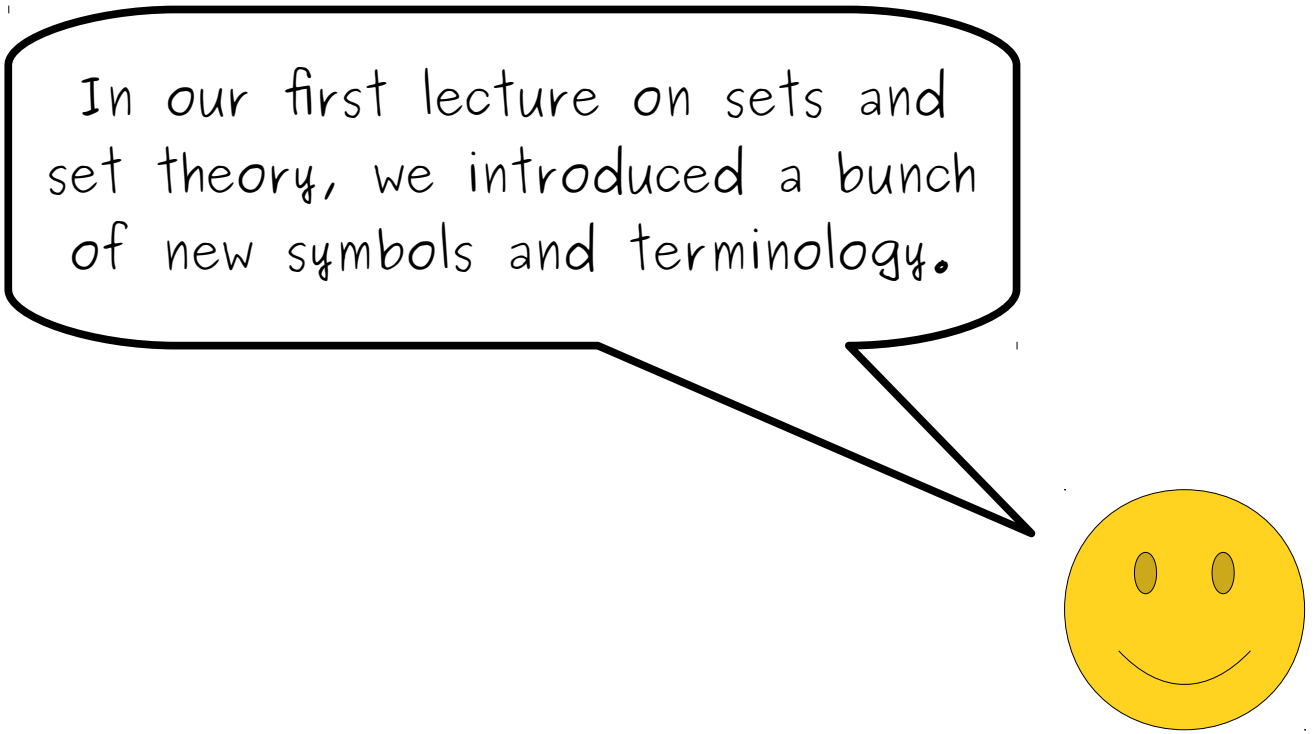
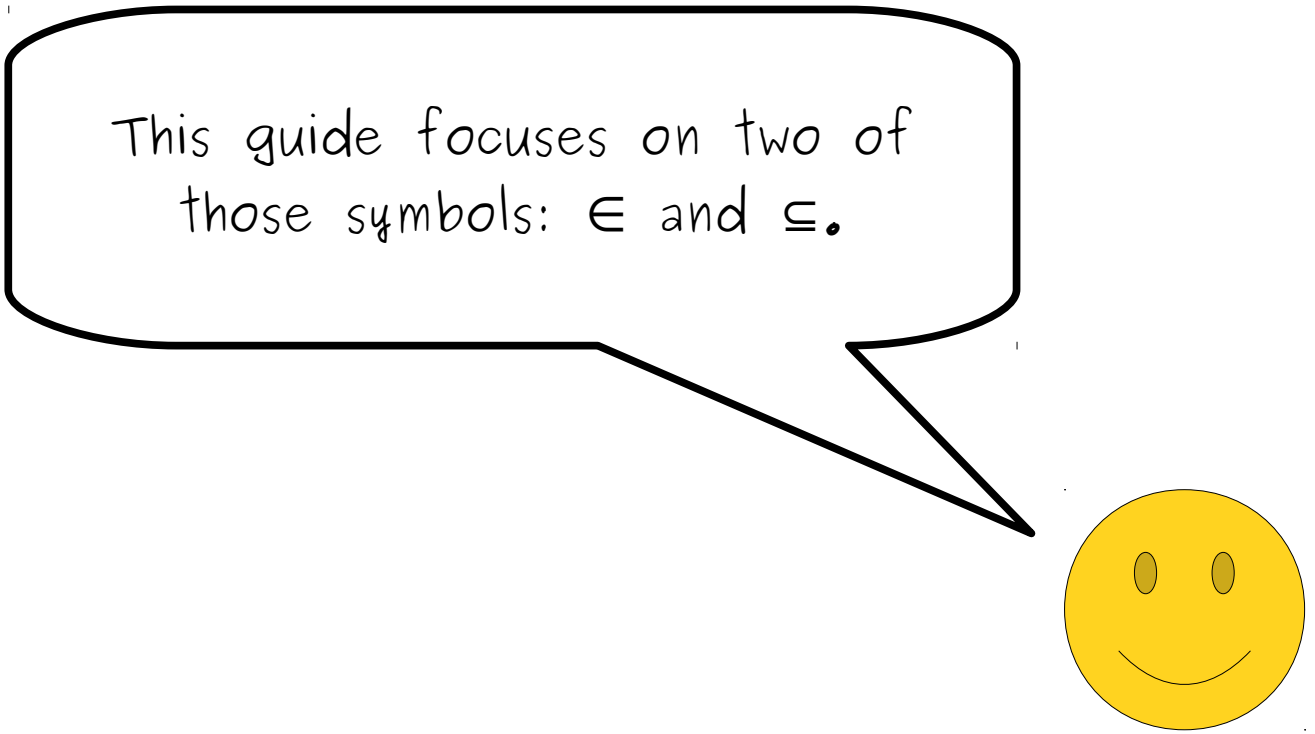
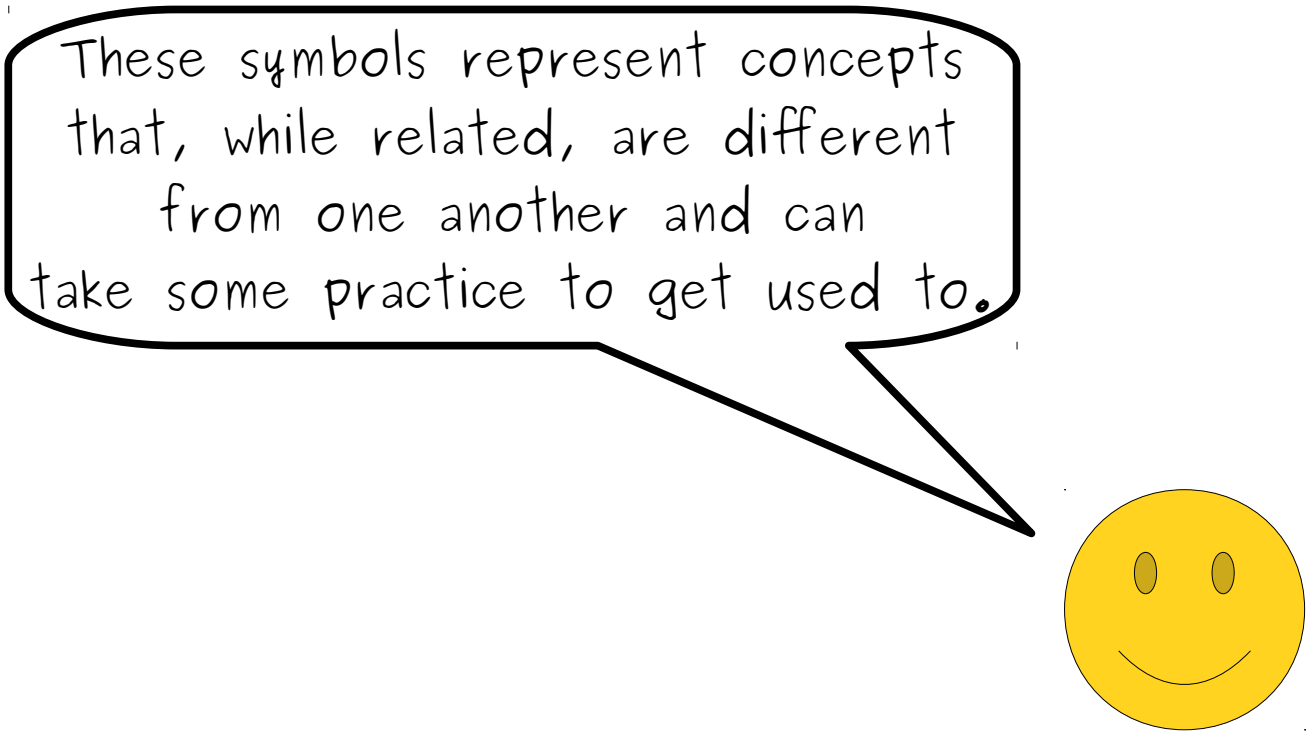
Guide to $\in$ and $\subseteq$


Hi everybody!

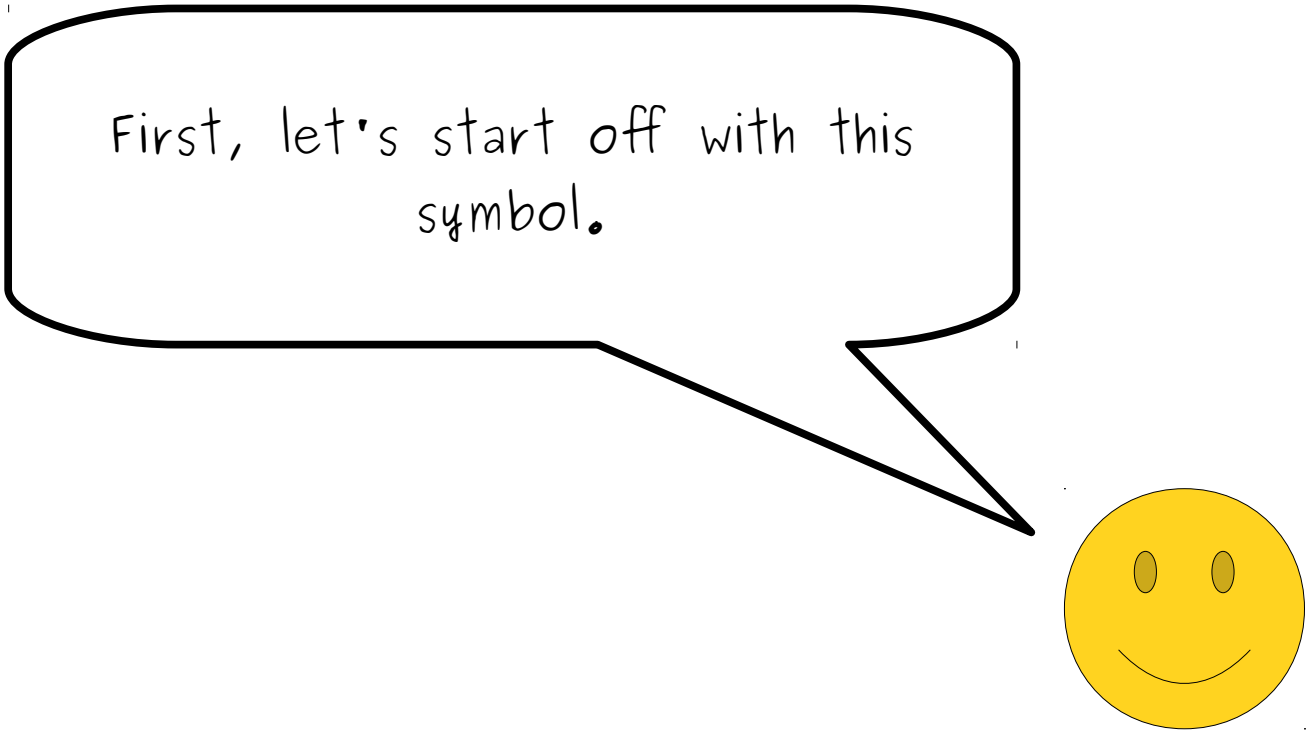

In our first lecture on sets and set theory, we introduced a bunch of new symbols and terminology.


This guide focuses on two of those symbols: ∈ and ⊆.

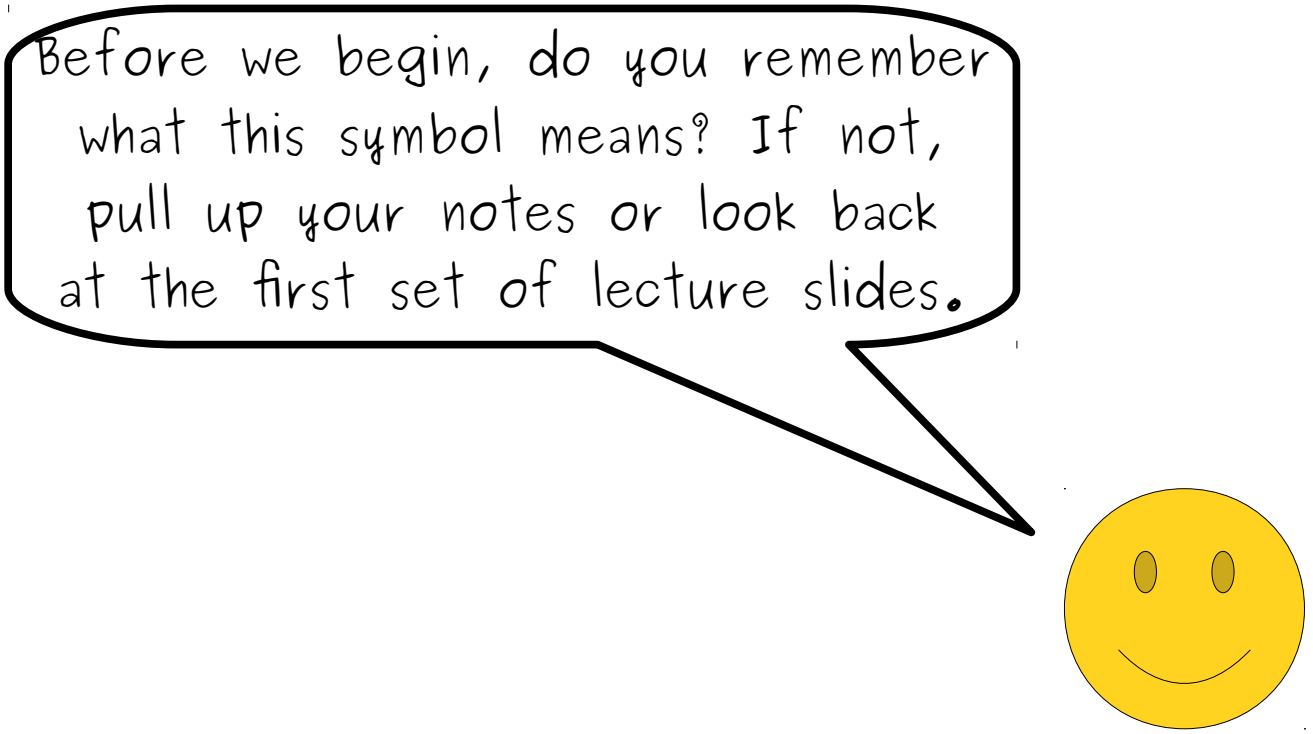

These symbols represent concepts
that, while related, are different
from one another and can
take some practice to get used to.

If you're still a bit confused, don't worry! Let's take some time to review them and see how they work and how they differ.

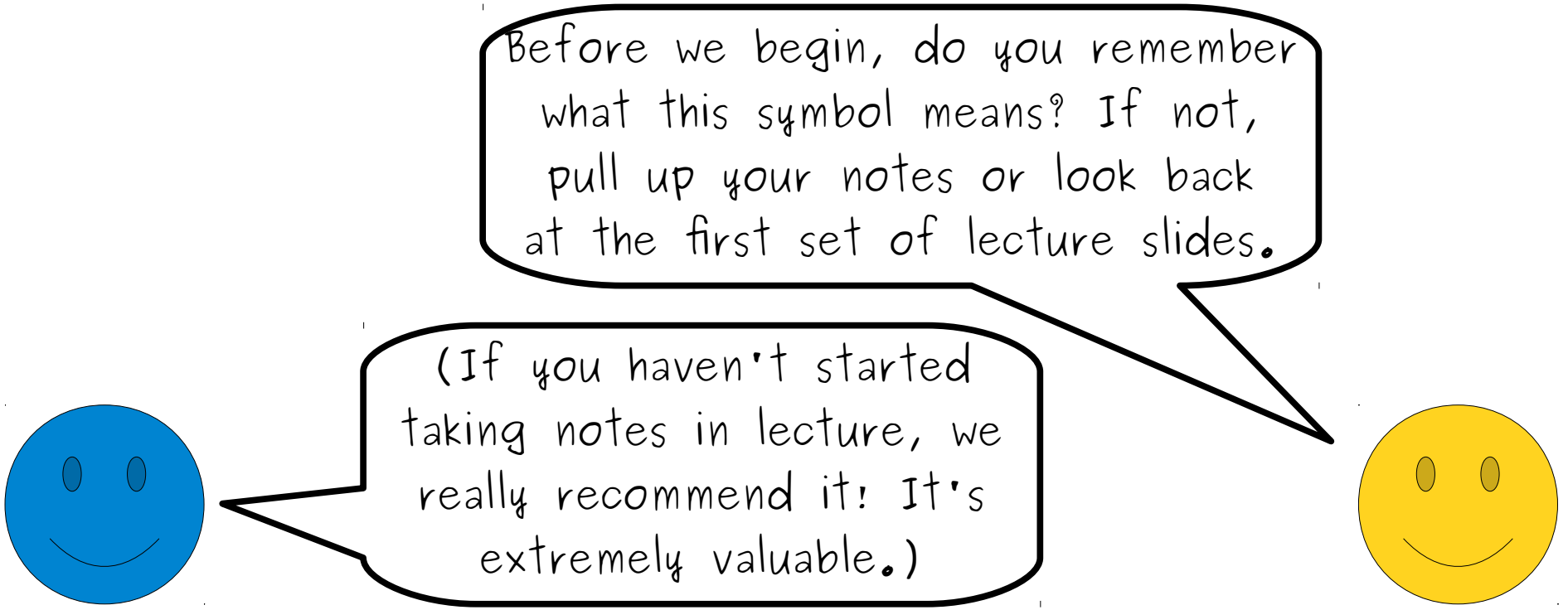
$\in$



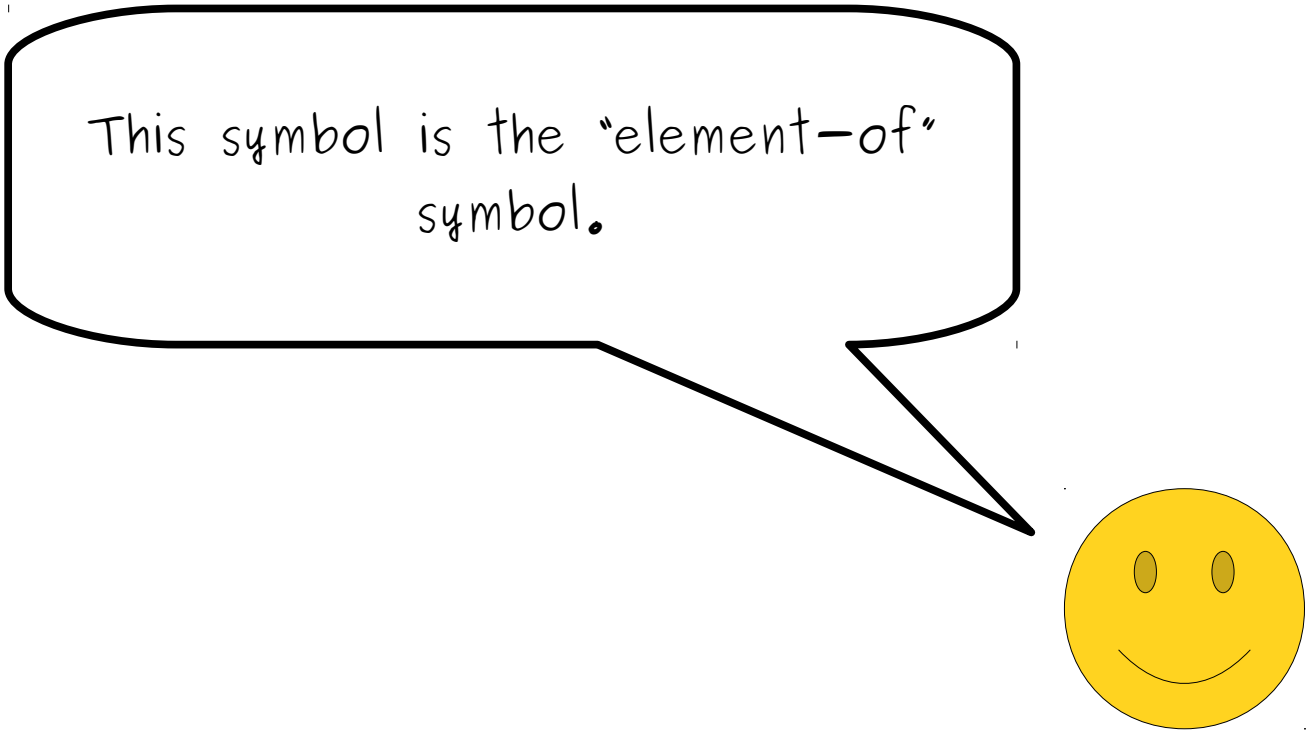
$\in$

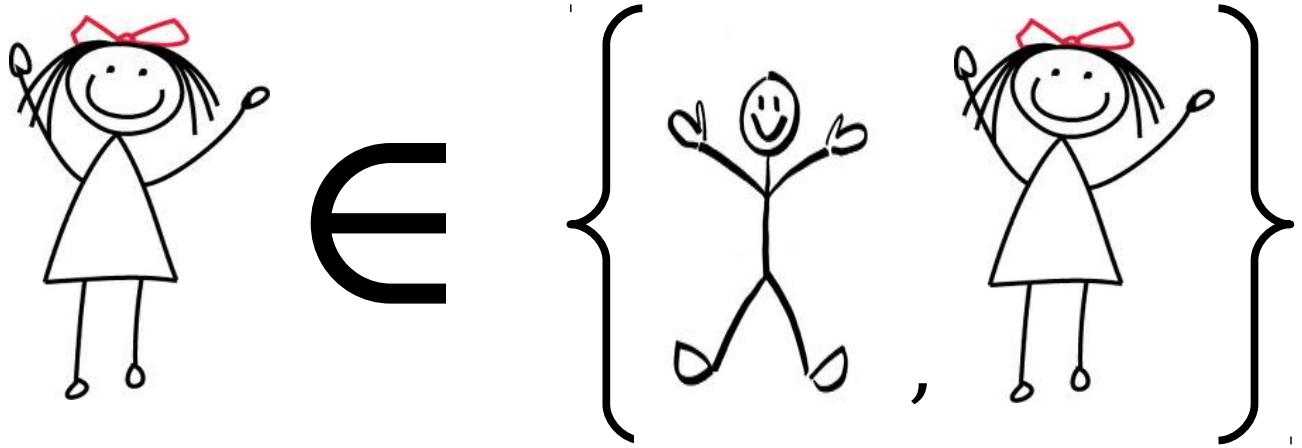

Before we begin, do you remember what this symbol means? If not, pull up your notes or look back at the first set of lecture slides.

$\in$

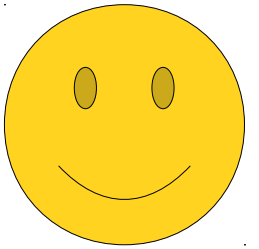

Before we begin, do you remember what this symbol means? If not, pull up your notes or look back at the first set of lecture slides.
(If you haven't started taking notes in lecture, we really recommend it! It's extremely valuable.)

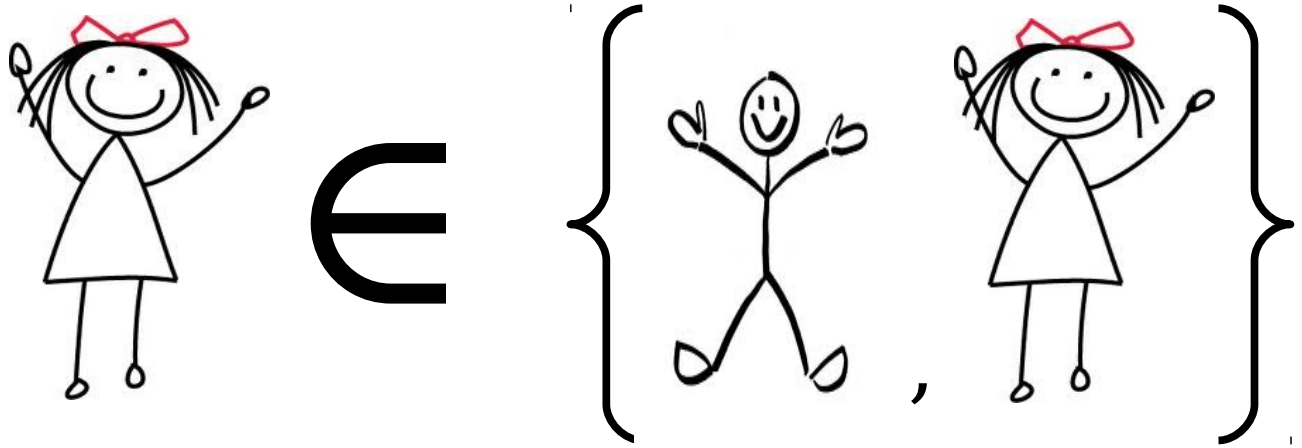
$\in$



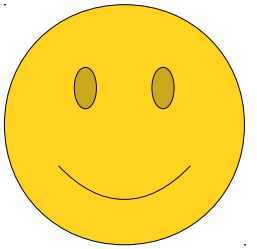


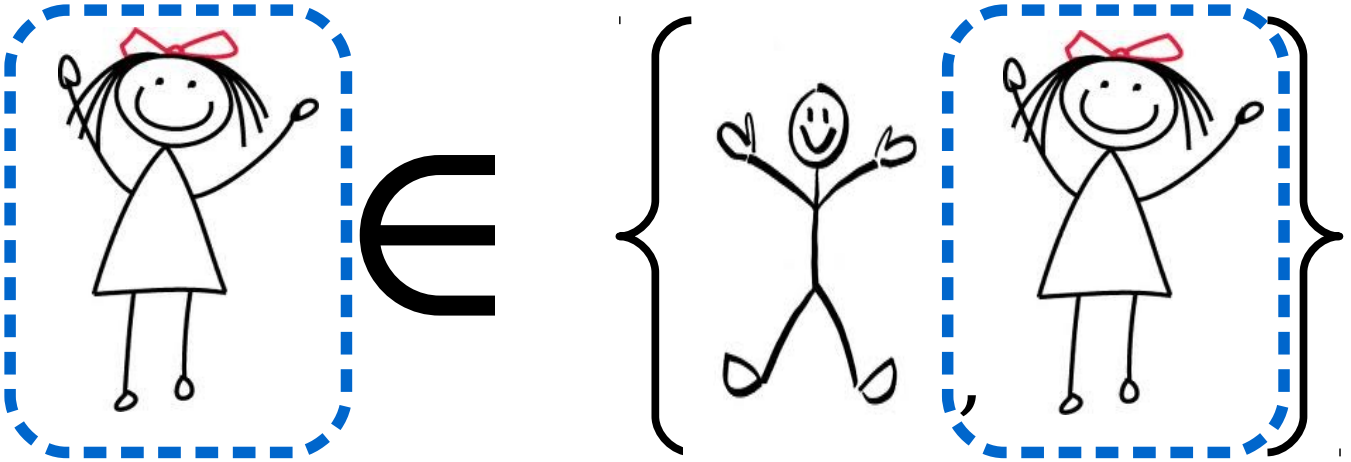
It's used to indicate that something is an *element* of a set.



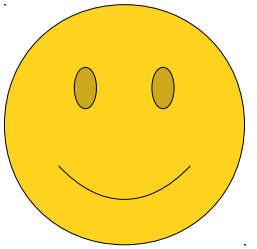


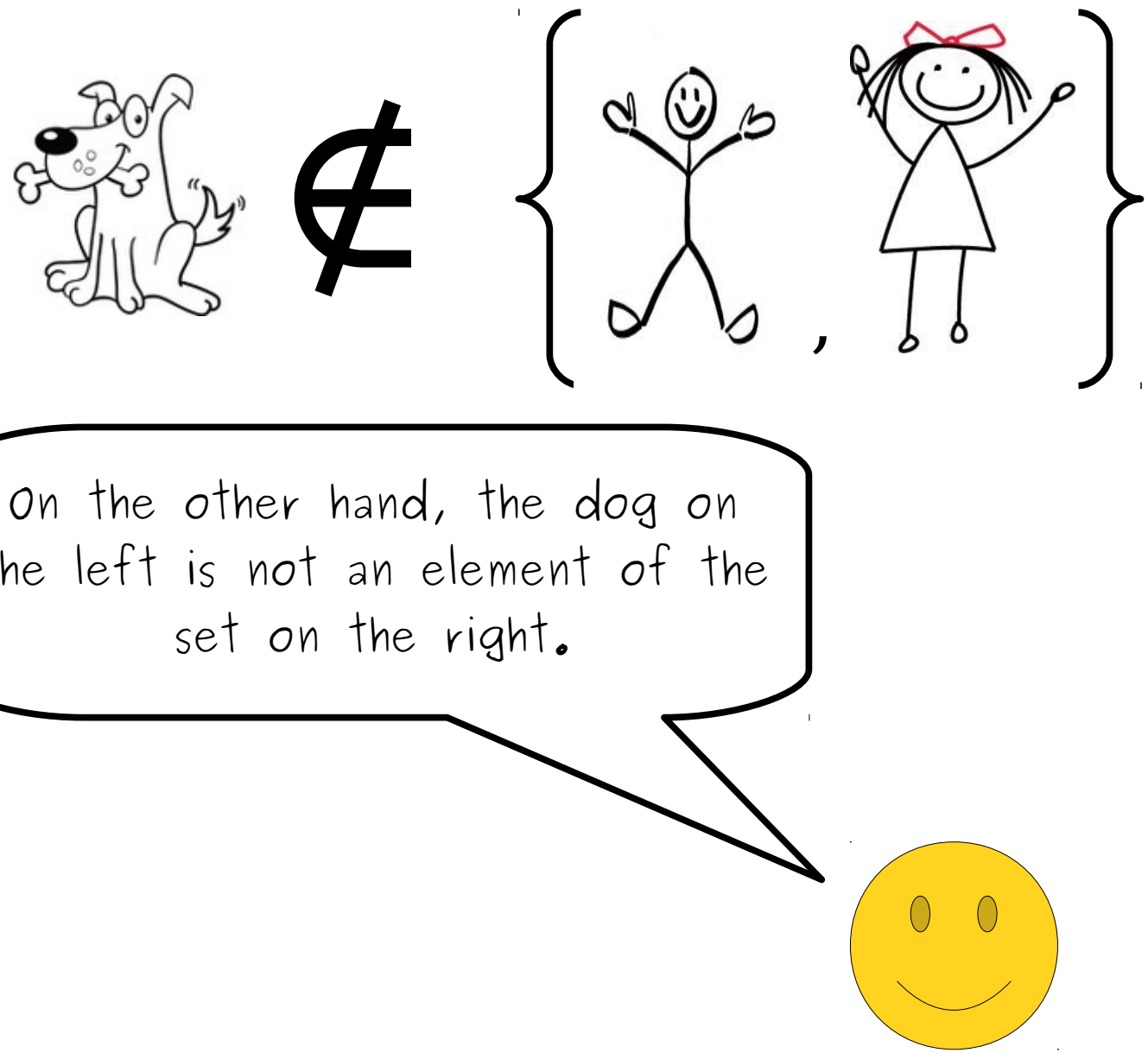
So, in this example, we're using the symbol to indicate that the girl belongs to the set containing the boy and the girl.



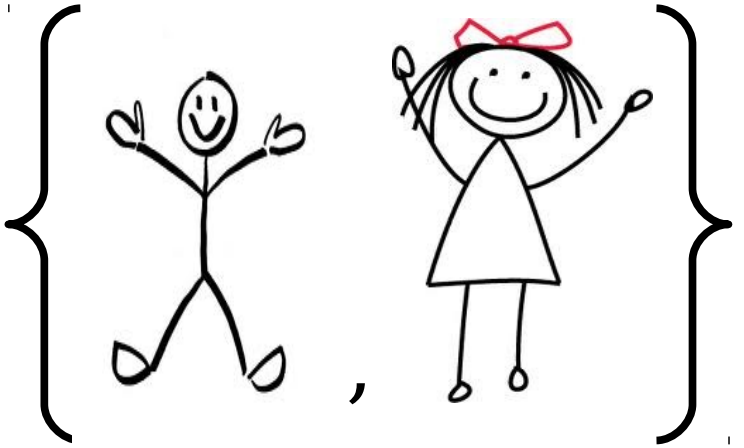


This statement is true because if you look inside the set on the right, you'll see the indicated person on the left.

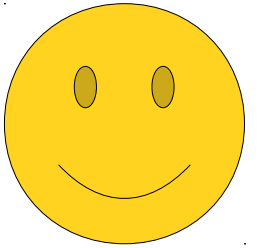


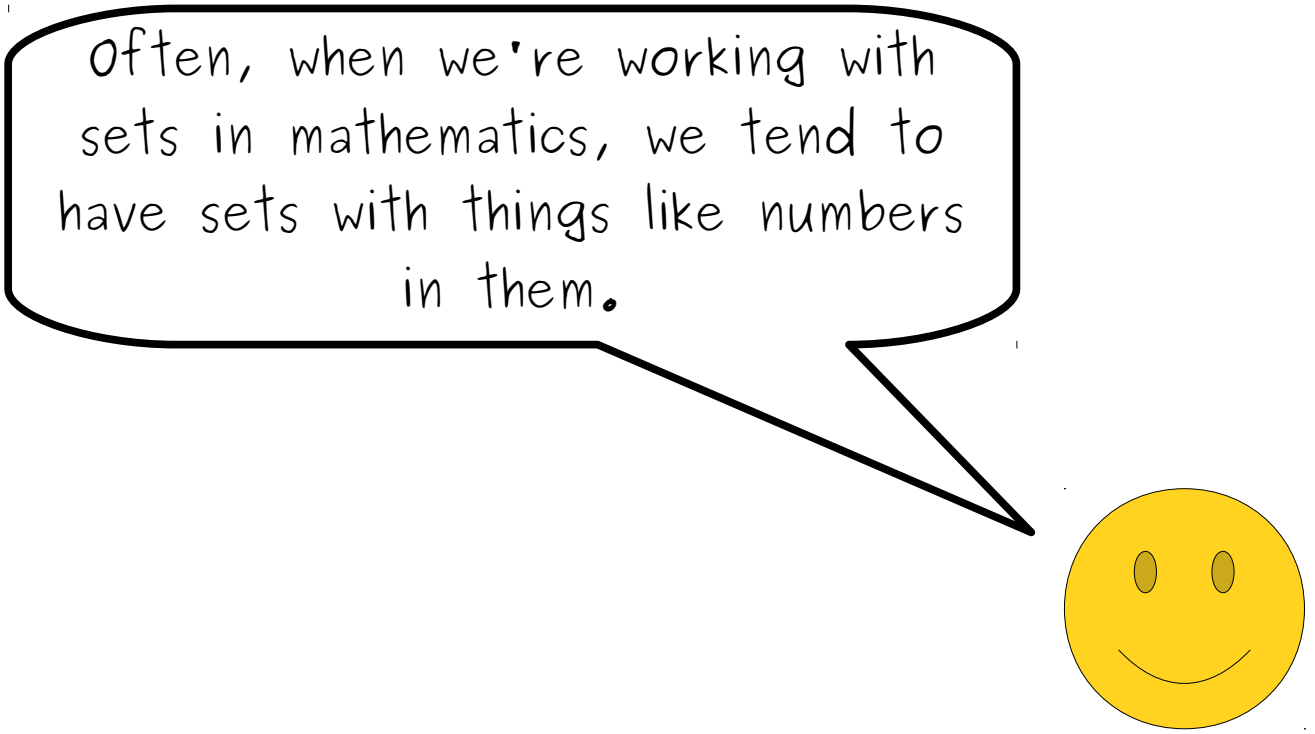

∉
On the other hand, the dog on the left is not an element of the set on the right.

$\notin$



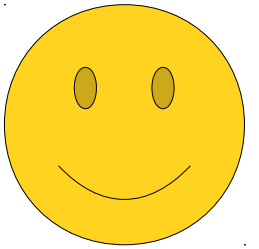
The reason why is that if we look inside the set on the right and take a look at what's inside of it, we won't find the dog anywhere.




Often, when we're working with sets in mathematics, we tend to have sets with things like numbers in them.

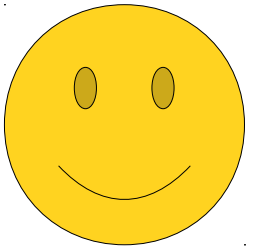
$$1 \in \{1, 2, 3, 4\}$$

So we'll typically see statements like this one, which is more mathematical in nature, even though the previous examples are perfectly correct uses of the $\in$ and $\notin$ symbols.

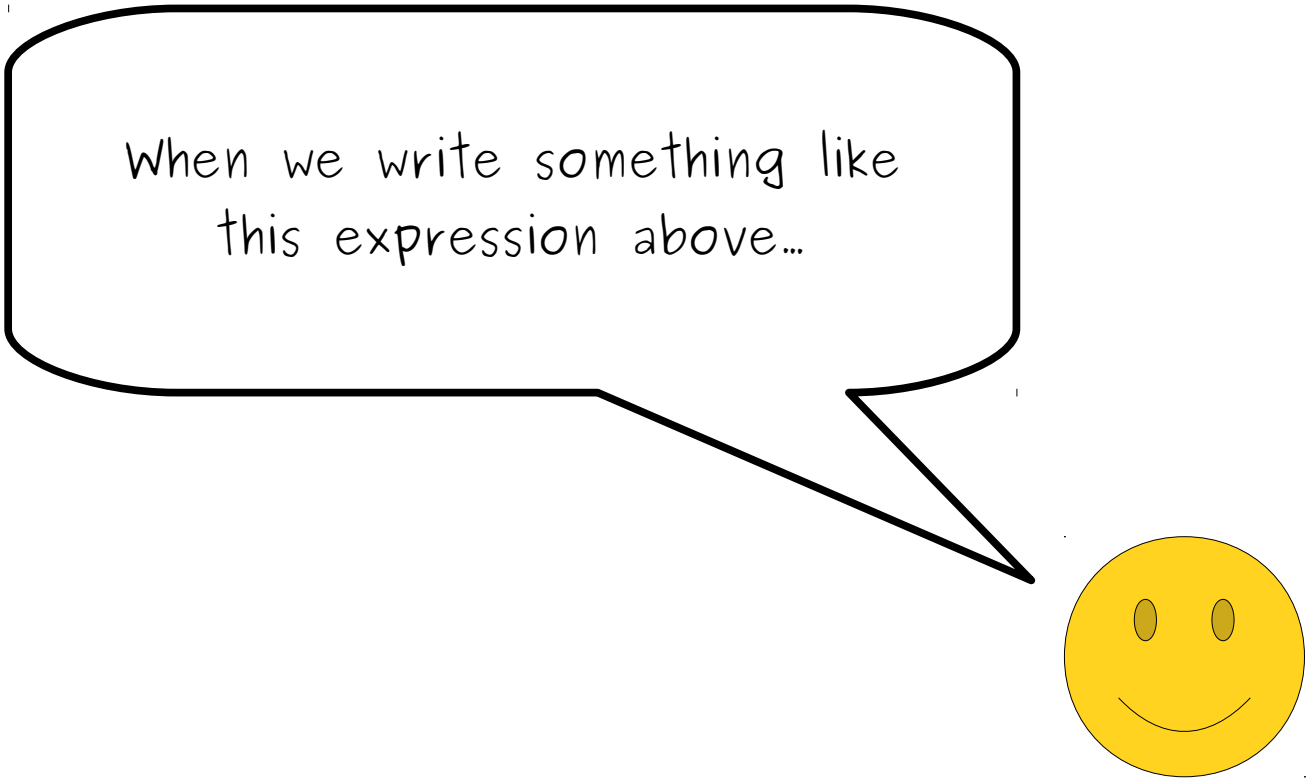


$\in$

When using the $\in$ symbol, it's important to keep in mind that there's a "sidedness" to it.

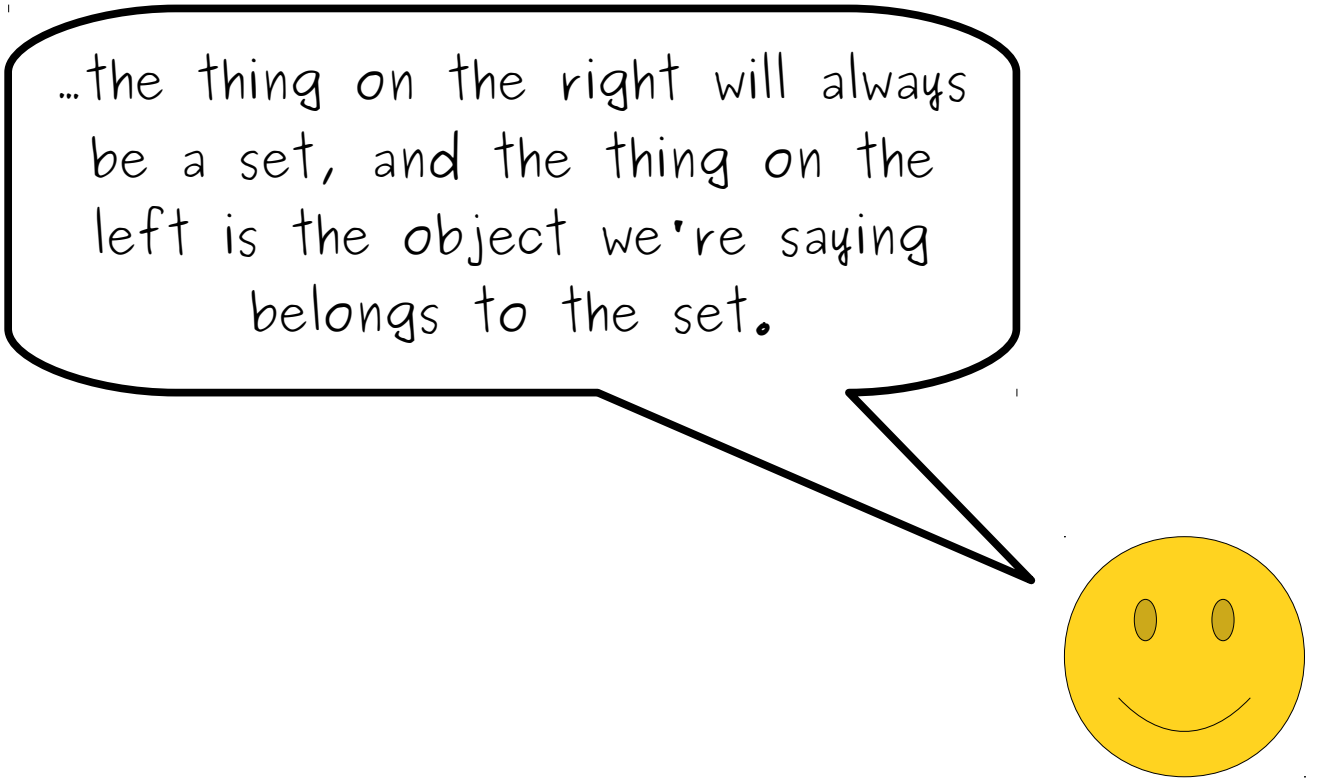


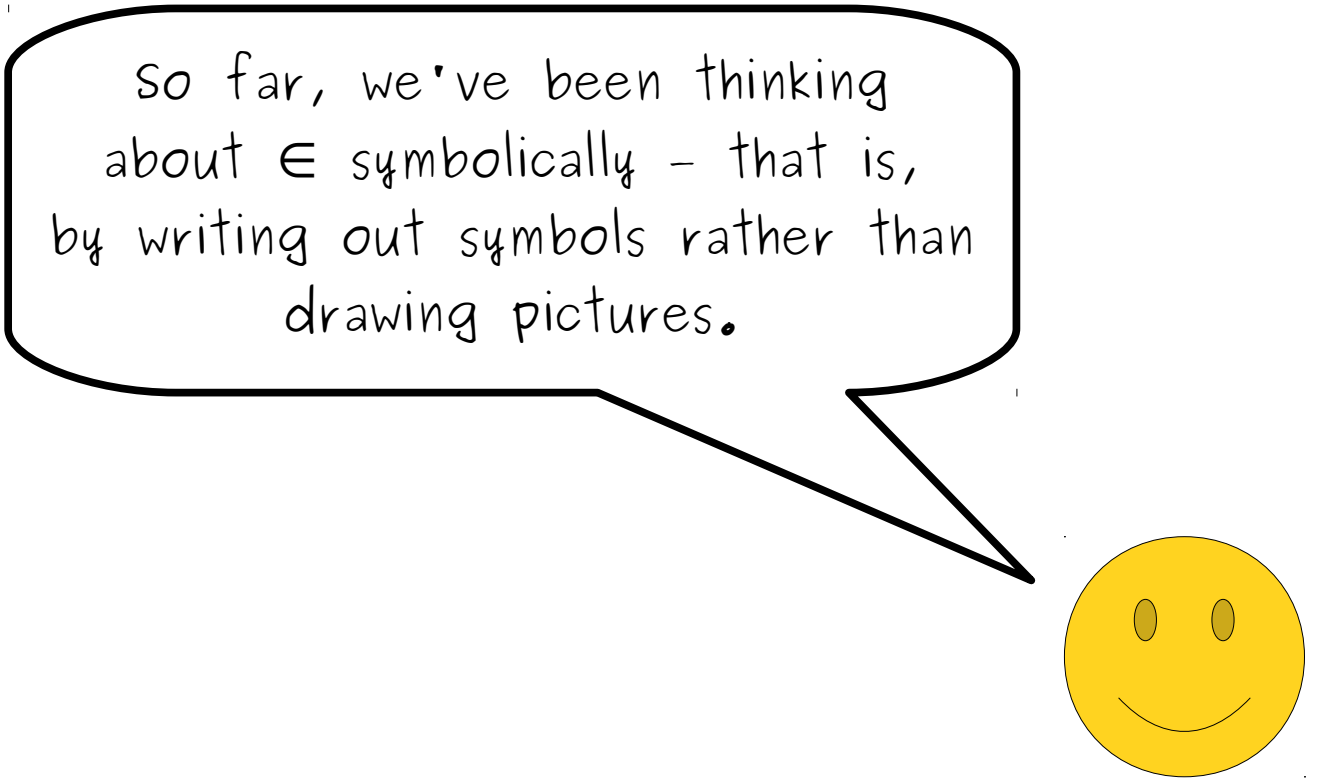
$$x \in S$$

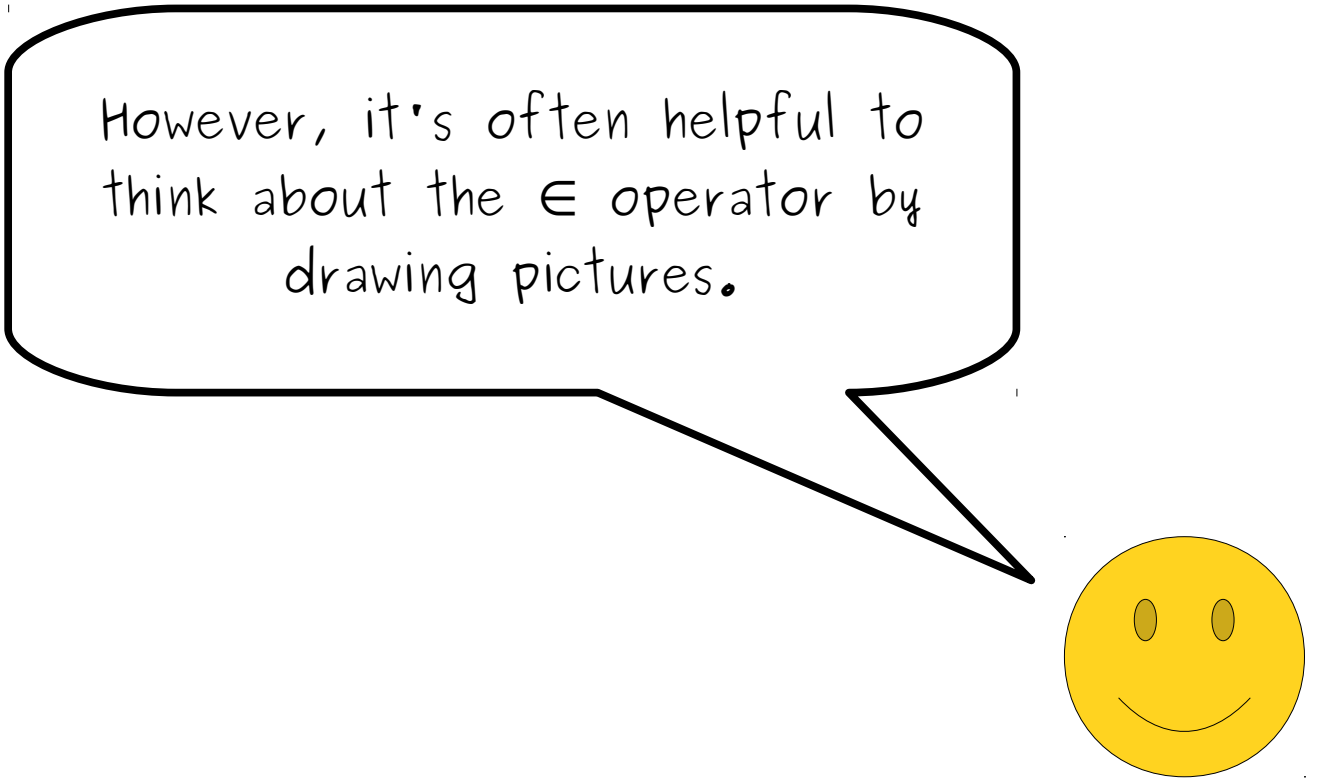


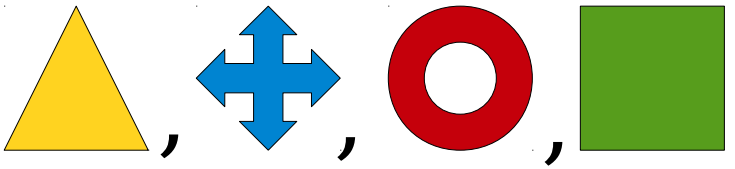
$$x \in S$$

This object is in this set

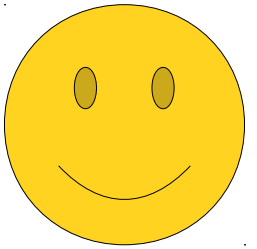


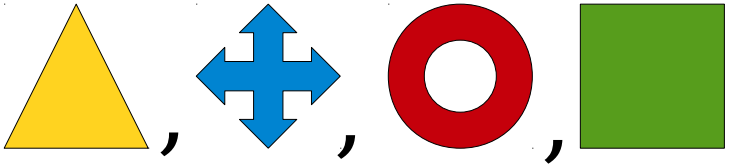

so far, we've been thinking
about ∈ symbolically - that is,
by writing out symbols rather than
drawing pictures.

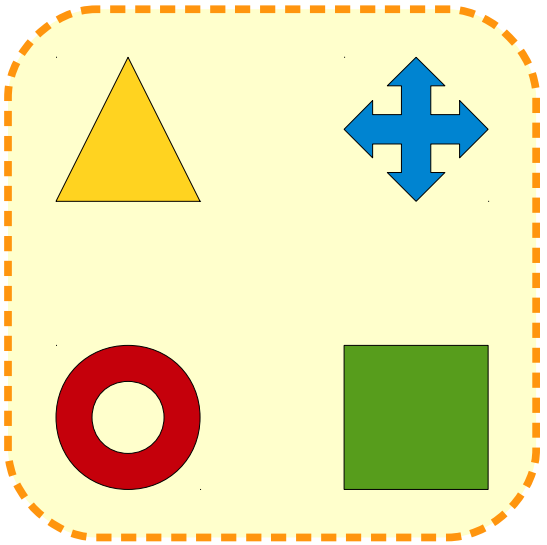

However, it's often helpful to think about the ∈ operator by drawing pictures.

$S = \{$  $\}$

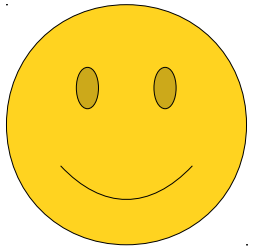
For example, let's imagine that we have this set S, which consists of shapes of different colors.

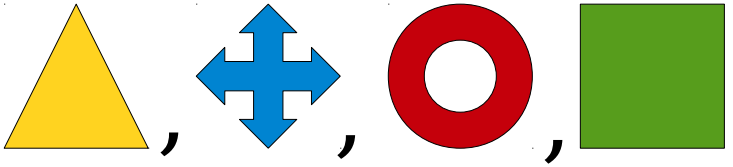


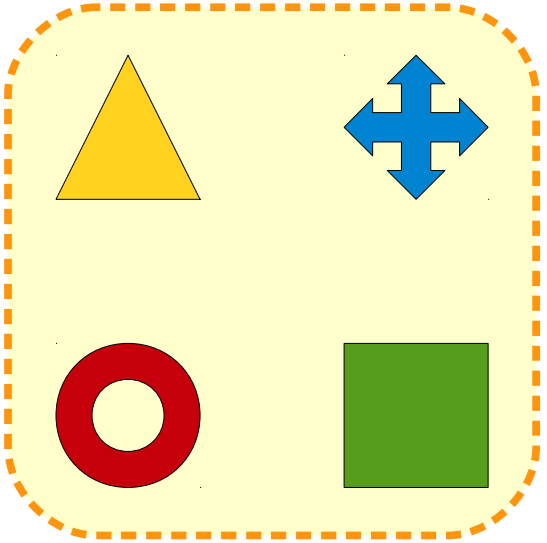
$S = \{$  $\}$

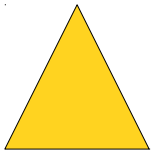


We can draw S as a blob containing each of its elements.

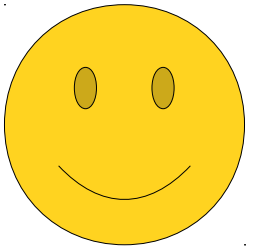


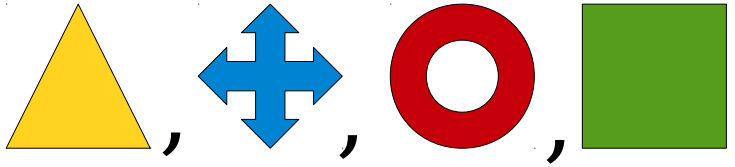
$S = \{$  $\}$

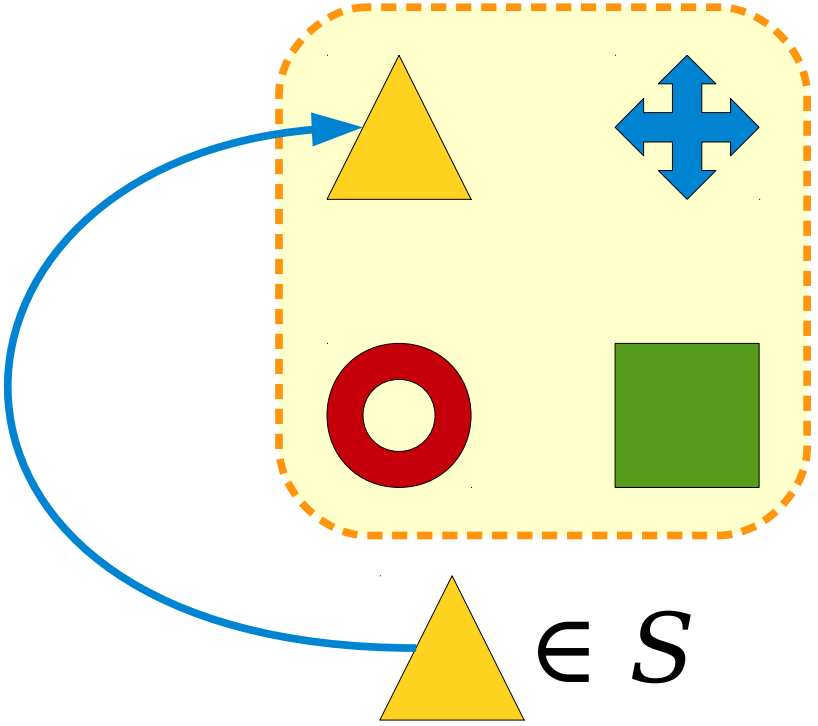


 $\in S$

Now, if we write something like the statement above...

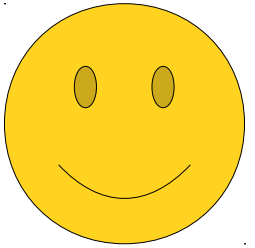


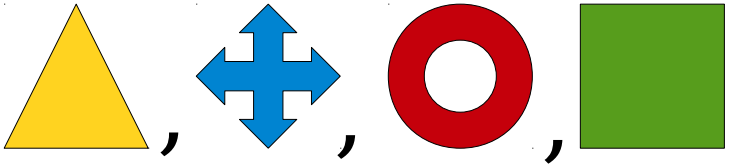
$S = \{$  $\}$

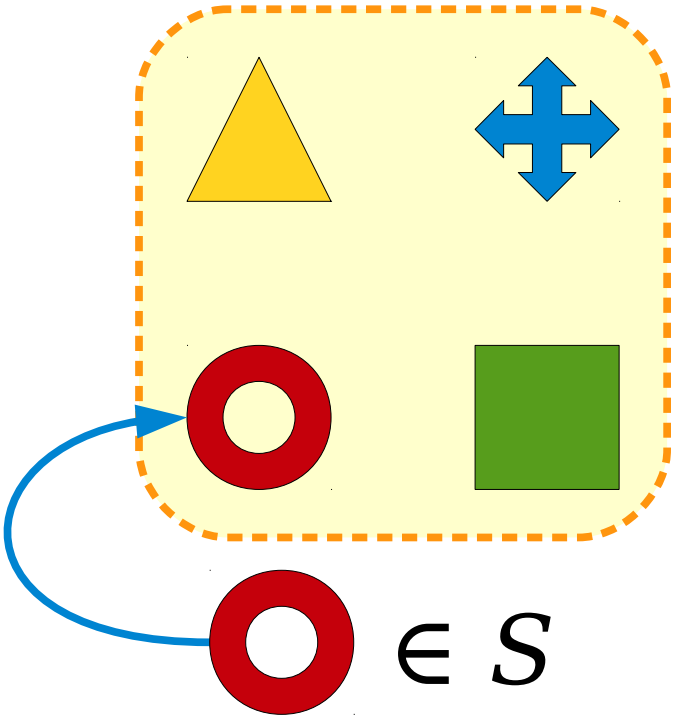


$\in S$

We can see that it's true because we can point at the yellow triangle inside of the blob for s.

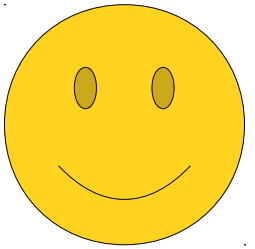


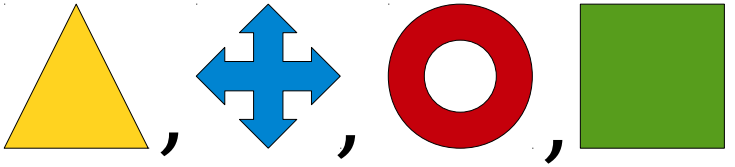
$S = \{$  $\}$

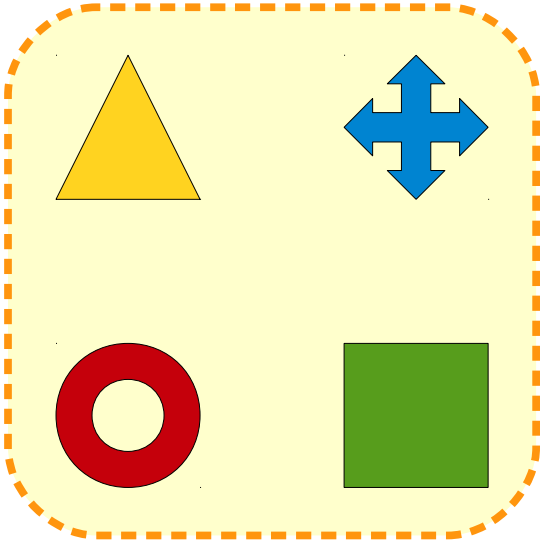


$\in S$

This other statement is also true because we can point out the element in question inside S.

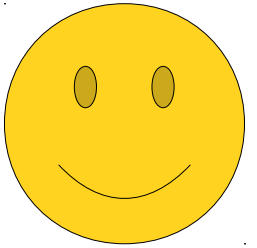


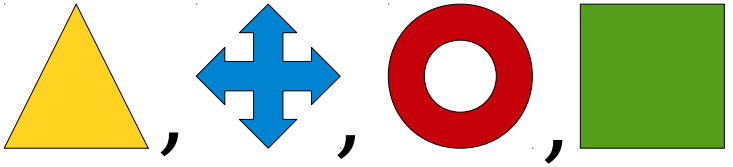
$S = \{$  $\}$

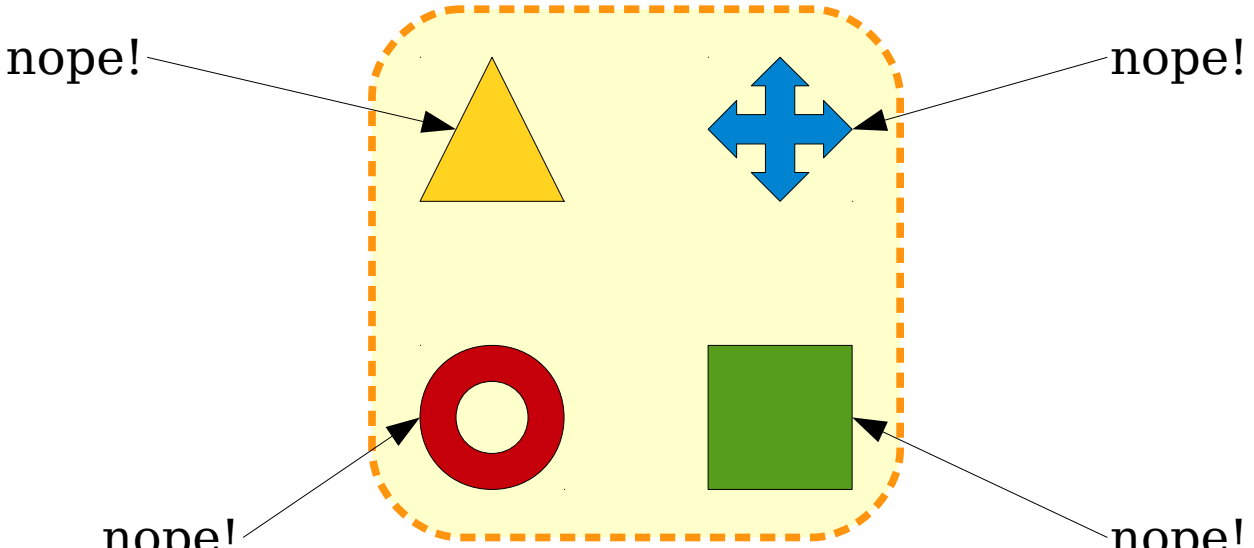


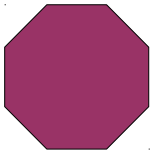
$\notin S$

On the other hand we can see that the purple octagon isn't in the set...

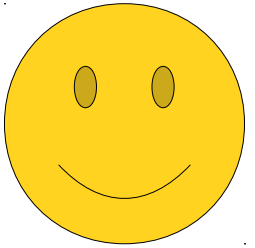


$S = \{$  $\}$



 $\notin S$

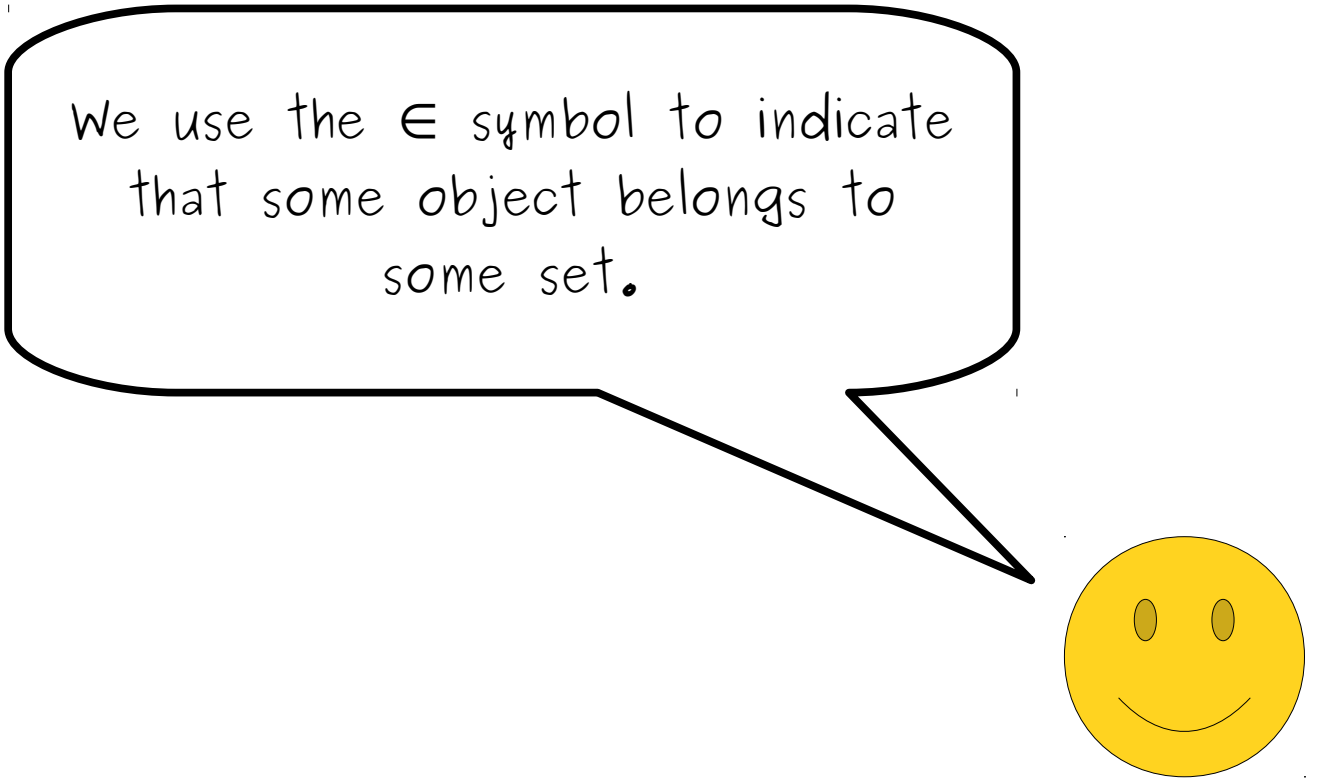
…because we can look at all the elements in S and we won't see it there.



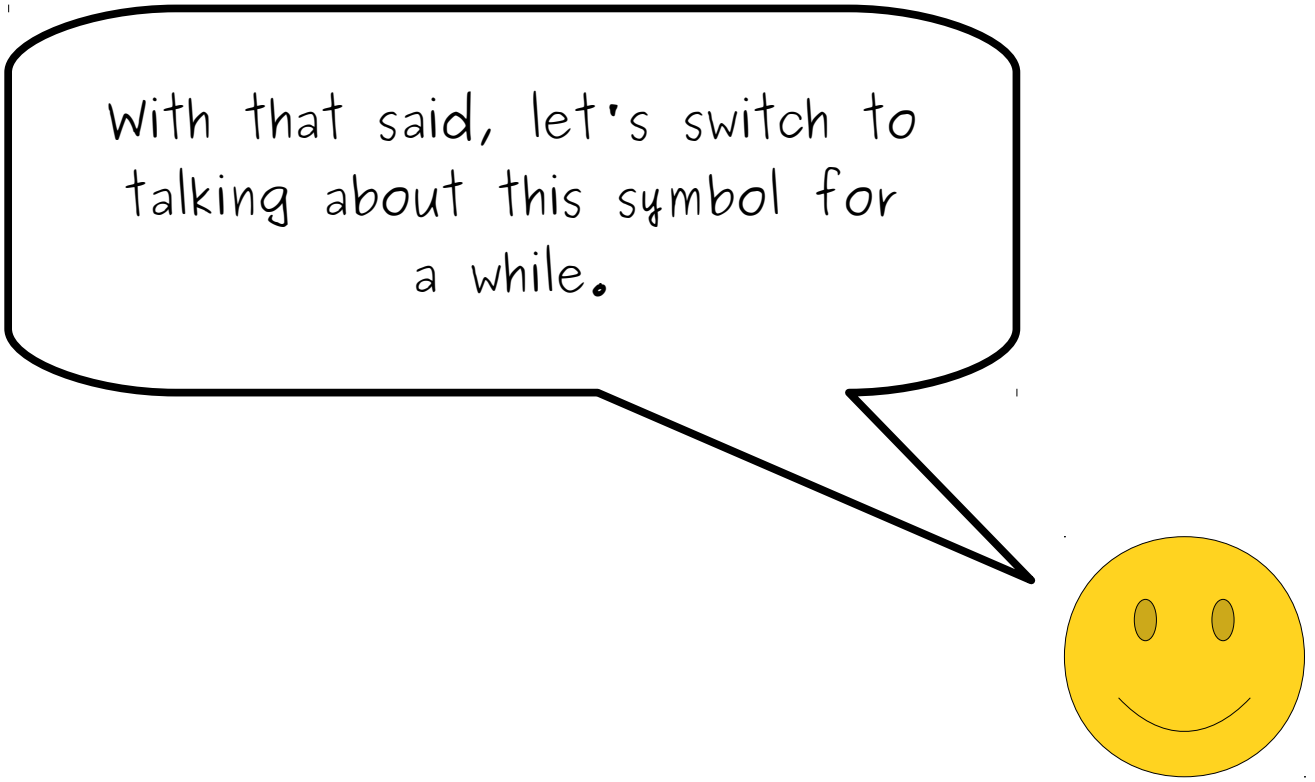
To recap things so far...

$$x \in S$$

This object is in this set

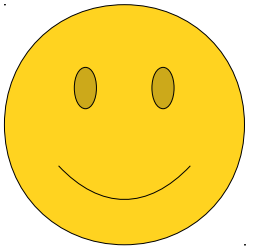


$\subseteq$

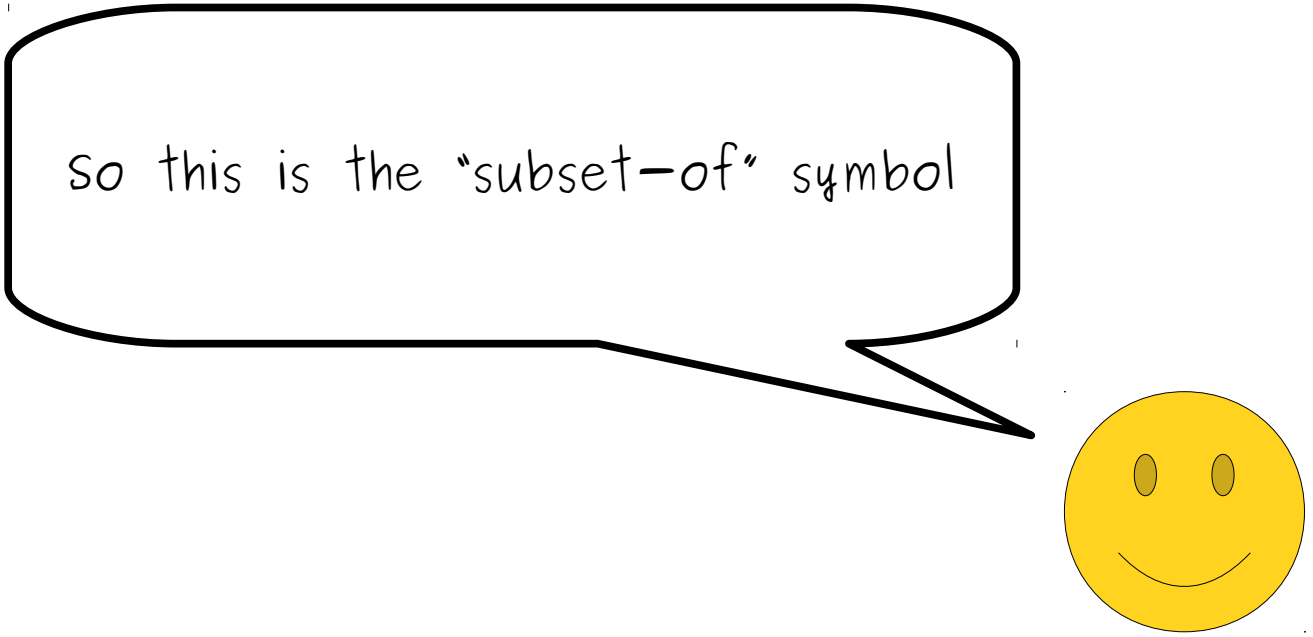

With that said, let's switch to talking about this symbol for a while.

$$\subseteq$$

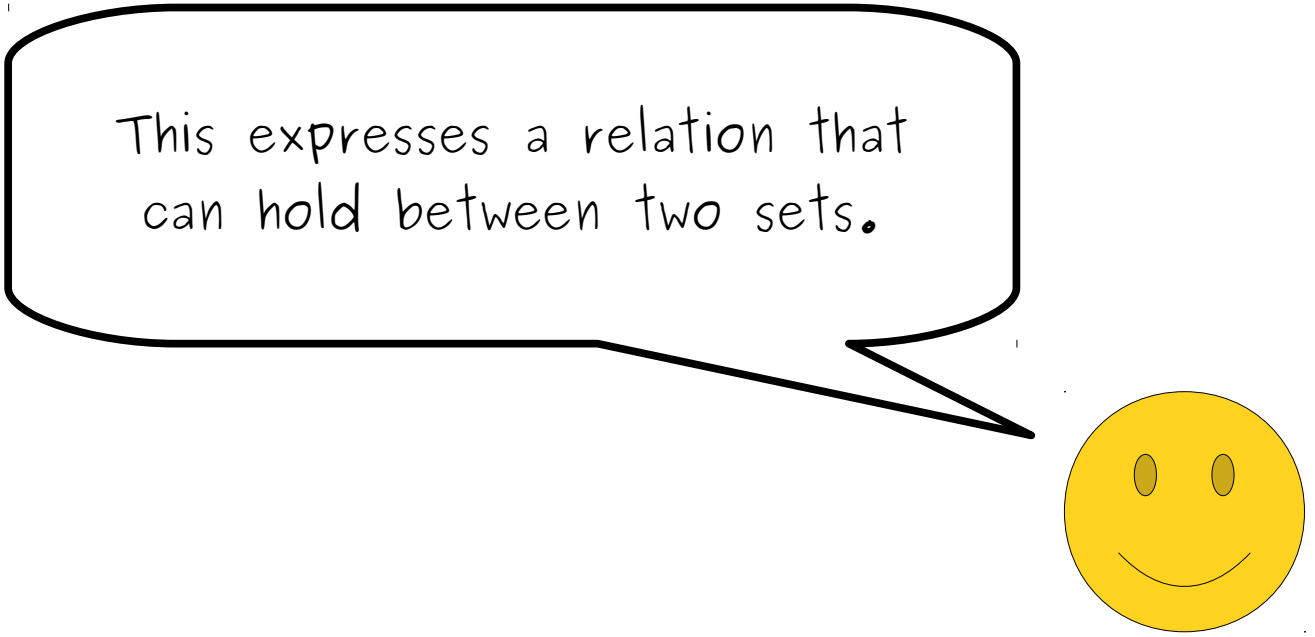
Before we talk about it, look at your notes and see what this symbol means. If you don't remember, quickly jump back to the lecture slides to get the answer.



$\subseteq$

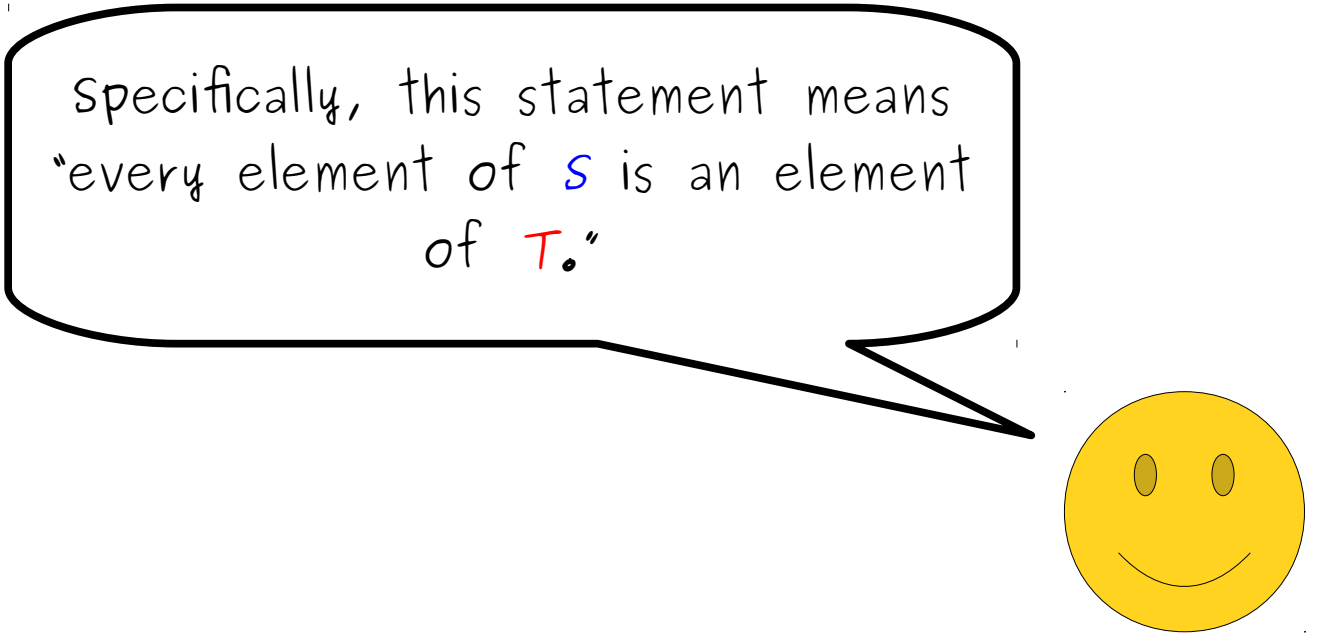


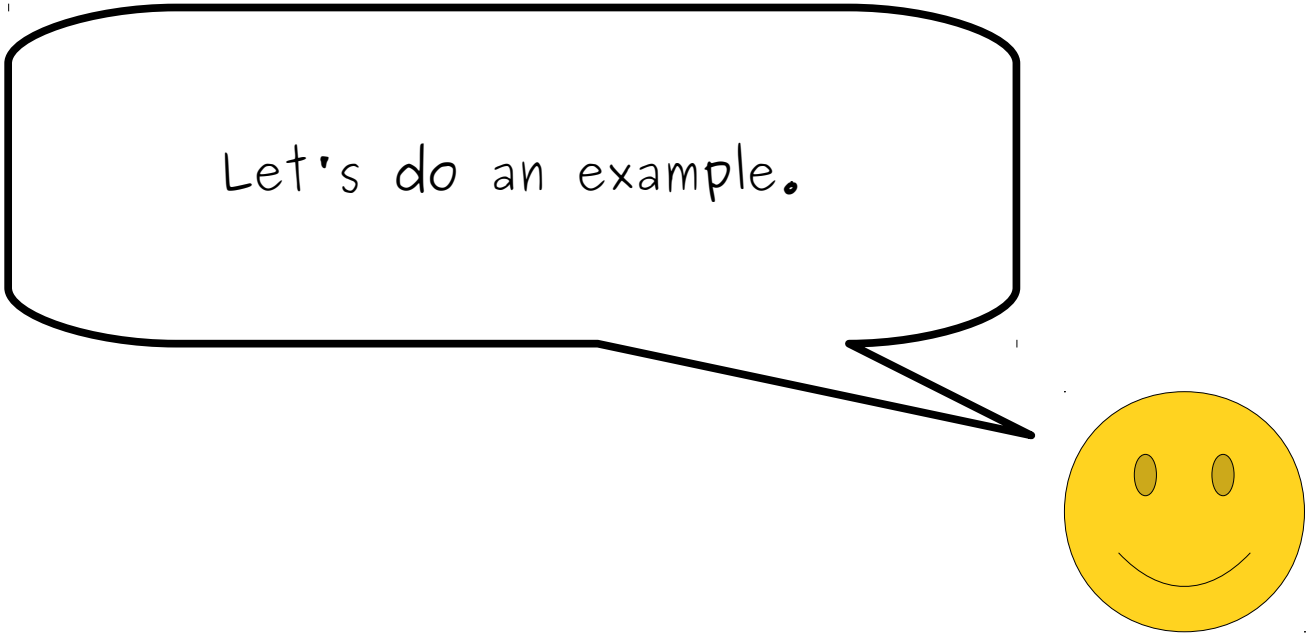
$$S \subseteq T$$

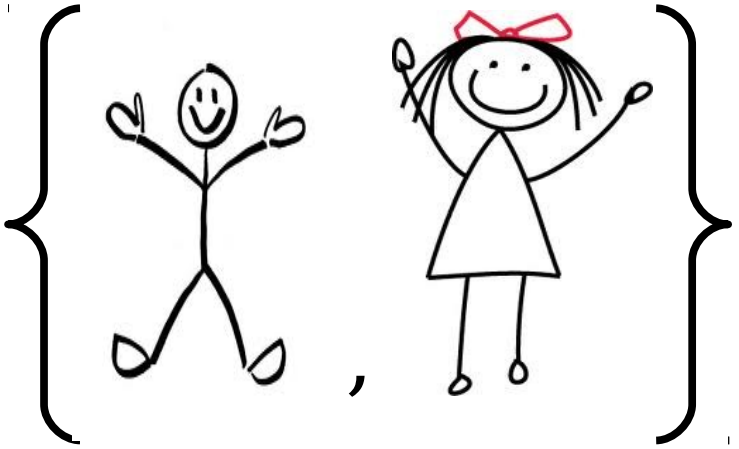


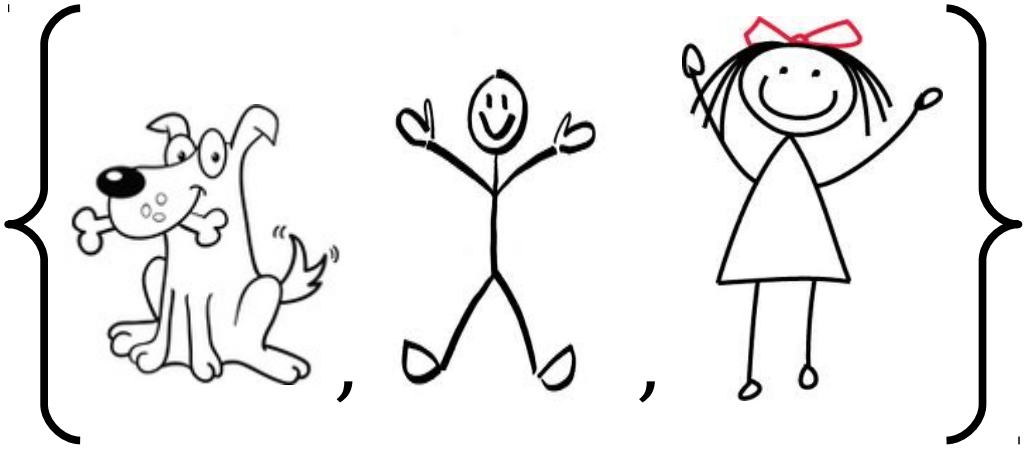
$$S \subseteq T$$

Every object in this set is in this set

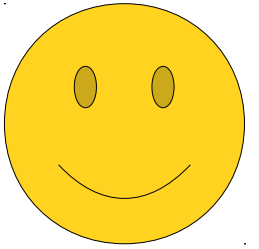


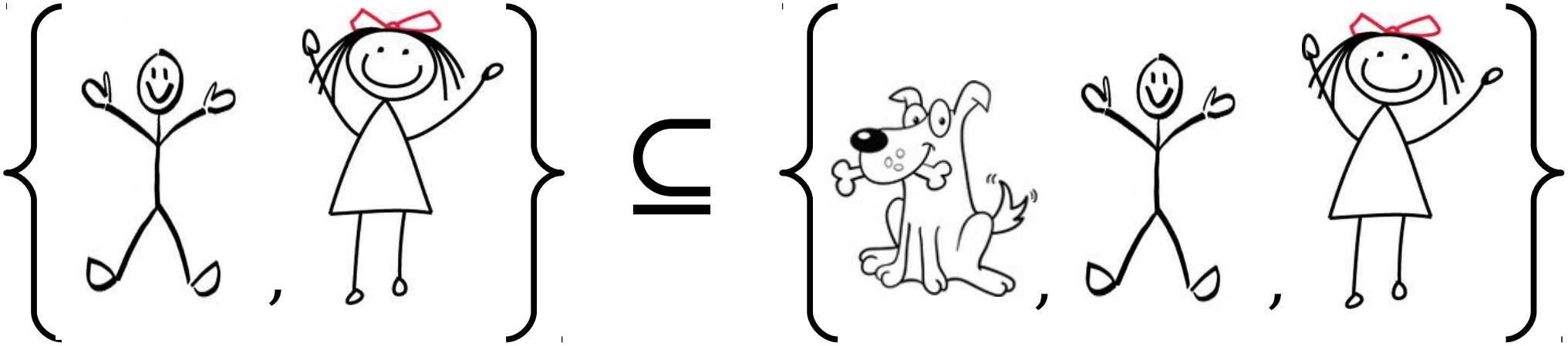

Let's do an example.



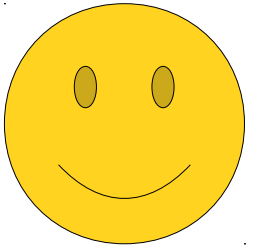


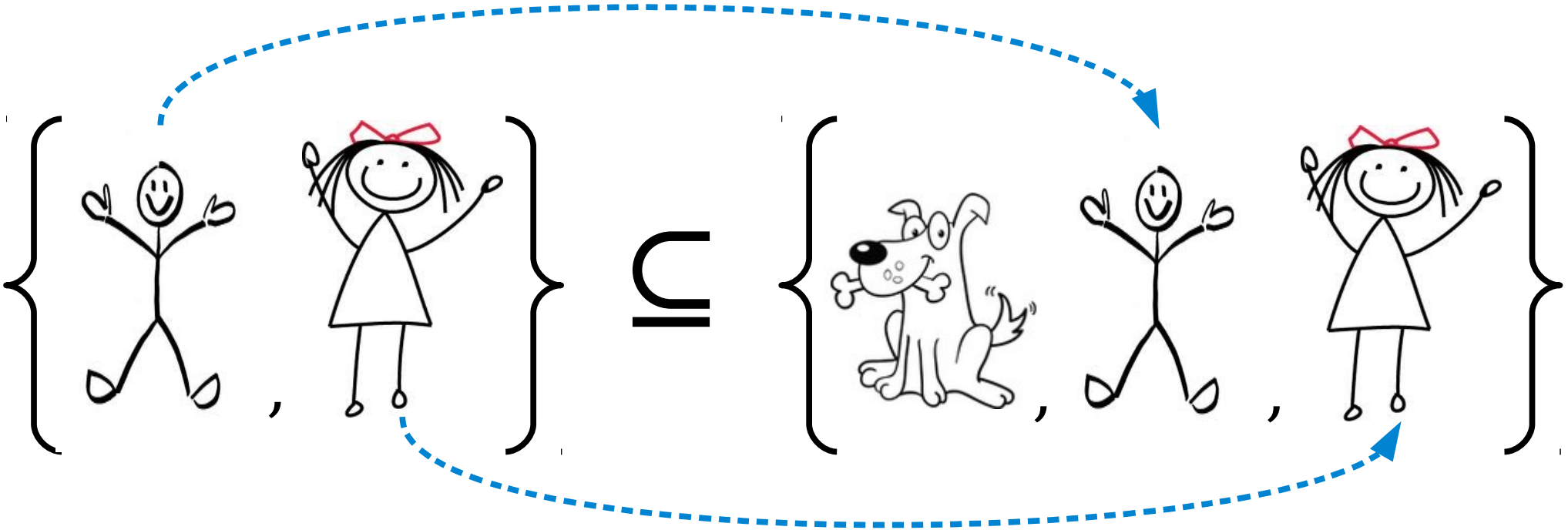
Here we have two different sets.



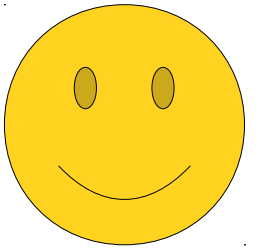


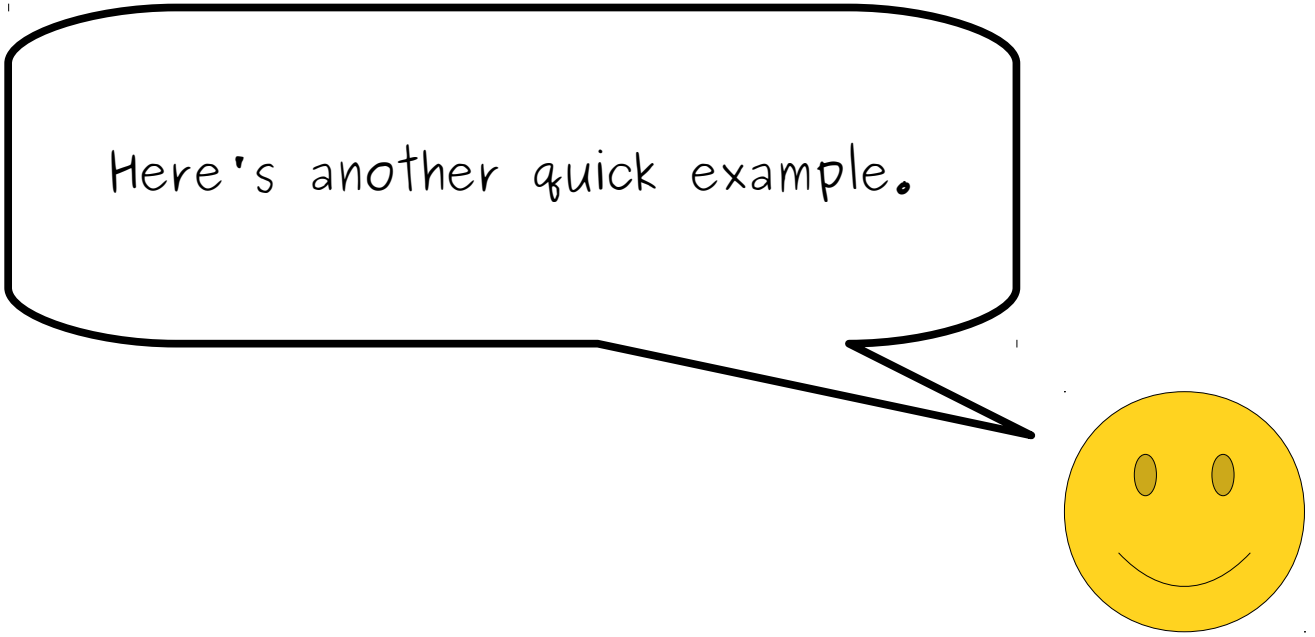
We can say that this first set is a subset of the second set...

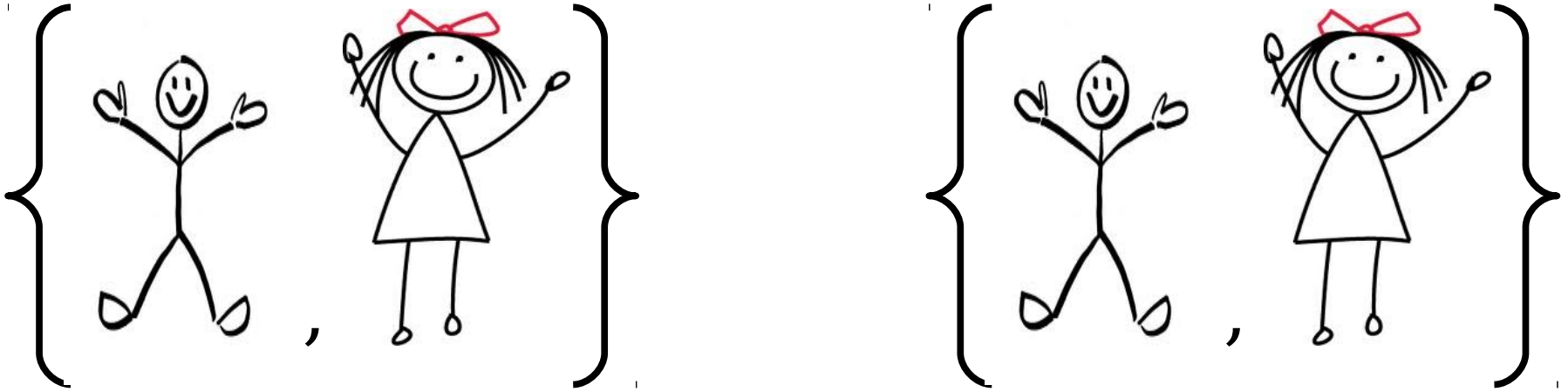


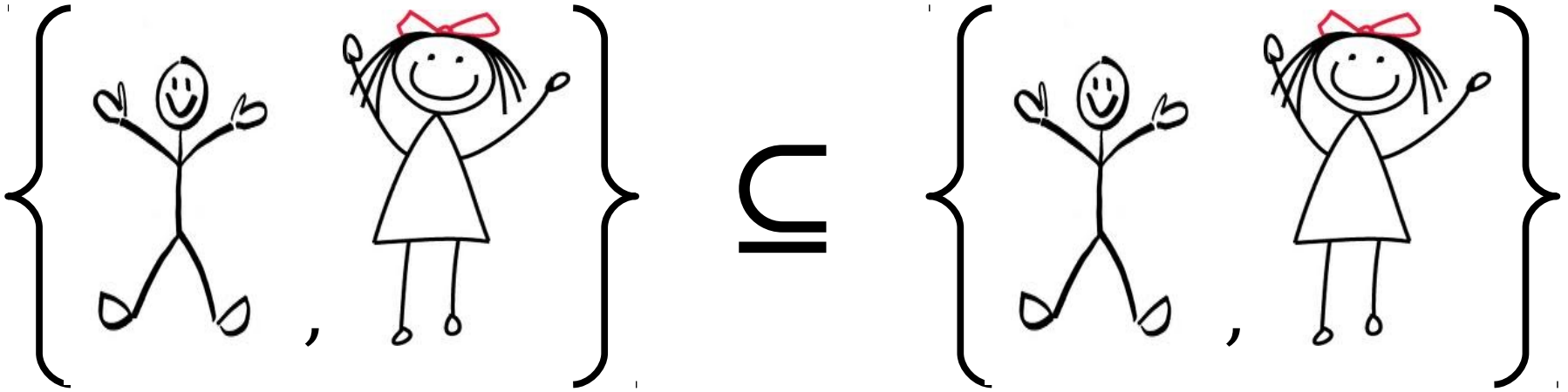


…because every element of the first set is also an element in the second set.

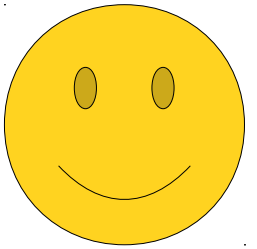


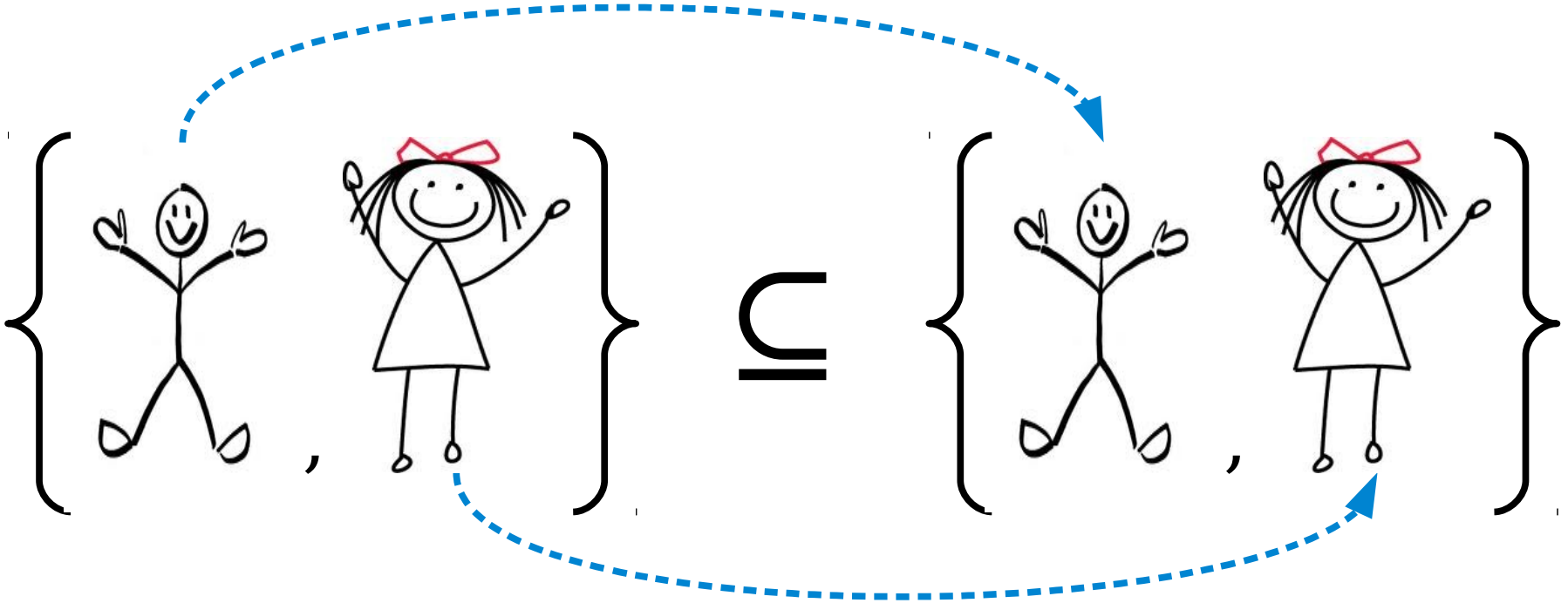

Here's another quick example.


Here are two sets, which just happen to be the same set.

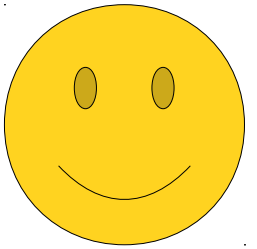


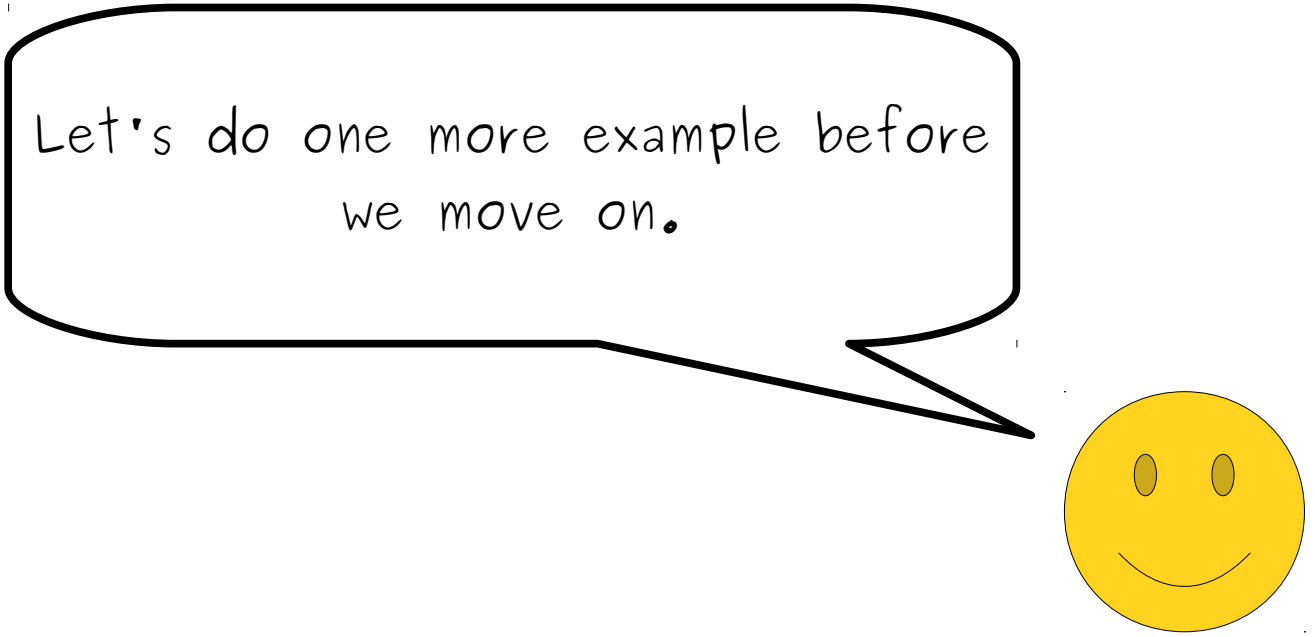
We say that the first set is a subset of the second set...

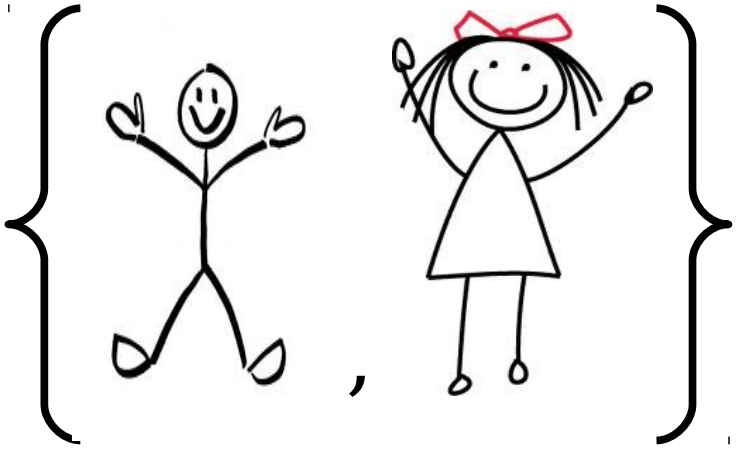




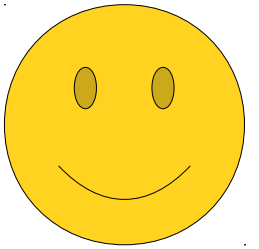
…because every element of the first set is also an element of the second set.

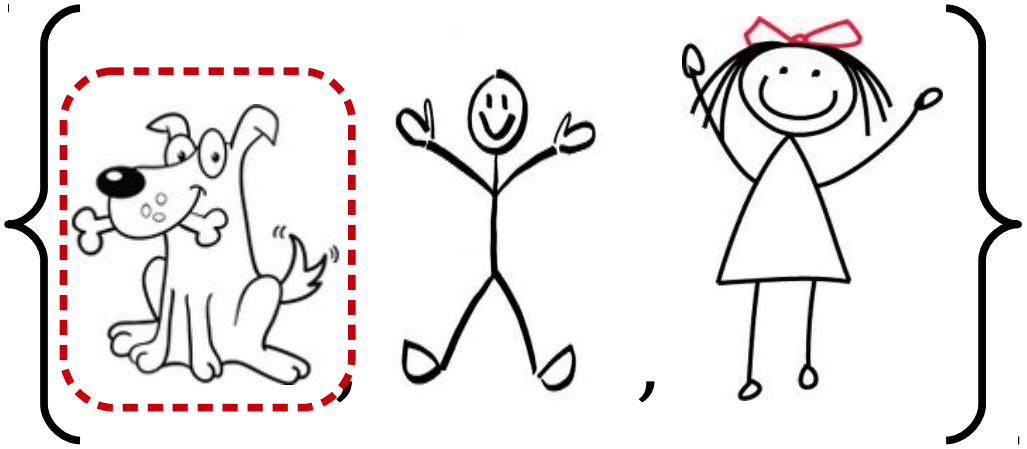


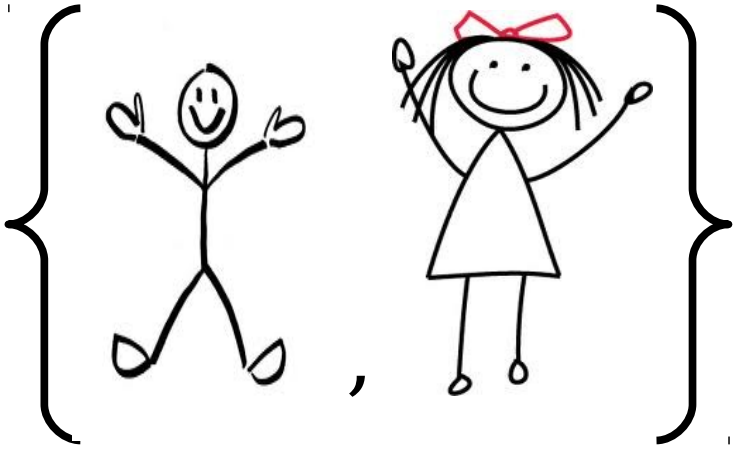

Let's do one more example before we move on.



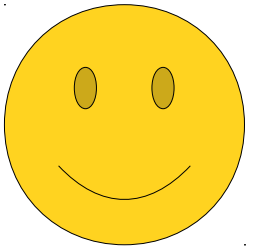
Here are two sets that we talked about earlier, but presented in the reverse order.

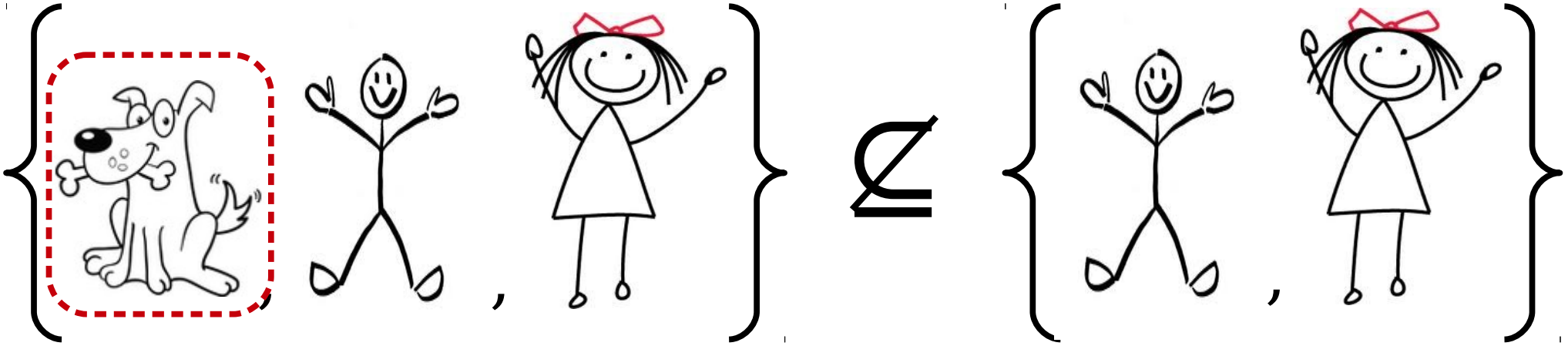




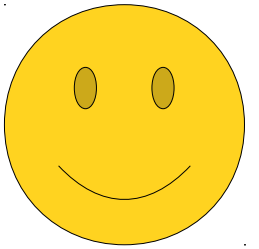


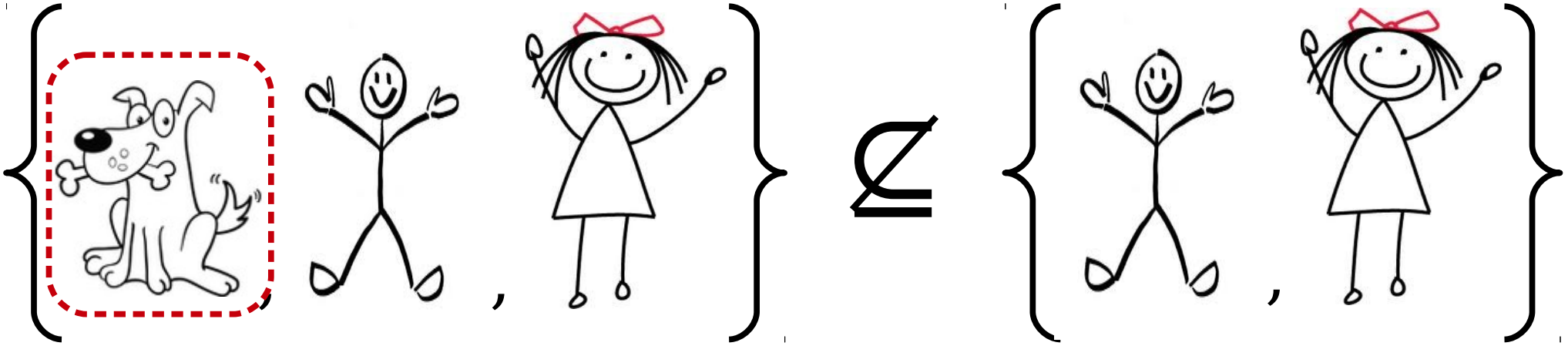
Notice how this first set contains an element that isn't present in the second set.



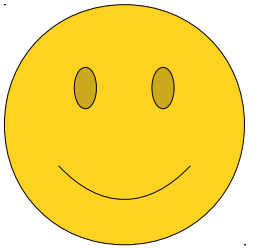


As a result, this first set is not a subset of the second set.

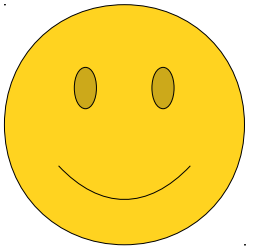


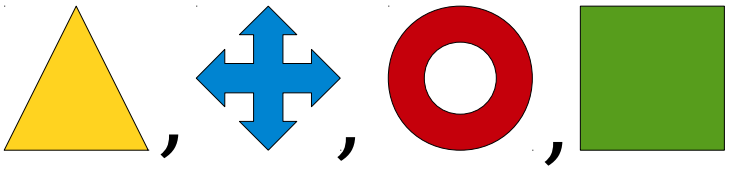


We denote this by using this special "not a subset" symbol. It's basically the subset symbol with a slash through it.

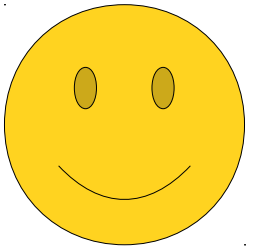


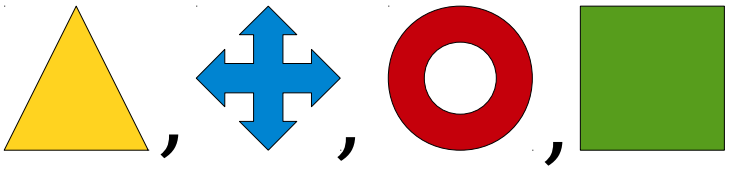
Earlier, we saw a visual intuition for what the $\in$ symbol meant. Let's take a few minutes to get a visual intuition for $\subseteq$.

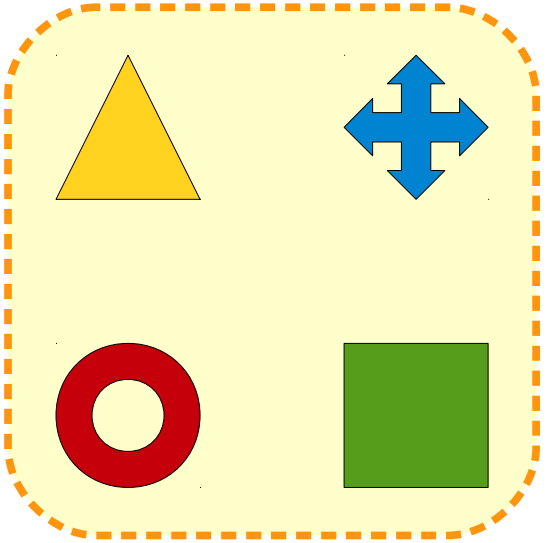


$S = \{$  $\}$

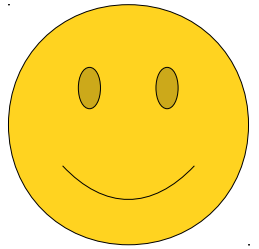
As before, let's look at this set S, which consists of a bunch of colorful shapes.

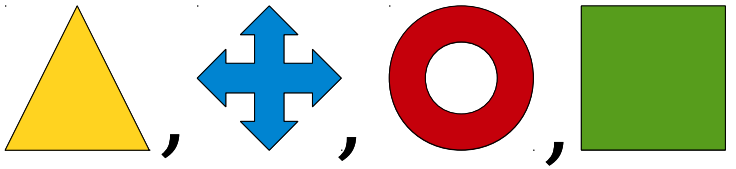


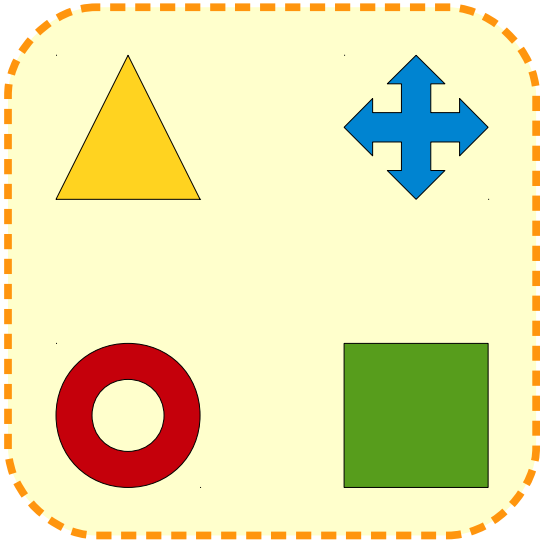
$S = \{$  $\}$

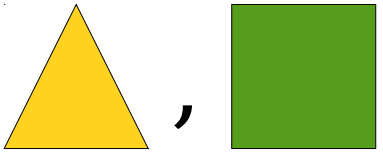


Also as before, we'll draw S as a blob containing each of its elements.

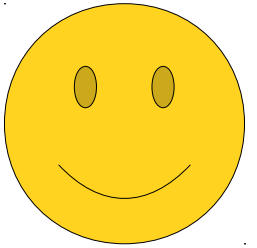


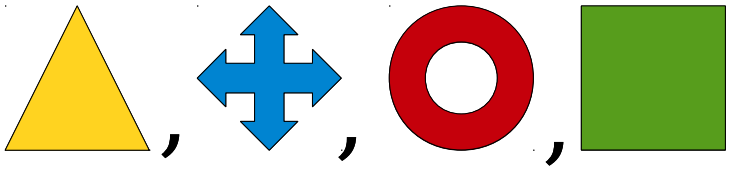
$S = \{$  $\}$

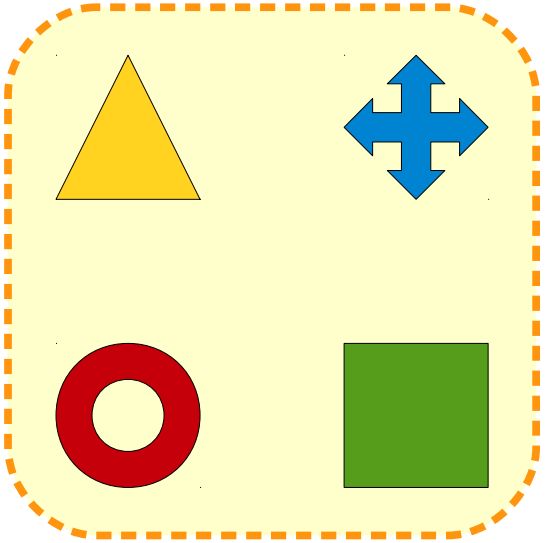


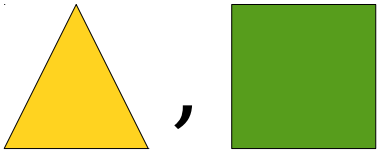
$\{$  $\} \subseteq S$

So imagine that we're given the above statement and asked to determine whether it's true.

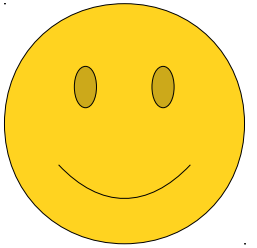


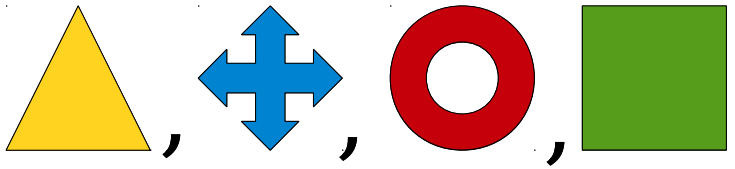
$S = \{$  $\}$

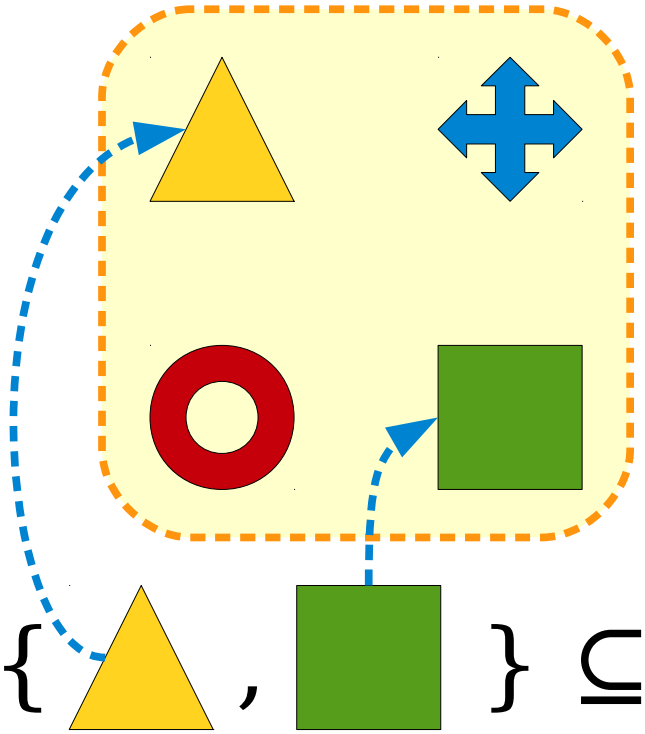


$\{$  $\} \subseteq S$

This is equivalent to asking whether every element in the set on the left happens to be in the set S.

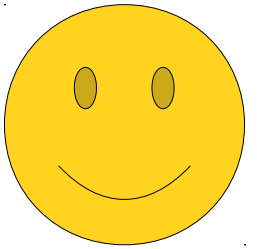


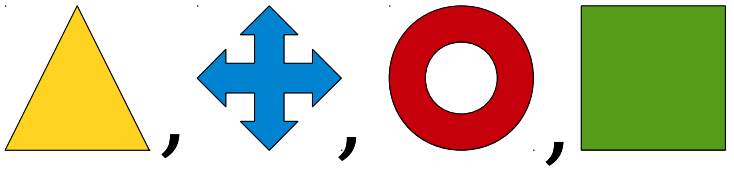
$S = \{$  $\}$

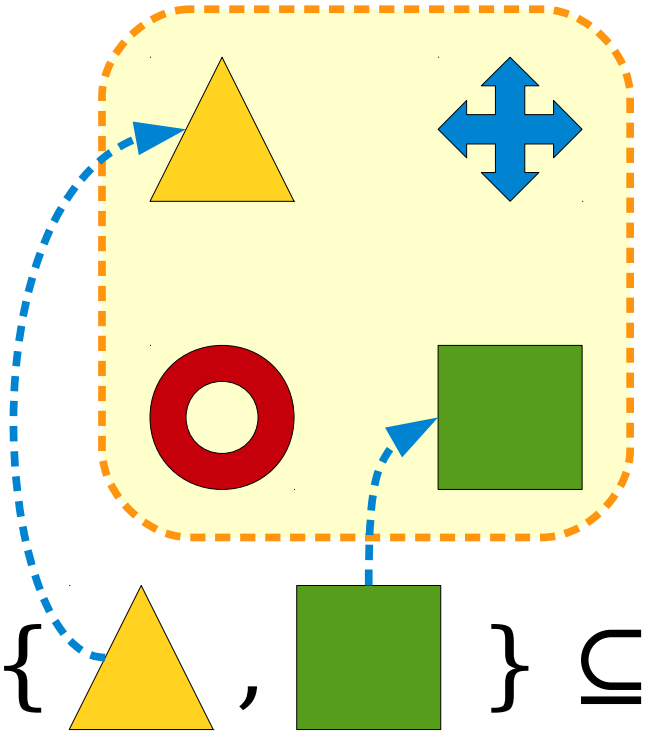


$\{$ $\} \subseteq S$

In this case, we can see that this is indeed the case. Notice that both objects are in S.

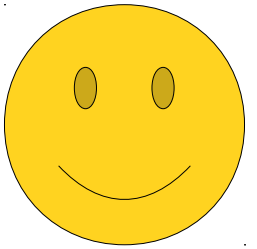


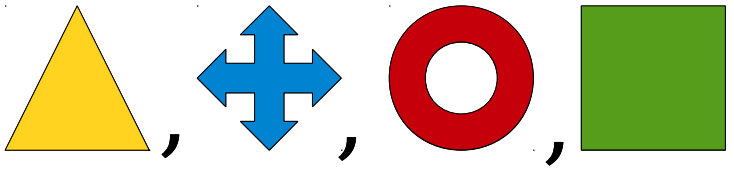
$S = \{$  $\}$

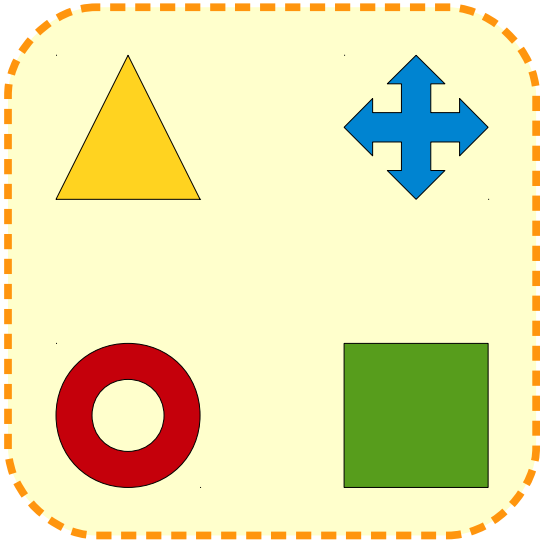


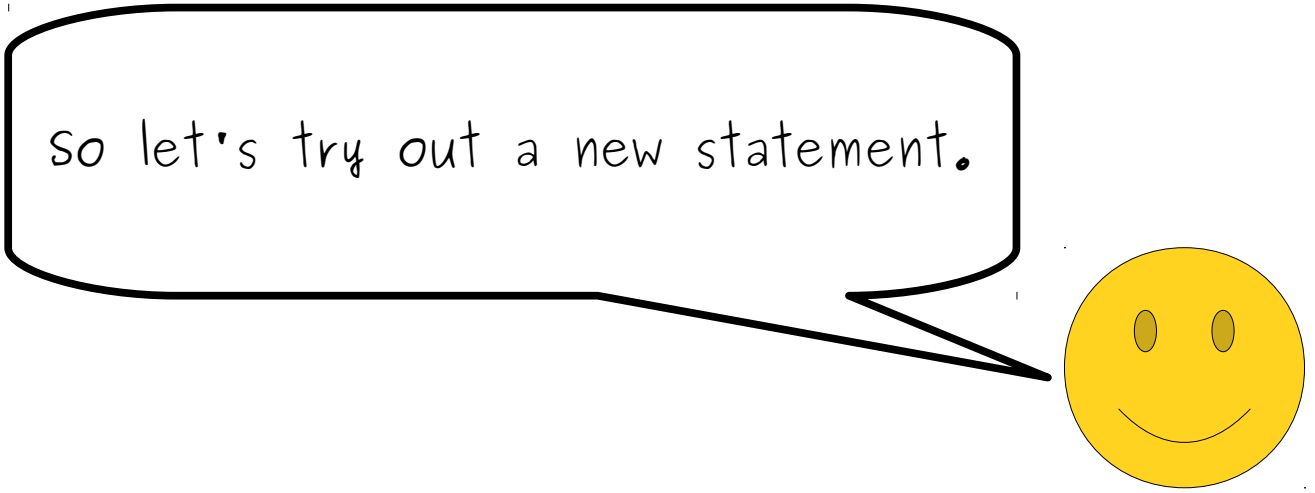
$\{$ $\} \subseteq S$

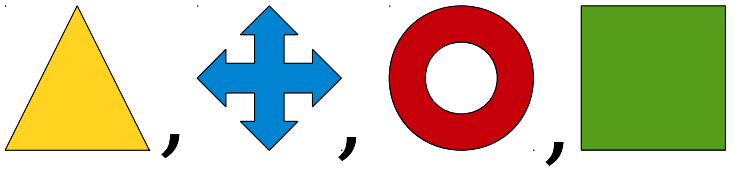
Therefore, the above statement is true.

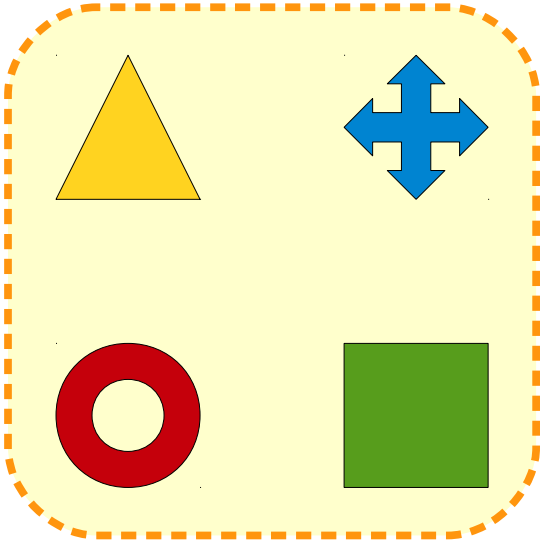


$S = \{$  $\}$



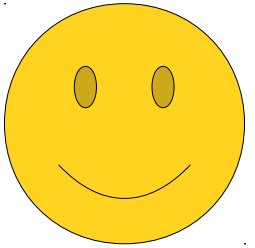


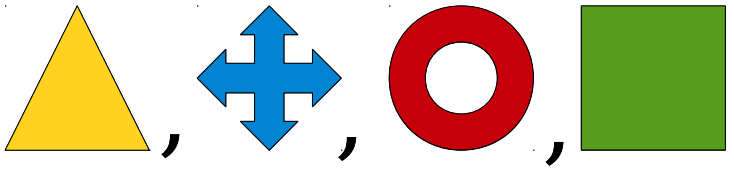
$S = \{$  $\}$

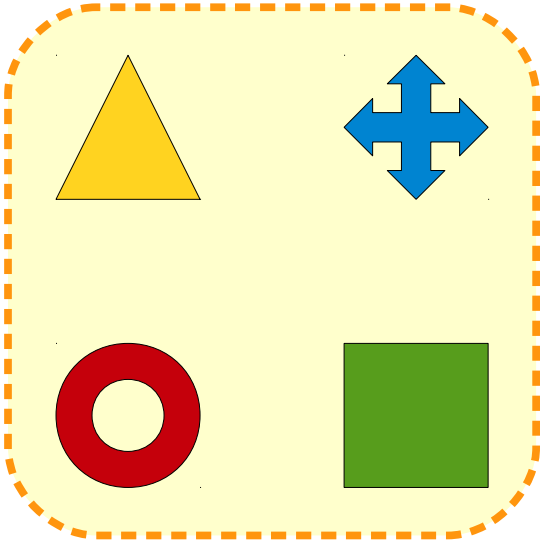


$\overset{?}{\subseteq} S$

Here's a statement that we're not sure about, which is why we have a question mark above the $\subseteq$ symbol.

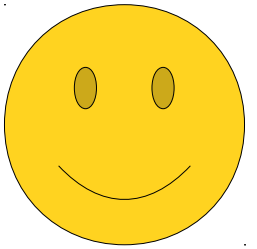


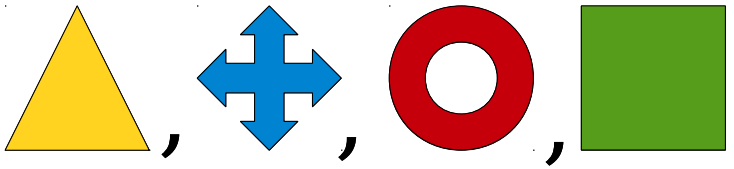
$S = \{$  $\}$

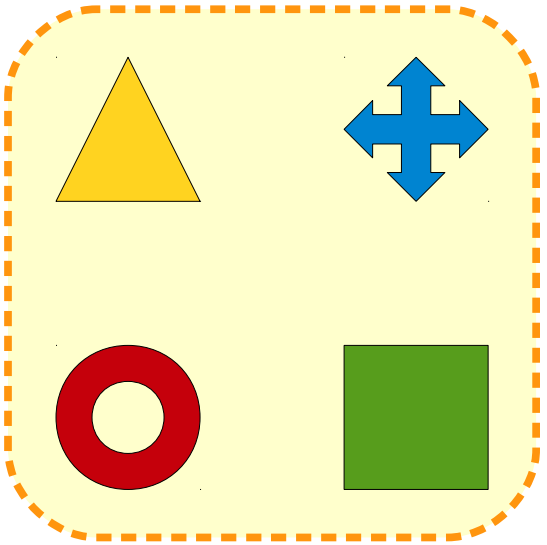


$\overset{?}{\subseteq} S$

so... is this statement true or false?

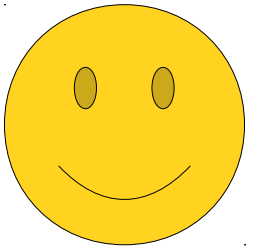


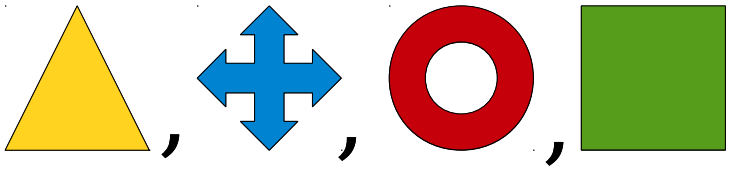
$S = \{$  $\}$

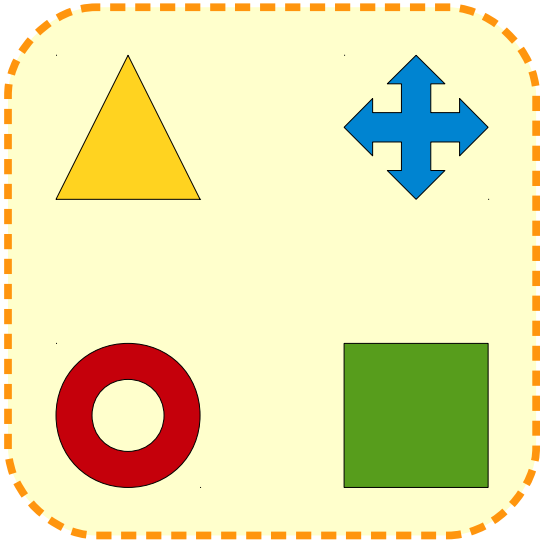


$\overset{?}{\subseteq} S$

Before you move on, make a guess! There's no risk involved - you're still learning!

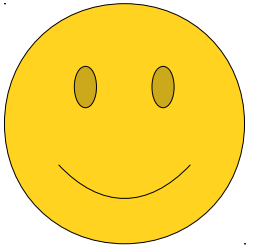


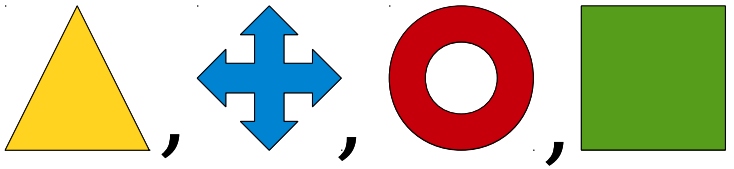
$S = \{$  $\}$

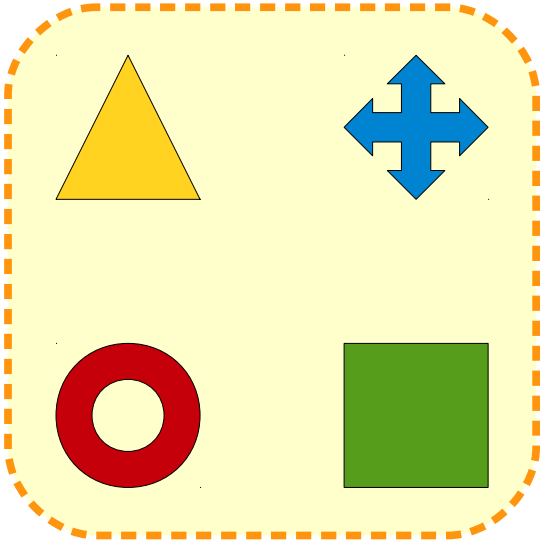


$\stackrel{?}{\subseteq} S$

So you have a guess about whether this statement is true or false? Like, really? Because if you haven't guessed yet, you totally should do that before moving on.

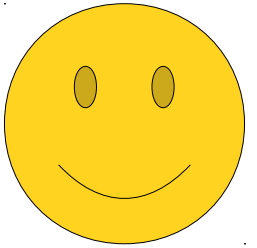


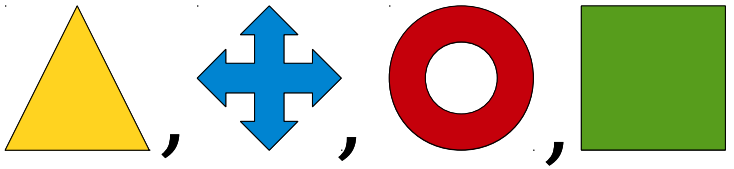
$S = \{$  $\}$



$\overset{?}{\subseteq} S$

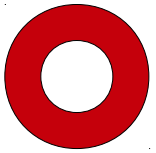
Okay, so let's see whether this is true or false.



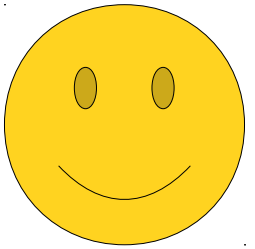
$S = \{$  $\}$

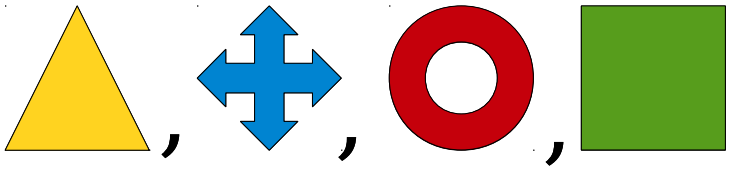
$S \subseteq T$

Every object in this set is in this set

 $\overset{?}{\subseteq} S$

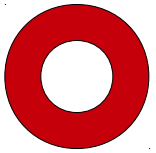
Earlier, we said that the ⊆ relation means that every object in the set on the left-hand side is an element of the set on the right-hand side.



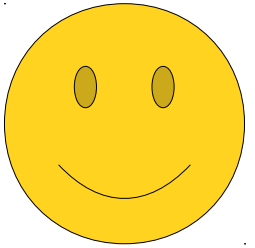
$S = \{$  $\}$

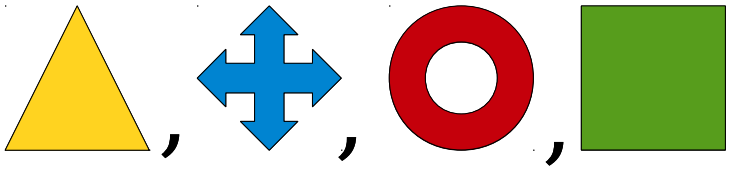
$S \subseteq T$

Every object in this set is in this set

 $\overset{?}{\subseteq} S$

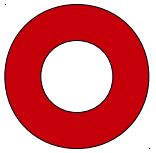
An important detail here is that the thing on the left-hand side of the $\subseteq$ relation has to be a set for the relation to even make sense.



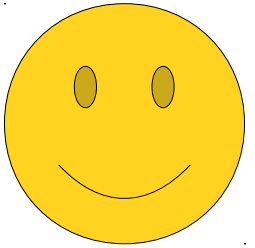
$S = \{$  $\}$

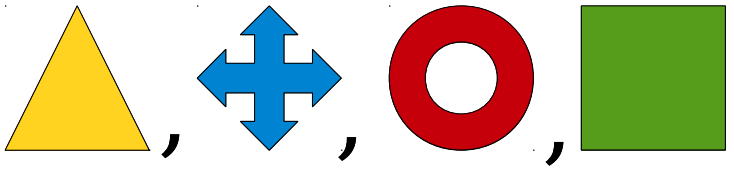
$S \subseteq T$

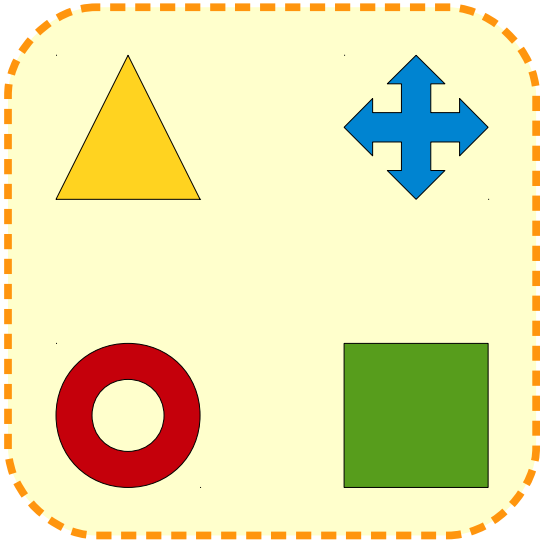
Every object in this set is in this set

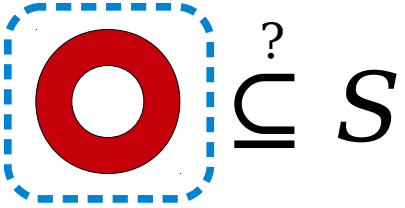
 $\overset{?}{\subseteq} S$

Otherwise, it would be like asking whether giraffe < 137 – it's a meaningless statement because you can't compare giraffes to numbers using less-than.

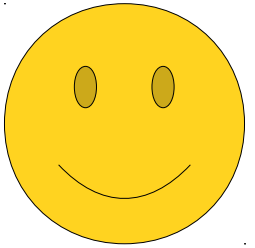


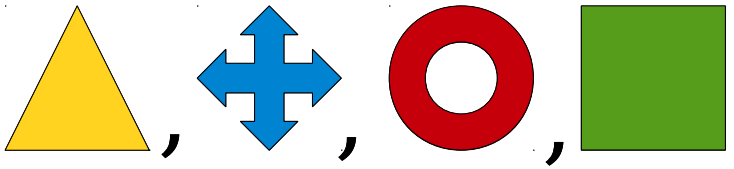
$S = \{$  $\}$

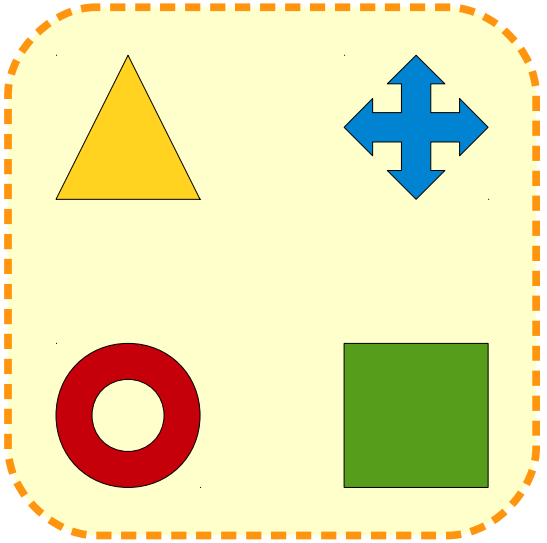


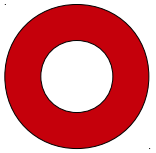
 $\overset{?}{\subseteq} S$

In this particular case, the object on the left-hand side of the subset symbol isn't a set...

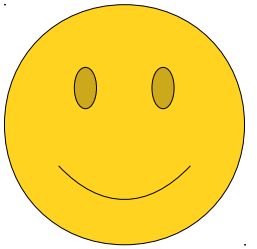


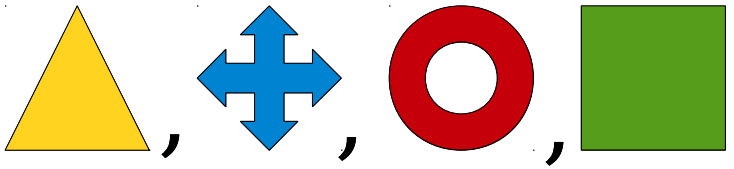
$S = \{$  $\}$

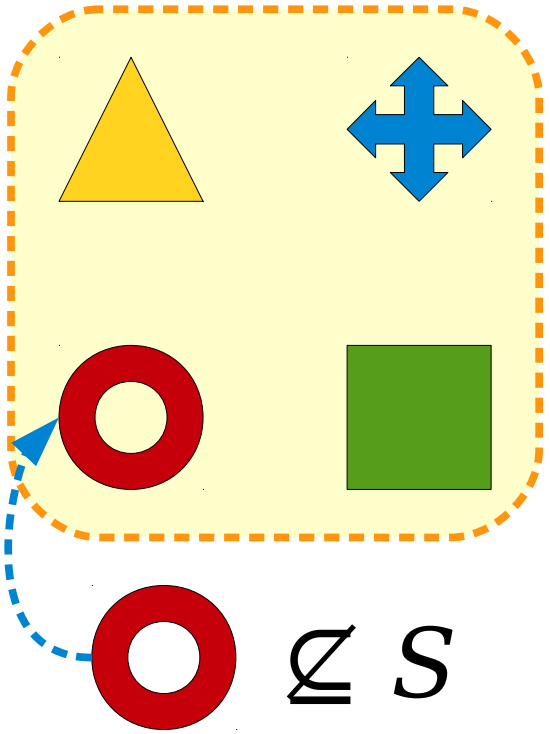


 $\notin S$

… so the initial statement was false.

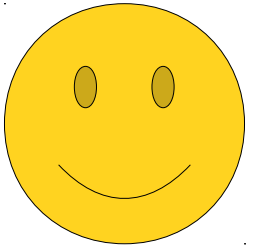


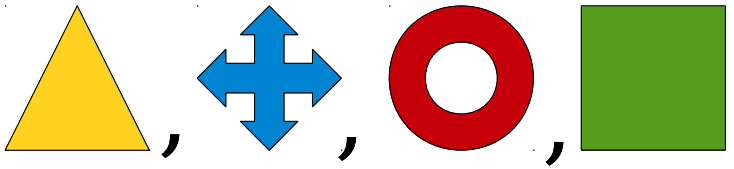
$S = \{$  $\}$

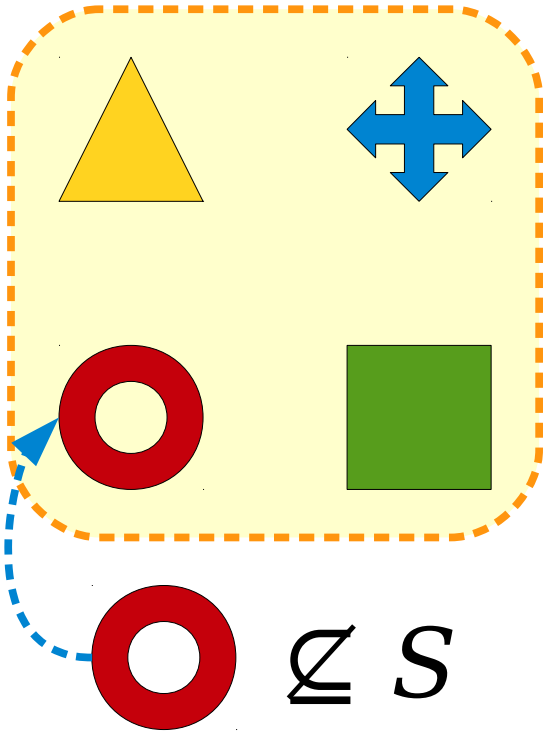


$\notin S$

This might seem a bit counterintuitive at first, because the red circle is in the set S.

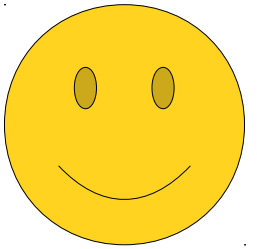


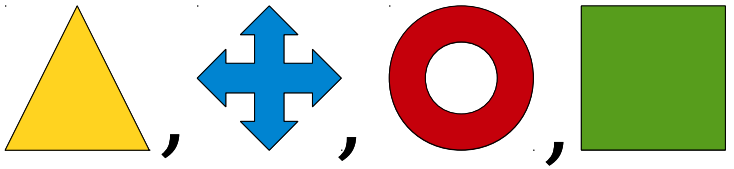
$S = \{$  $\}$

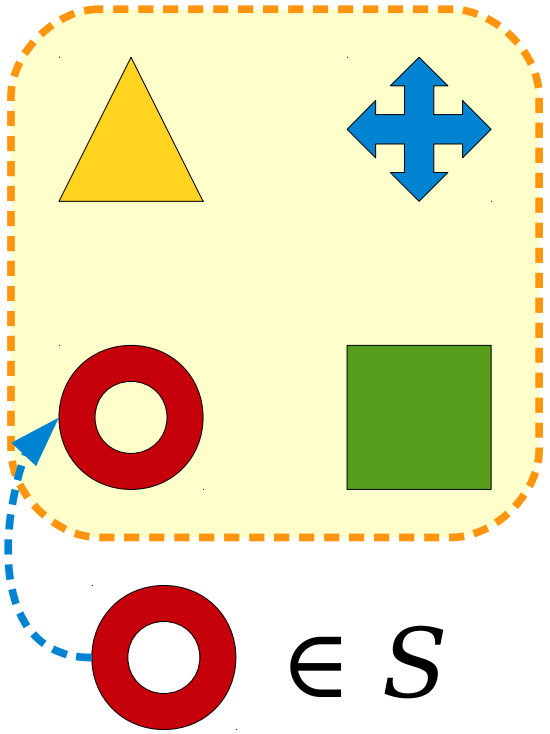


$\notin S$

But we have a different way of expressing that idea!

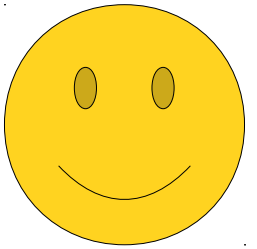


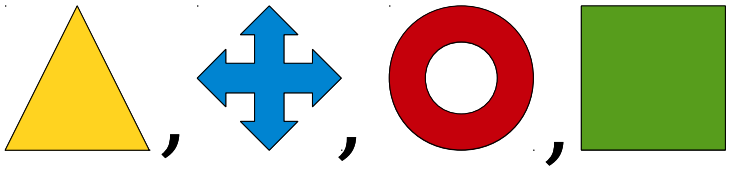
$S = \{$  $\}$

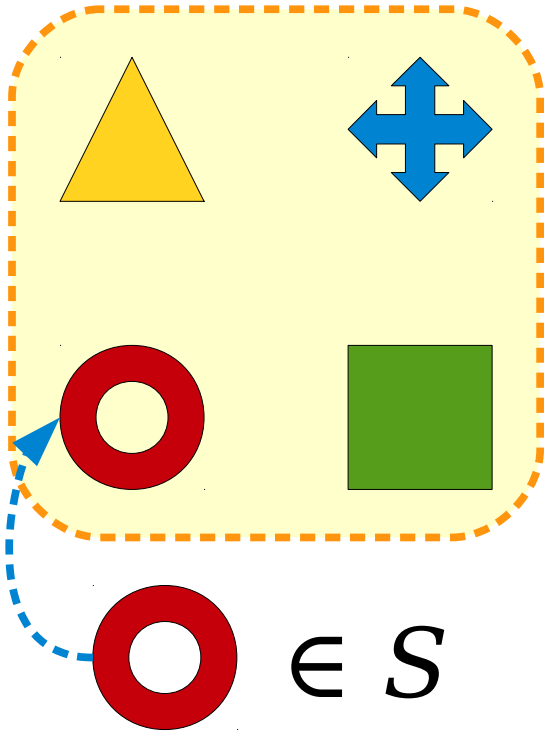


$\in S$

We can always say that the red circle is an element of the set S, even if it's not a subset of the set S.

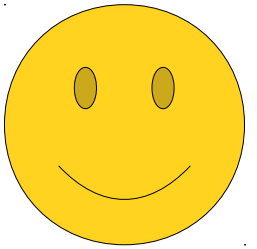


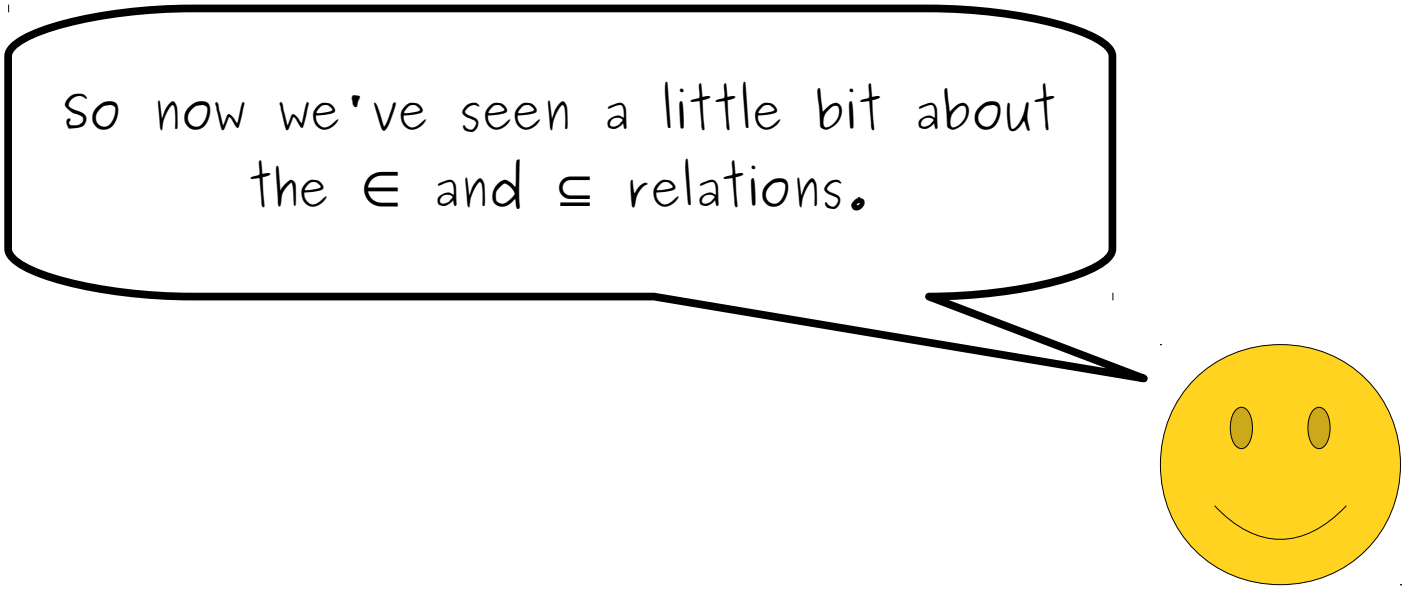
$S = \{$  $\}$



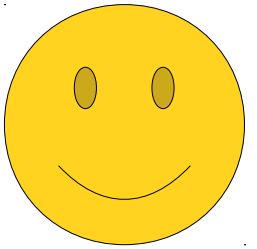
$\in S$

So remember that $\in$ and $\subseteq$ aren't the same thing. You can be an element of a set without being a subset and vice-versa.

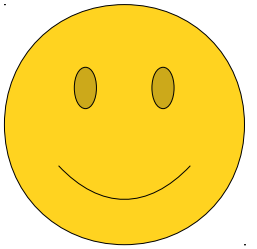



So now we've seen a little bit about the ∈ and ⊆ relations.

We've seen that you use $\in$ to ask whether objects belong to sets and $\subseteq$ to ask whether one set is a subset of another.

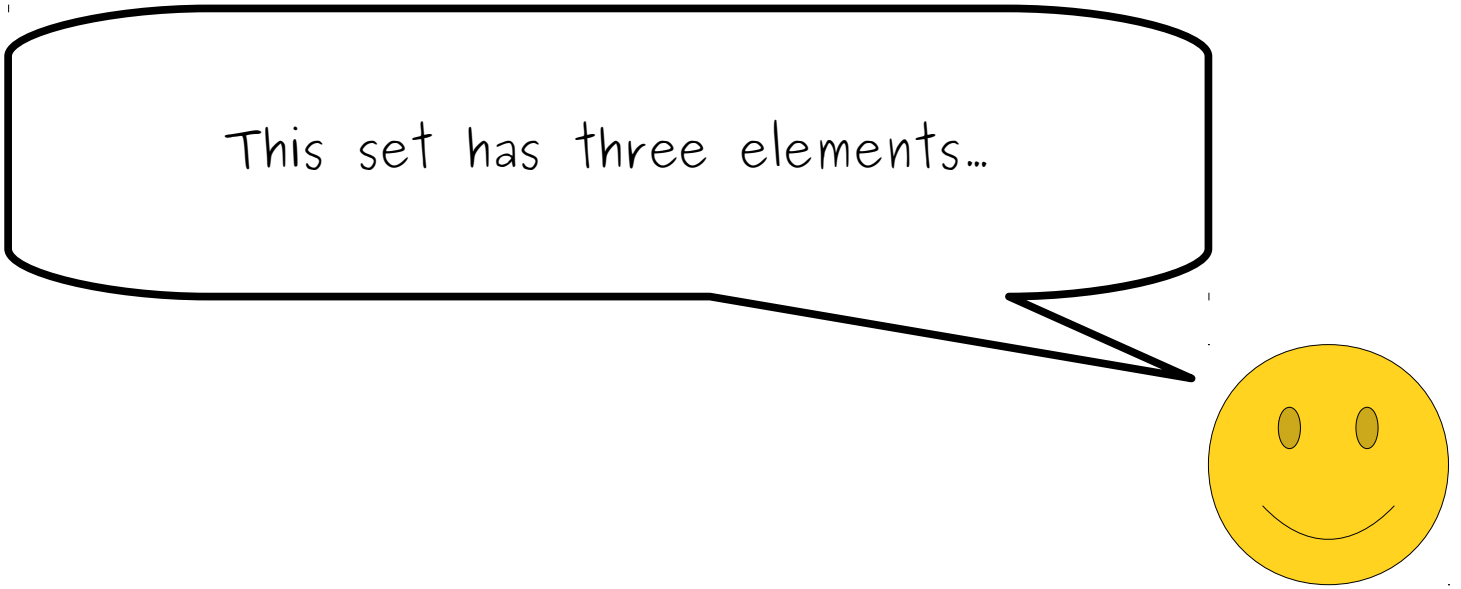


Things get a little bit more interesting when we start talking about sets that contain other sets.

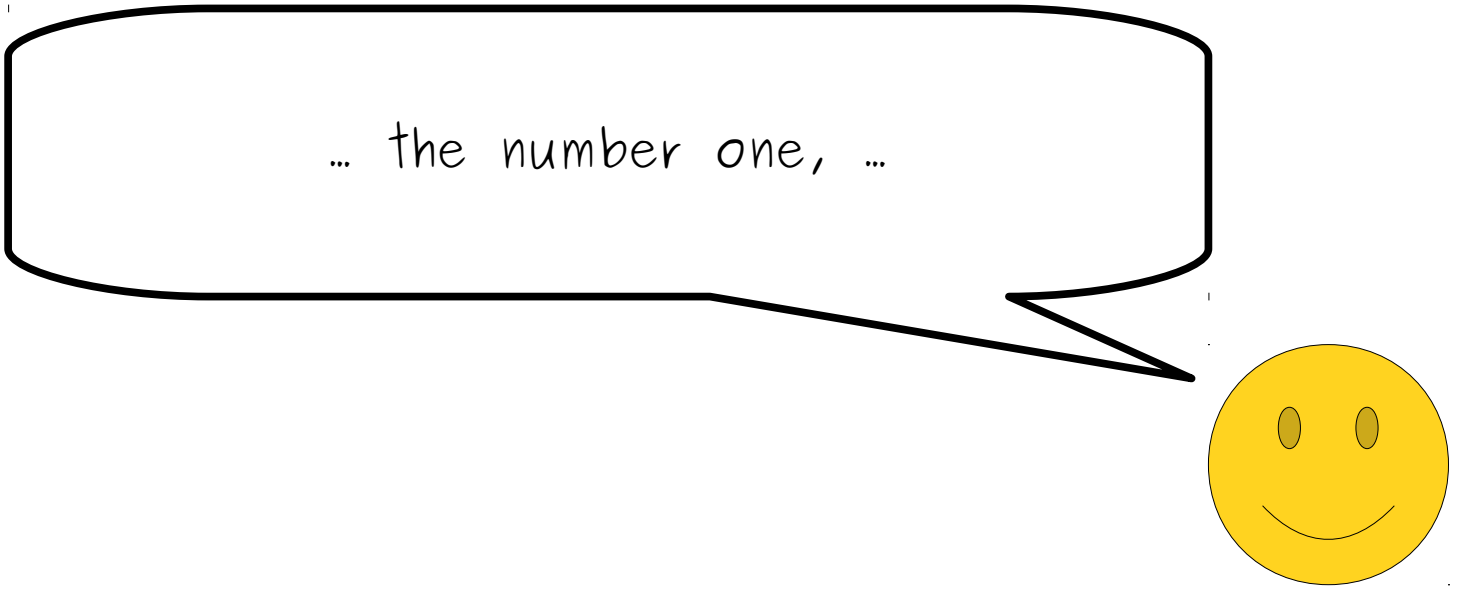


$$\{ 1, \{2, 3\}, 4 \}$$

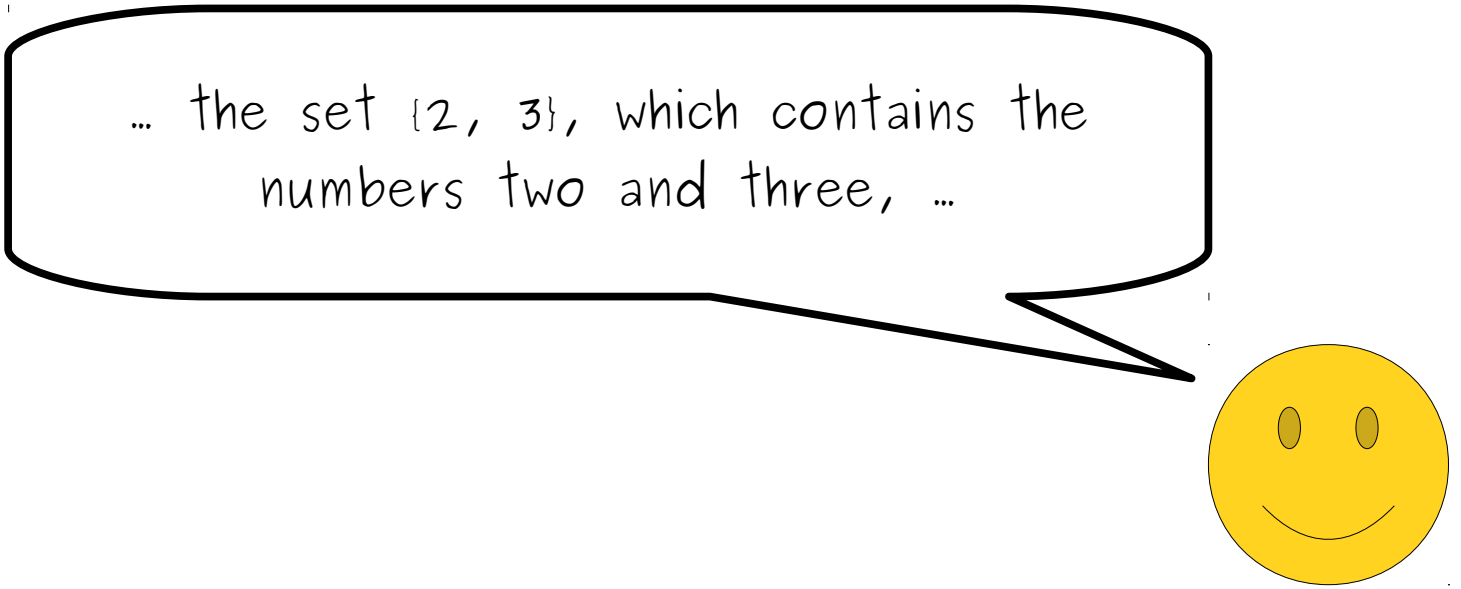
$$\{ 1, \{2, 3\}, 4 \}$$



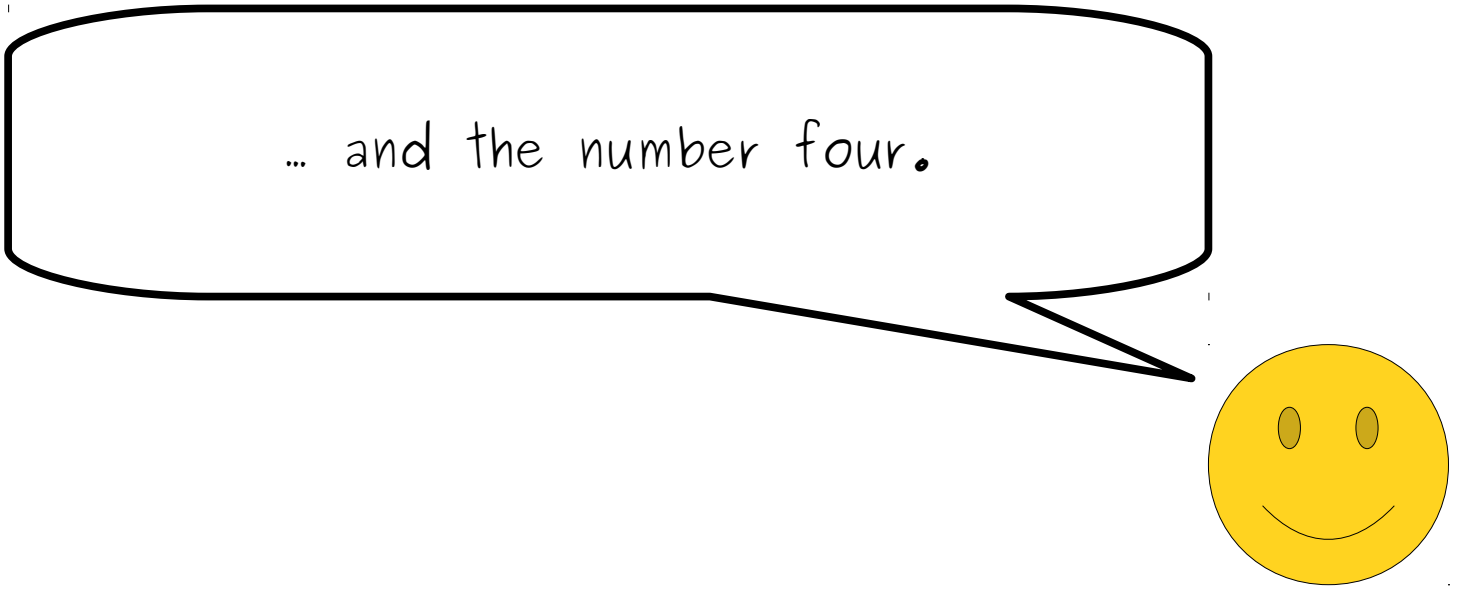
$$\{\ 1, \{2, 3\}, 4\ \}$$



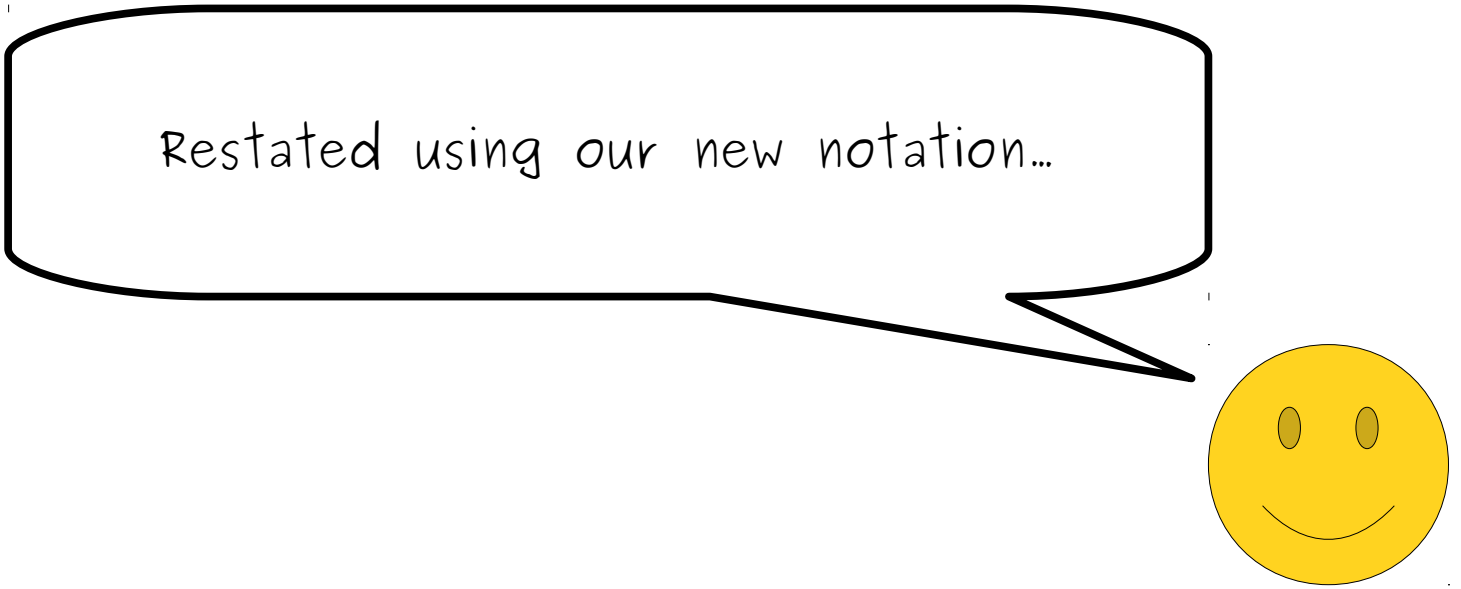
$$\{\ 1,\ \{2,\ 3\},\ 4\ \}$$



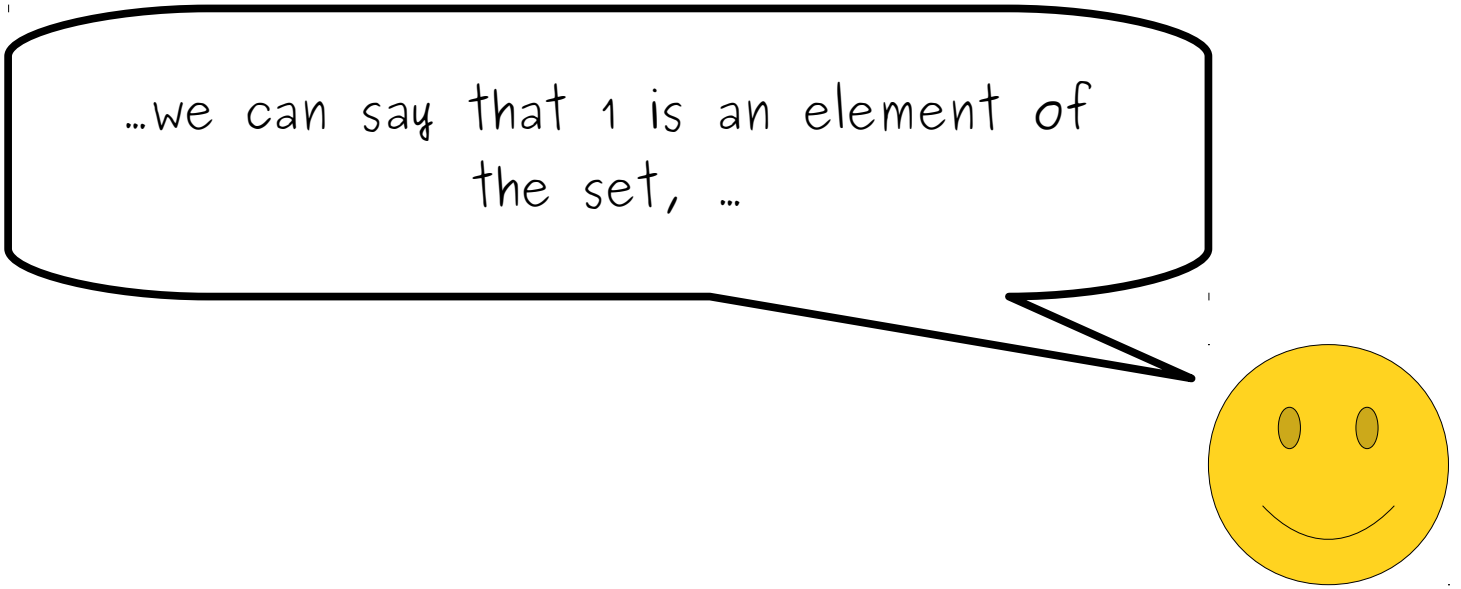
$$\{\ 1, \{2, 3\}, 4\ \}$$



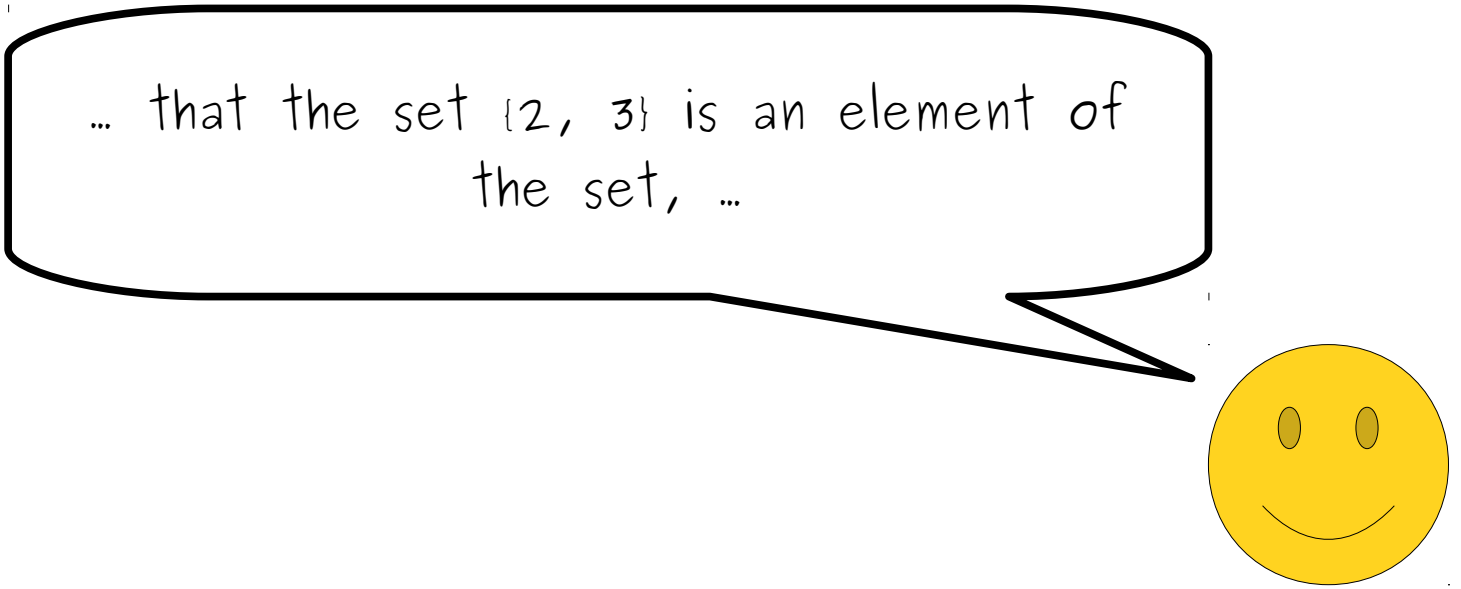
$$\{\ 1, \{2, 3\}, 4\ \}$$



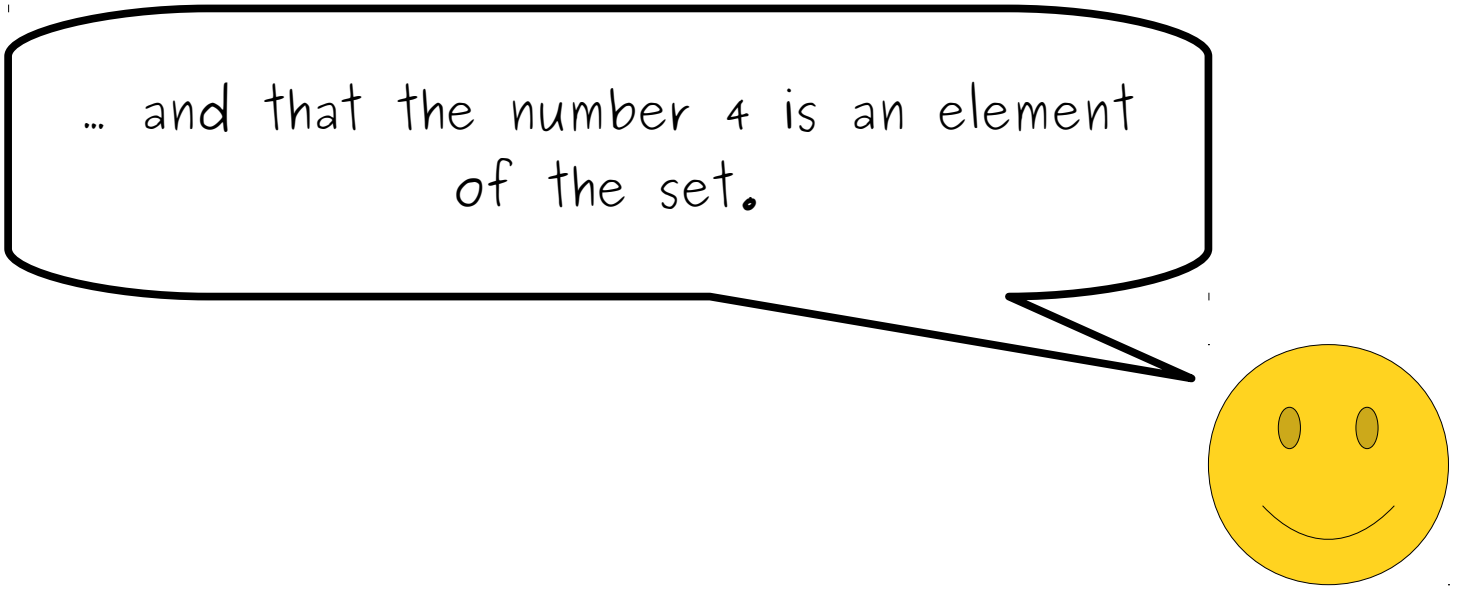
$$1 \in \{ 1, \{2, 3\}, 4 \}$$



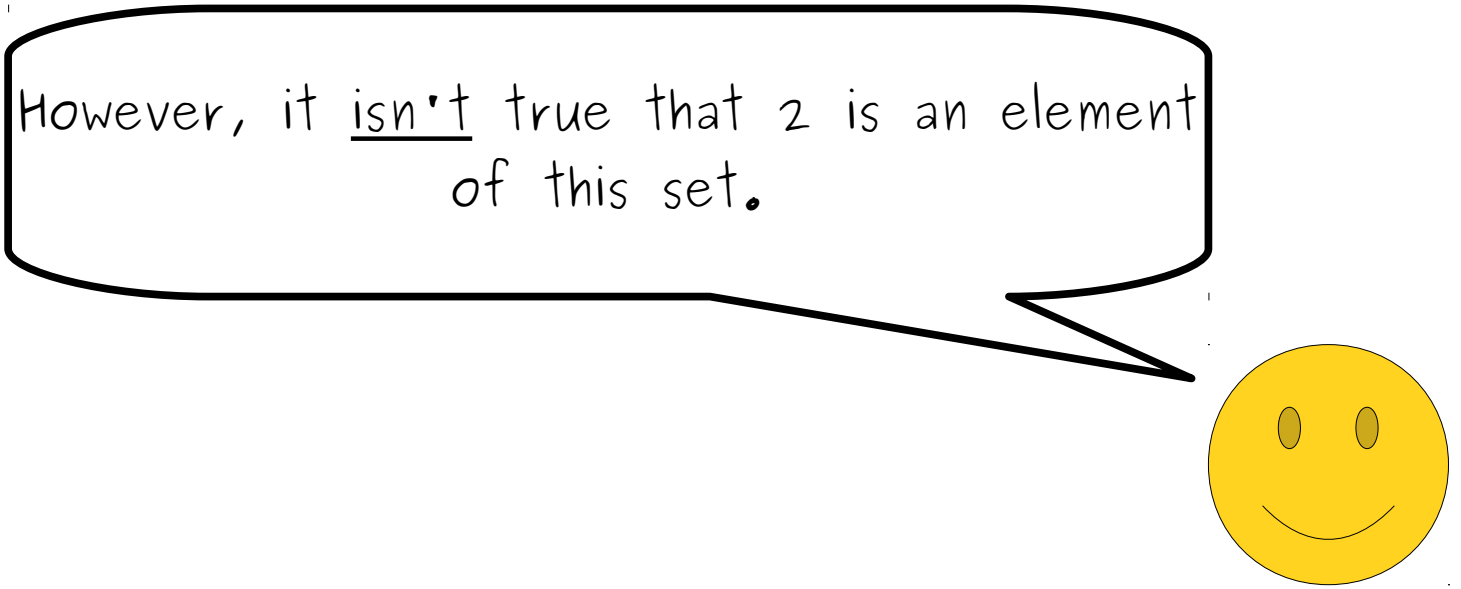
$$\{2, 3\} \in \{ 1, \{2, 3\}, 4 \}$$



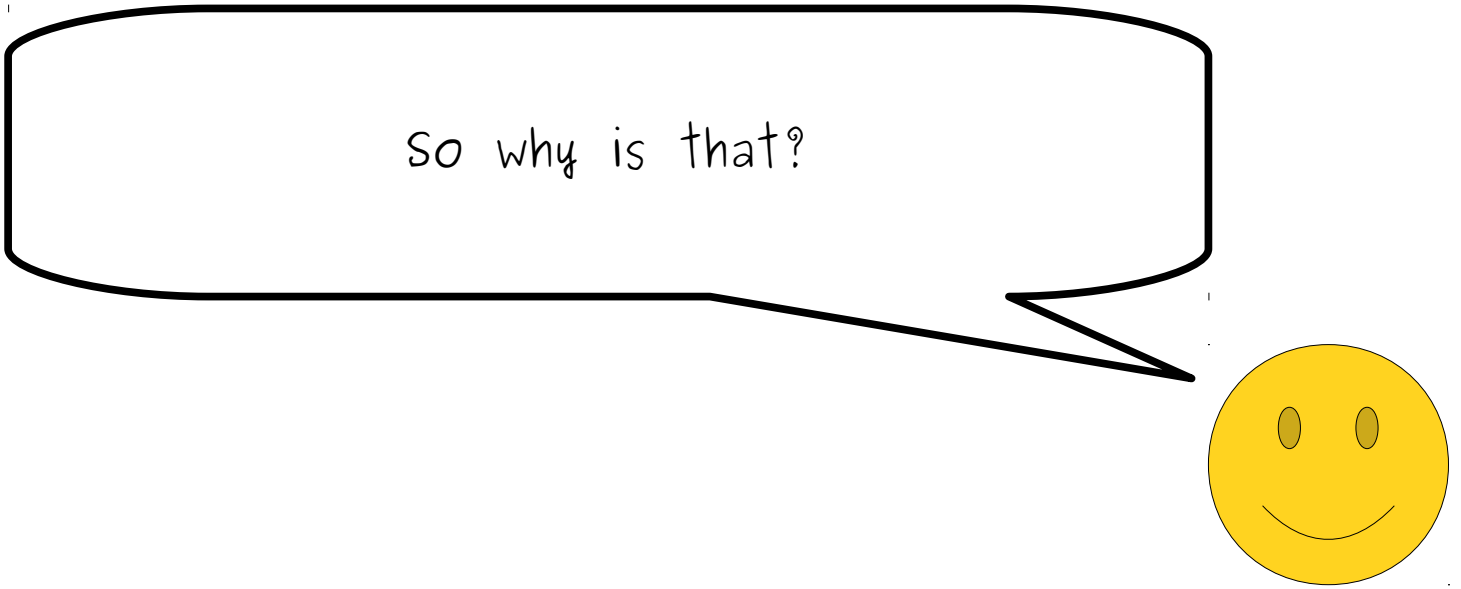
$$4 \in \{ 1, \{2, 3\}, 4 \}$$



$$2 \notin \{ 1, \{2, 3\}, 4 \}$$

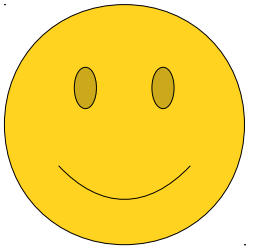


$$2 \notin \{ 1, \{2, 3\}, 4 \}$$



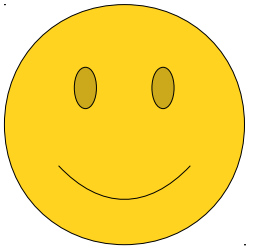
$$2 \notin \{ 1, \{2, 3\}, 4 \}$$

Remember, the set on the right has three different elements: the number 1, the set {2, 3}, and the number 4.

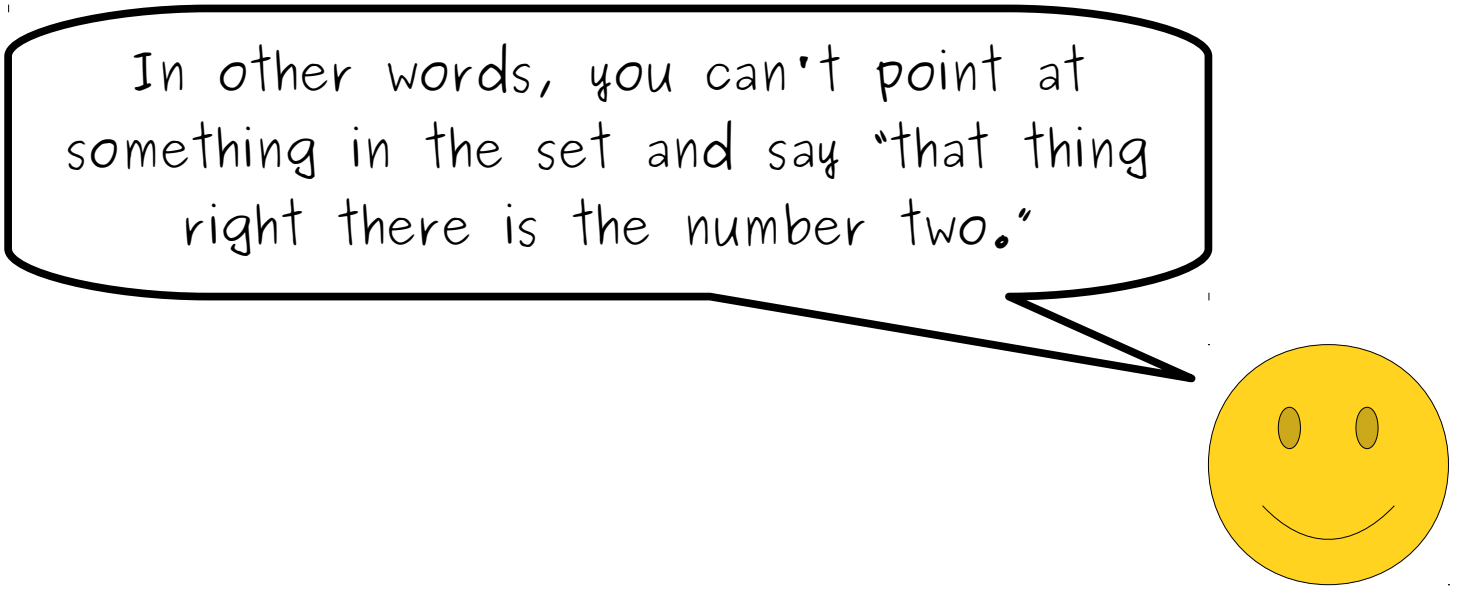


$$2 \notin \{\, 1, \{2, 3\}, 4 \,\}$$

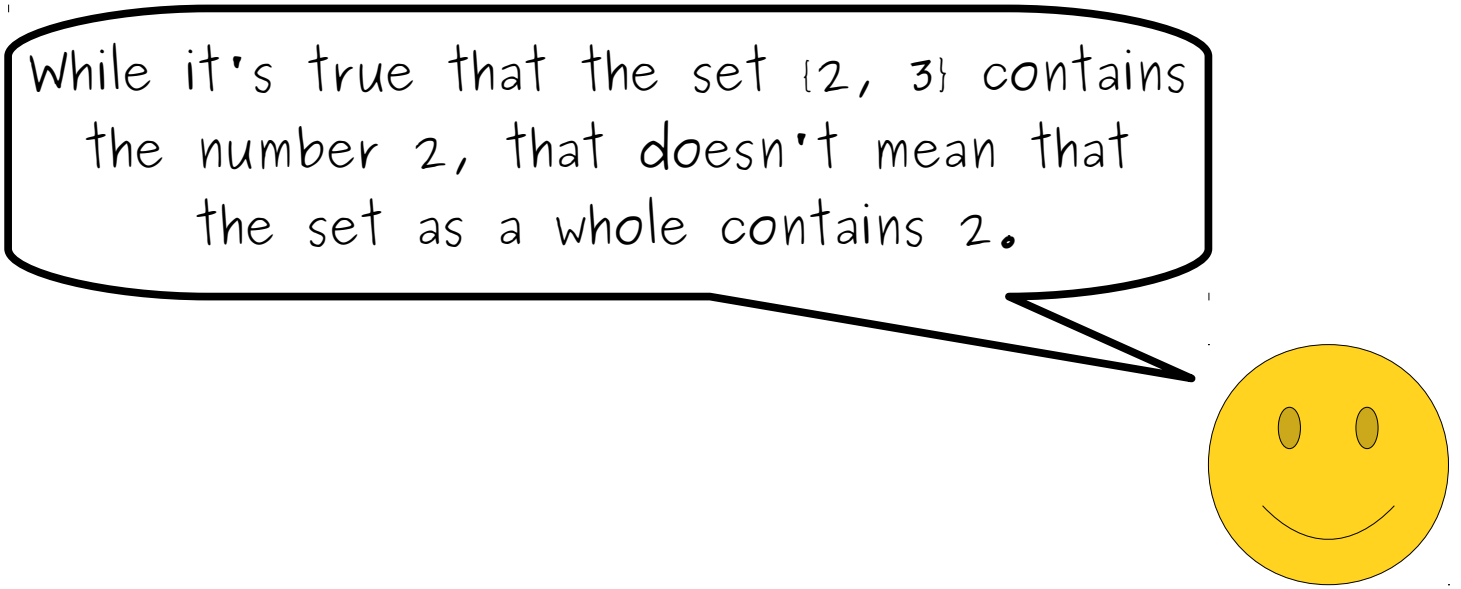
None of the terms highlighted in blue is exactly equal to the number two.



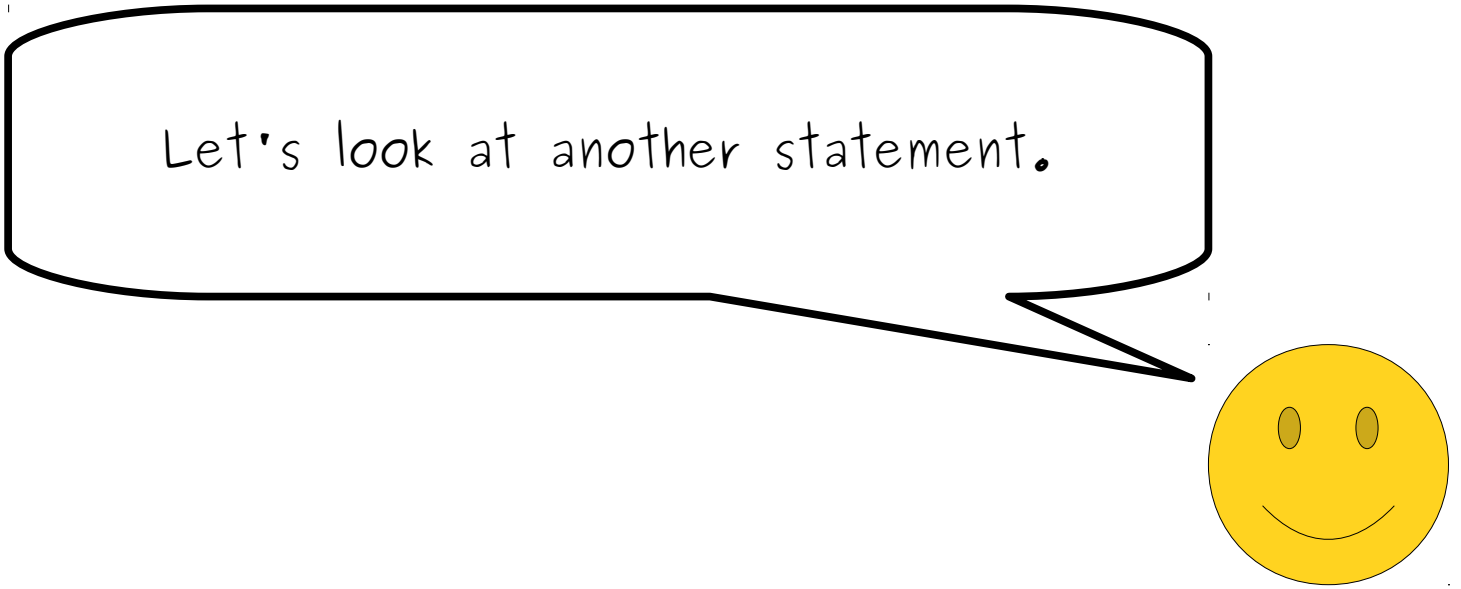
$$2 \notin \{ 1, \{2, 3\}, 4 \}$$


In other words, you can't point at something in the set and say "that thing right there is the number two."

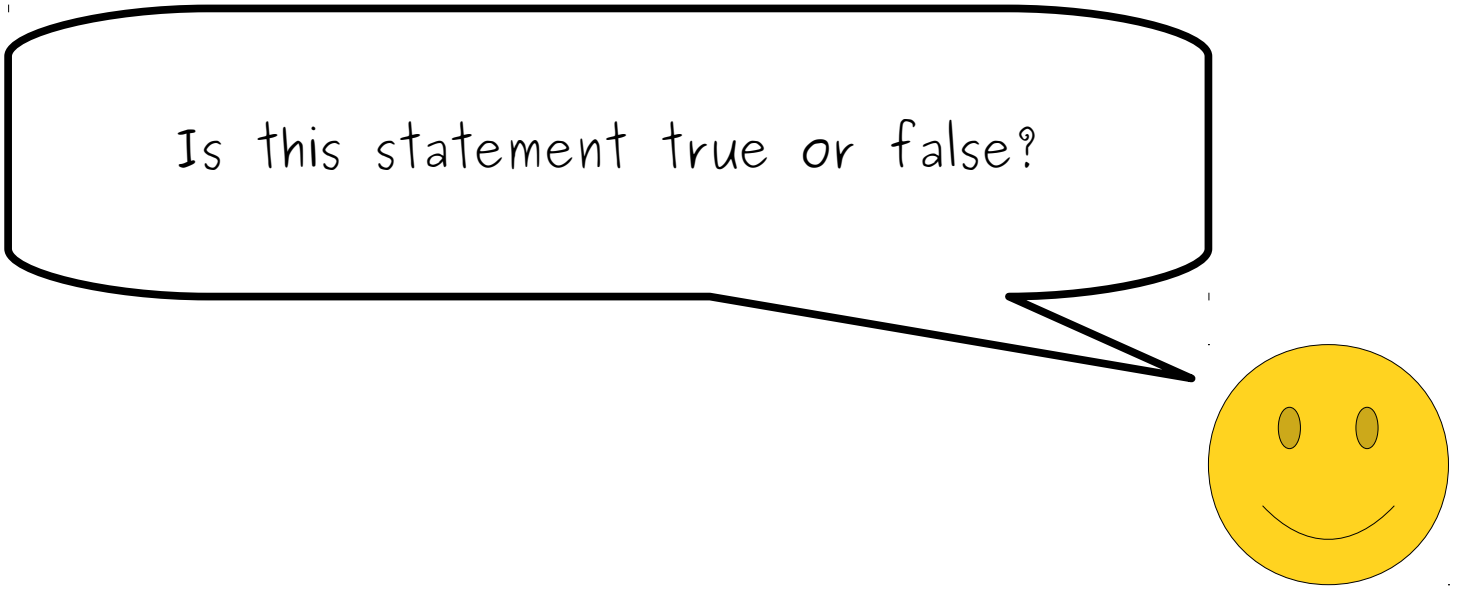
$$2 \notin \{ 1, \{2, 3\}, 4 \}$$


While it's true that the set {2, 3} contains the number 2, that doesn't mean that the set as a whole contains 2.

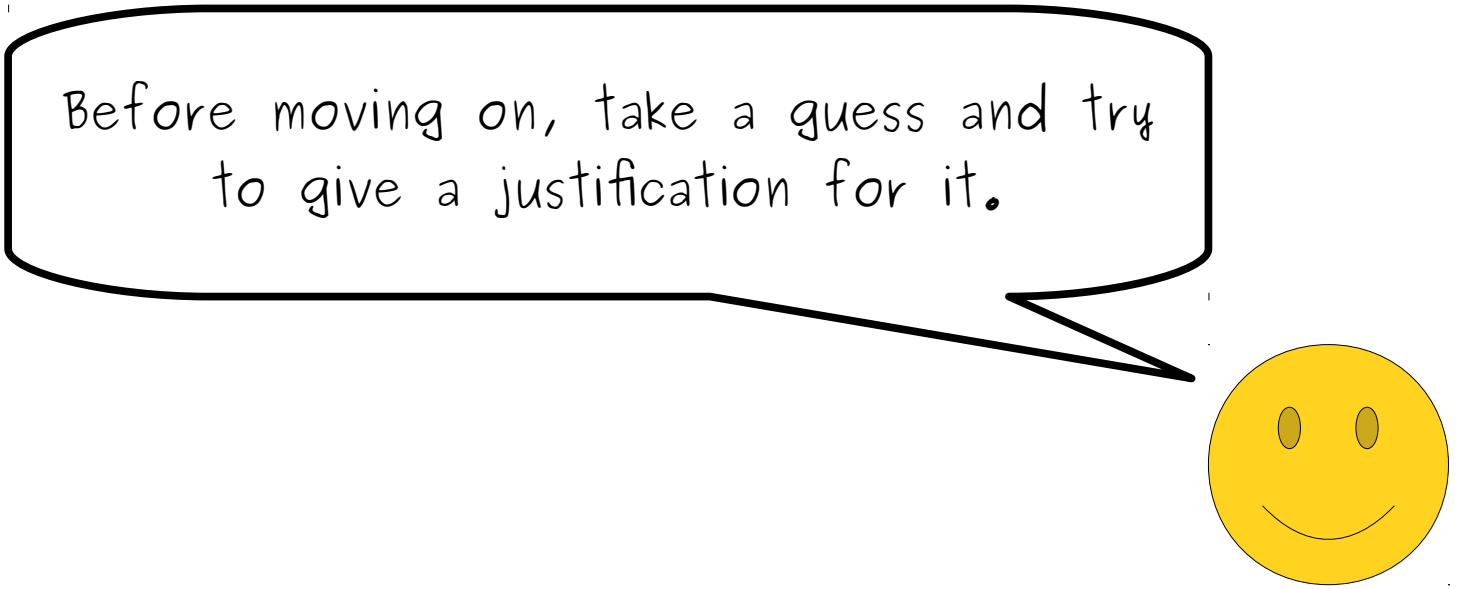
{ 1, {2, 3}, 4 }



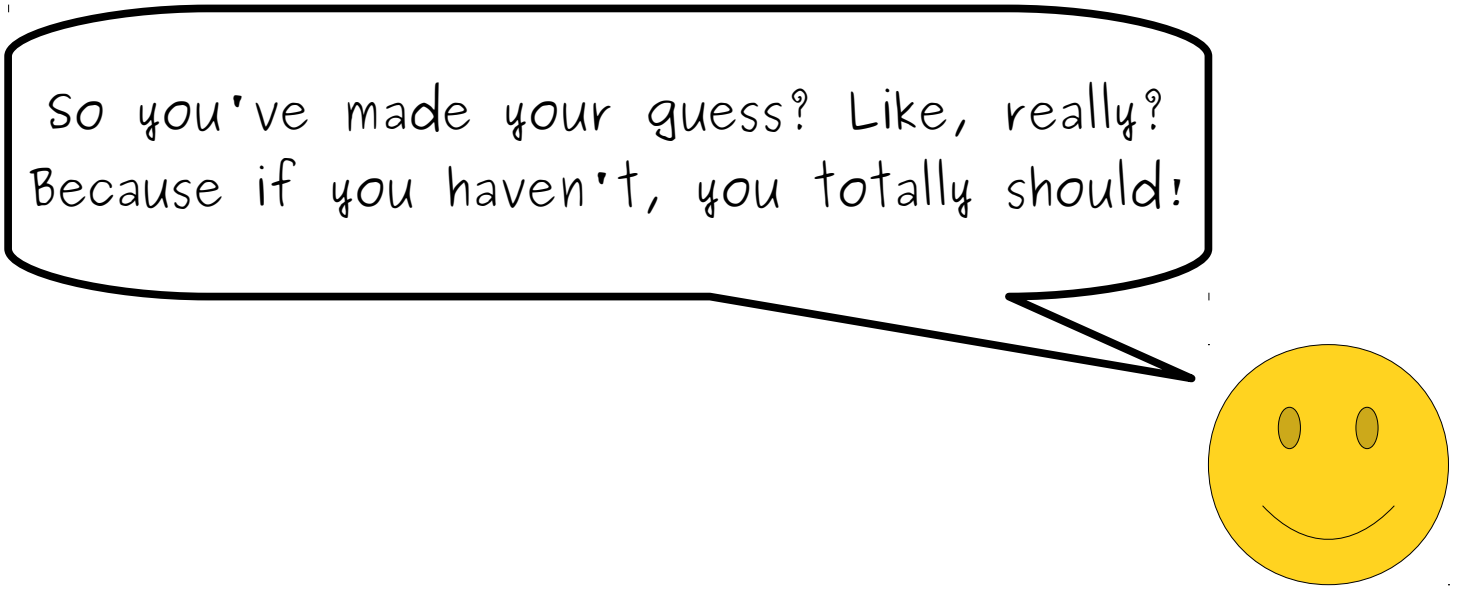
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



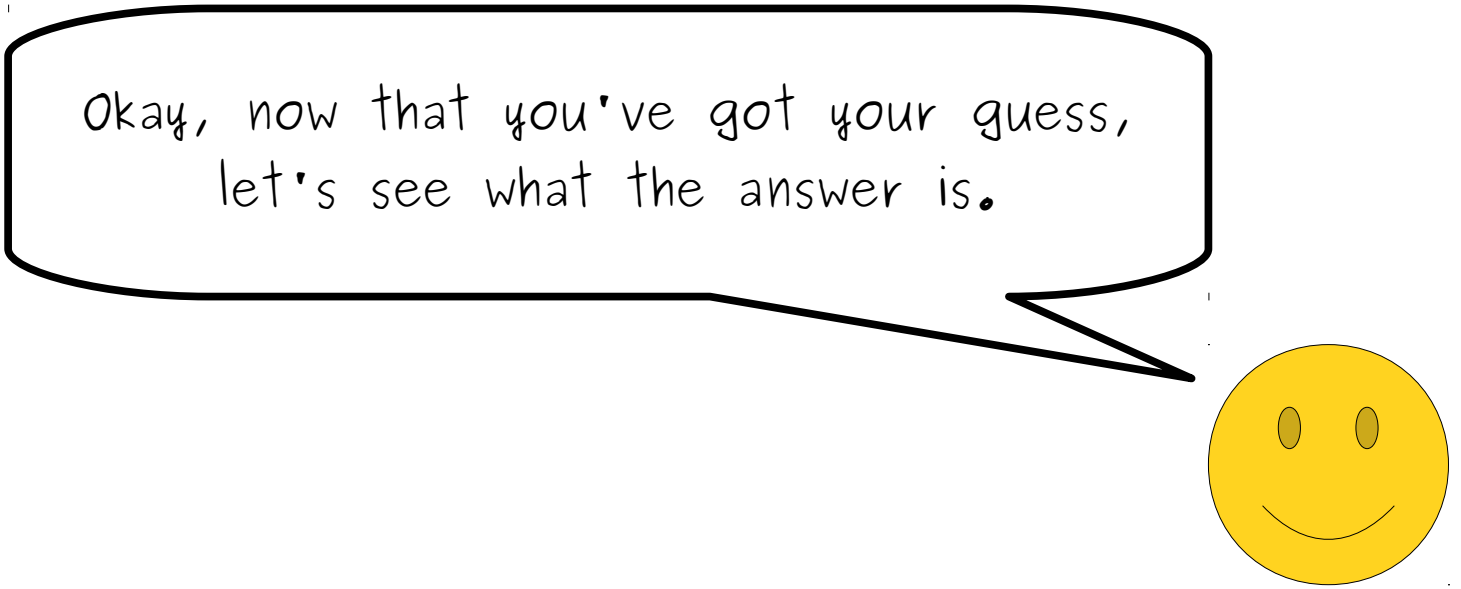
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



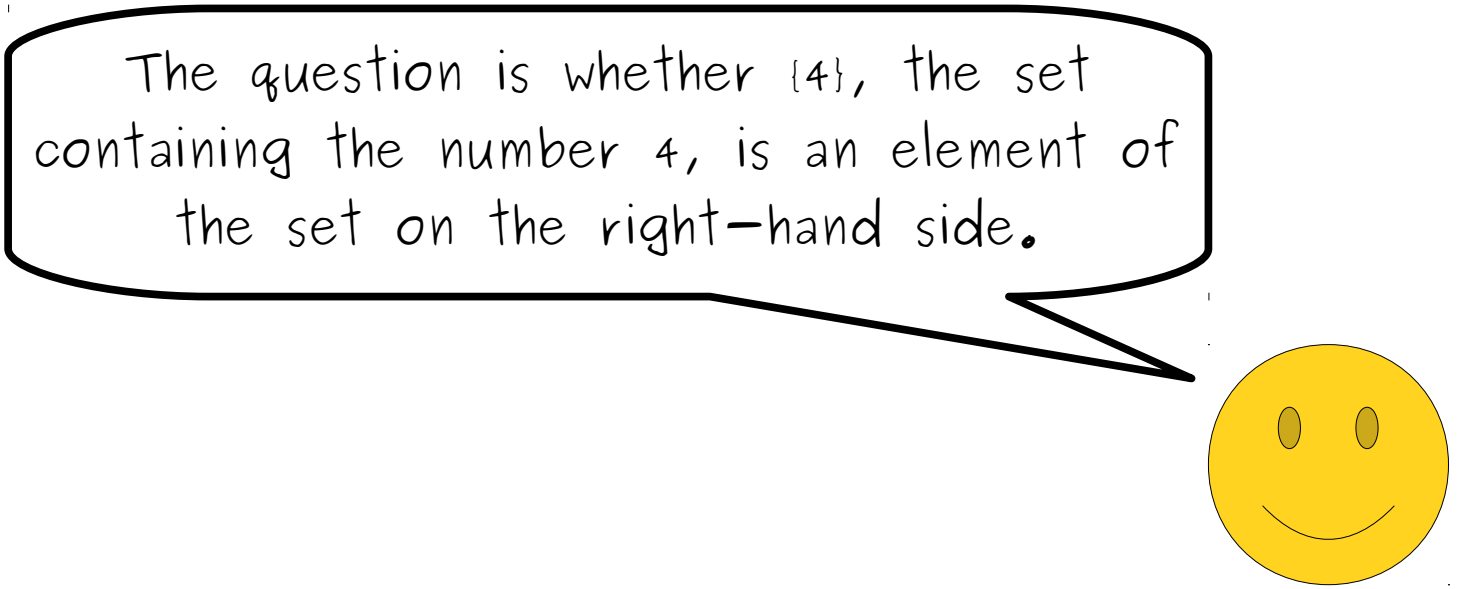
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



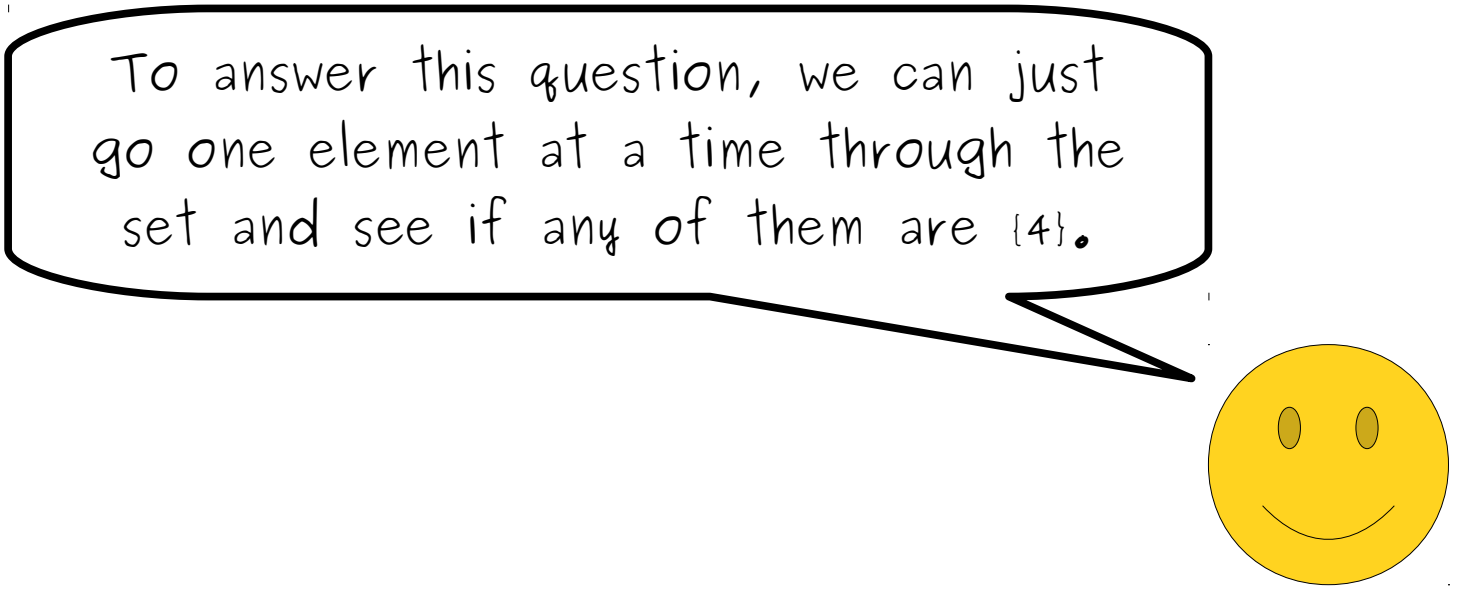
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



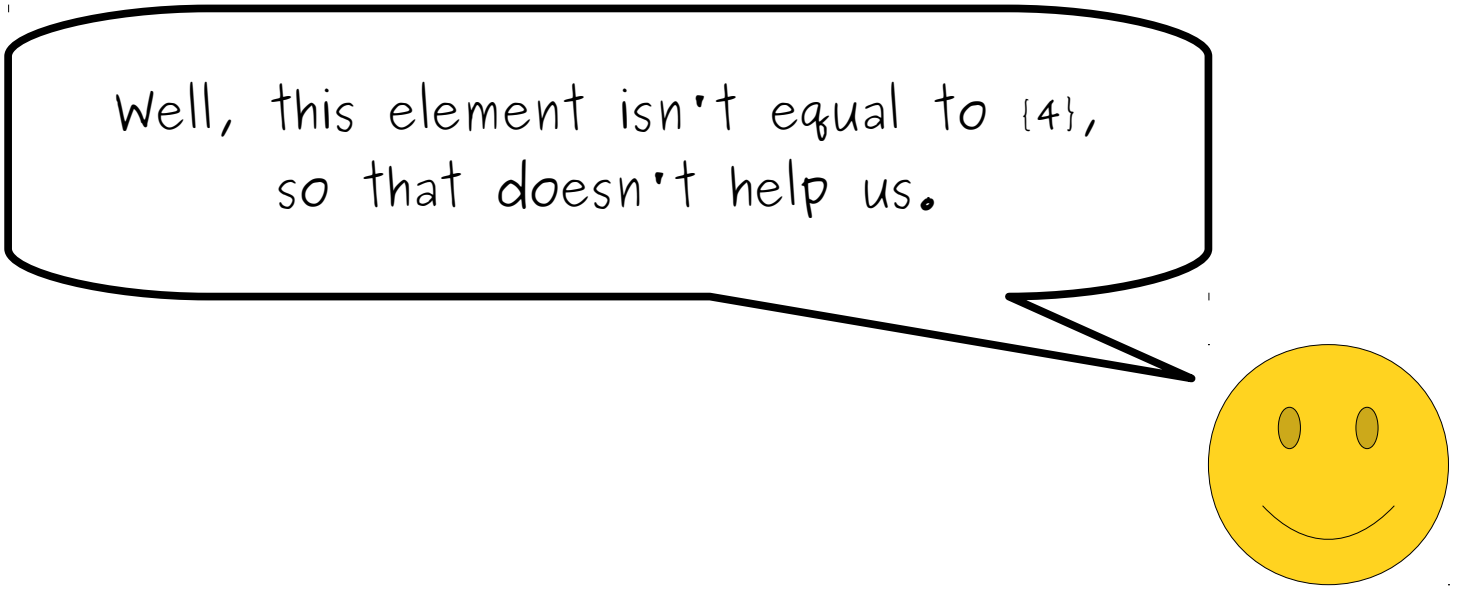
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



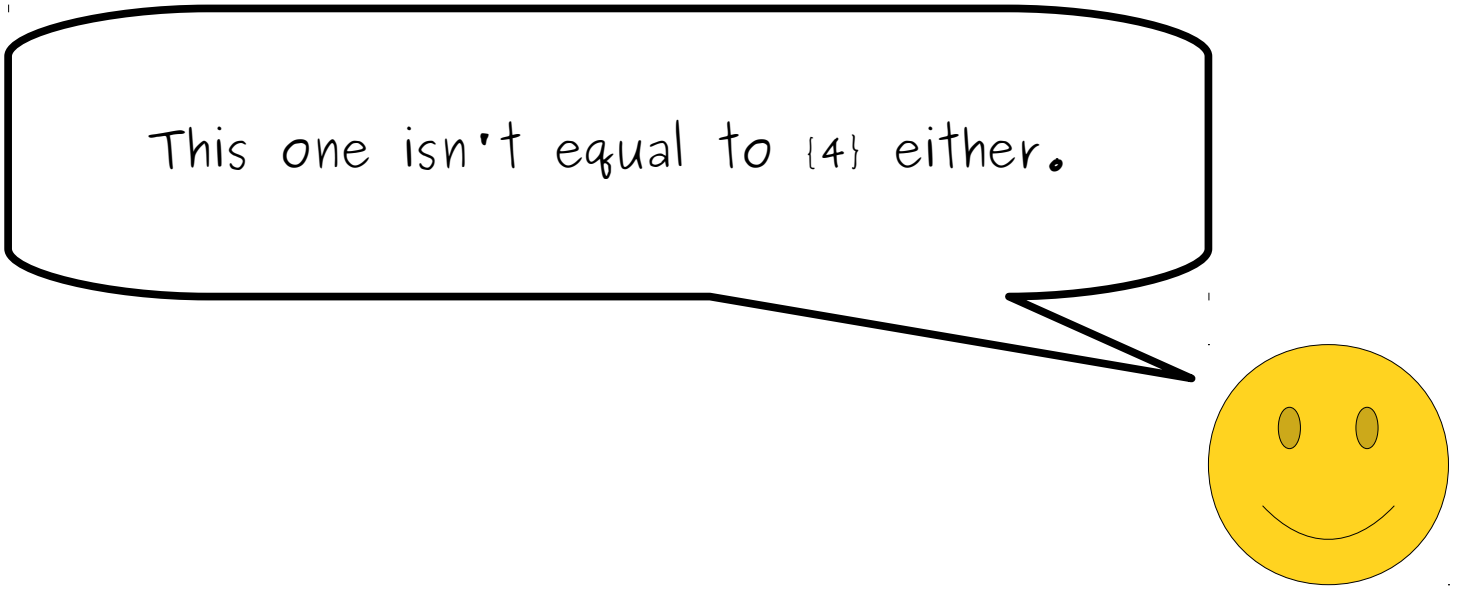
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



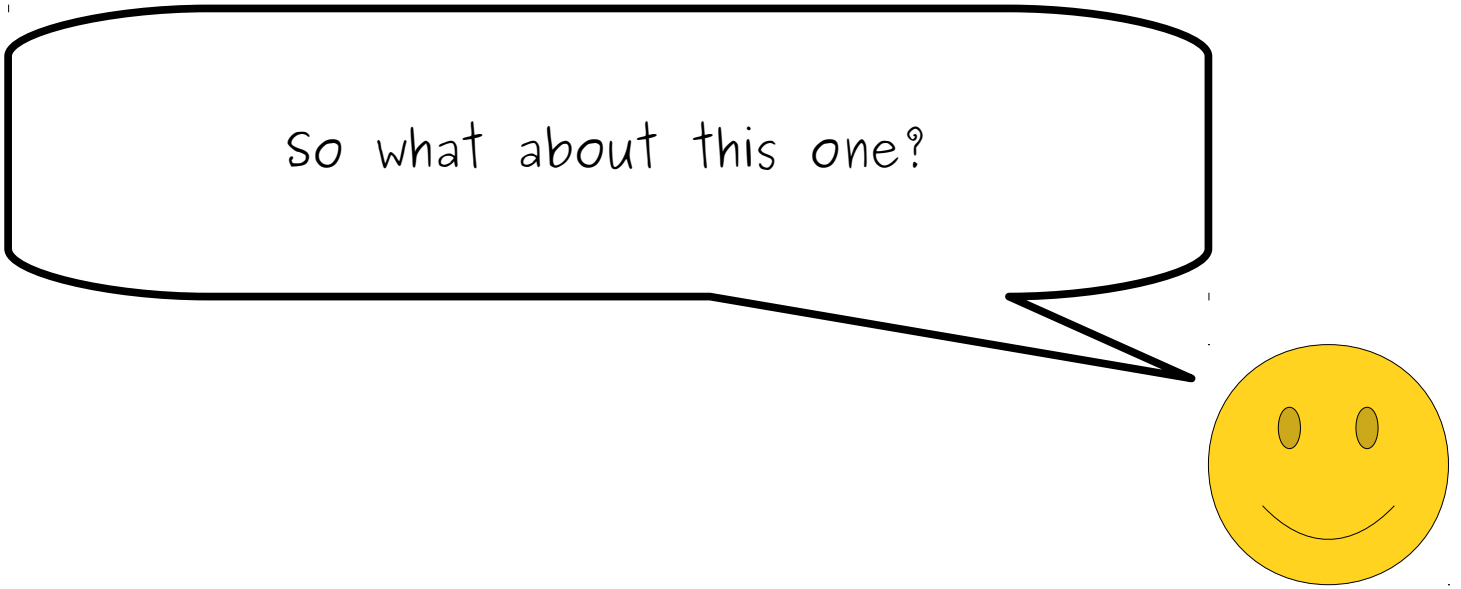
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



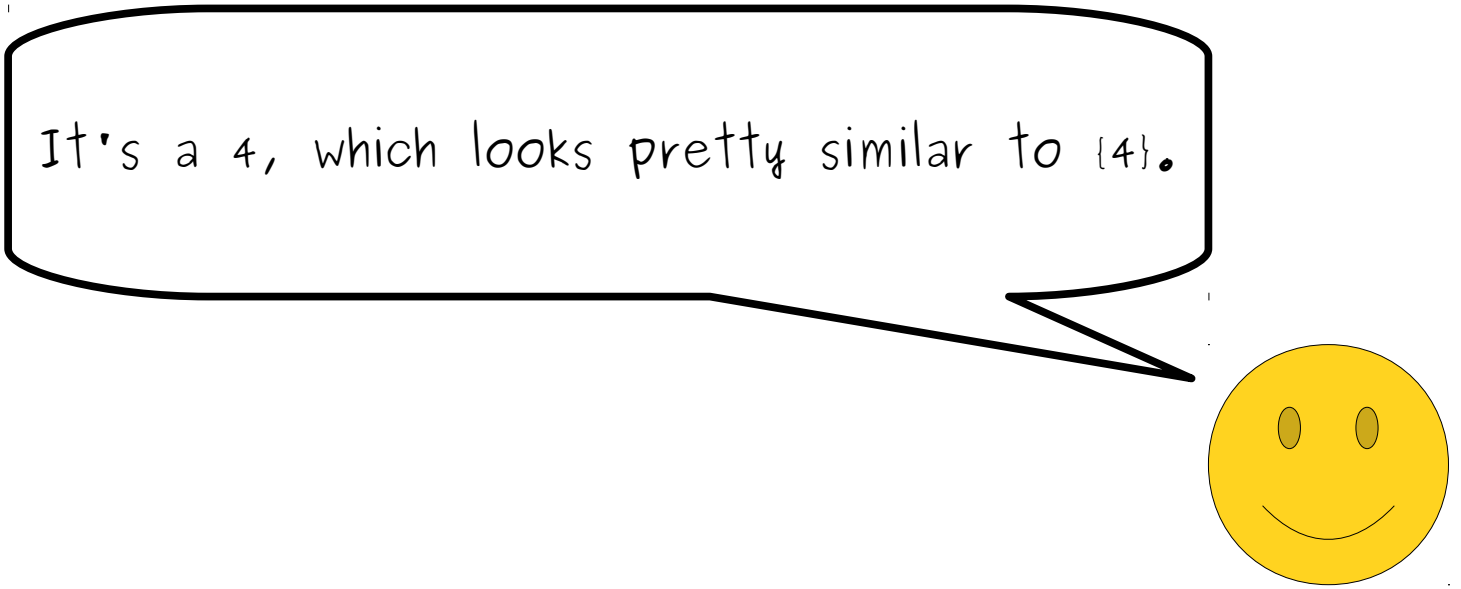
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



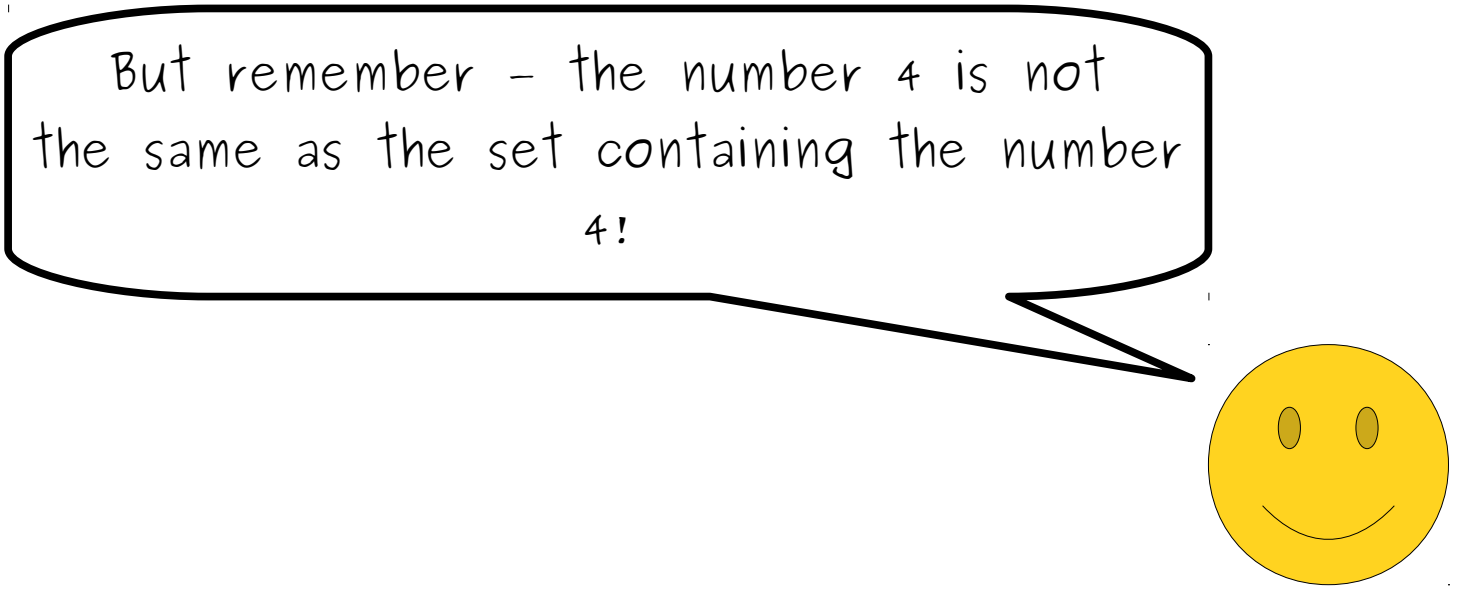
$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

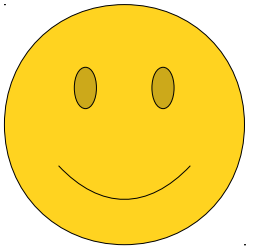


$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

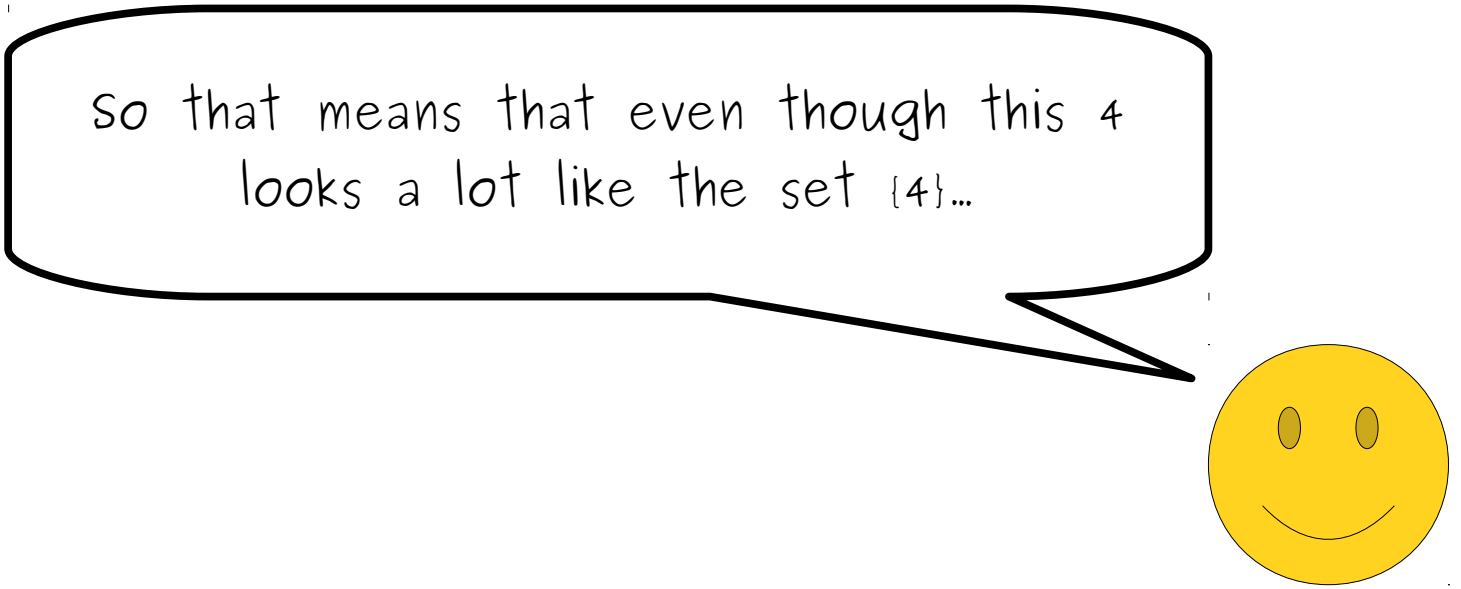


$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

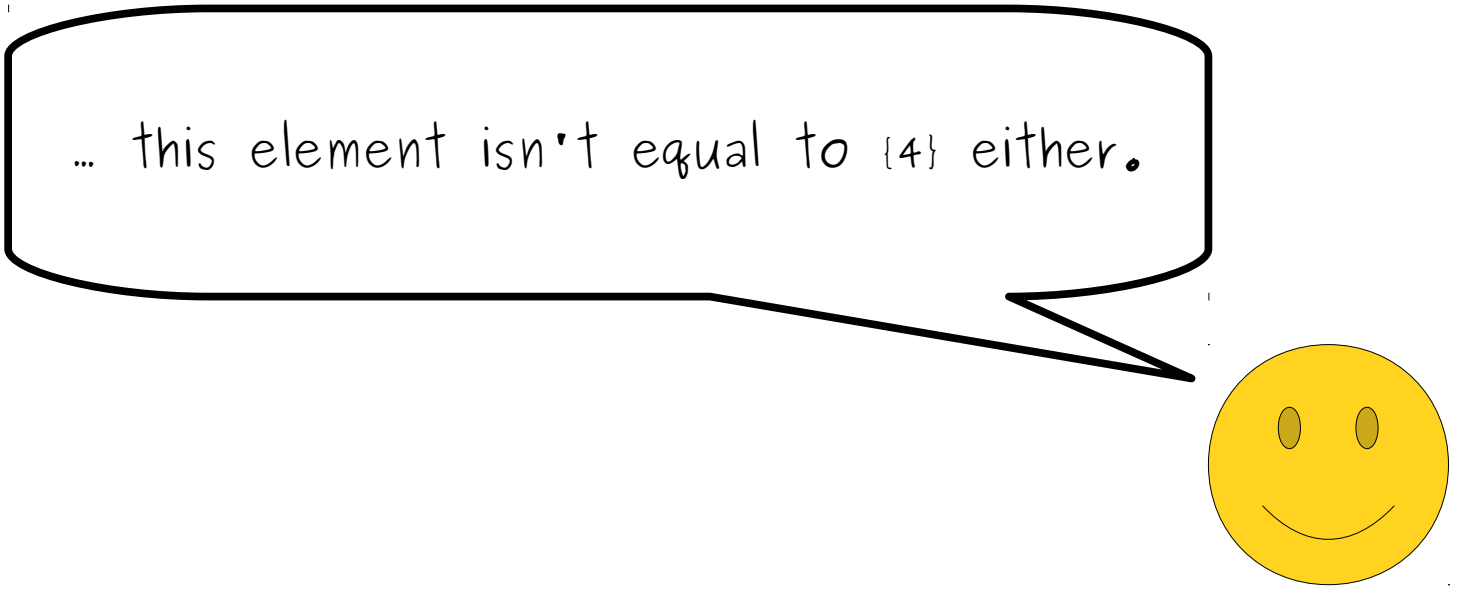
Generally speaking, no object is ever equal to the set that contains it.



$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

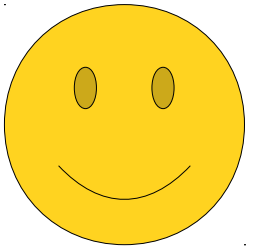


$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

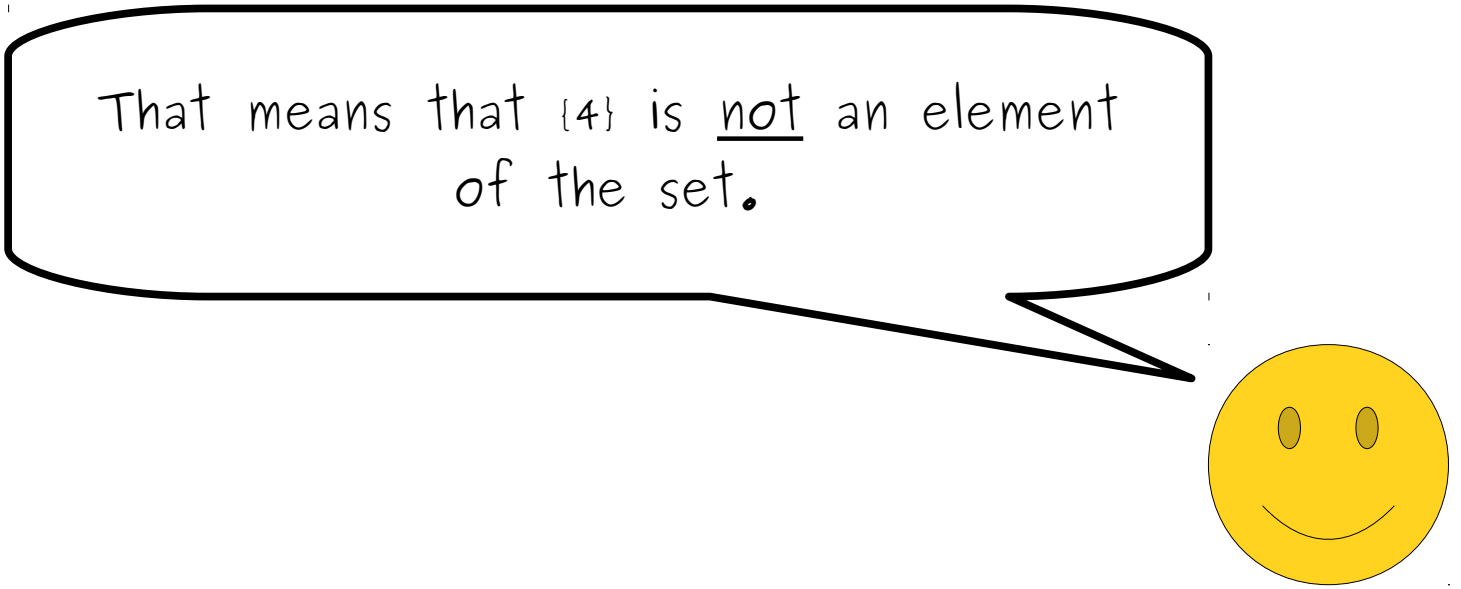


$$\{4\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

So we've gone through all the elements of the set, trying to see if any of them are equal to {4}, and it looks like none of them are.

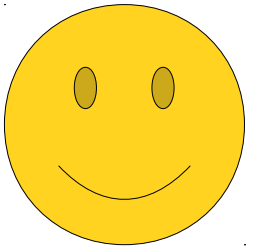


$$\{4\} \notin \{\, 1, \{2, 3\}, 4 \,\}$$



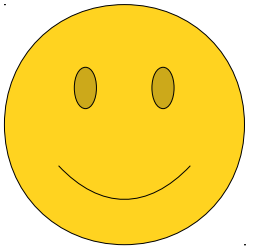
$$\{4\} \notin \{\, 1, \{2, 3\}, 4 \,\}$$

If you thought that $\{4\}$ was an element of the set, don't worry! It's a reasonable thought. It just happens to not work given how we've chosen to define sets and the $\in$ symbol.

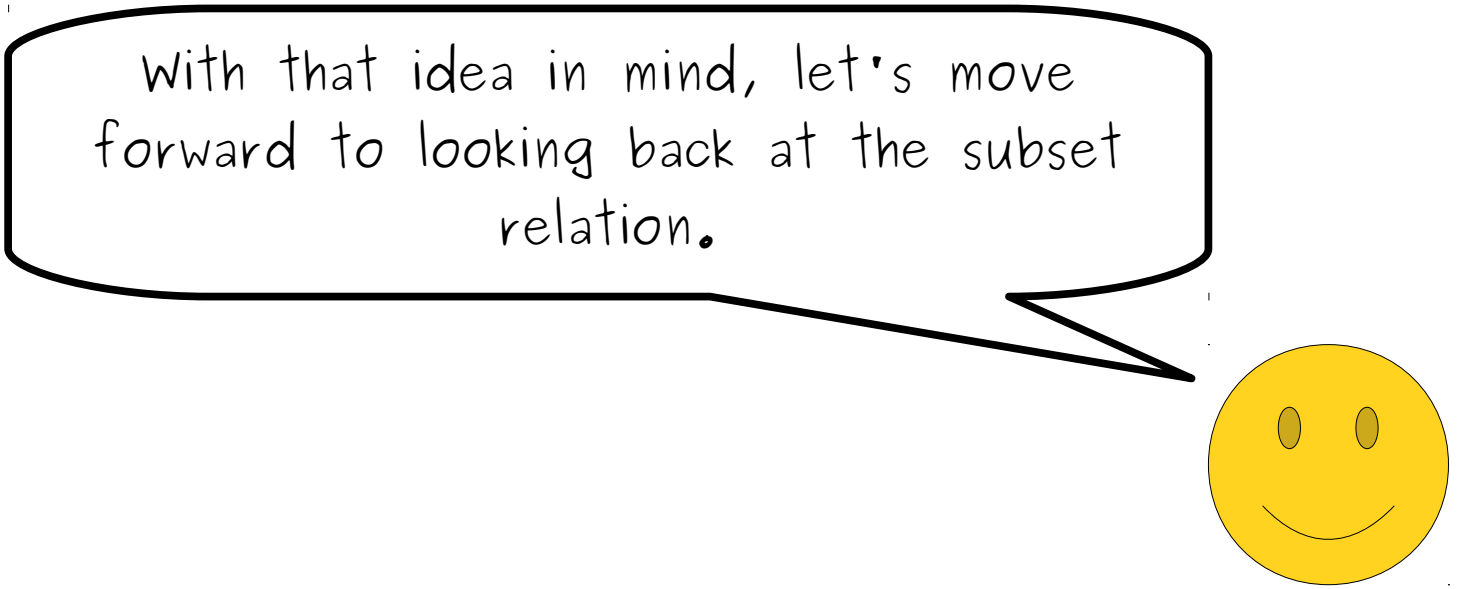


$$\{4\} \notin \{\, 1, \{2, 3\}, 4 \,\}$$

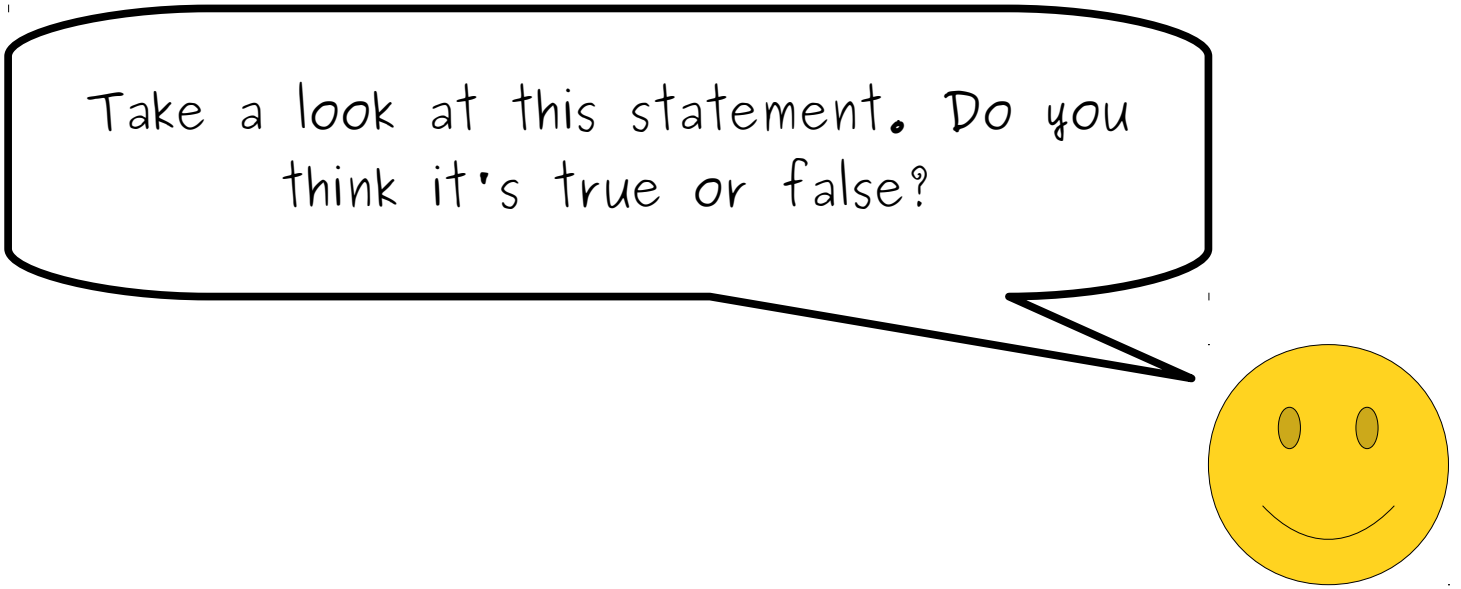
Going forward, remember that x and {x} are different things – x is an individual object, while {x} is a set containing the object x.



$$\{ 1, \{2, 3\}, 4 \}$$

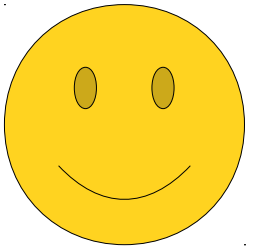

With that idea in mind, let's move forward to looking back at the subset relation.

$$\{1\} \stackrel{?}{\subseteq} \{\, 1, \{2, 3\}, 4 \,\}$$

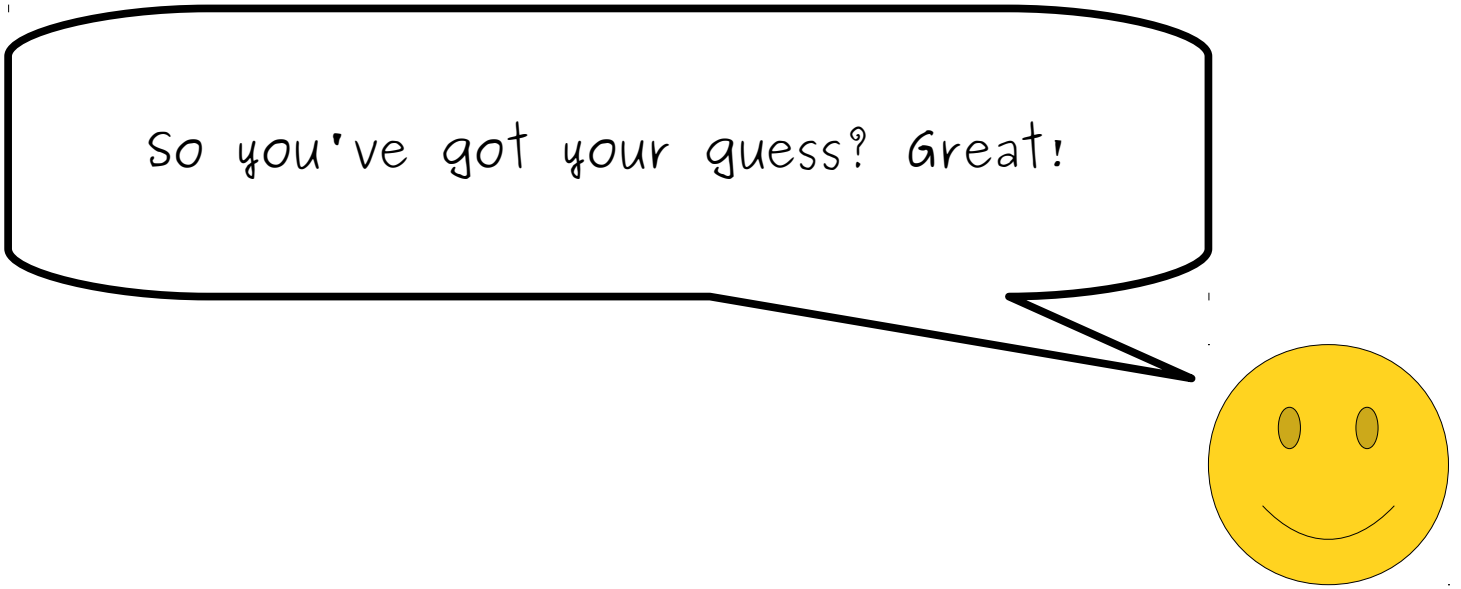


$$\{1\} \stackrel{?}{\subseteq} \{\, 1, \{2, 3\}, 4 \,\}$$

As before, take a minute to come up with a guess before moving on. You'll be happier if you do.

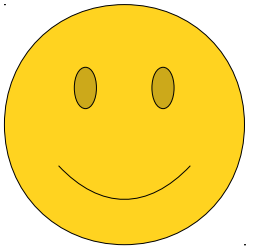


$$\{1\} \stackrel{?}{\subseteq} \{\, 1, \{2, 3\}, 4 \,\}$$



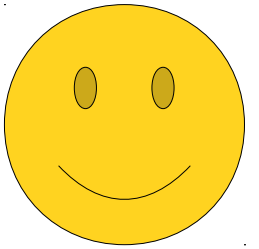
$$\{1\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

To determine whether {1} is a subset of {1, {2,3}, 4}, we need to ask whether every element of {1} is also an element of {1, {2,3}, 4}

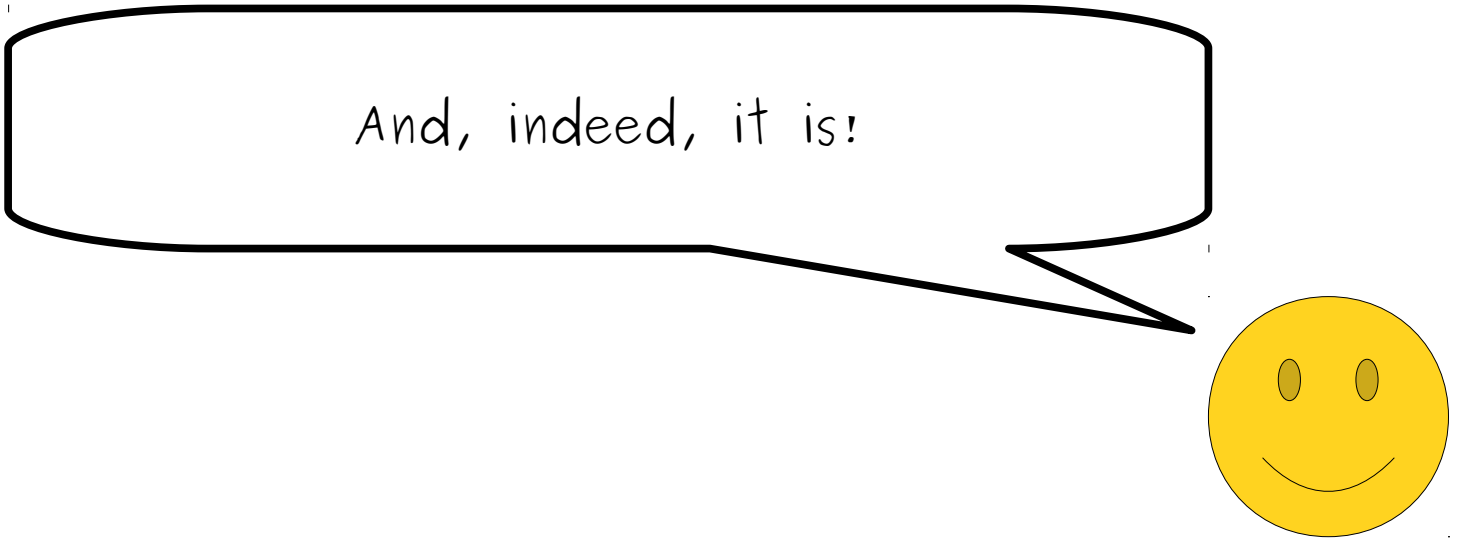


$$\{1\} \stackrel{?}{\subseteq} \{\, 1, \{2, 3\}, 4 \,\}$$

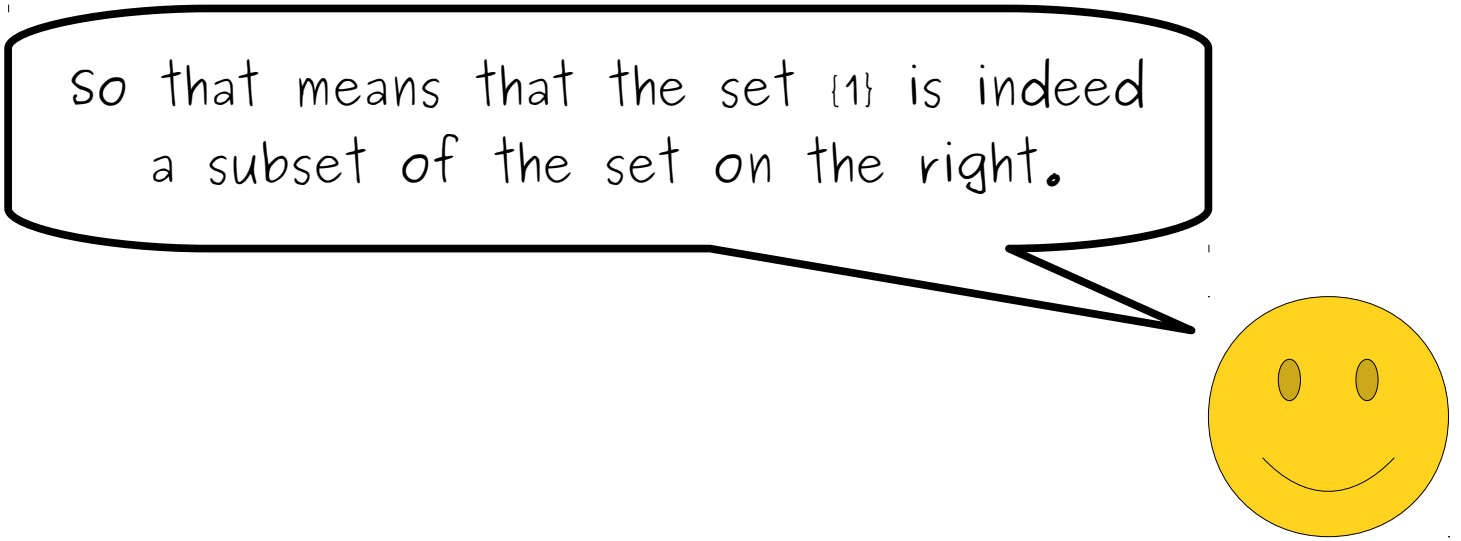
The set {1} only has one element (the number 1), so we just need to see whether the number 1 is contained in the set on the right.



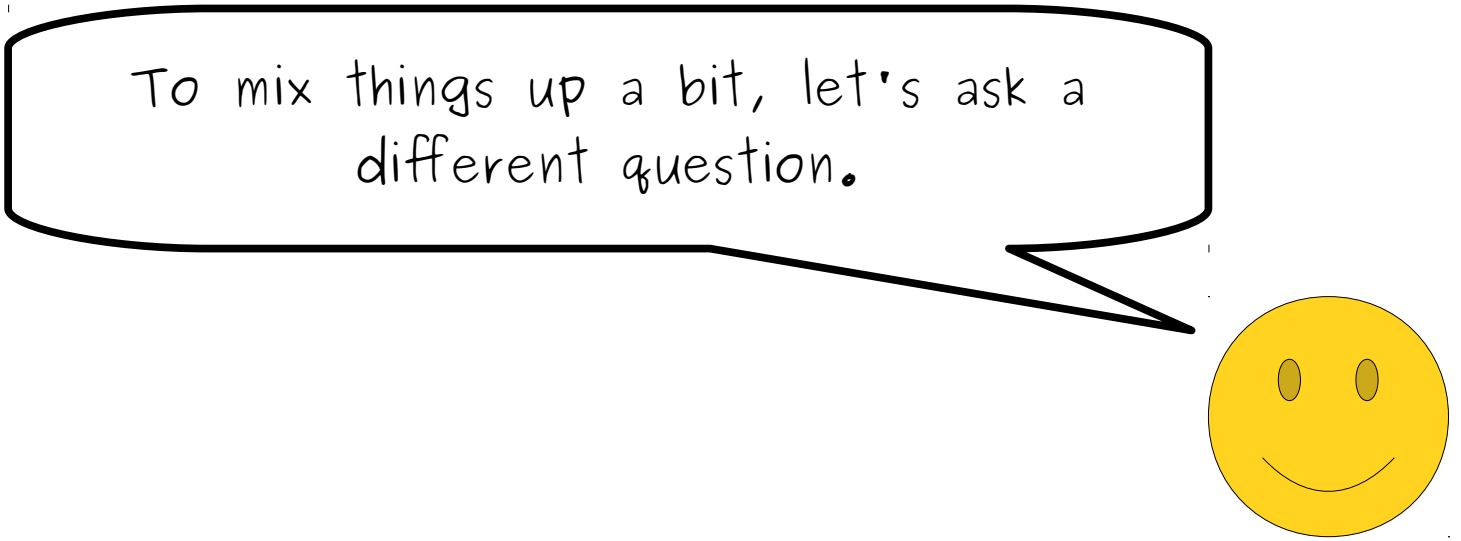
$$\{1\} \stackrel{?}{\subseteq} \{1, \{2, 3\}, 4\}$$



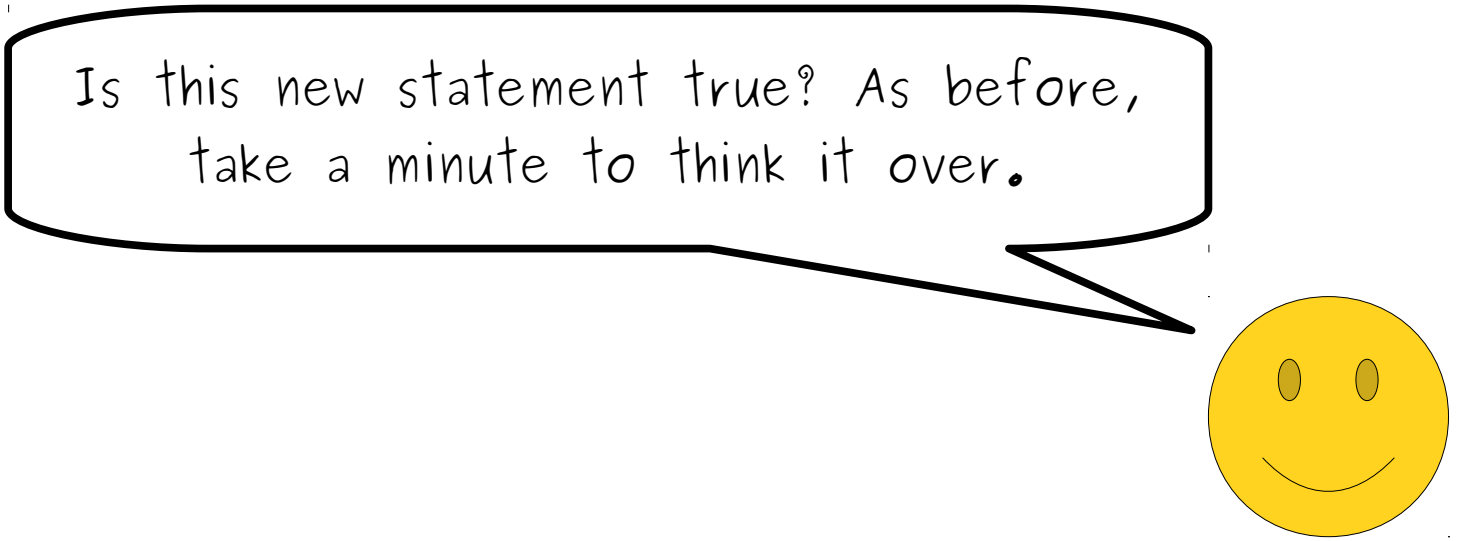
$$\{1\} \subseteq \{\, 1, \{2, 3\}, 4 \,\}$$



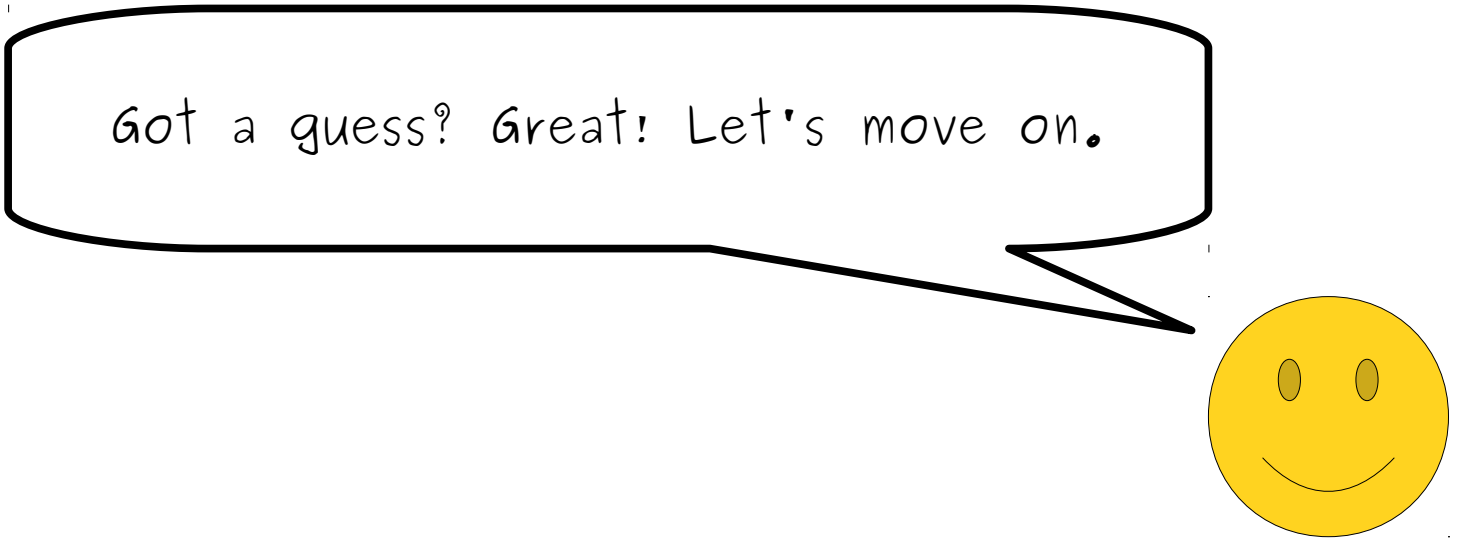
$$\{1\} \subseteq \{\, 1, \{2, 3\}, 4 \,\}$$



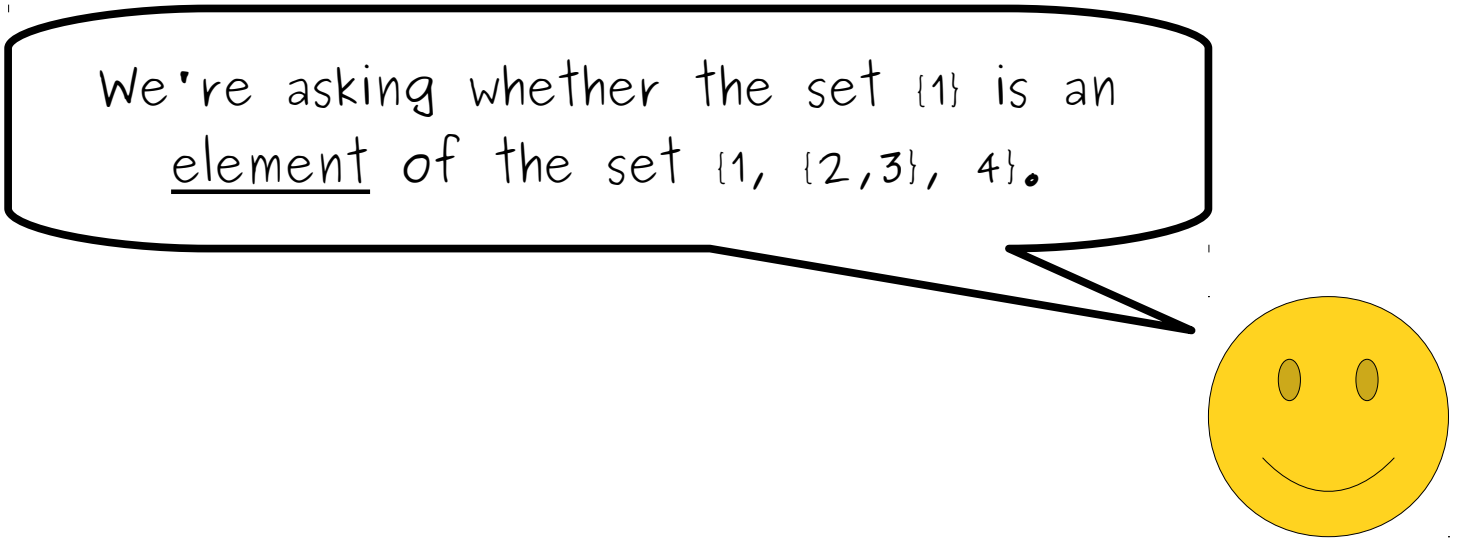
$$\{1\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



$$\{1\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

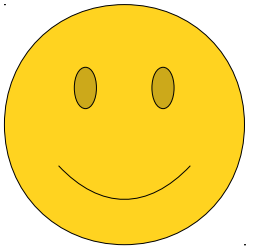


$$\{1\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$



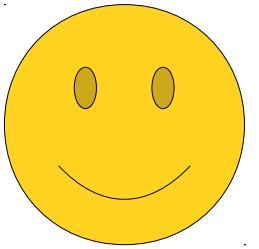
$$\{1\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

That means that we need to see if anything in the set {1, {2,3}, 4} is equal to the set {1}.

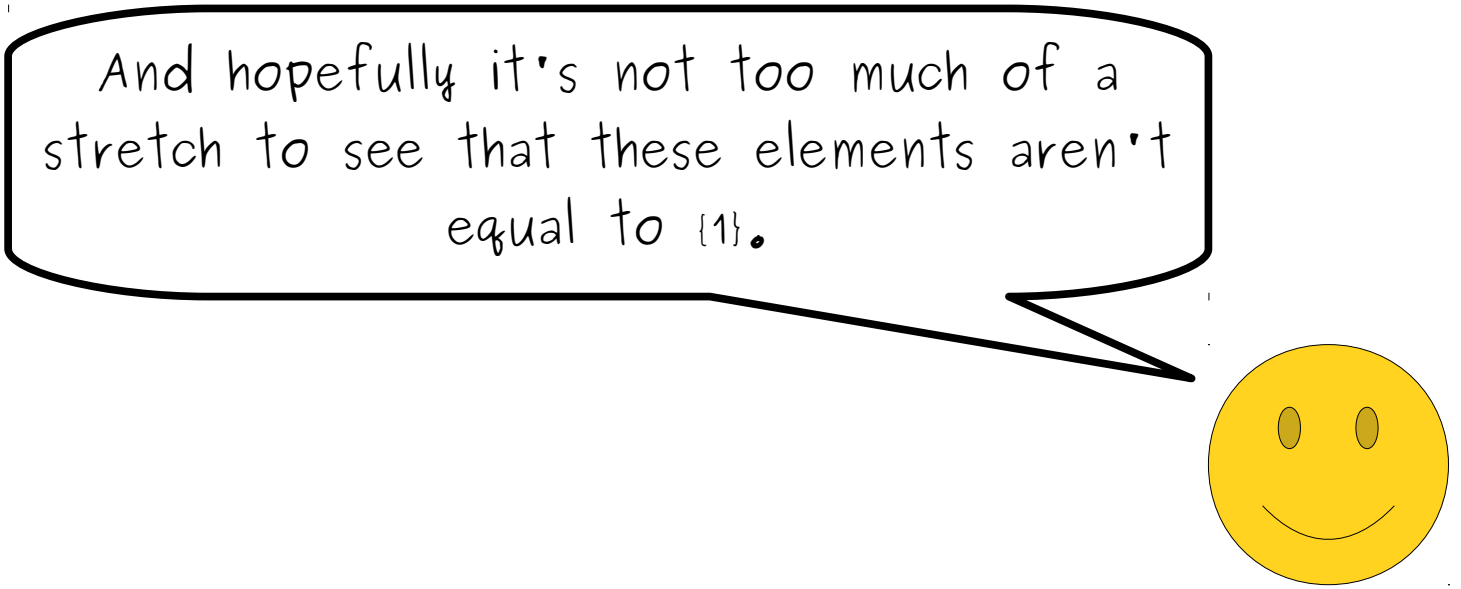


$$\{1\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

Using our reasoning from before, we know that 1 and {1} aren't the same thing, so this element isn't the set {1}.

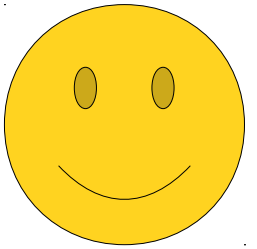


$$\{1\} \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

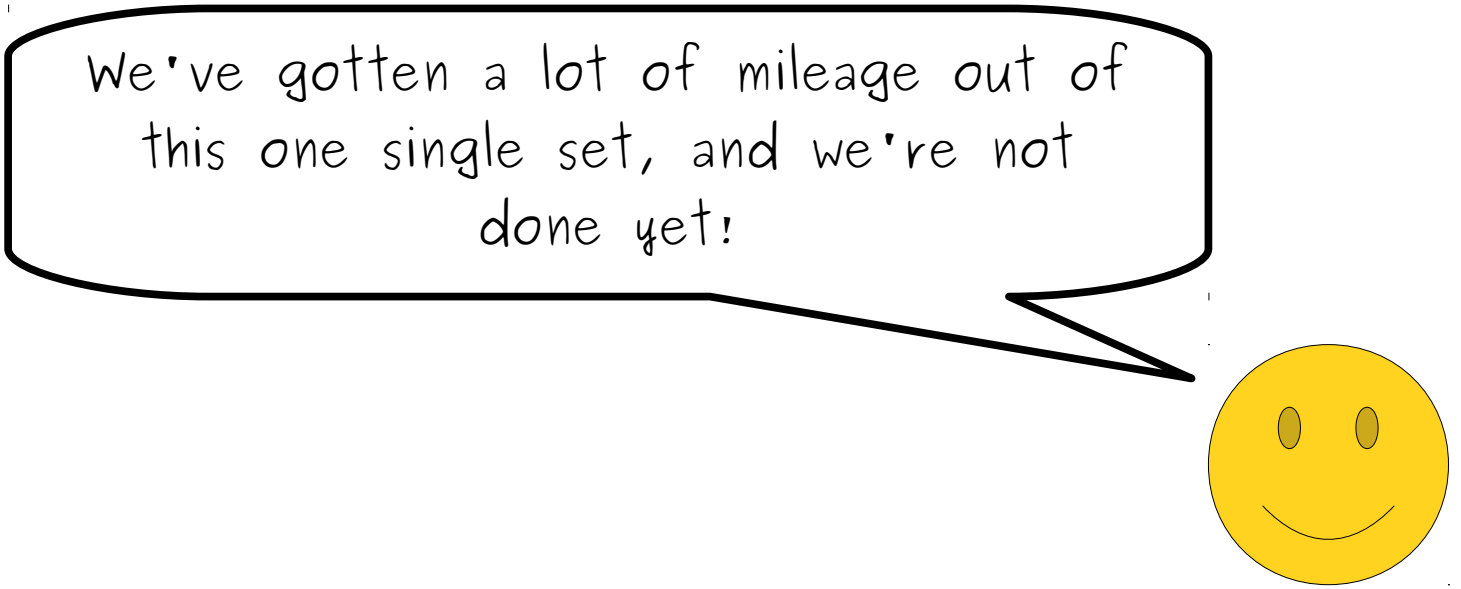


$$\{1\} \notin \{\, 1, \{2, 3\}, 4 \,\}$$

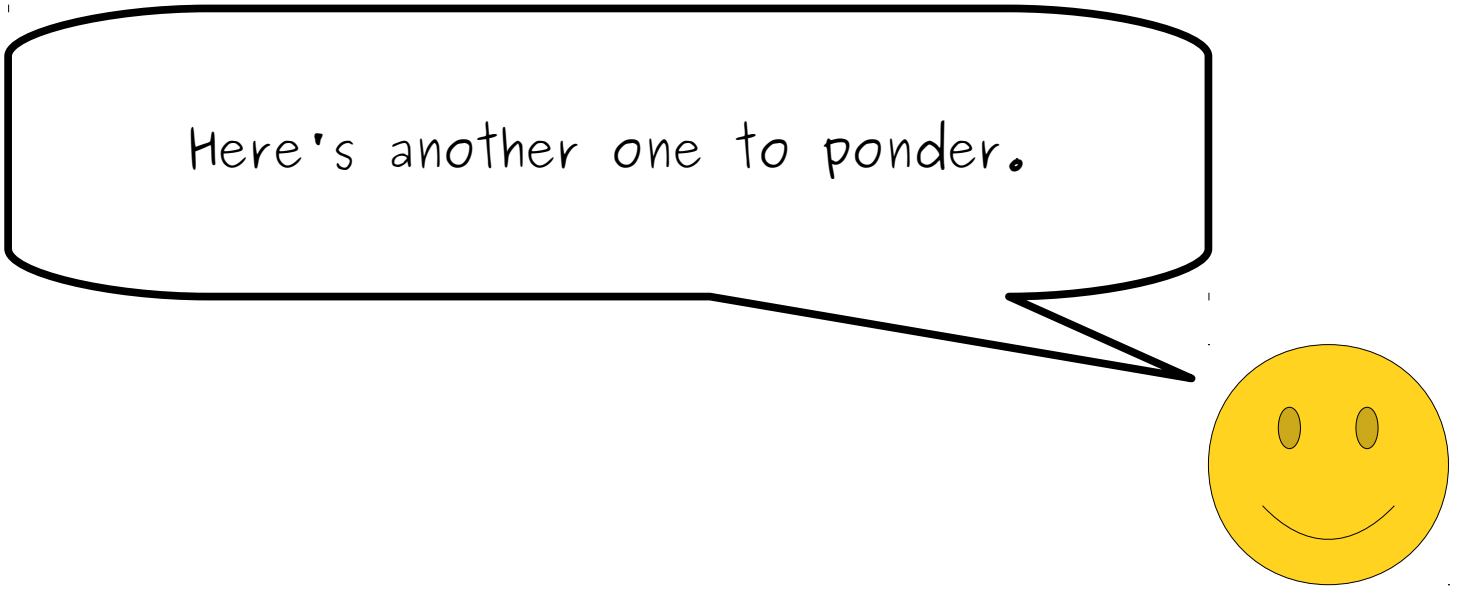
So overall we see that nothing in the set on the right is equal to {1}, so {1} is not an element of {1, {2,3}, 4}.



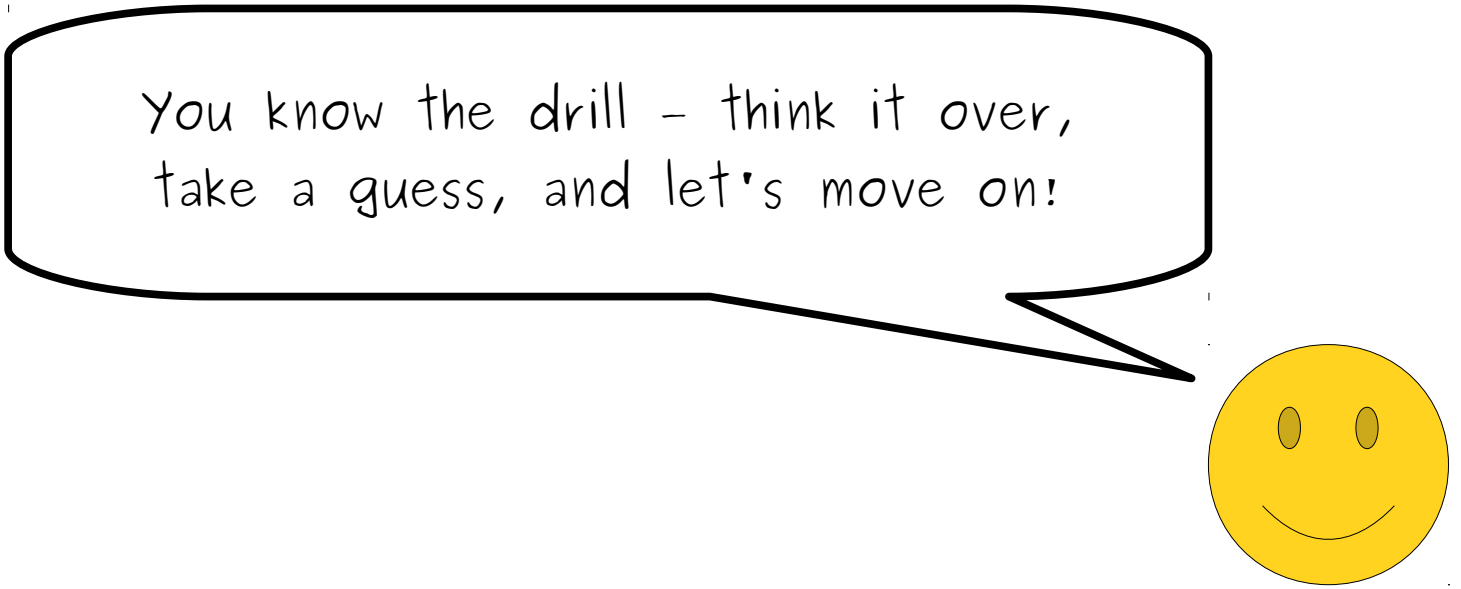
$$\{ 1, \{2, 3\}, 4 \}$$


We've gotten a lot of mileage out of this one single set, and we're not done yet!

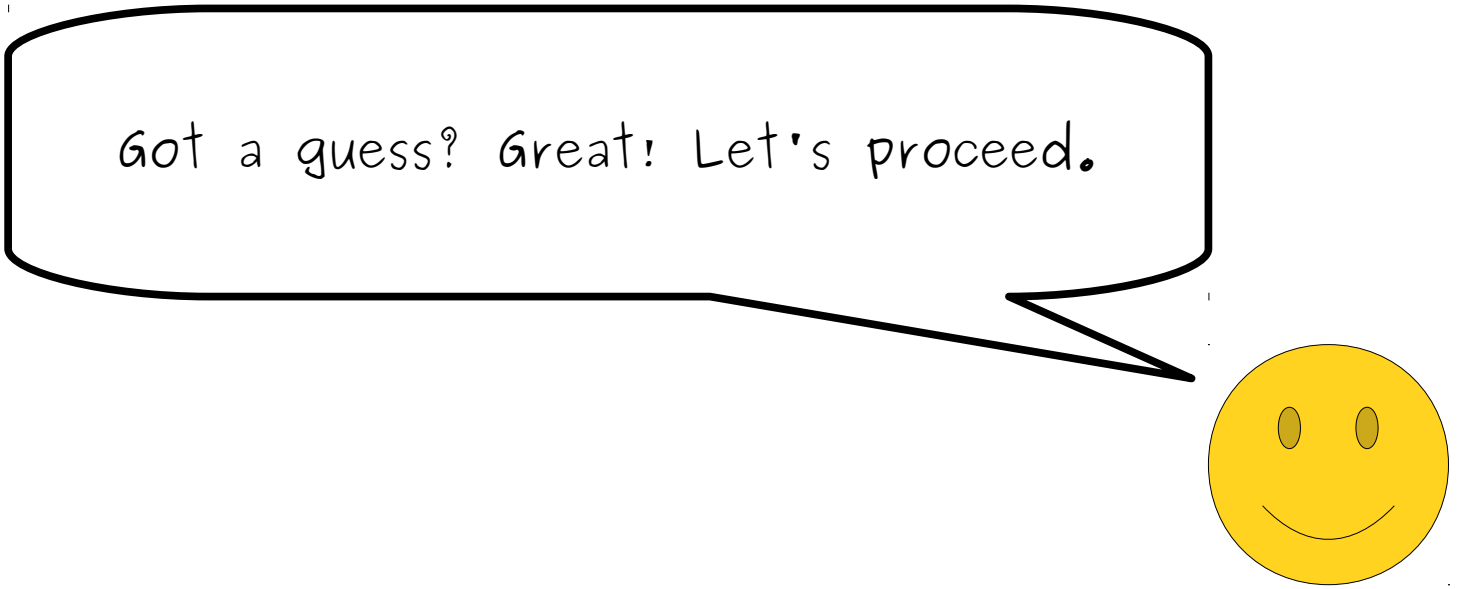
$$\{2, 3\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$



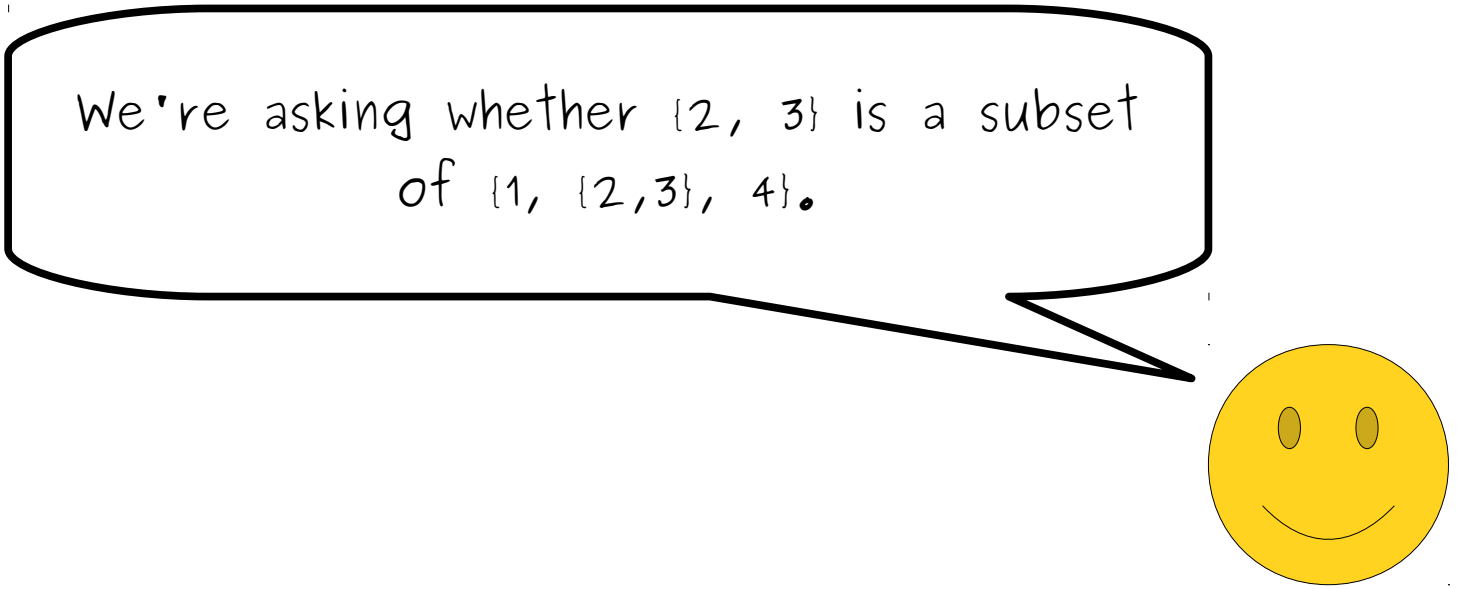
$$\{2, 3\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$



$$\{2, 3\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

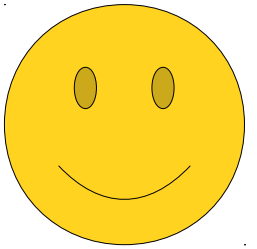


$$\{2, 3\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

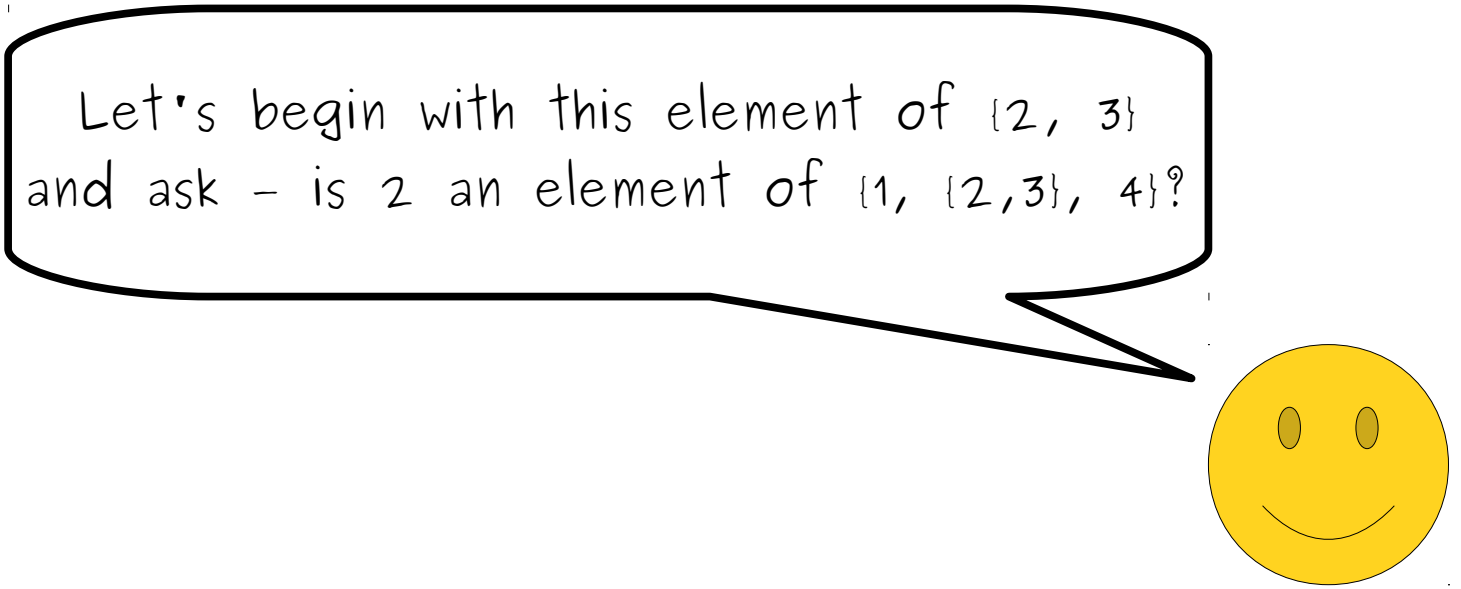


$$\{2, 3\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

To answer that question, we have to see whether every element of {2, 3} is an element of {1, {2,3}, 4}.



$$\{2, 3\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$



$$\{2, 3\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

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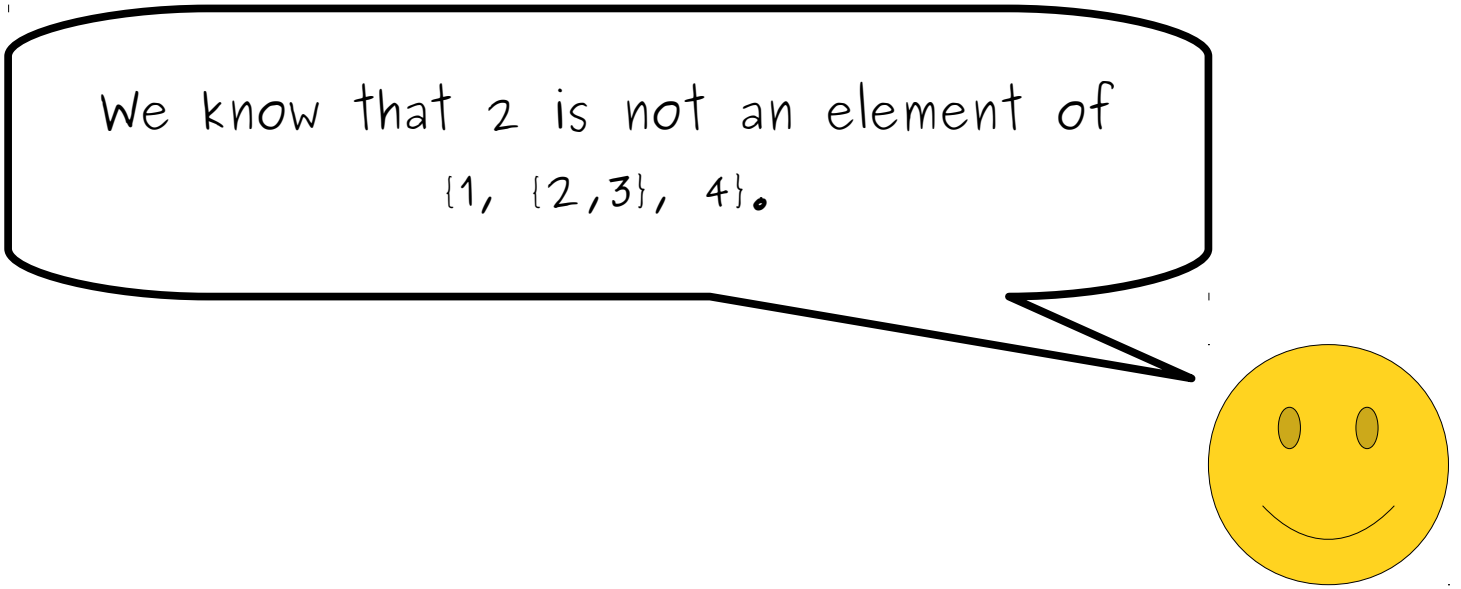
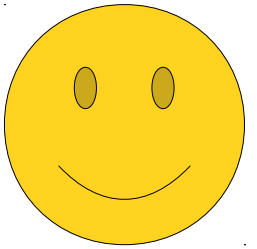
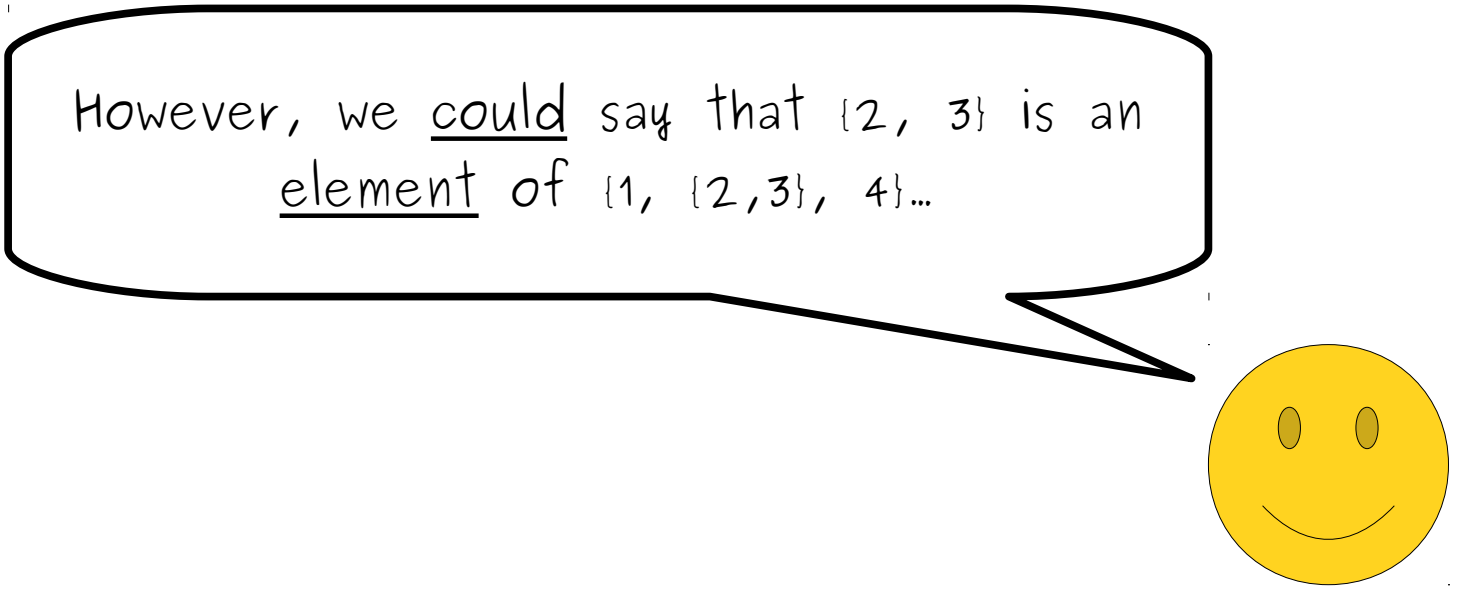


$$\{2, 3\} \not\subseteq \{ 1, \{2, 3\}, 4 \}$$

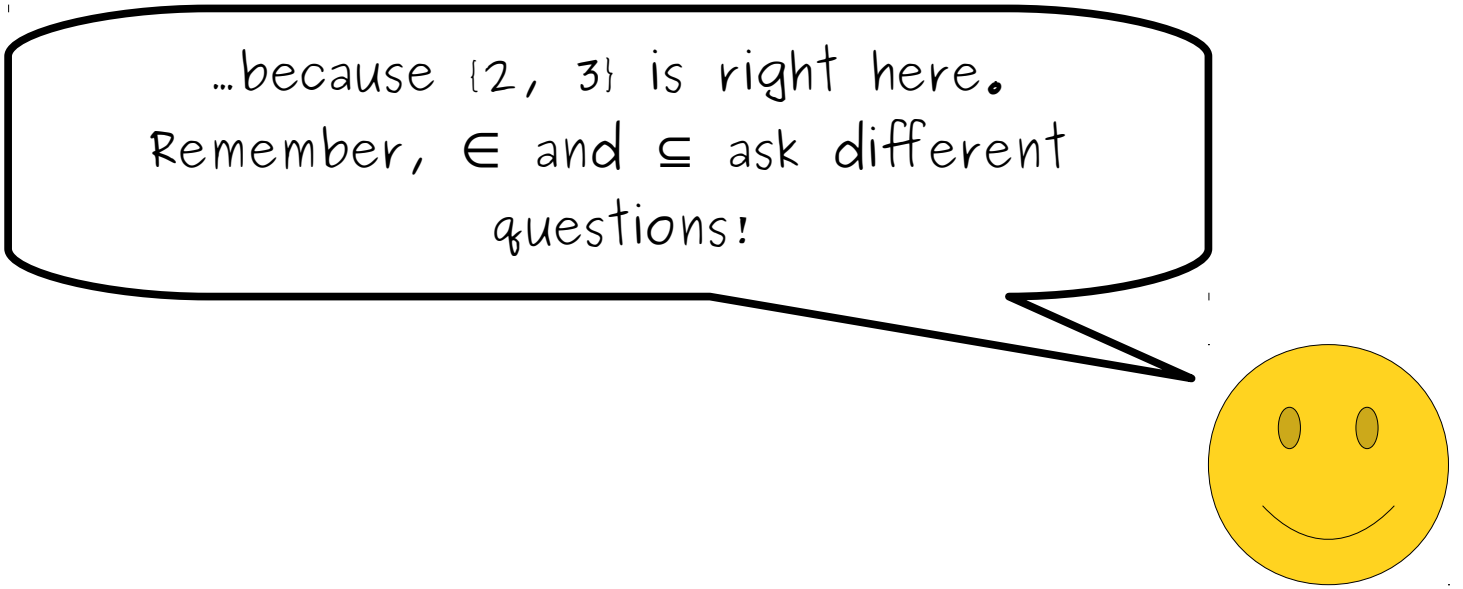
That means that the set {2, 3} isn't a subset of {1, {2,3}, 4}, because the set {1, {2,3}, 4} doesn't contain 2.



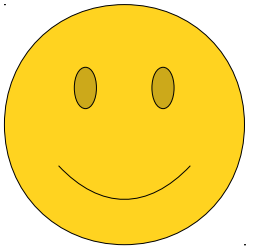
$$\{2, 3\} \in \{ 1, \{2, 3\}, 4 \}$$



$$\{2, 3\} \in \{ 1, \{2, 3\}, 4 \}$$

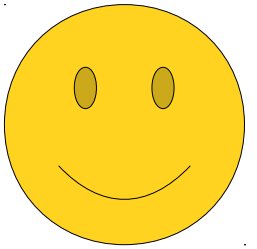

…because {2, 3} is right here.
Remember, ∈ and ⊆ ask different questions!

To conclude our little tour of $\in$ and $\subseteq$, let's talk about our friend the empty set.



$$\emptyset$$

The empty set – which we denote using the symbol shown above – is a set that contains no elements.

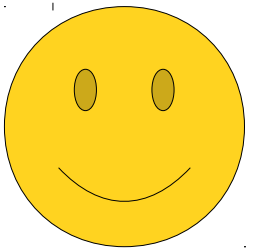


$\emptyset$

The empty set has a few nice properties
that are worth keeping in mind.

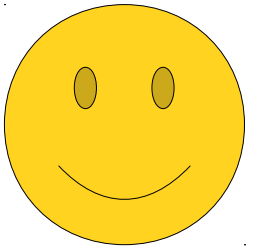
$$x \notin \emptyset$$

First, since the empty set has no elements, if you take any object x, you'll know it's not an element of the empty set. In fact, that's how the empty set is sometimes defined!



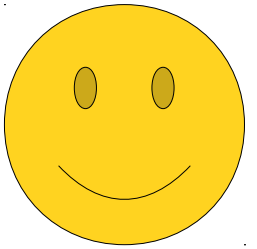
$$\emptyset \subseteq S$$

Second, as mentioned in class, we mentioned that the empty set is a subset of <u>any</u> set S. This is a consequence of <u>vacuous truth</u>.

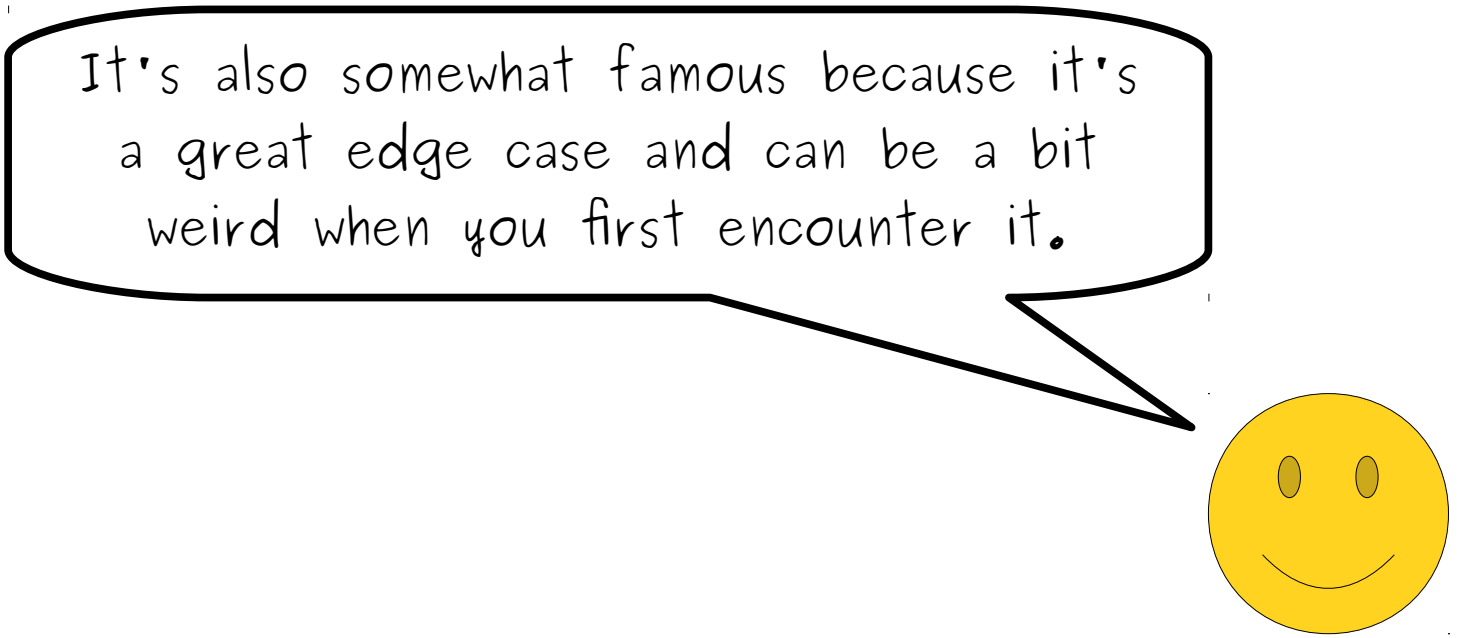


$\emptyset$

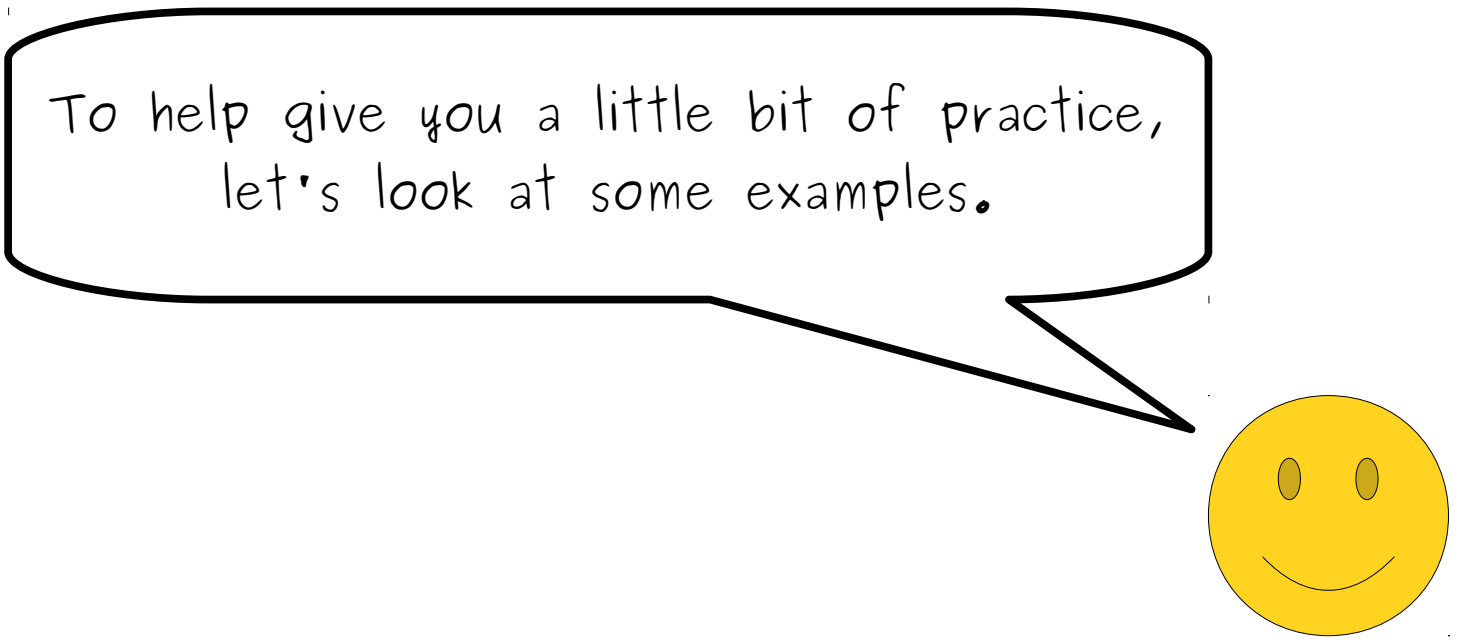
Although the empty set is probably most famous because of these two properties, it's just a set like any other and it obeys all the regular rules of the $\in$ and $\subseteq$ relations.



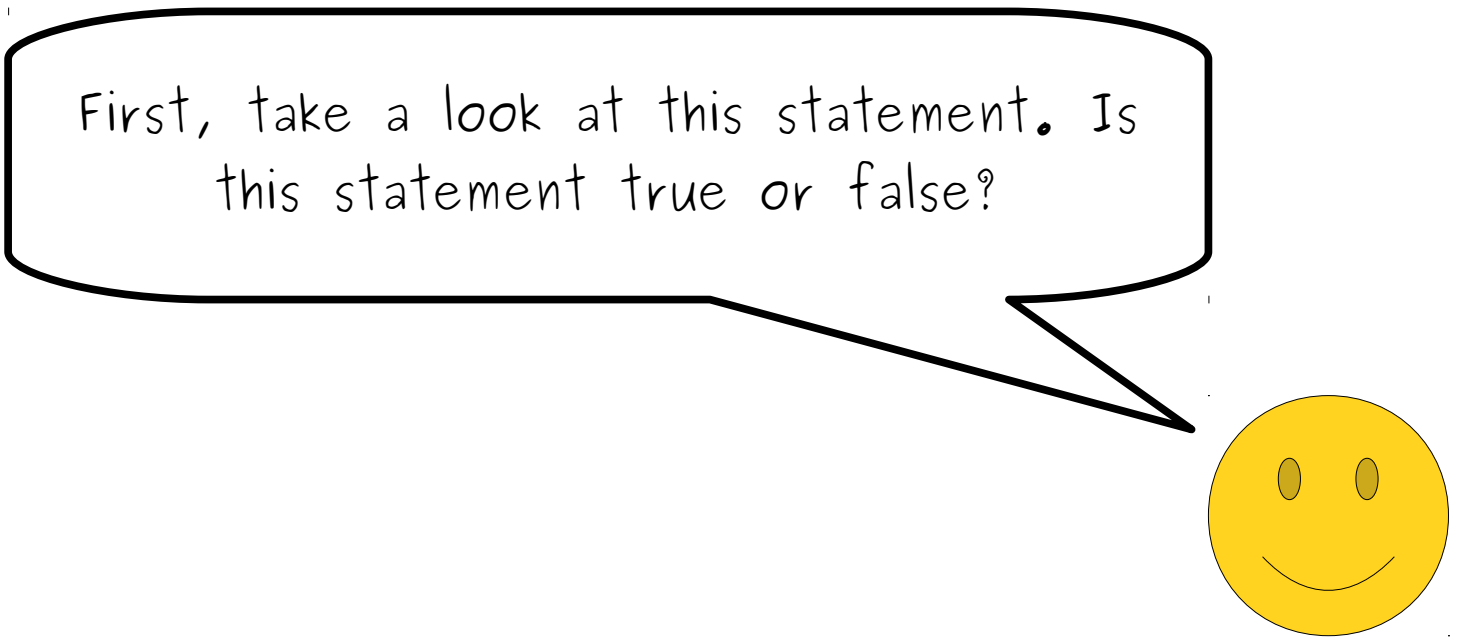
$\emptyset$


It's also somewhat famous because it's a great edge case and can be a bit weird when you first encounter it.

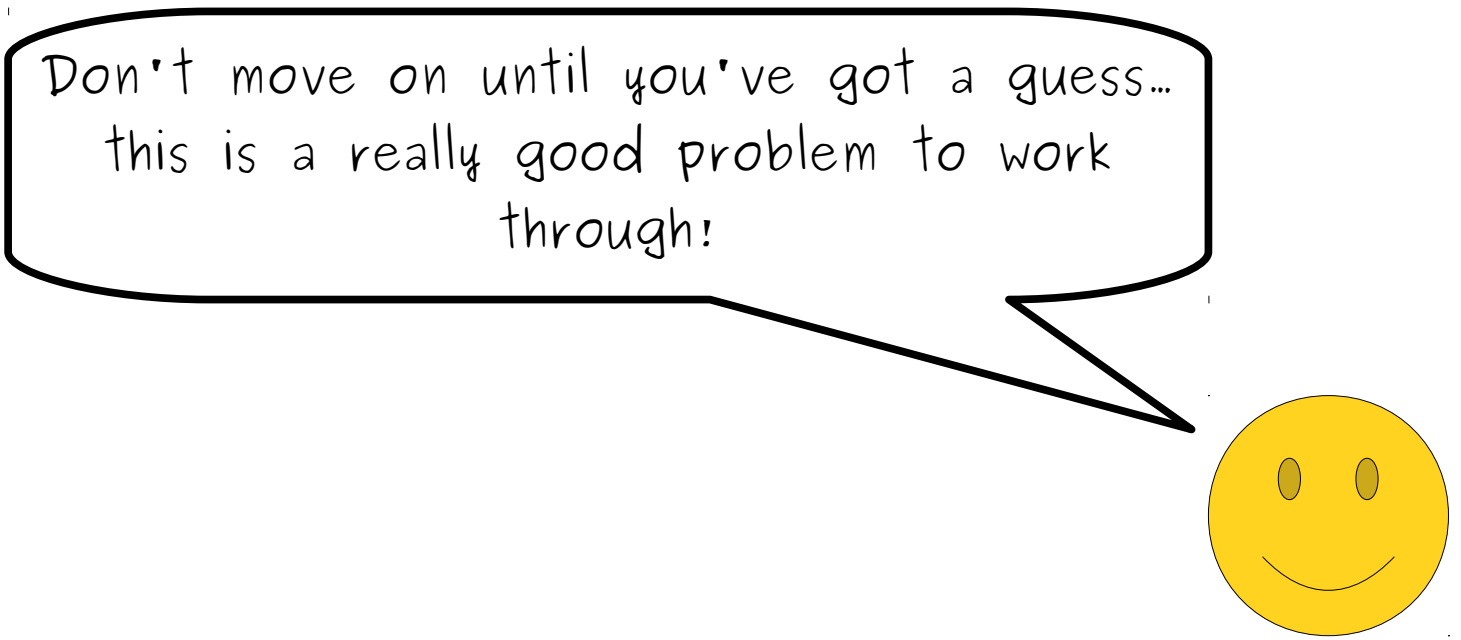
$\emptyset$


To help give you a little bit of practice,
let's look at some examples.

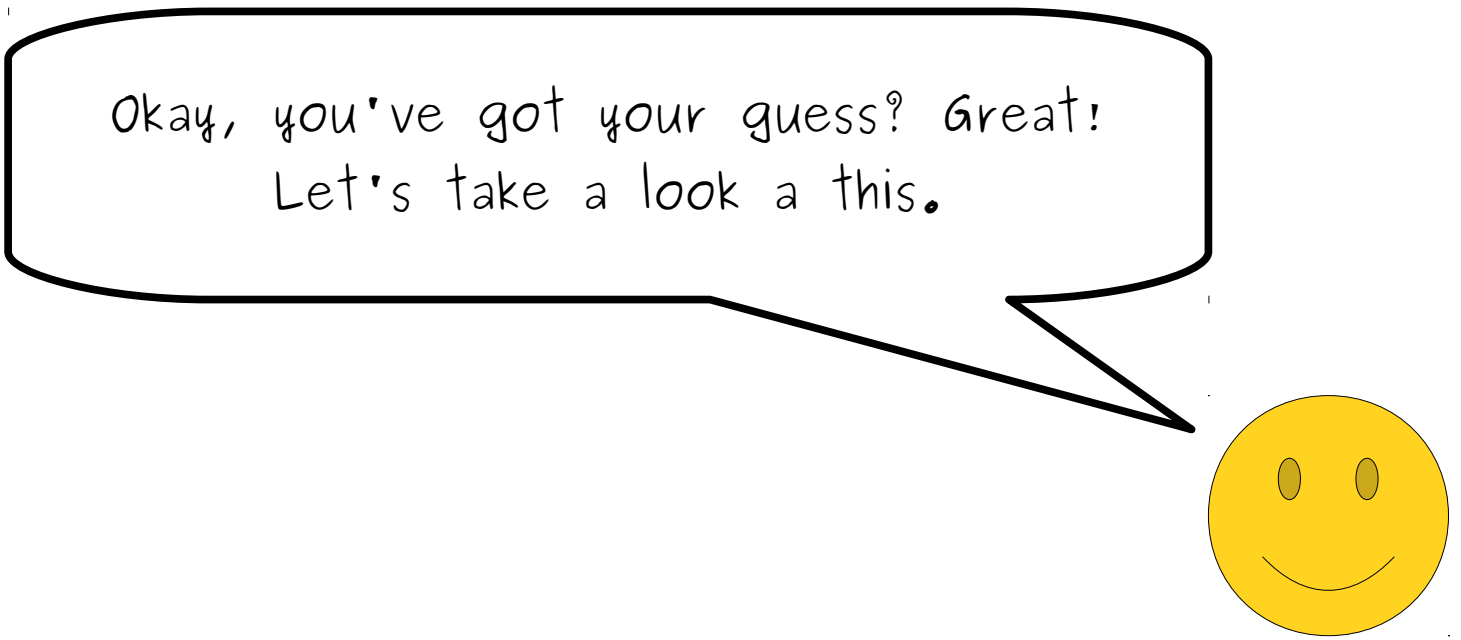
$$\emptyset \stackrel{?}{\in} \{ 1, \{2, 3\}, 4 \}$$



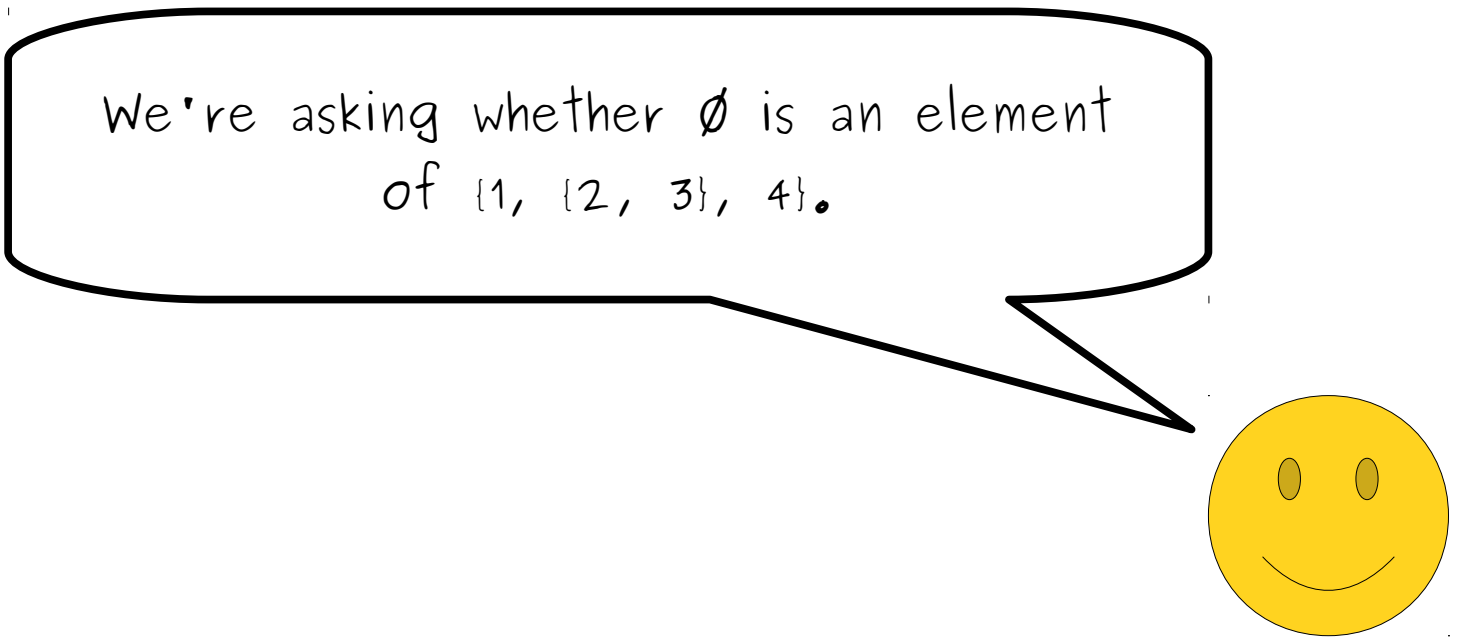
$$\emptyset \stackrel{?}{\in} \{ 1, \{2, 3\}, 4 \}$$



$$\emptyset \stackrel{?}{\in} \{ 1, \{2, 3\}, 4 \}$$

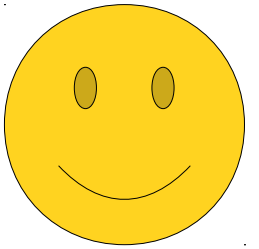


$$\emptyset \stackrel{?}{\in} \{ 1, \{2, 3\}, 4 \}$$

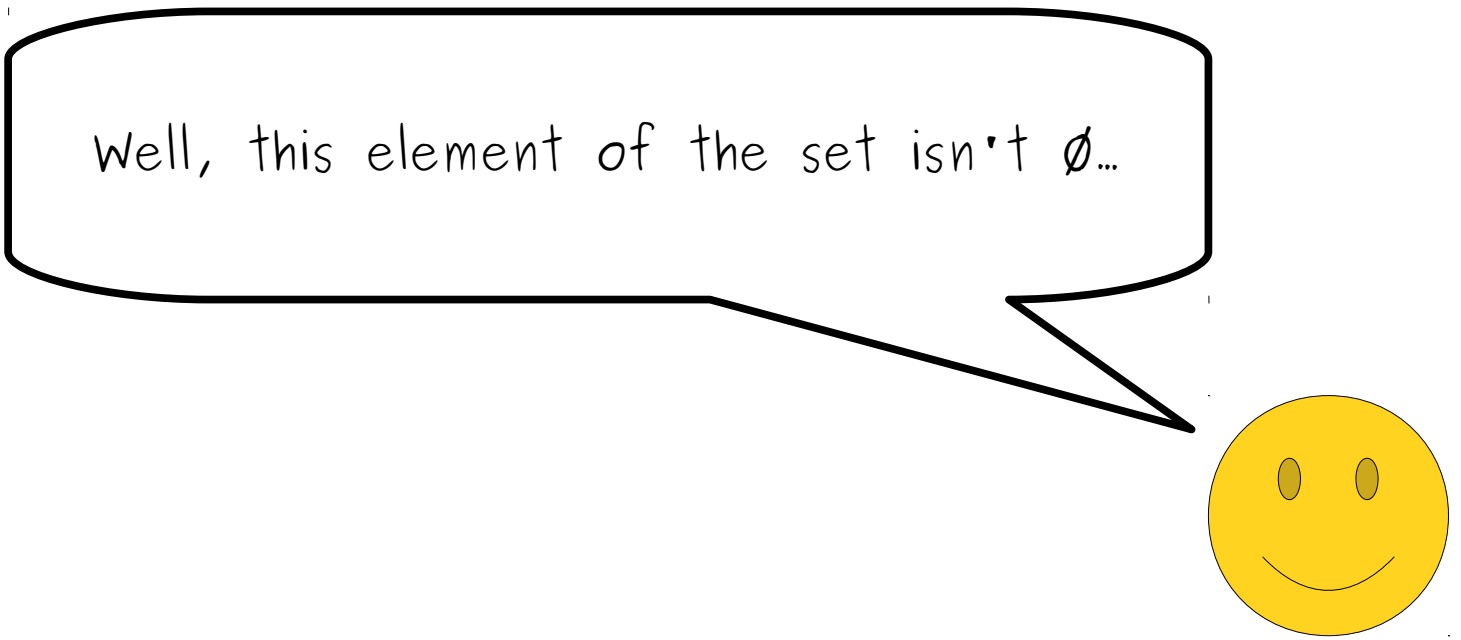


$$\emptyset \stackrel{?}{\in} \{ 1, \{2, 3\}, 4 \}$$

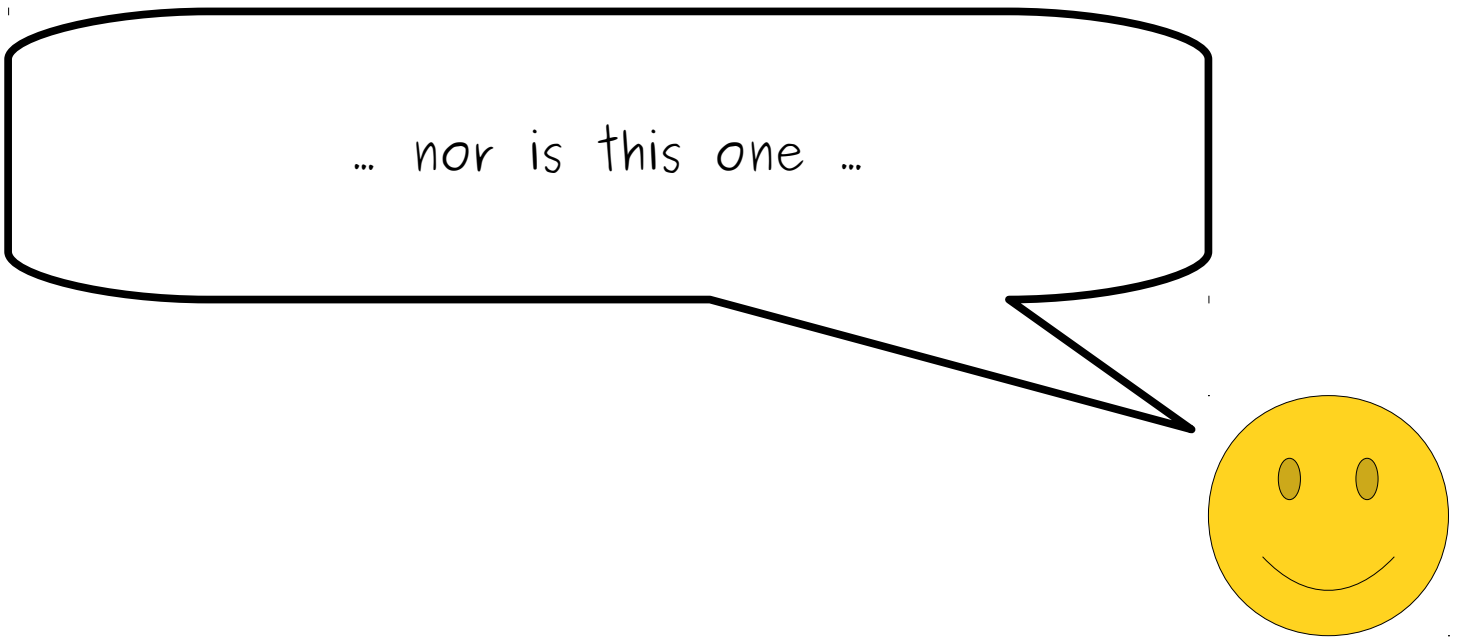
That means we need to see whether any of the elements of {1, {2,3}, 4} happen to be equal to ∅.



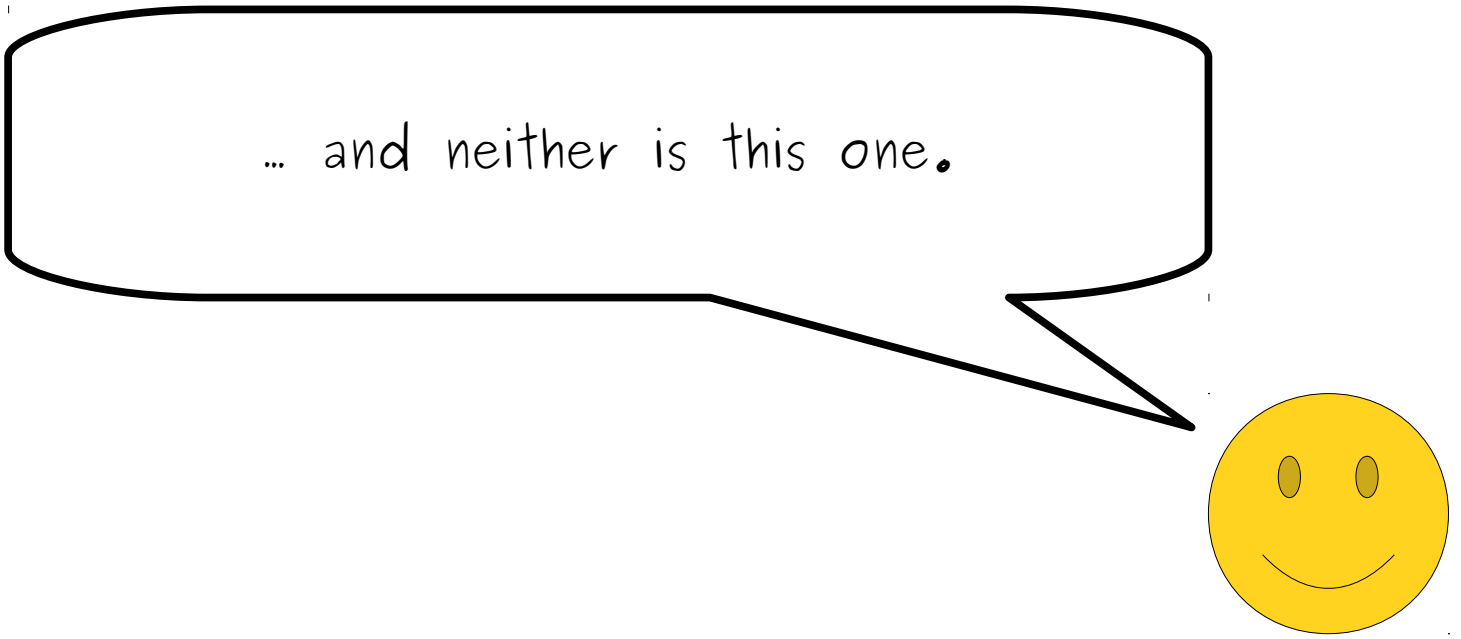
$$\emptyset \stackrel{?}{\in} \{ 1, \{2, 3\}, 4 \}$$



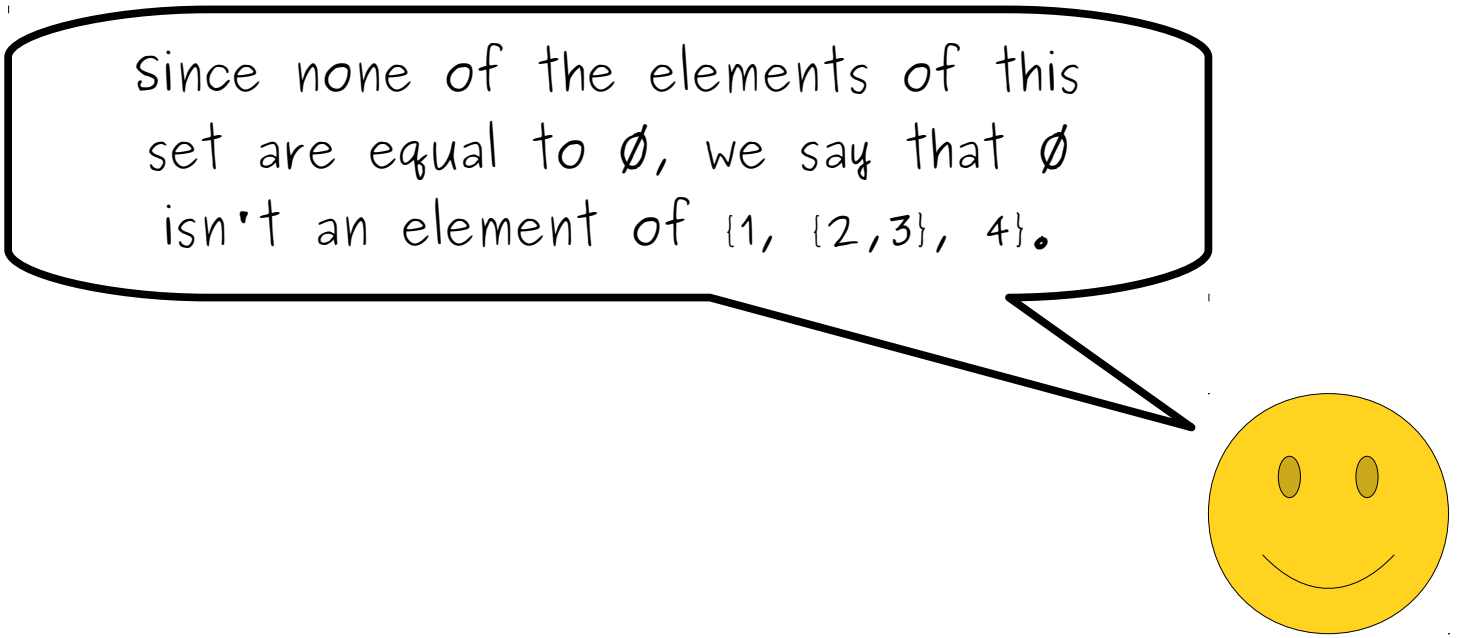
$$\emptyset \stackrel{?}{\in} \{ 1, \{2, 3\}, 4 \}$$



$$\emptyset \stackrel{?}{\in} \{\, 1, \{2, 3\}, 4 \,\}$$

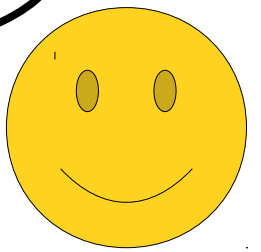


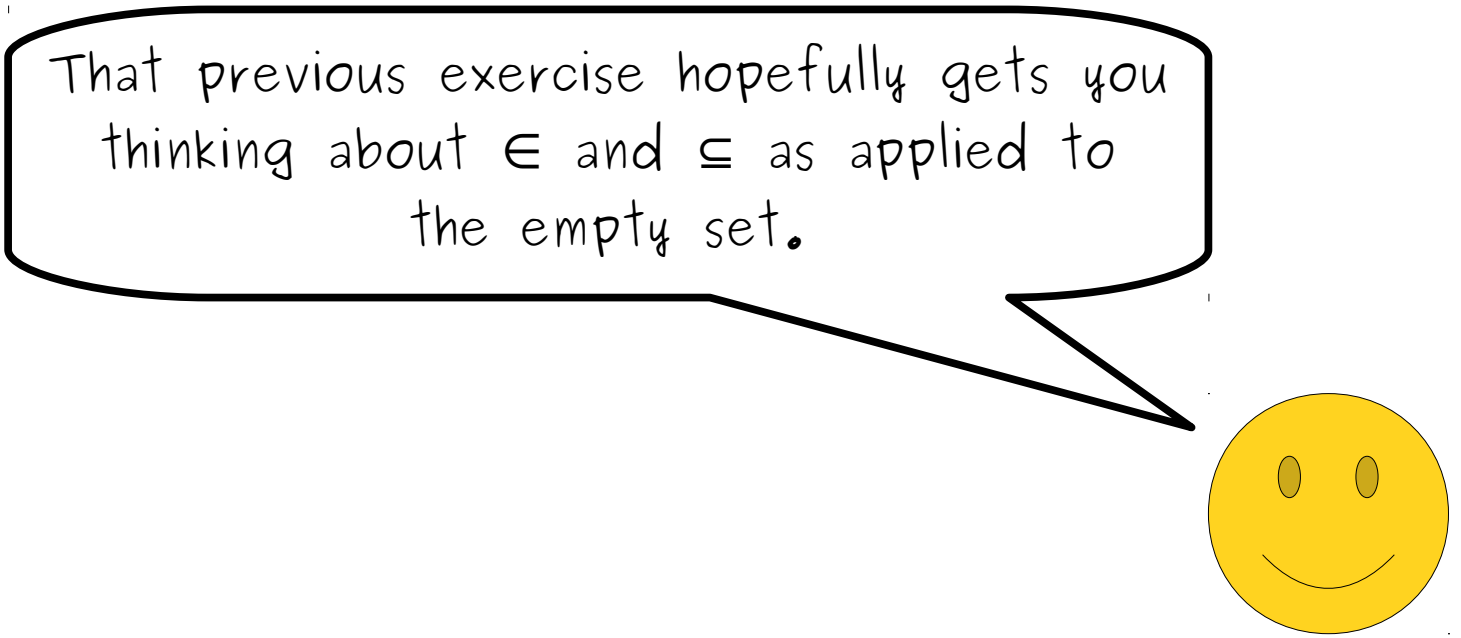
$$\emptyset \notin \{ 1, \{2, 3\}, 4 \}$$



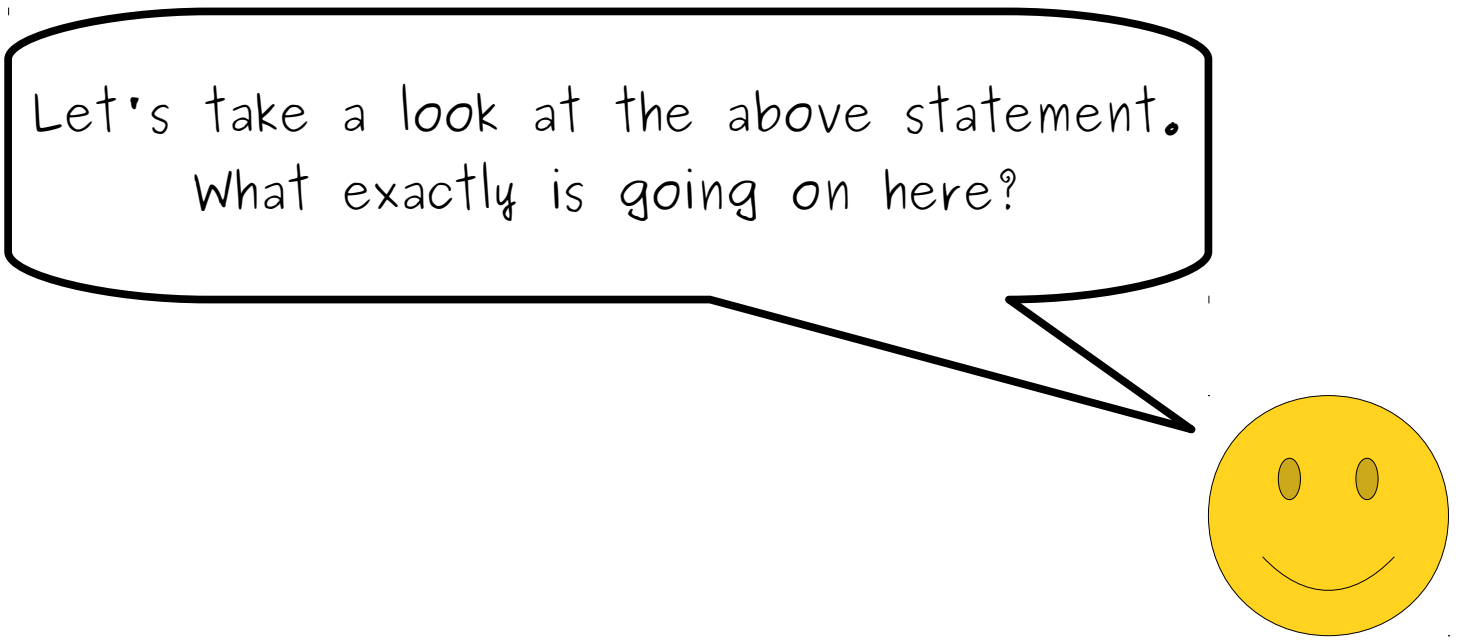
$$\emptyset \subseteq \{ 1, \{2, 3\}, 4 \}$$

Now, we could correctly say that the empty set is a subset of {1, {2,3}, 4}, because it's a subset of every set. But, as you've seen before, $\in$ and $\subseteq$ represent different concepts, so that doesn't mean that $\emptyset$ would be an element of this set.

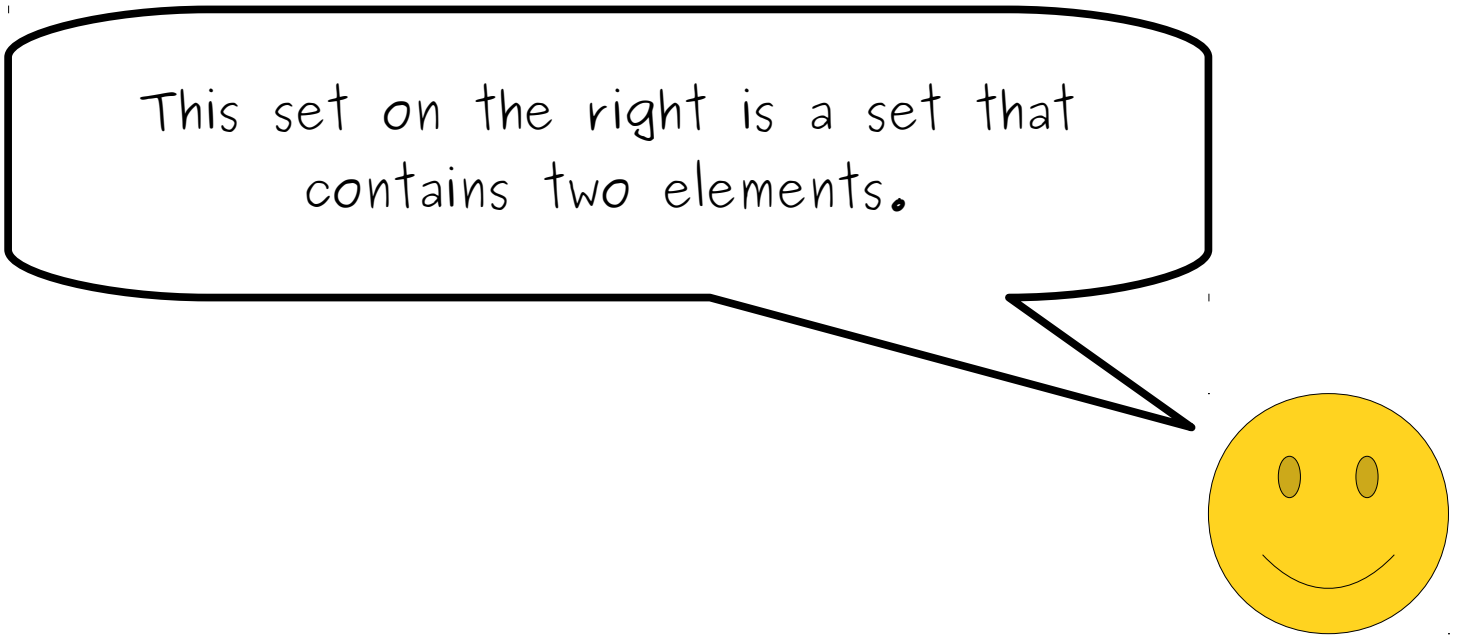



That previous exercise hopefully gets you thinking about ∈ and ⊆ as applied to the empty set.

$$\emptyset \stackrel{?}{\in} \{ \emptyset, 137 \}$$

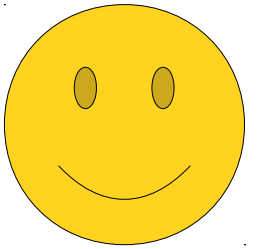


$$\emptyset \stackrel{?}{\in} \{ \emptyset, 137 \}$$

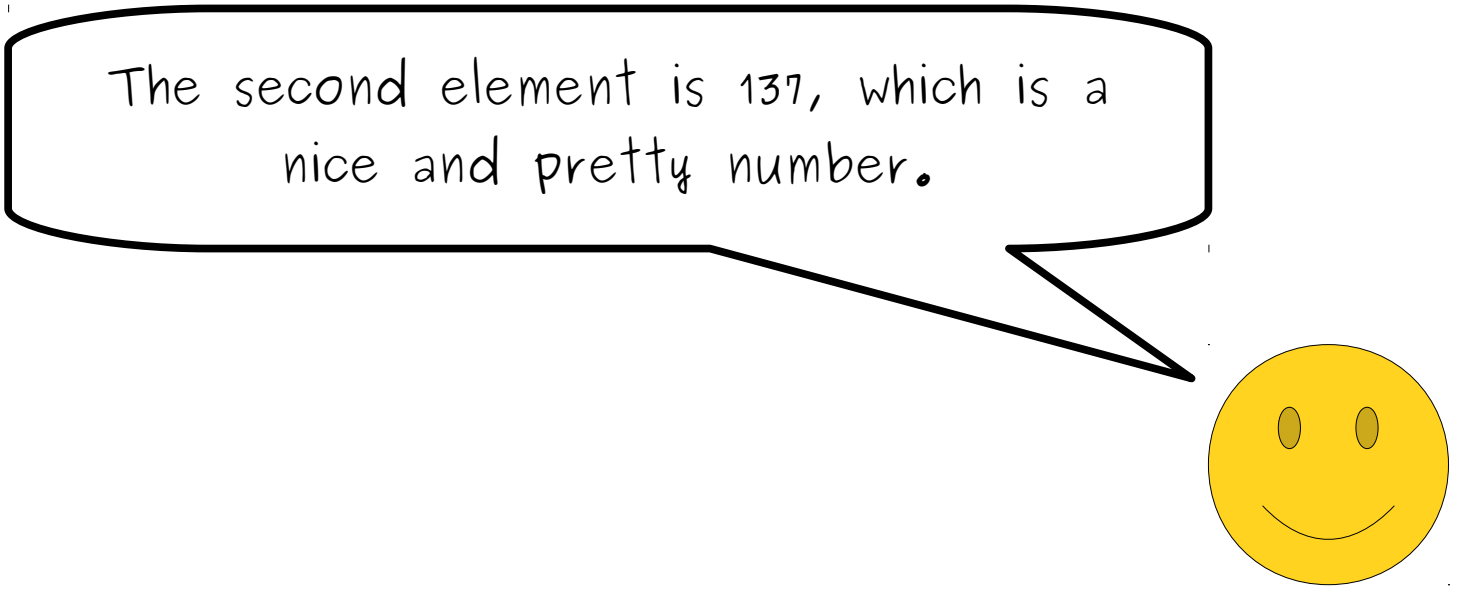


$$\emptyset \stackrel{?}{\in} \{ \emptyset, 137 \}$$

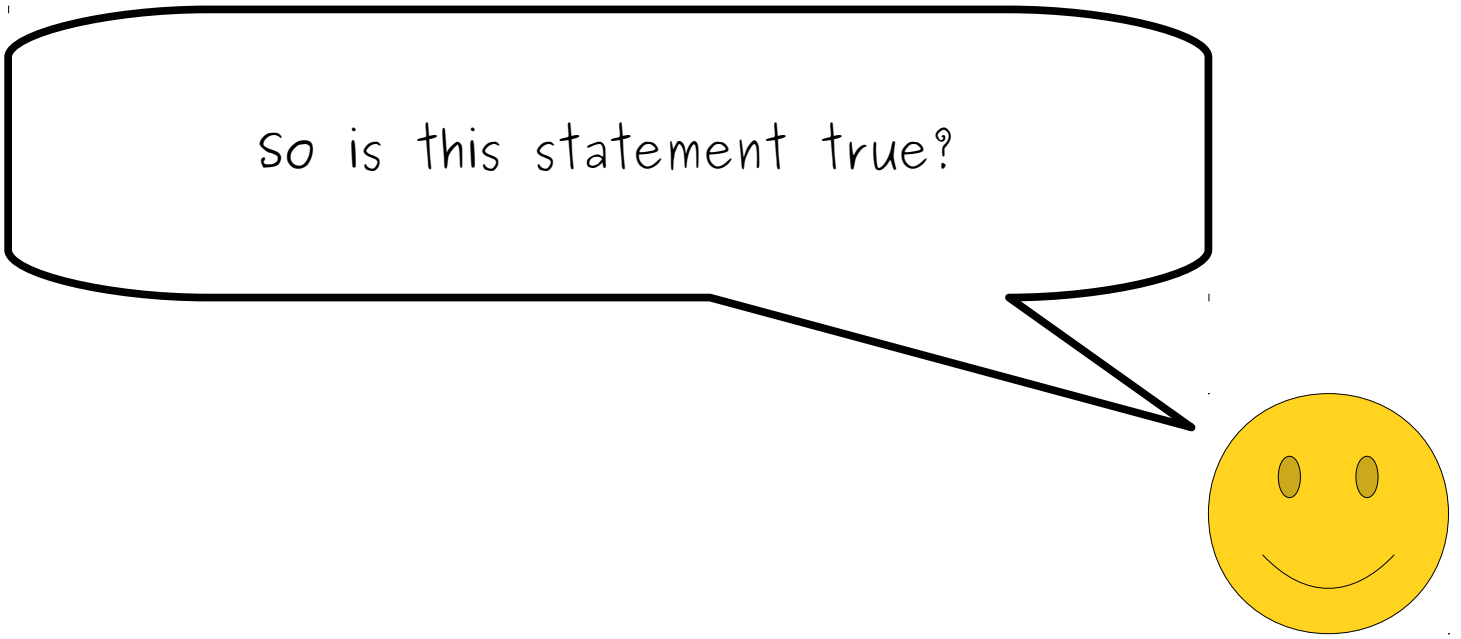
The first element is the empty set. You've seen examples of sets containing other sets before, and in this case this set just happens to contain ∅.



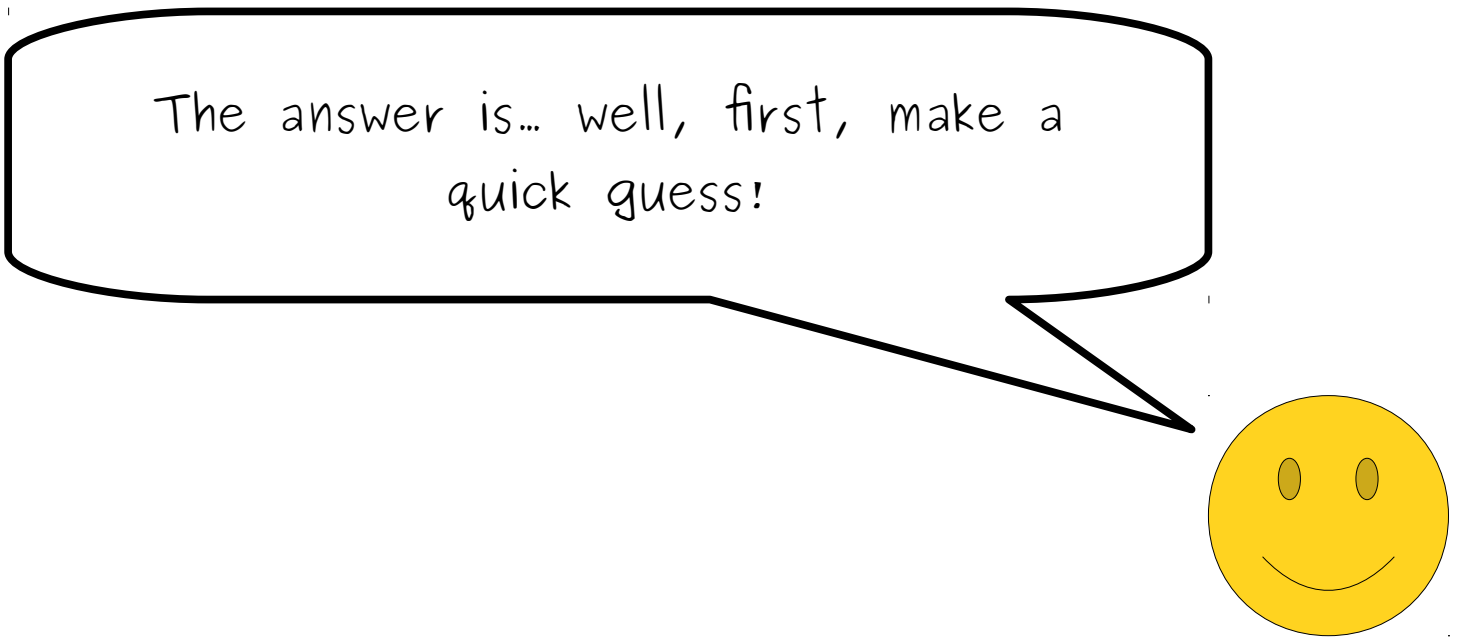
$$\emptyset \stackrel{?}{\in} \{ \emptyset, 137 \}$$



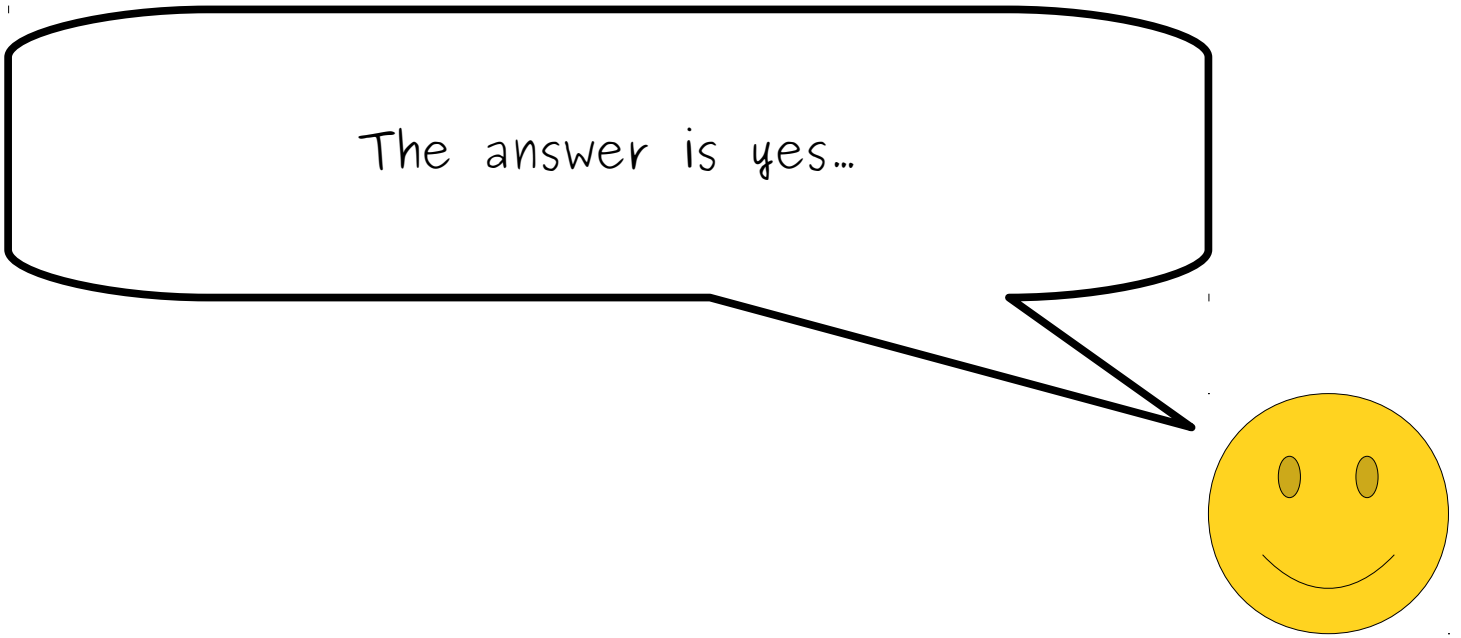
$$\emptyset \stackrel{?}{\in} \{ \emptyset, 137 \}$$



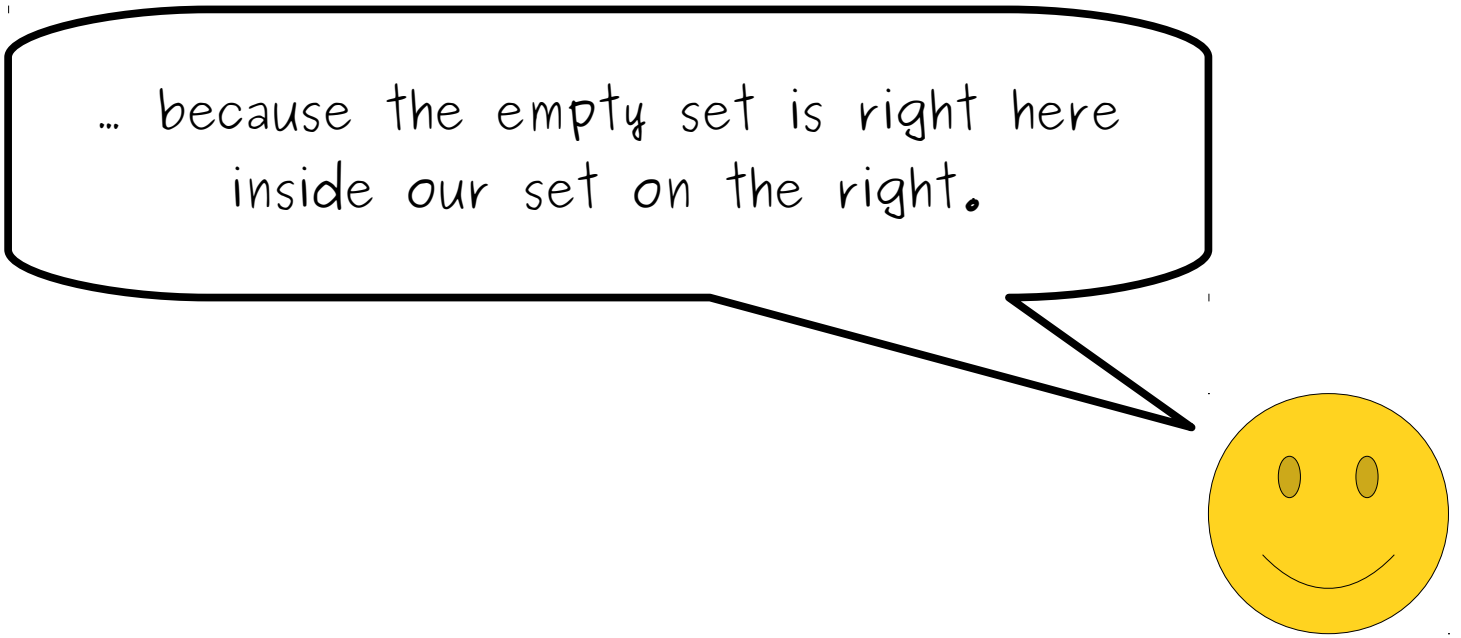
$$\emptyset \stackrel{?}{\in} \{ \emptyset, 137 \}$$



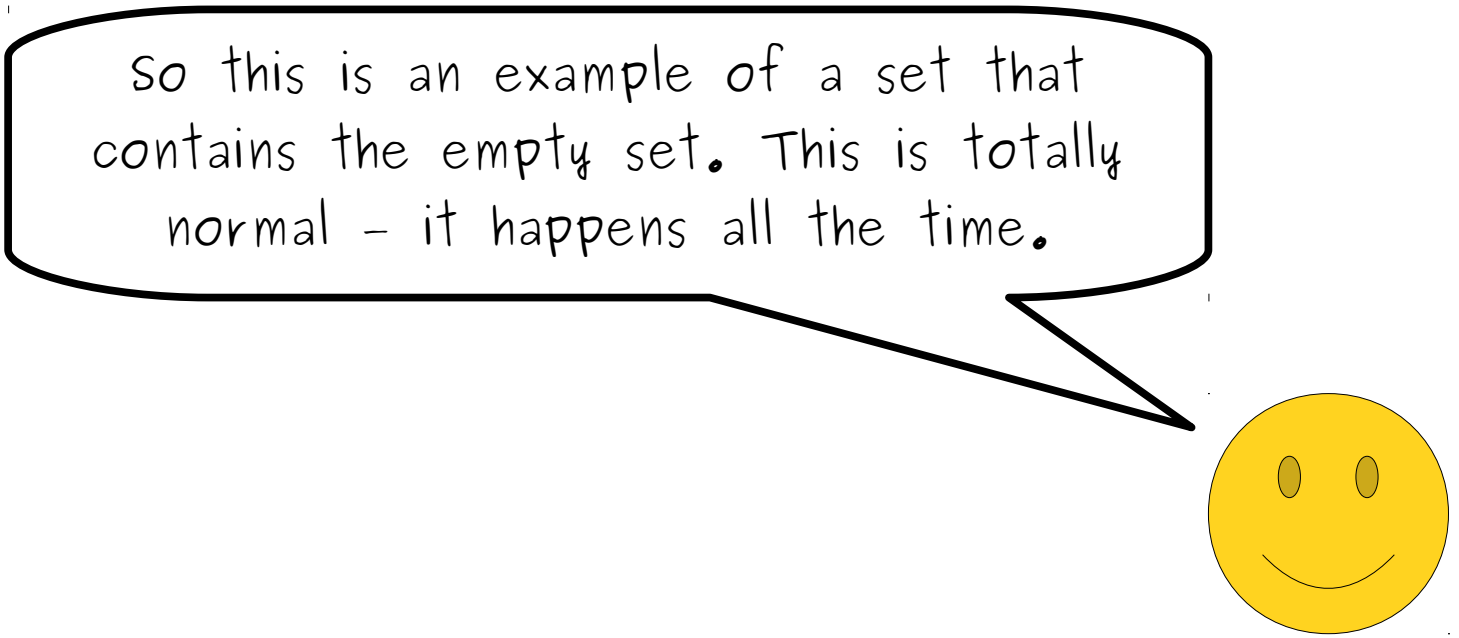
$$\emptyset \in \{ \emptyset, 137 \}$$

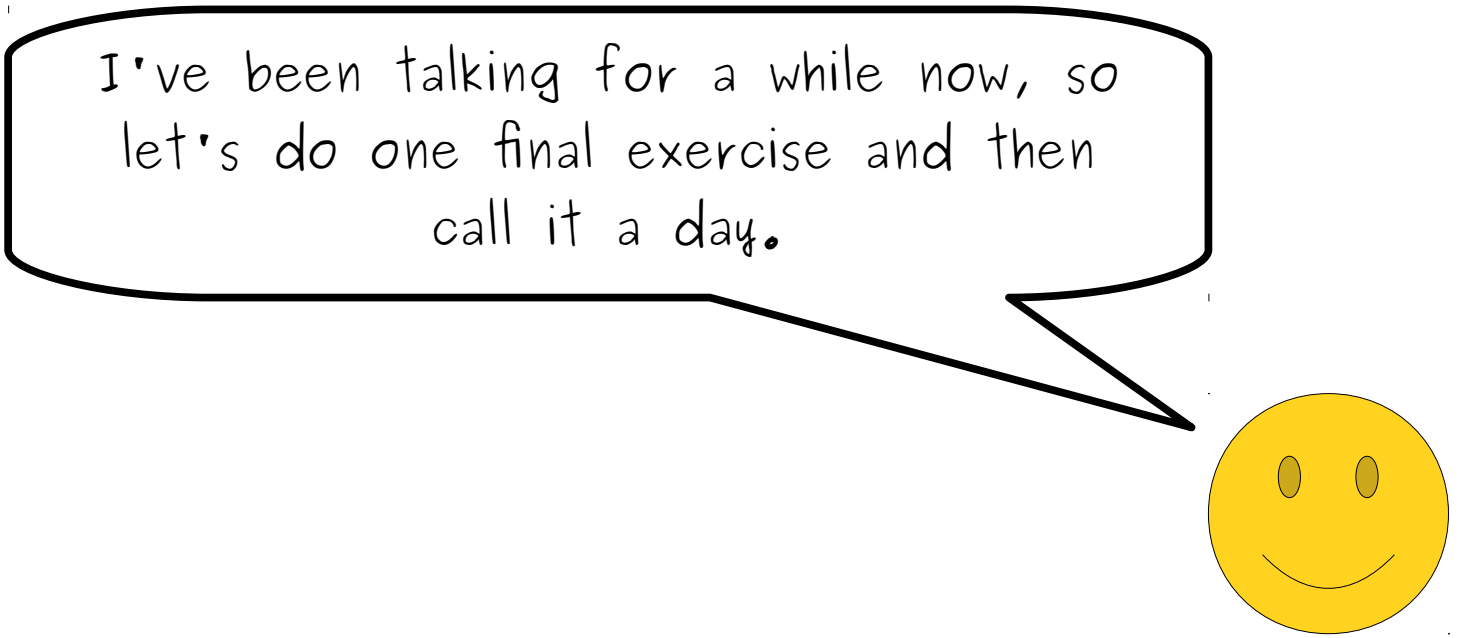


$$\emptyset \in \{ \emptyset, 137 \}$$

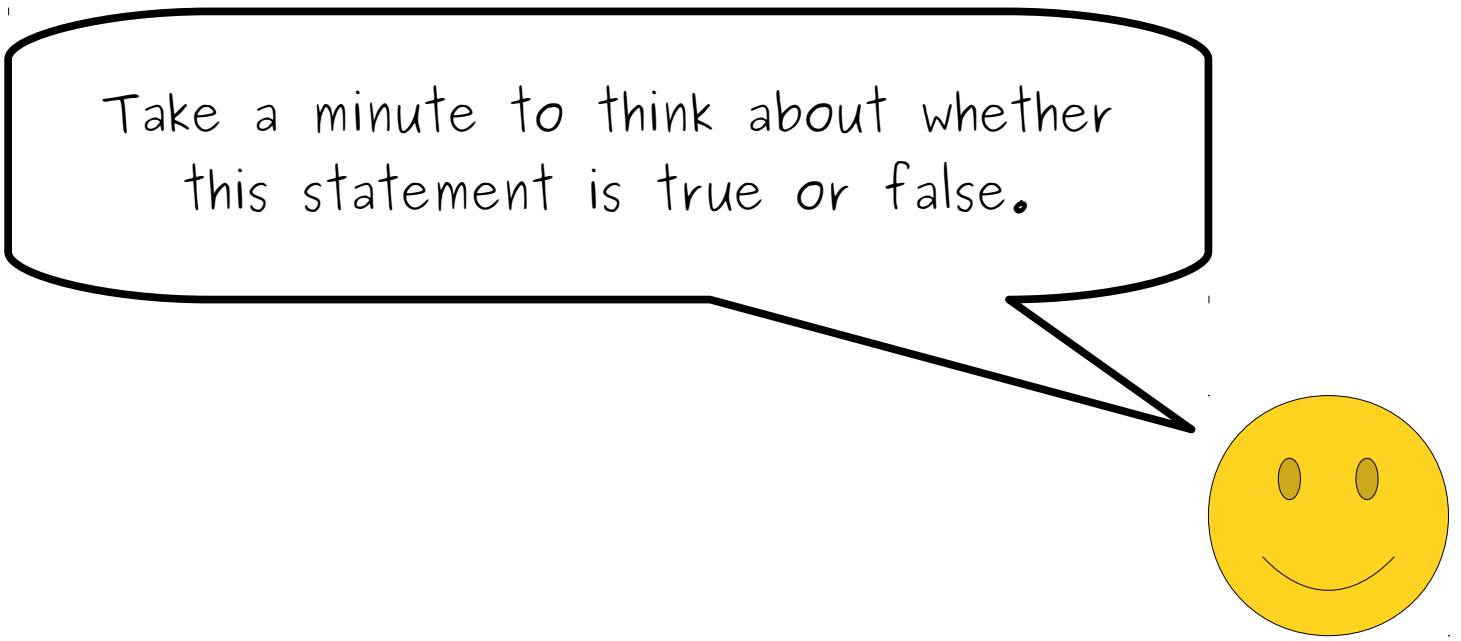


$$\emptyset \in \{ \emptyset, 137 \}$$

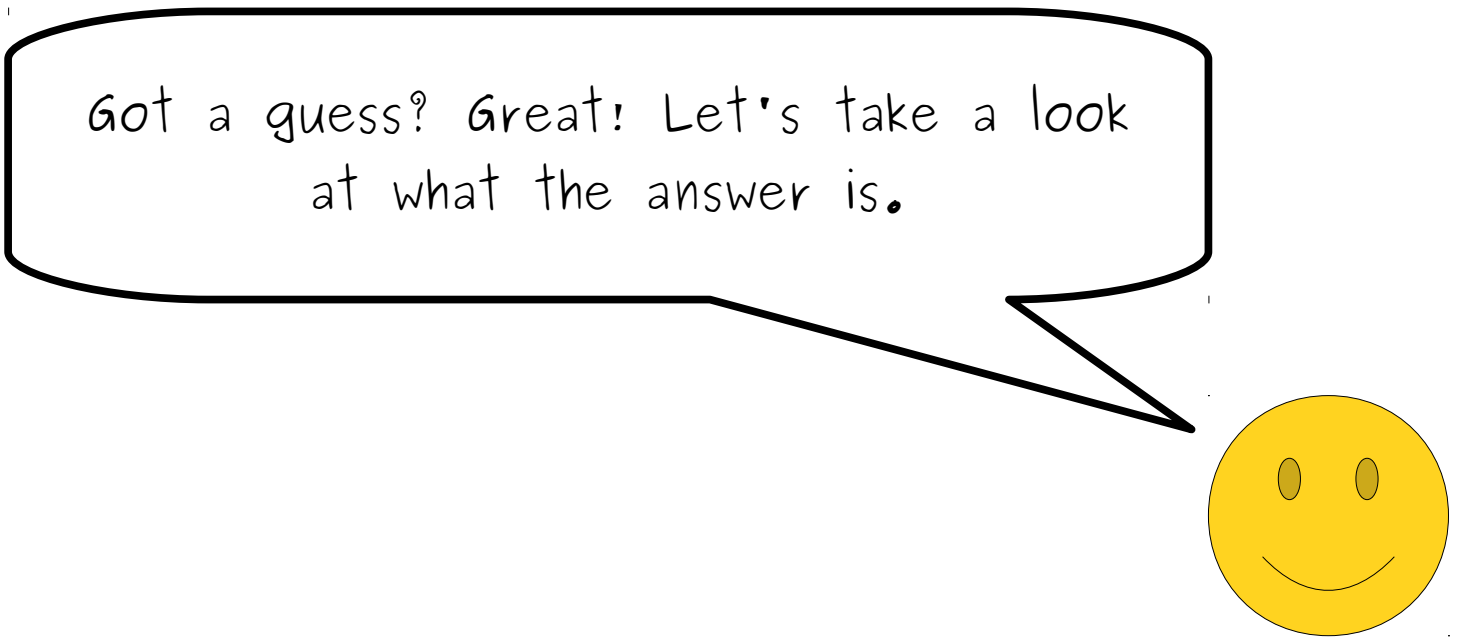



I've been talking for a while now, so let's do one final exercise and then call it a day.

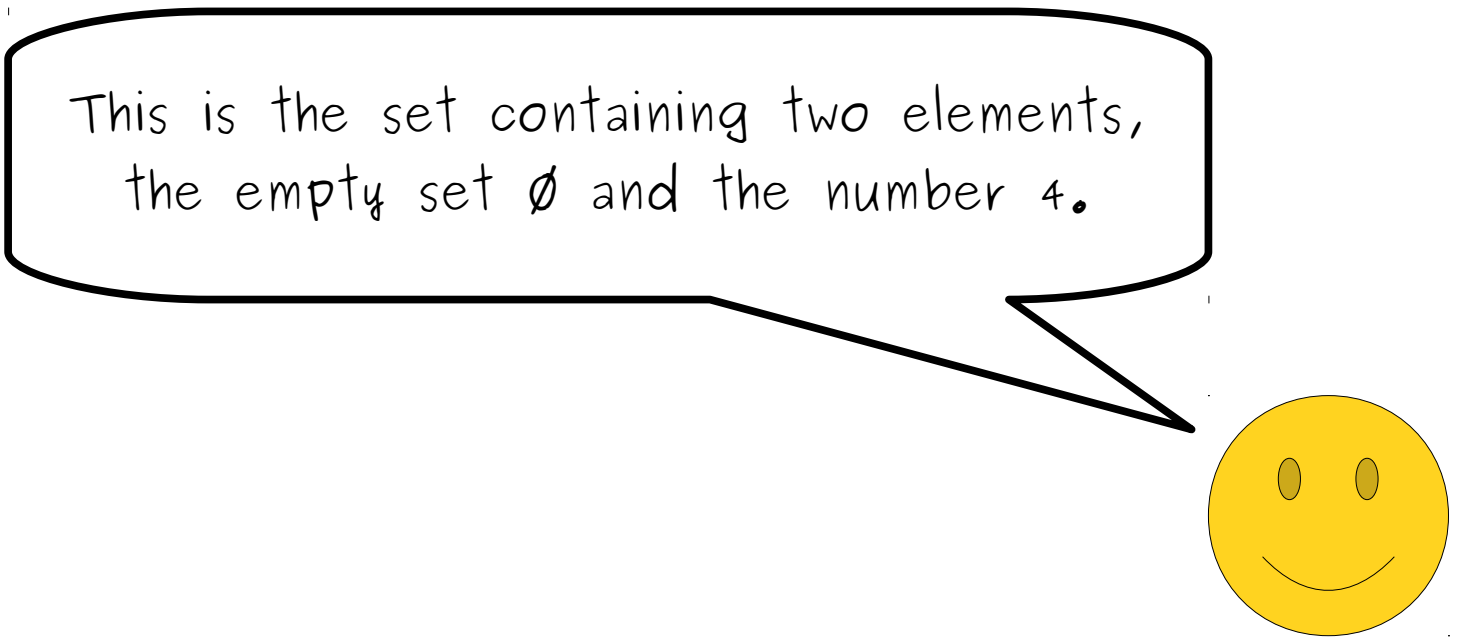
$$\{\emptyset, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$



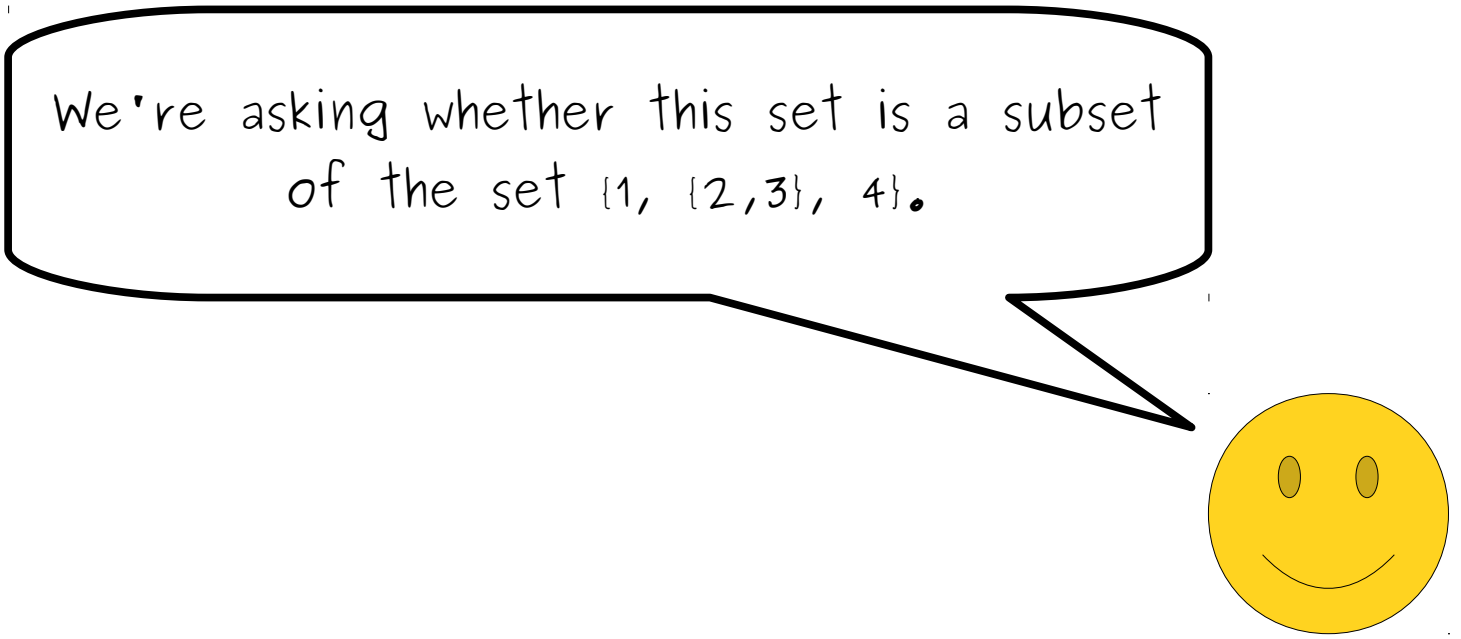
$$\{\emptyset, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$



$$\{\emptyset, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

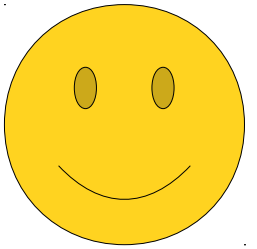


$$\{\varnothing, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

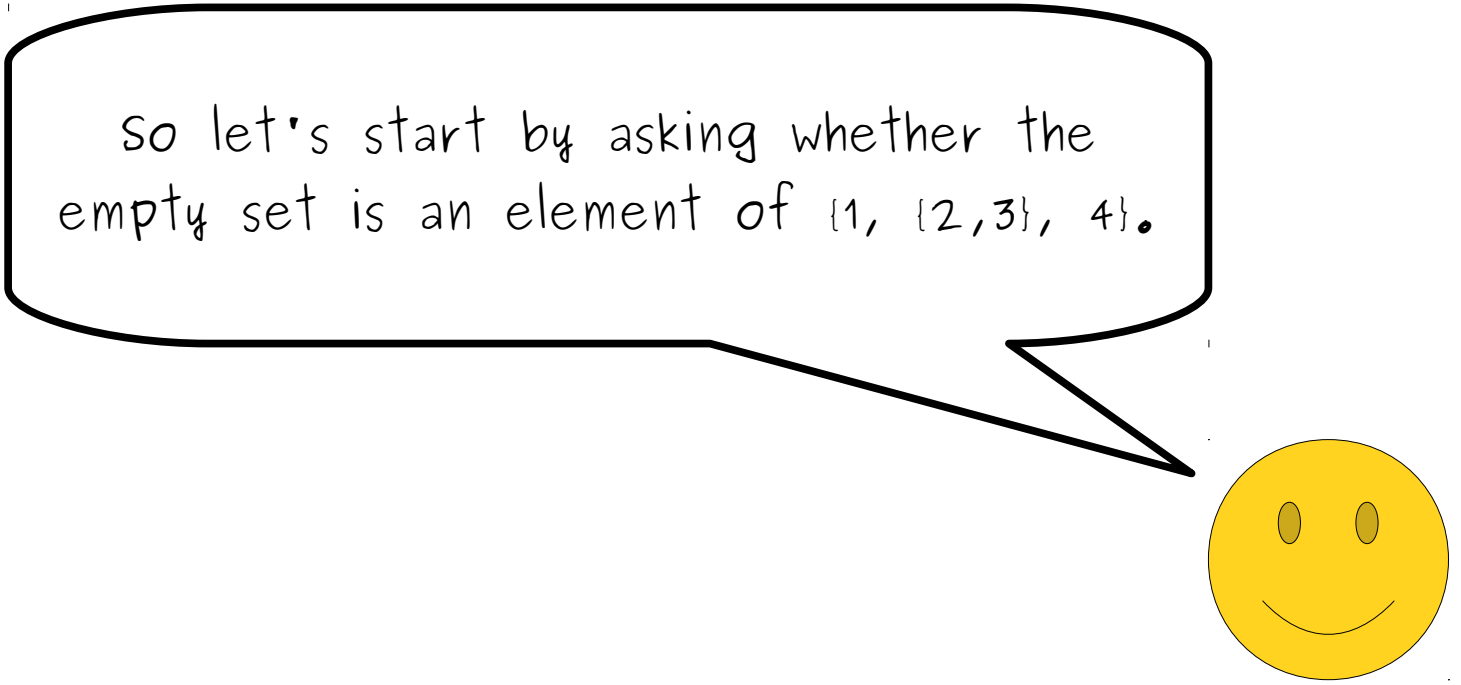


$$\{\emptyset, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

As you've seen before, this means that we need to check whether every element of the set on the left happens to be an element of the set on the right.

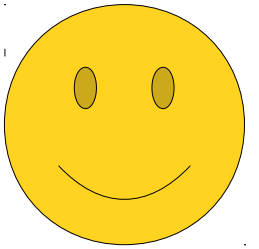


$$\{\emptyset, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$



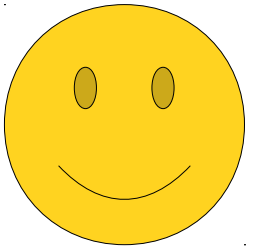
$$\{\emptyset, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

Something to notice here is that even though the "top-level" question asks about the $\subseteq$ relation, right now we're asking whether this particular thing (the empty set) is an element of another set, and that involves the $\in$ relation.

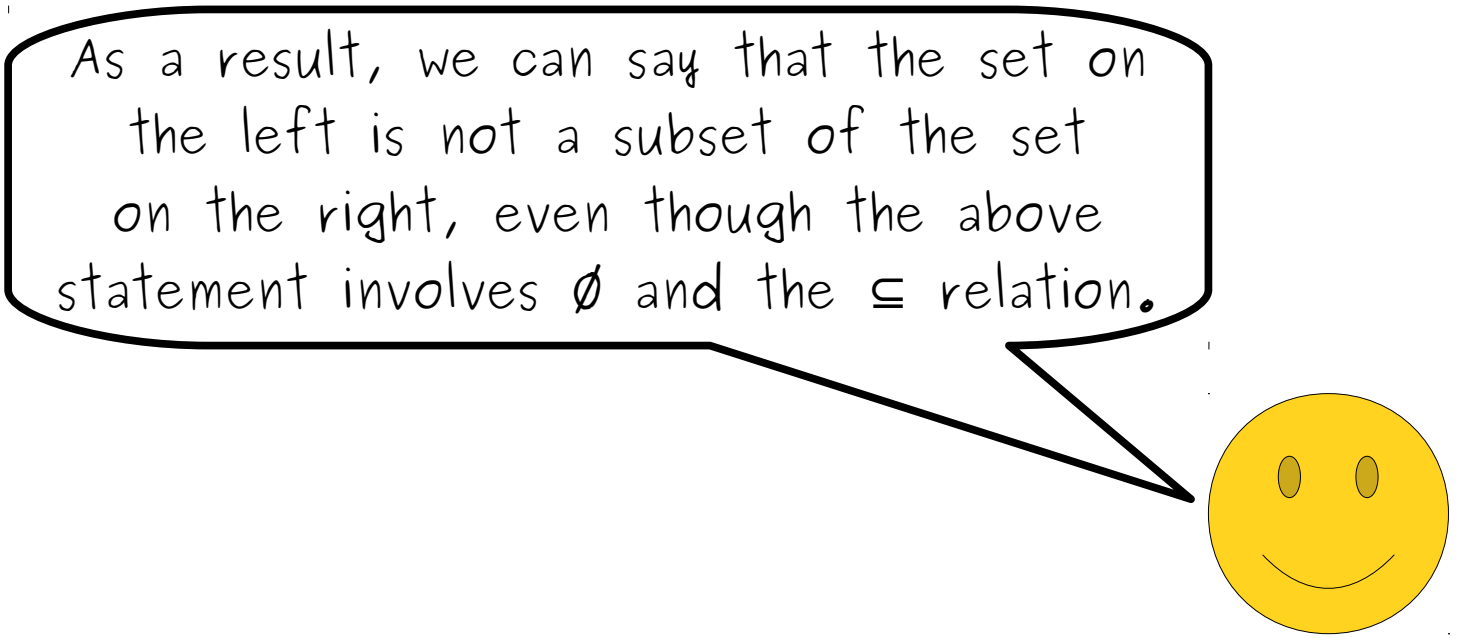


$$\{\emptyset, 4\} \stackrel{?}{\subseteq} \{ 1, \{2, 3\}, 4 \}$$

And, as we saw before, we know that the empty set is not an element of $\{1, \{2,3\}, 4\}$, because none of the elements of that set are equal to $\emptyset$.



$$\{\emptyset, 4\} \not\subseteq \{ 1, \{2, 3\}, 4 \}$$



Hope this helps!
Please feel free to ask questions if you have them.

Did you find this useful? If so, let us know! We can go and make more guides like these.