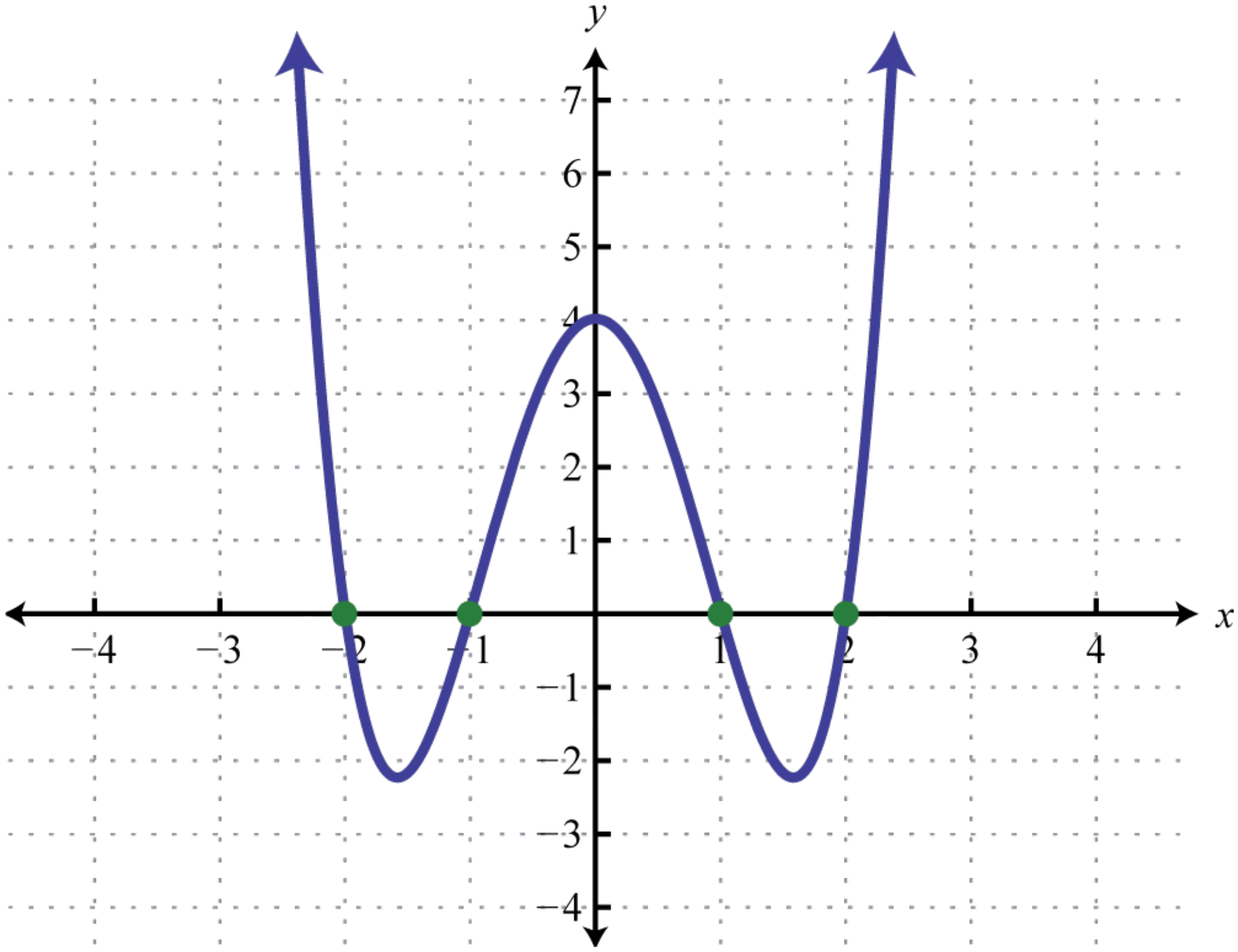
Functions

What is a function?



$$f(x) = x^4 - 5x^2 + 4$$

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) numHeads++;
    numTries++;
  }

  return numTries;
}
```

Functions, CS Edition

- In programming, functions
  - might take in inputs,
  - might return values,
  - might have side effects,
  - might never return anything,
  - might crash, and
  - might return different values when called multiple times.

High School versus CS Functions

- In high school, functions usually were given by a rule:

$$f(x) = 4x + 15$$

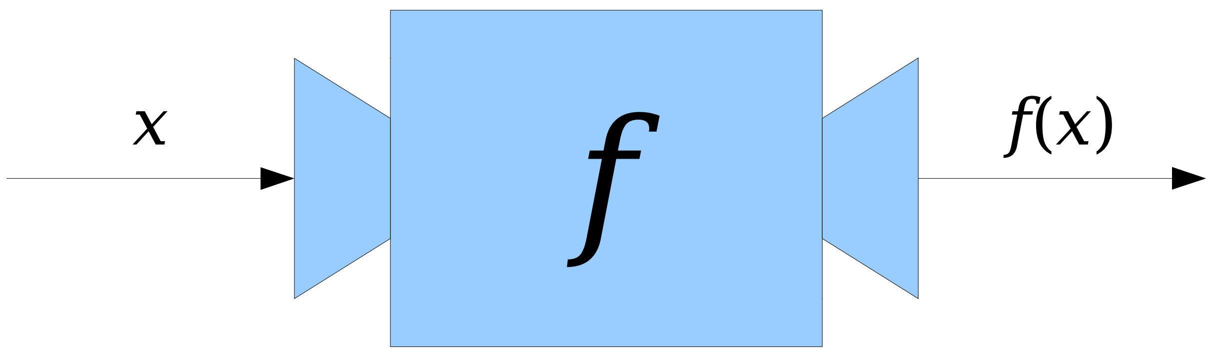
- In CS, functions are usually given by code:

```
int factorial(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++) {
        result *= i;
    }
    return result;
}
```

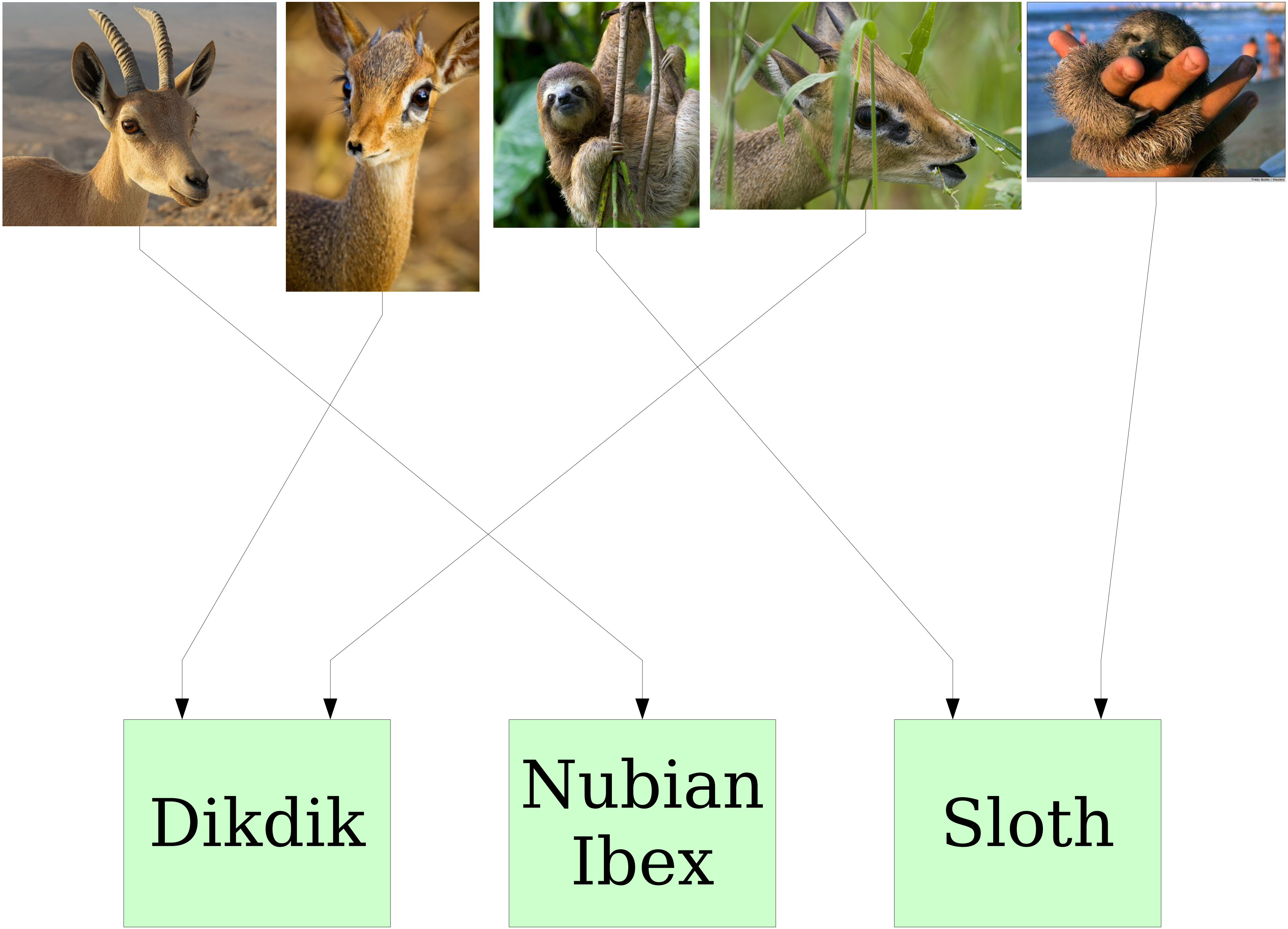
- What sorts of functions are we going to allow from a mathematical perspective?

Rough Idea of a Function:

A function is an object f that takes in an input and produces exactly one output.



(This is not a complete definition – we'll revisit this in a bit.)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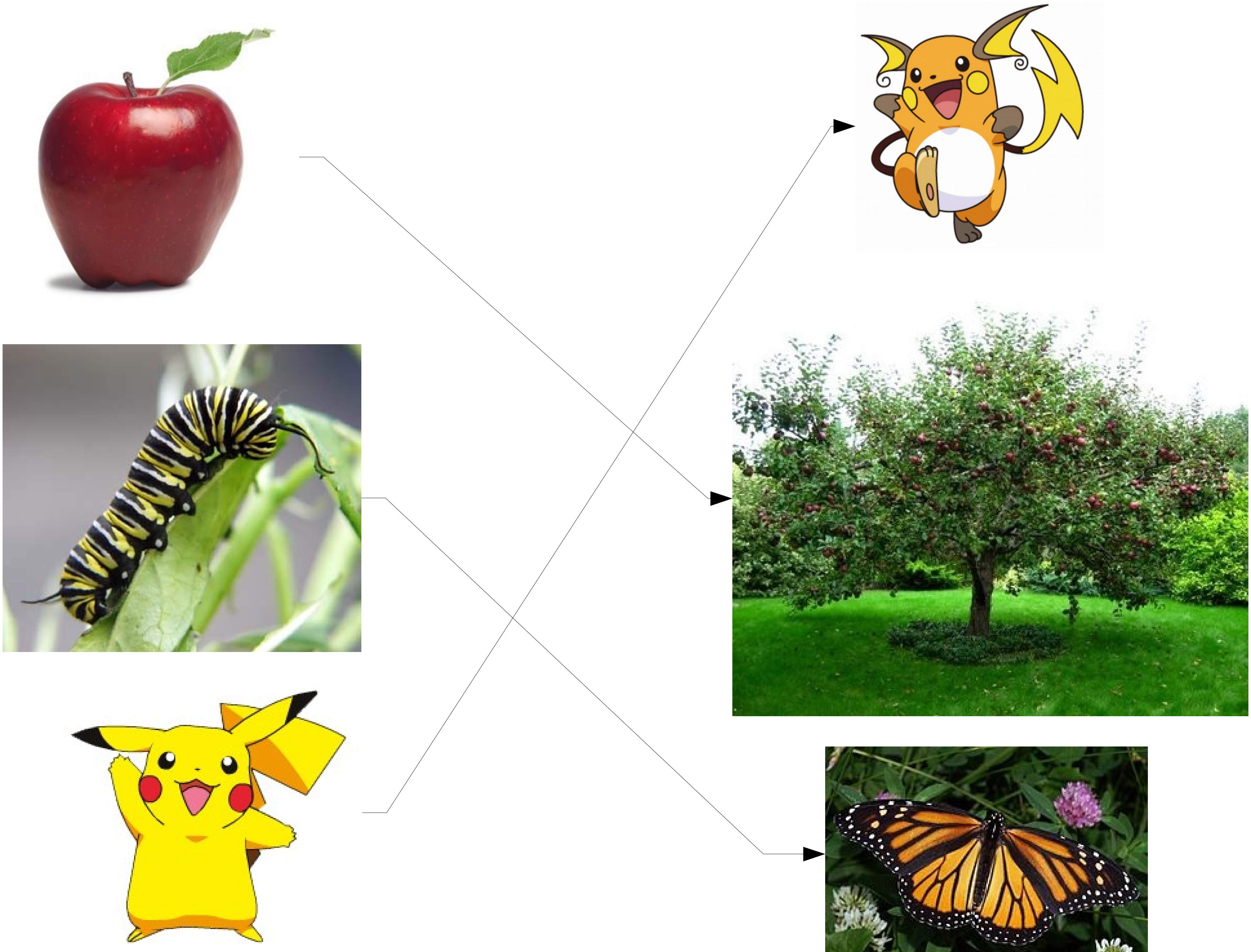
Nubian Ibex
Sloth



… but also …

$$f(x) = x^2 + 3x - 15$$

$$f(n)=\begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{otherwise} \end{cases}$$

Functions like these are called ***piecewise functions***.

To define a function, you will typically either

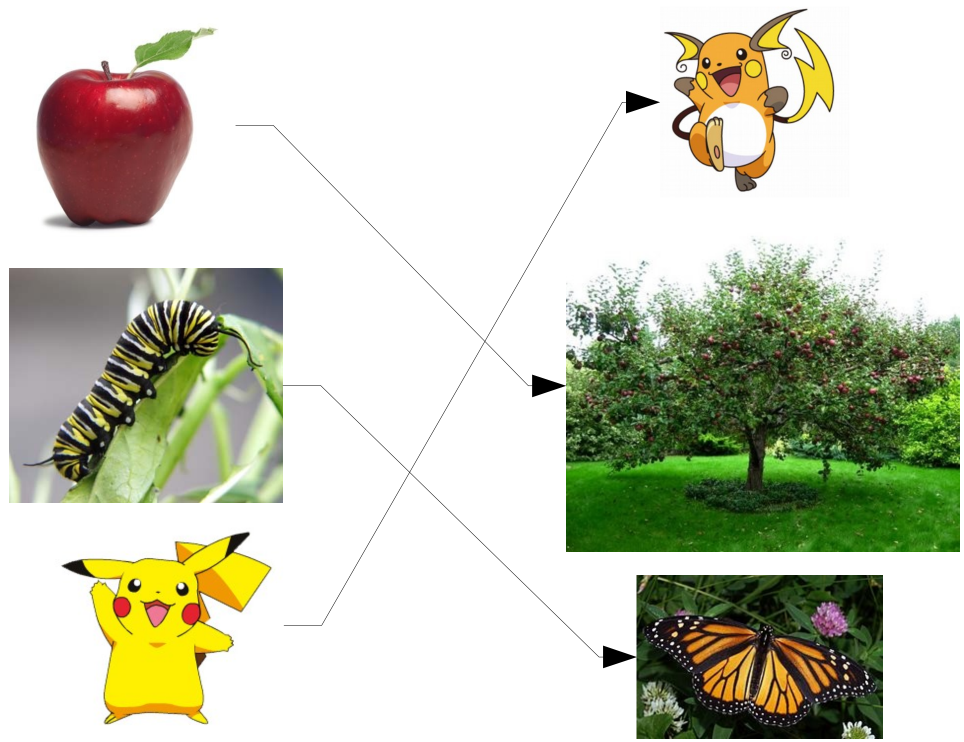
- draw a picture, or
- give a rule for determining the output.

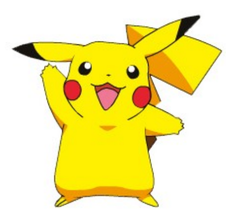
In mathematics, functions are ***deterministic***.

That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it's not a valid function under our definition:

```
int randomNumber(int numOutcomes) {
    return rand() % numOutcomes;
}
```



$f($  $) =$ 

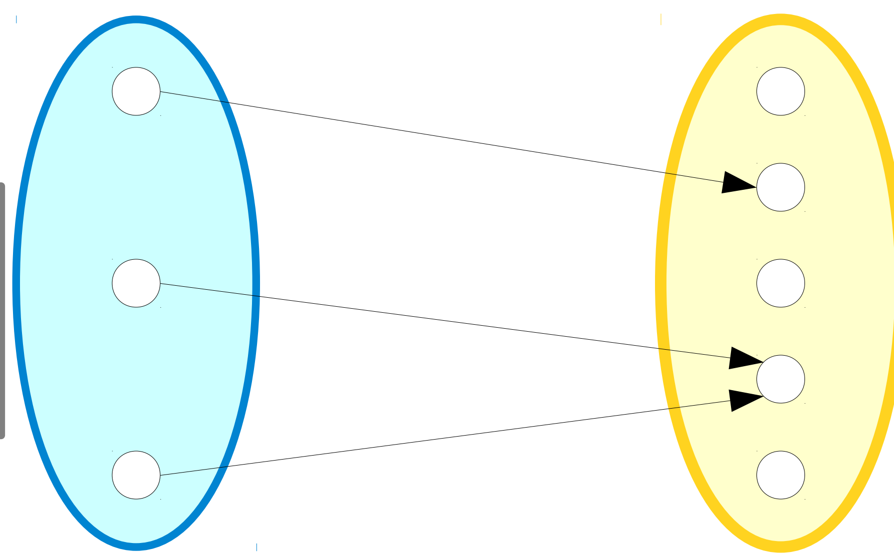
$f(137) = \ldots?$

We need to make sure we can't apply functions to meaningless inputs.

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

The function must be defined for every element of the domain.



Domain

Codomain

The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.

Domains and Codomains

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- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

The codomain of this function is $\mathbb{R}$. Everything produced is a real number, but not all real numbers can be produced.

The domain of this function is $\mathbb{R}$. Any real number can be provided as input.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- This notation just says what the domain and codomain of the function are. It doesn't say how the function is evaluated.
- Think of it like a "function prototype" in C or C++. The notation $f : ArgType \to RetType$ is like writing

  RetType *f*(*ArgType* `argument);`

  We know that f takes in an *ArgType* and returns a *RetType*, but we don't know exactly which *RetType* it's going to return for a given *ArgType*.

The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function with a nonempty domain have an empty codomain?

Defining Functions

- Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.
- Examples:
    - $f(n) = n + 1$, where $f : \mathbb{Z} \to \mathbb{Z}$
    - $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
    - $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$
- Notice that we're giving both a rule and the domain/codomain.

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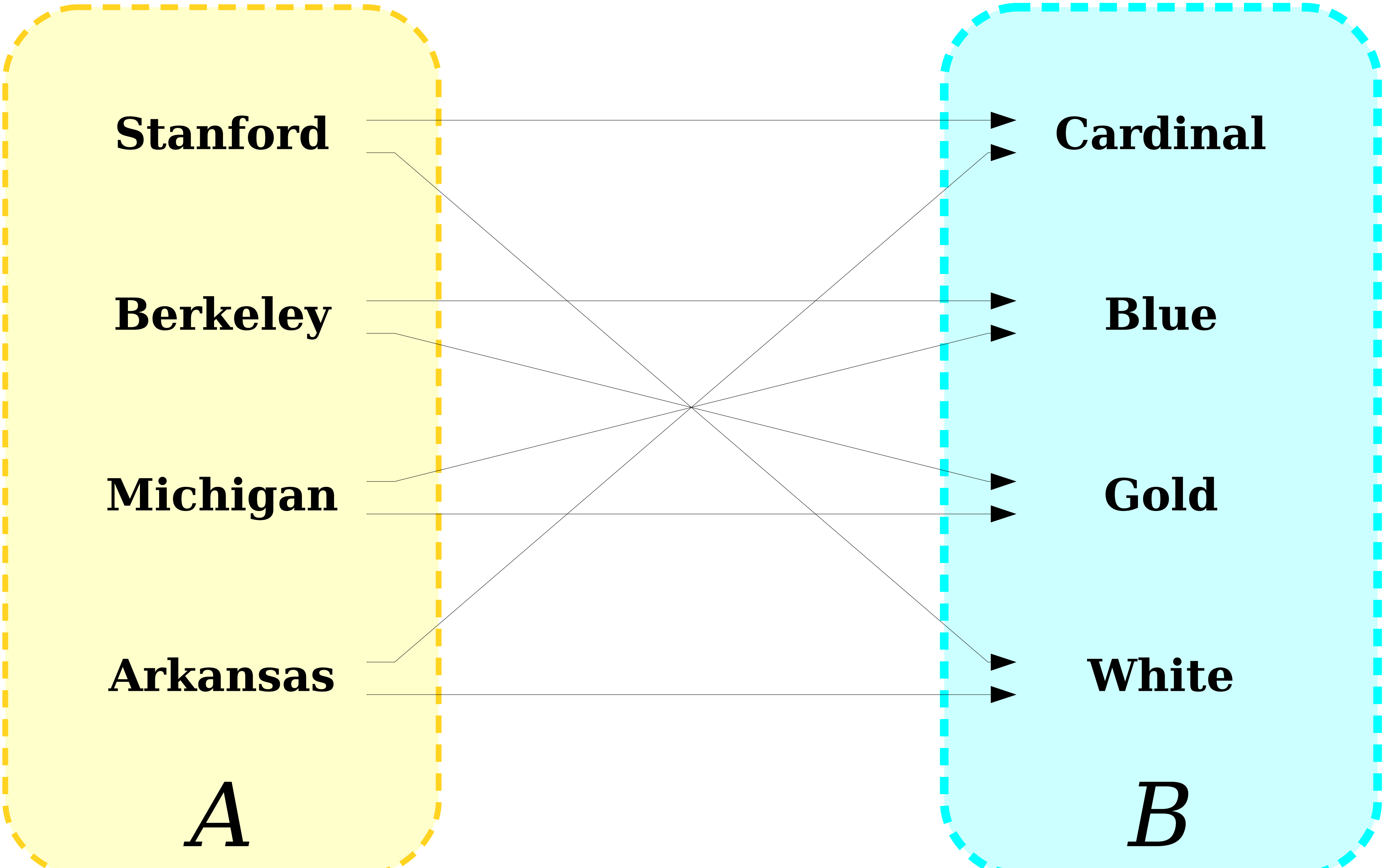
- $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \to \mathbb{Z}$

This is the ceiling function – the smallest integer greater than or equal to *x*. For example, $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil \pi \rceil = 4$.

Notice that we're giving both a rule and the domain/codomain.

Is This a Function From A to B?



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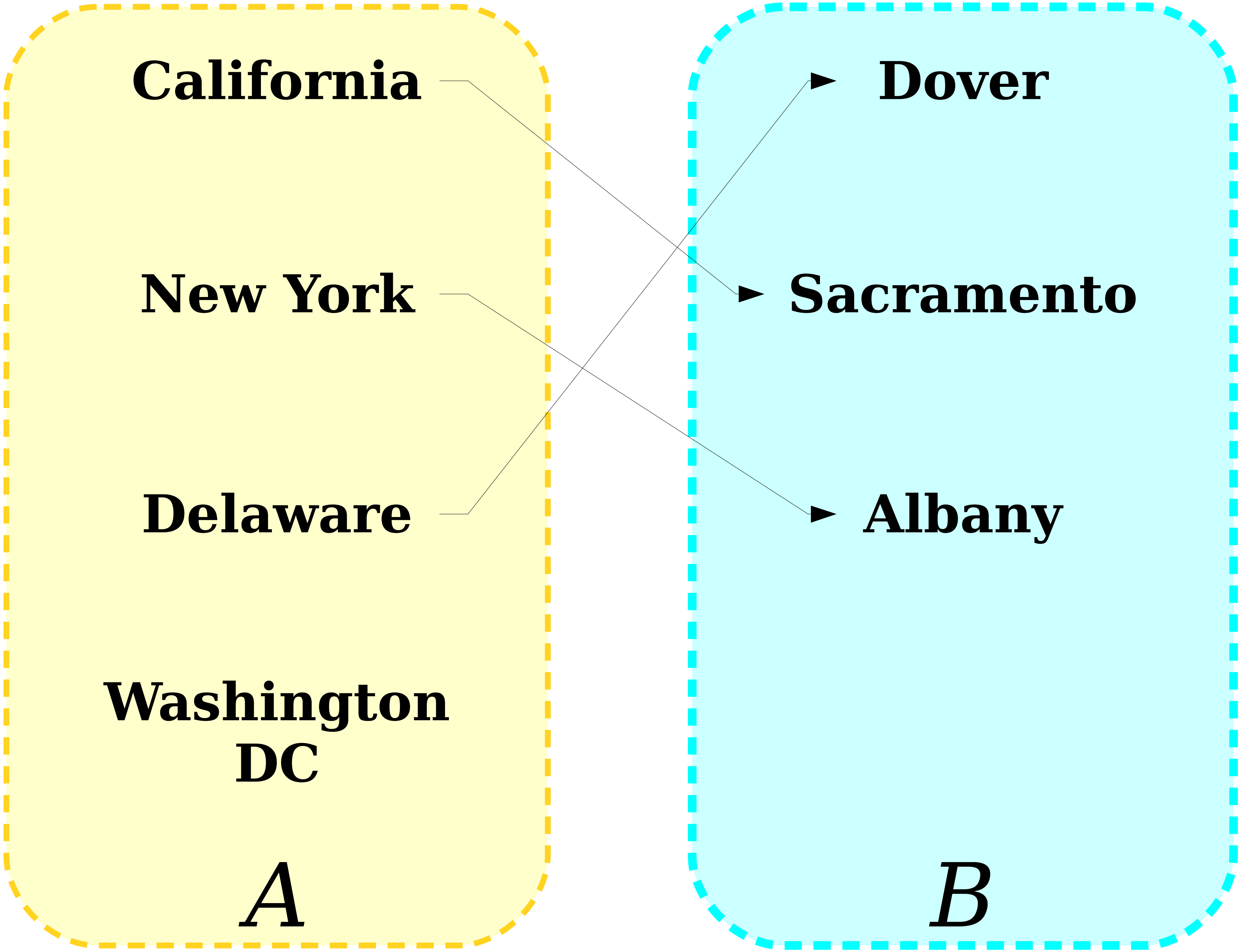


Is This a Function From A to B?

عيد الفطر

عيد الأضحى

A

مُحَرَّم

صَفَر

رَبيع الأوّل

رَبيع الثاني

جُمادى الأولى

جُمادى الآخرة

رَجَب

شَعْبان

رَمَضان

شَوّال

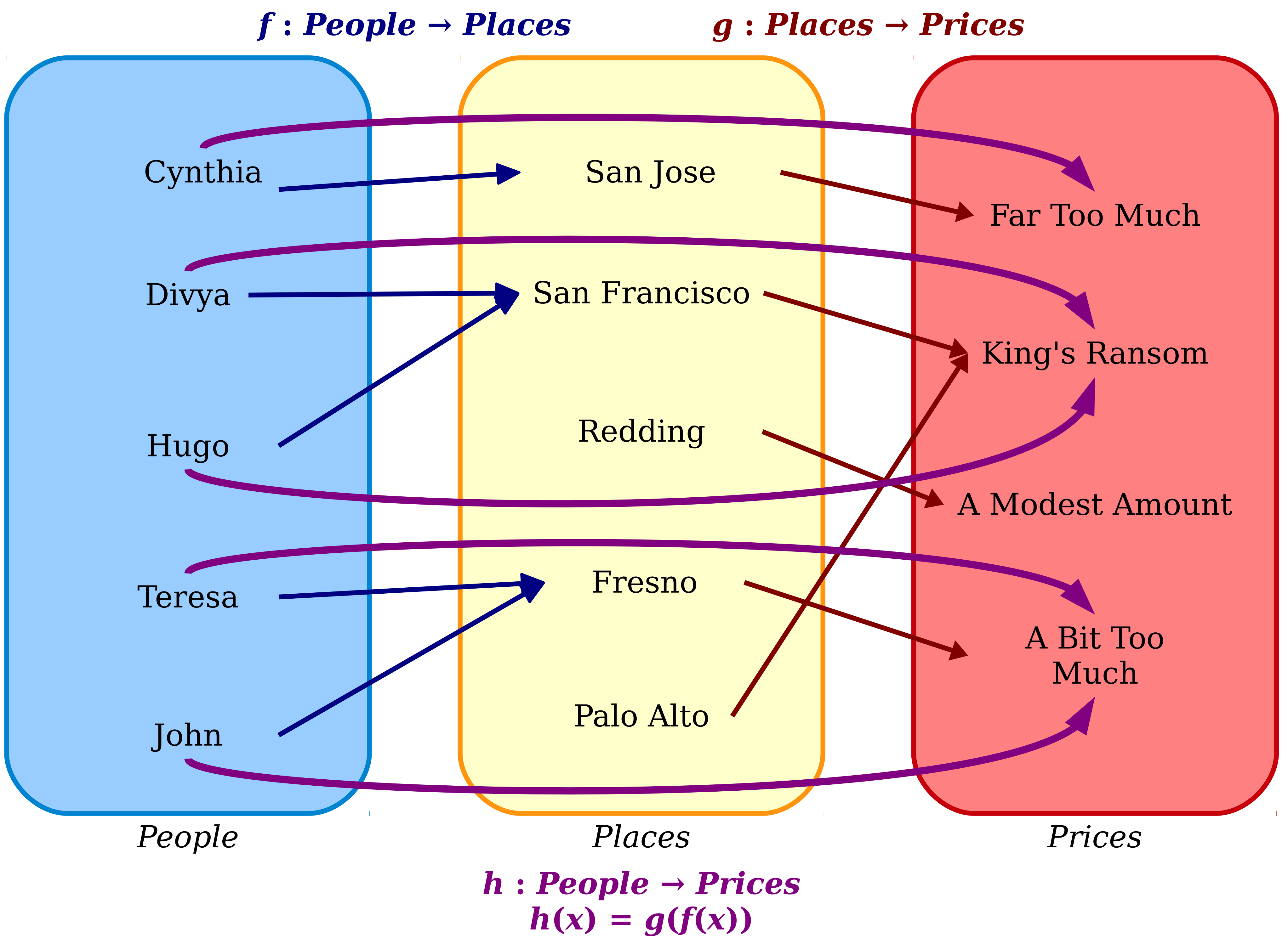
ذو القعدة

ذو الحجة

B

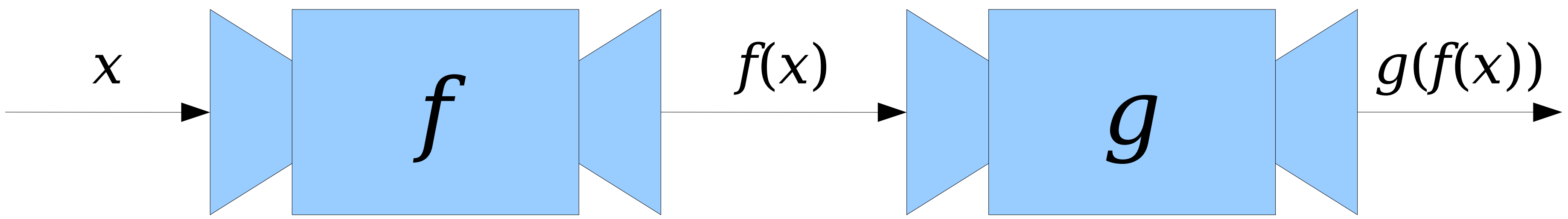
Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **Y** or **N**.

Combining Functions


f : People → Places
g : Places → Prices
Cynthia
Divya
Hugo
Teresa
John
San Jose
San Francisco
Redding
Fresno
Palo Alto
Far Too Much
King's Ransom
A Modest Amount
A Bit Too Much
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



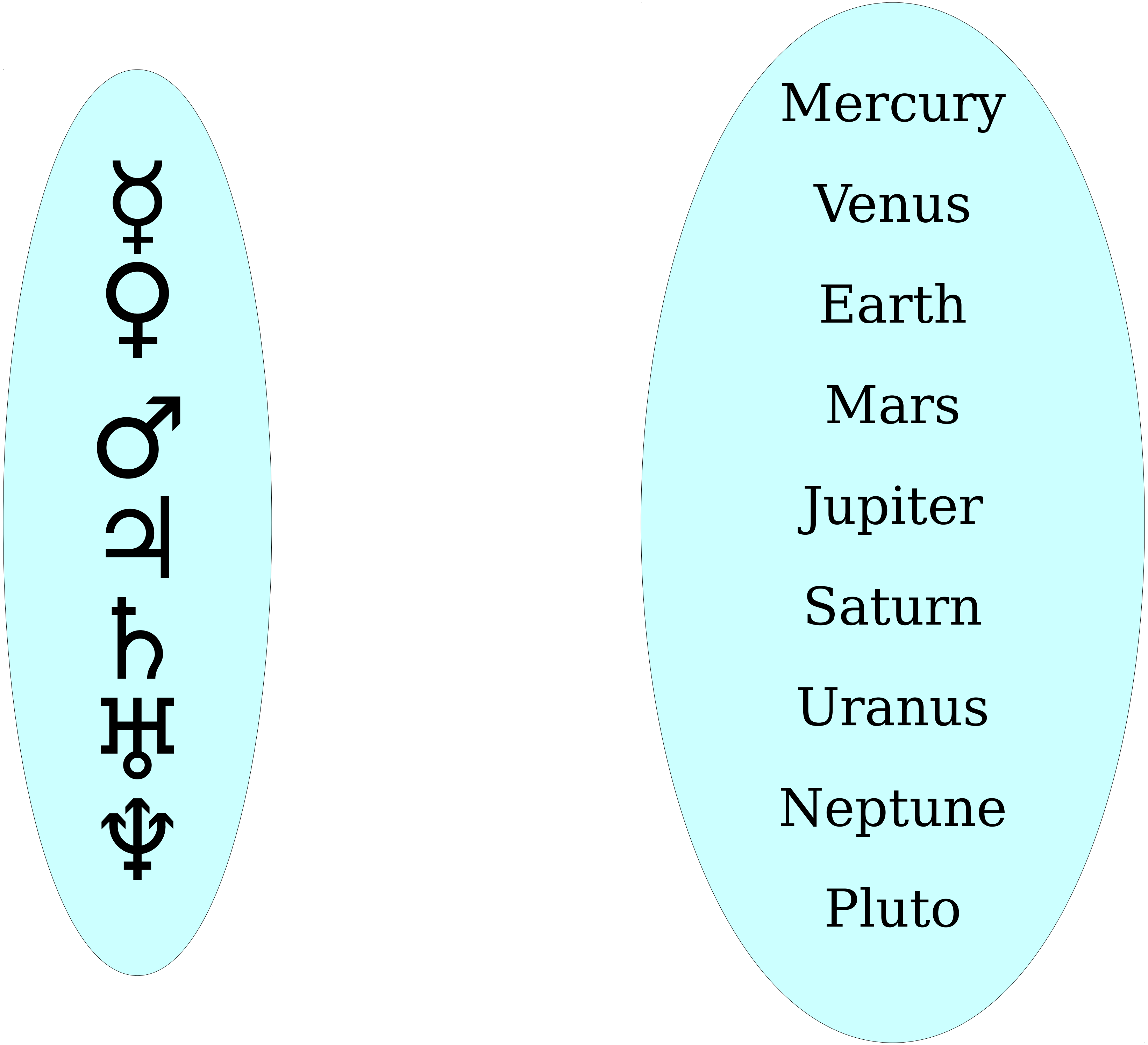
Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Special Types of Functions



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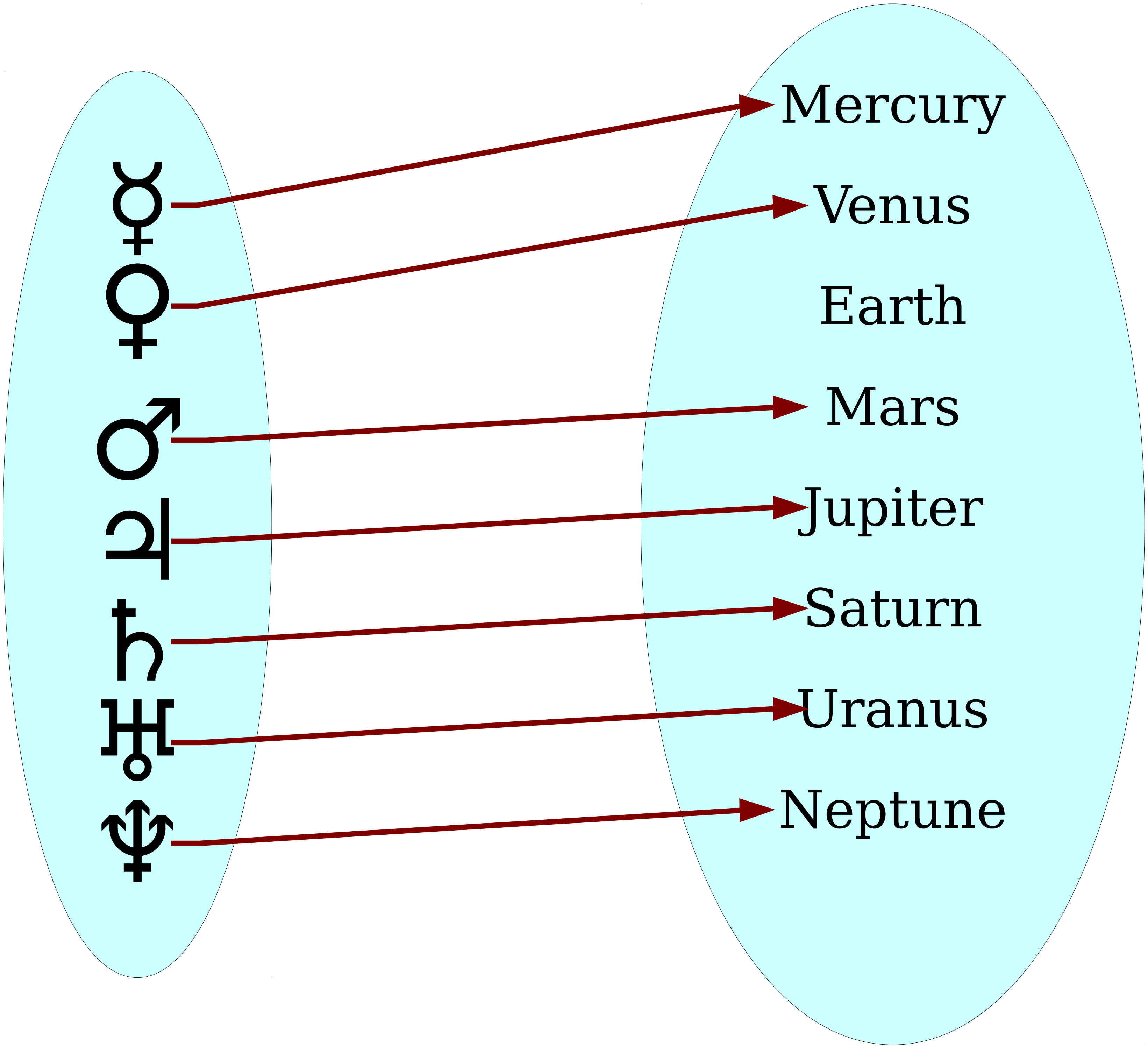
Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune
Pluto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Jupiter
Saturn
Uranus
Neptune

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

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Proof:

How many of the following are correct ways of starting off this proof?

Consider any $n_1, n_2 \in \mathbb{N}$ where $n_1 = n_2$. We will prove that $f(n_1) = f(n_2)$.
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Answer at **PollEv.com/cs103** or
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What does it mean for the function f to be injective?

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Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

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Proof:

How many of the following are correct ways of starting off this proof?

Assume for the sake of contradiction that f is not injective.

Assume for the sake of contradiction that there are integers x_1 and x_2 where $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Consider arbitrary integers x_1 and x_2 where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.

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Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

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so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required.

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How many of the following are correct ways of starting off this proof?

Assume for the sake of contradiction that f is not injective.

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Consider arbitrary integers x_1 and x_2 where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.

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Injections and Composition

Injections and Composition

- ***Theorem:*** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.
- Our goal will be to prove this result. To do so, we're going to have to call back to the formal definitions of injectivity and function composition.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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There are two definitions of injectivity that we can use here:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ ((g \circ f)(a_1) = (g \circ f)(a_2) \to a_1 = a_2)$$

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Therefore, we'll choose an arbitrary $a_1, a_2 \in A$ where $a_1 \neq a_2$, then prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$.

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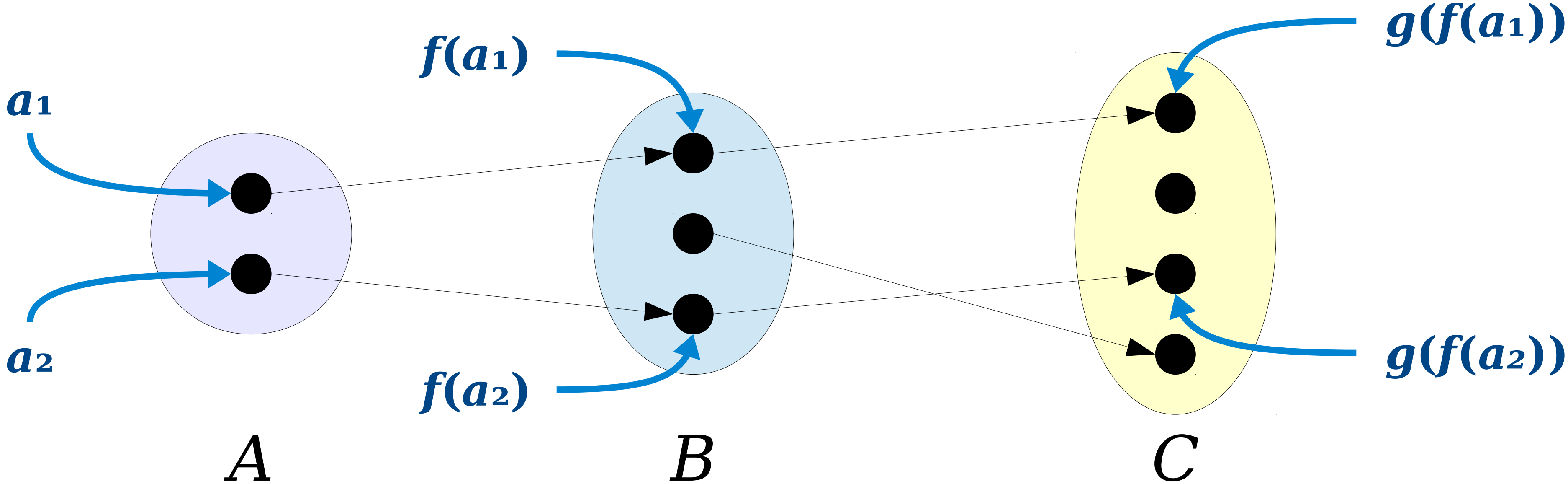

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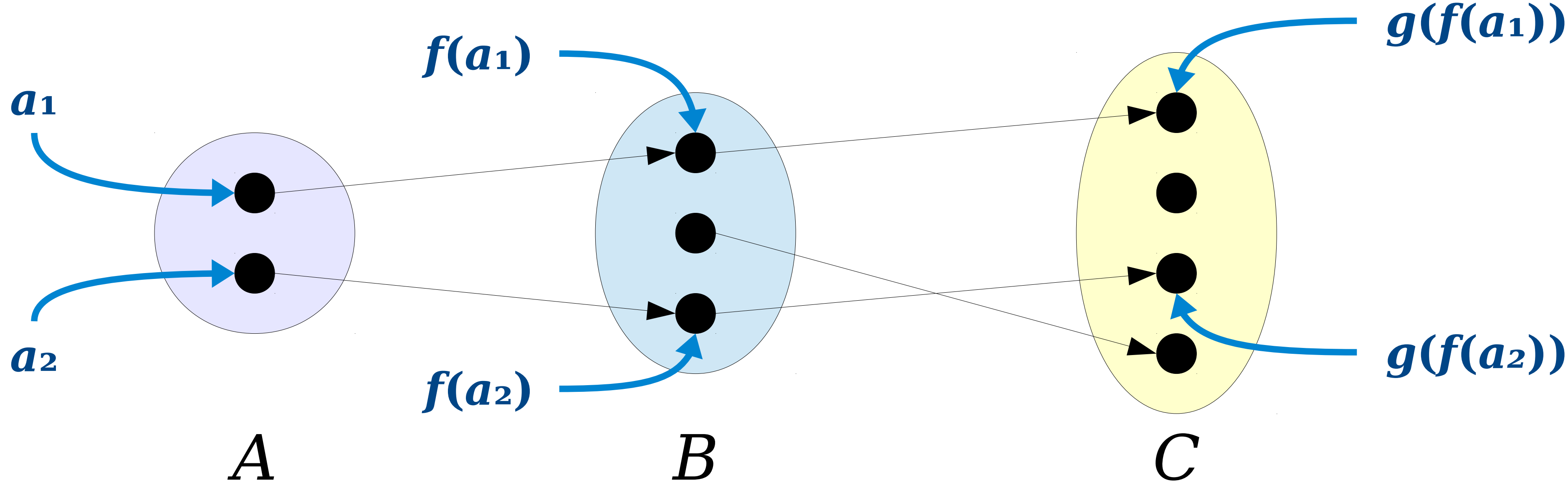

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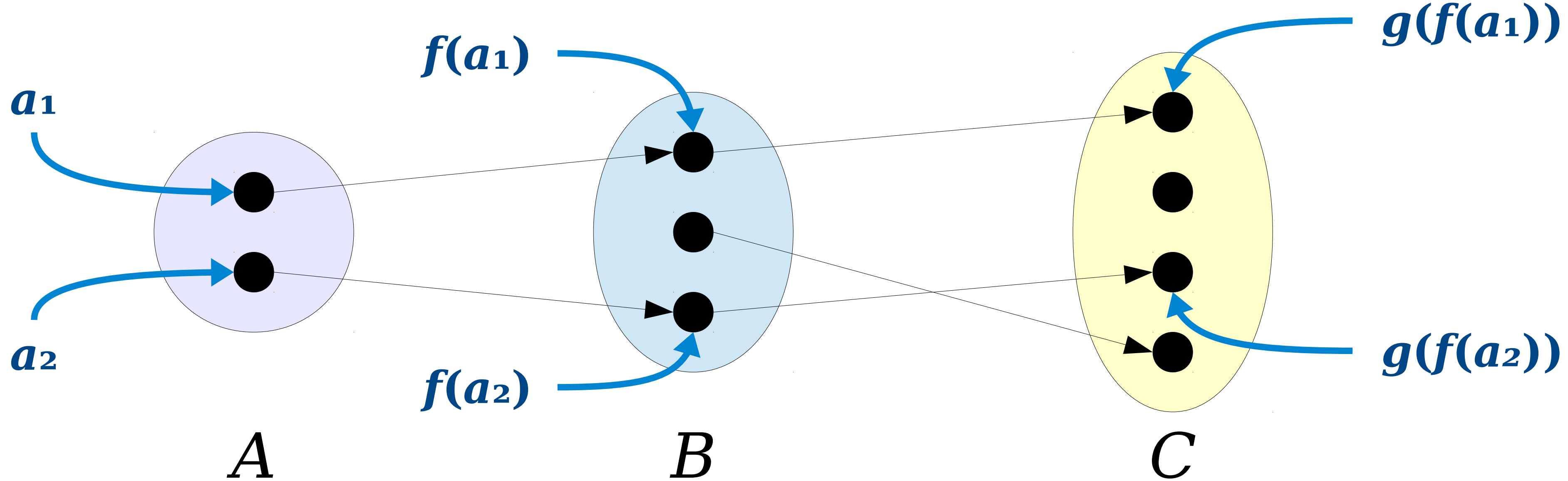

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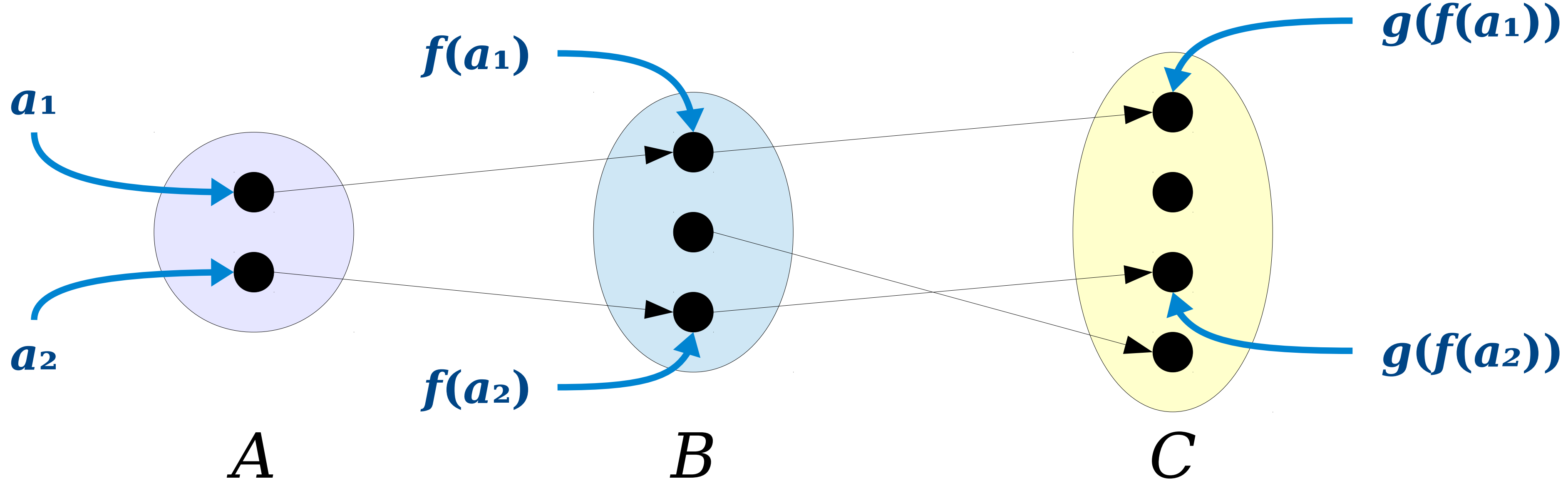

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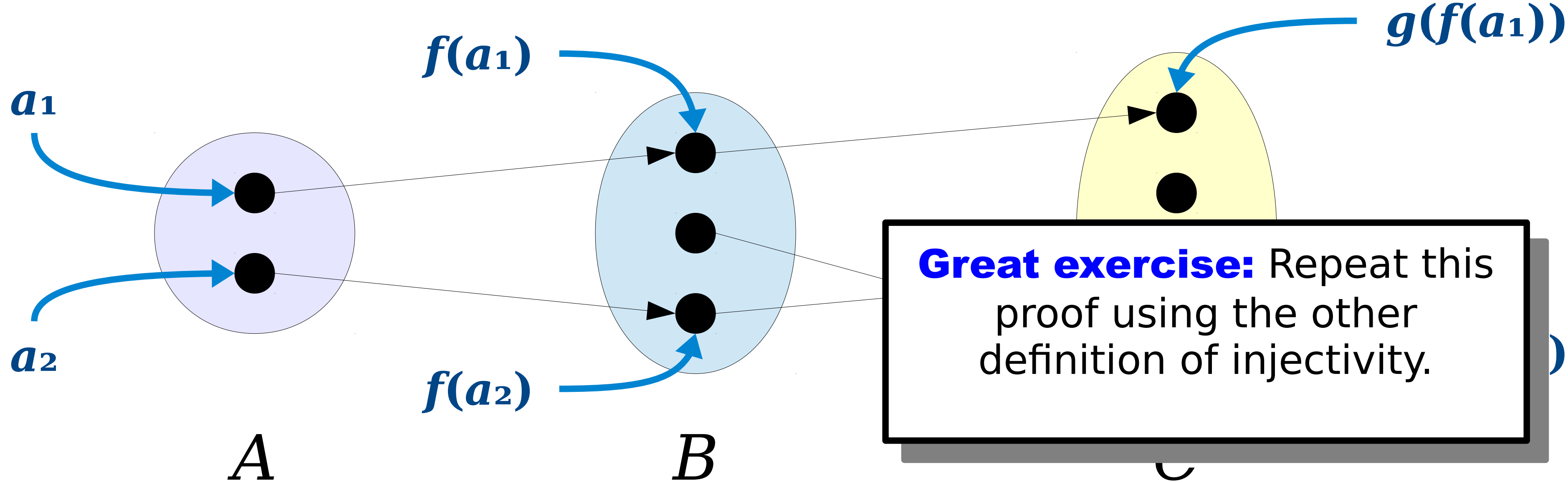
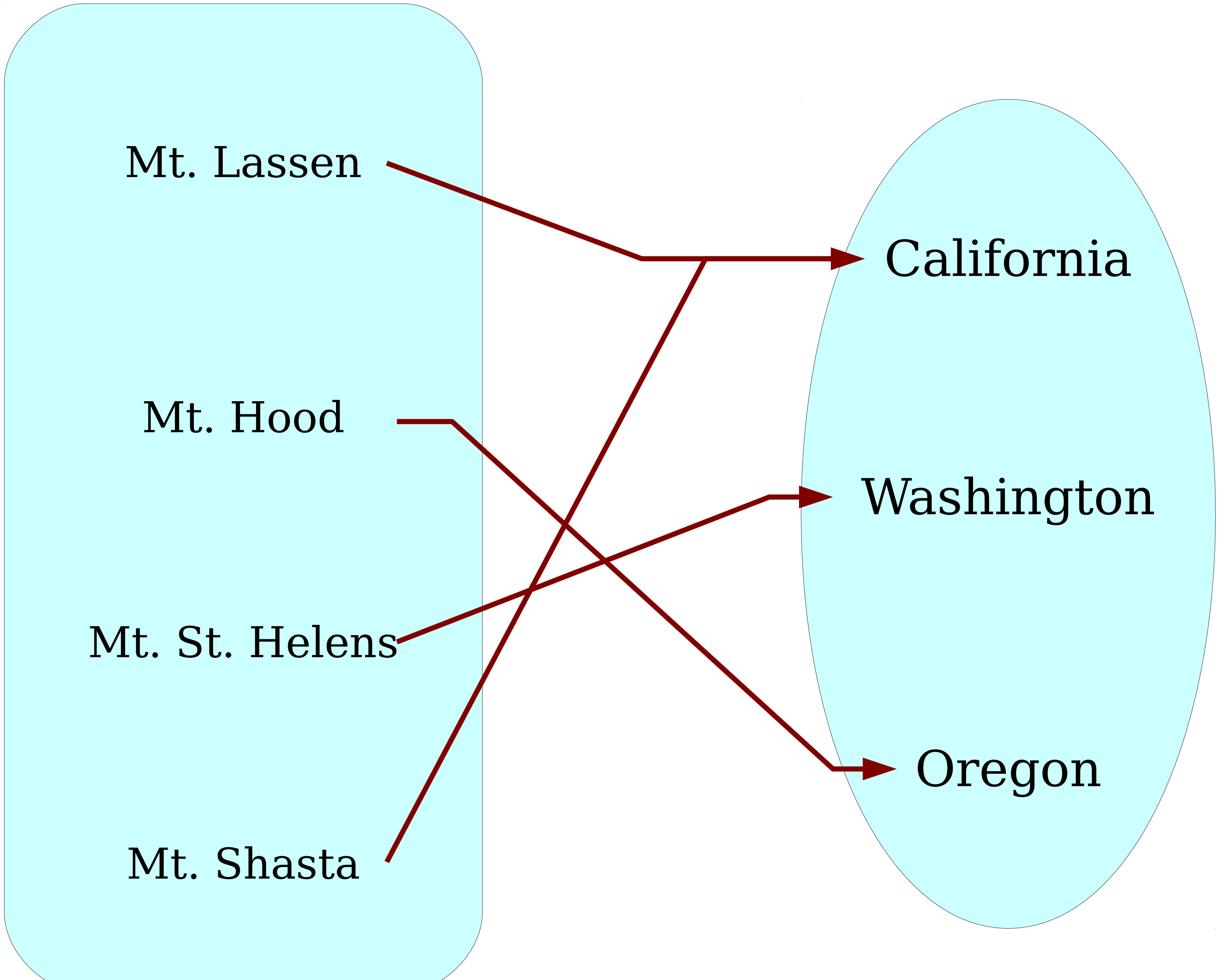


Great exercise: Repeat this proof using the other definition of injectivity.

Another Class of Functions



Mt. Lassen
Mt. Hood
Mt. St. Helens
Mt. Shasta
California
Washington
Oregon

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

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Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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What does it mean for f to be surjective?

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Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Composing Surjections

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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What does it mean for g $\circ$ $f : A \to C$ to be surjective?

$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

Therefore, we'll choose arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

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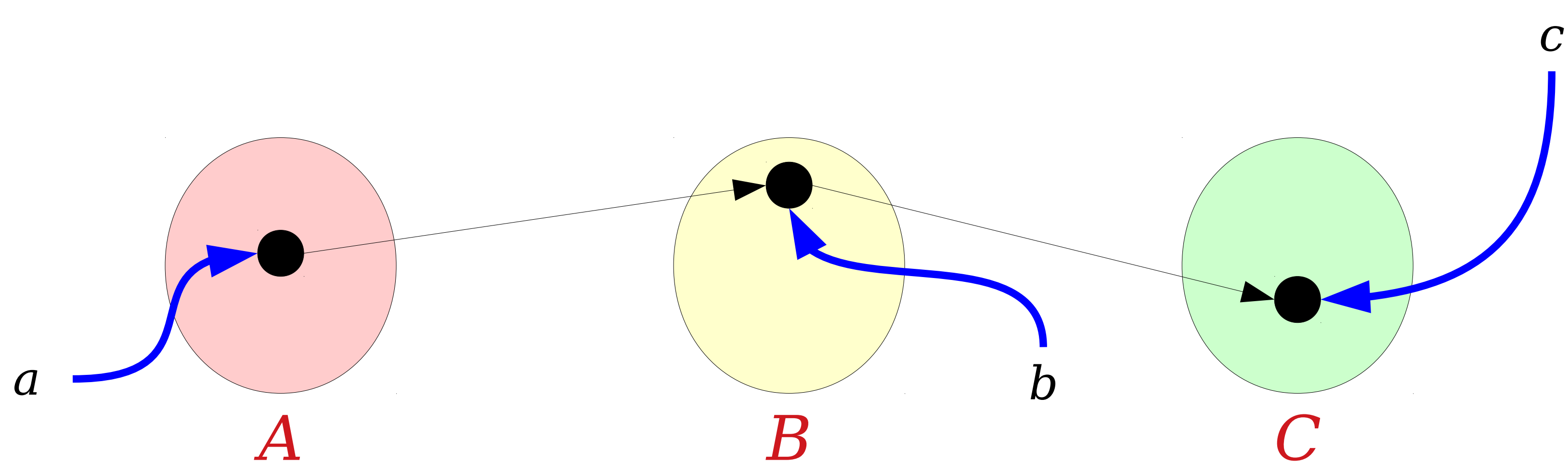

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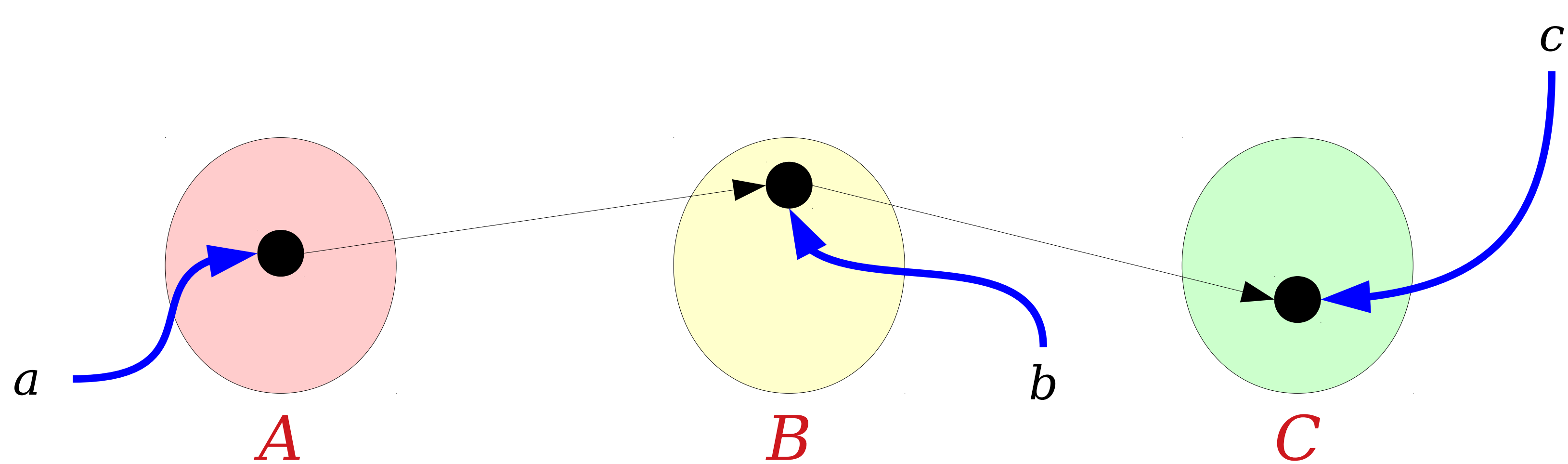

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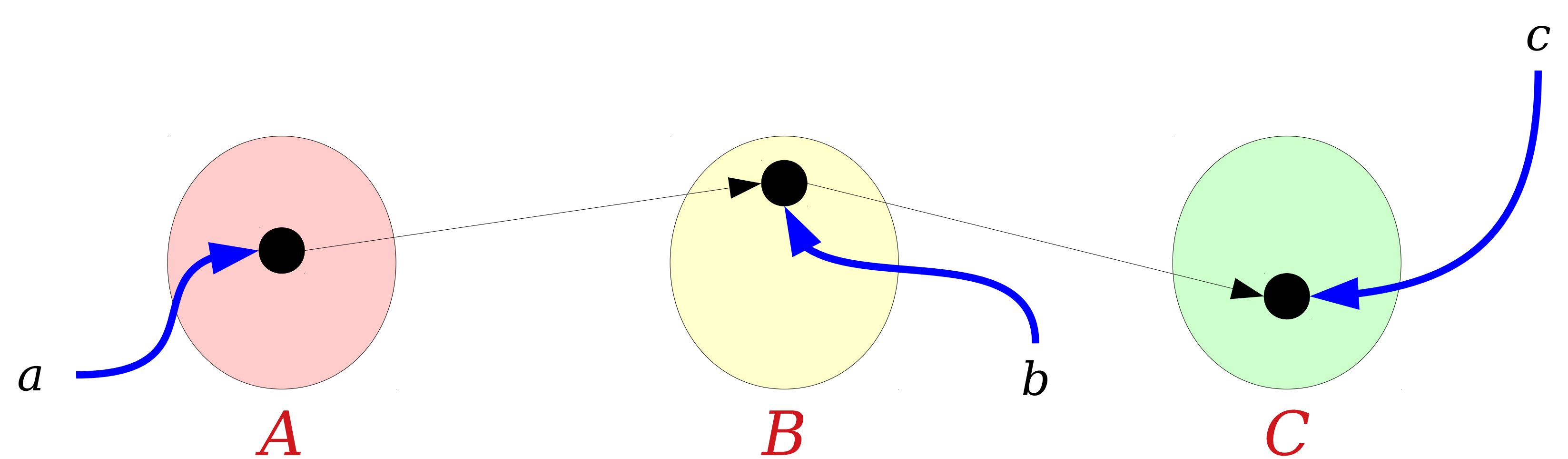

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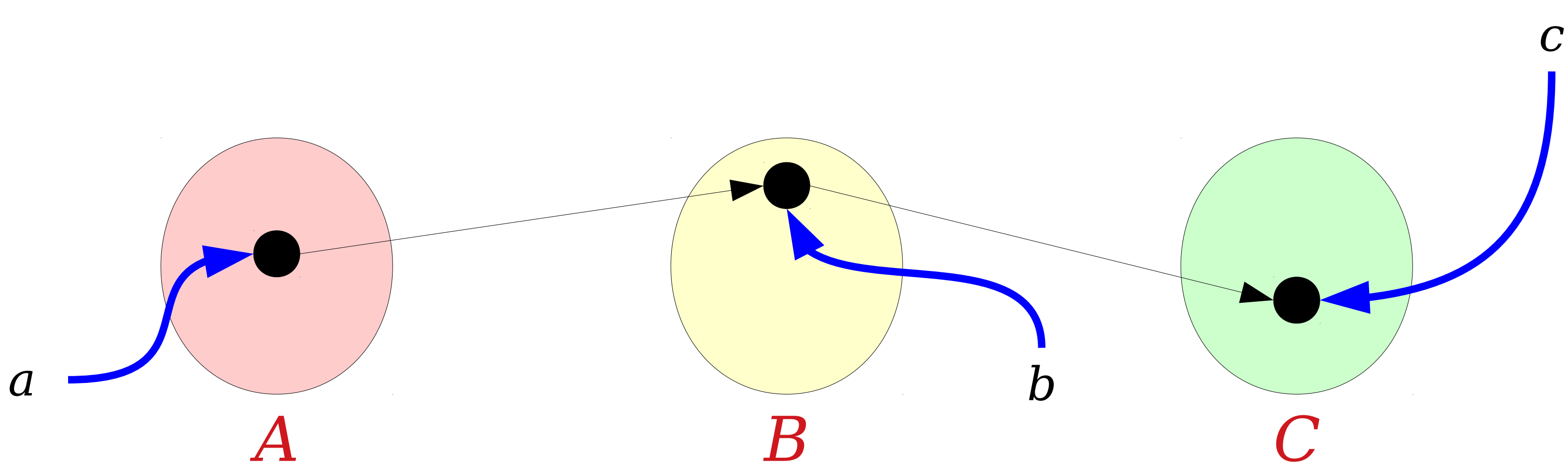

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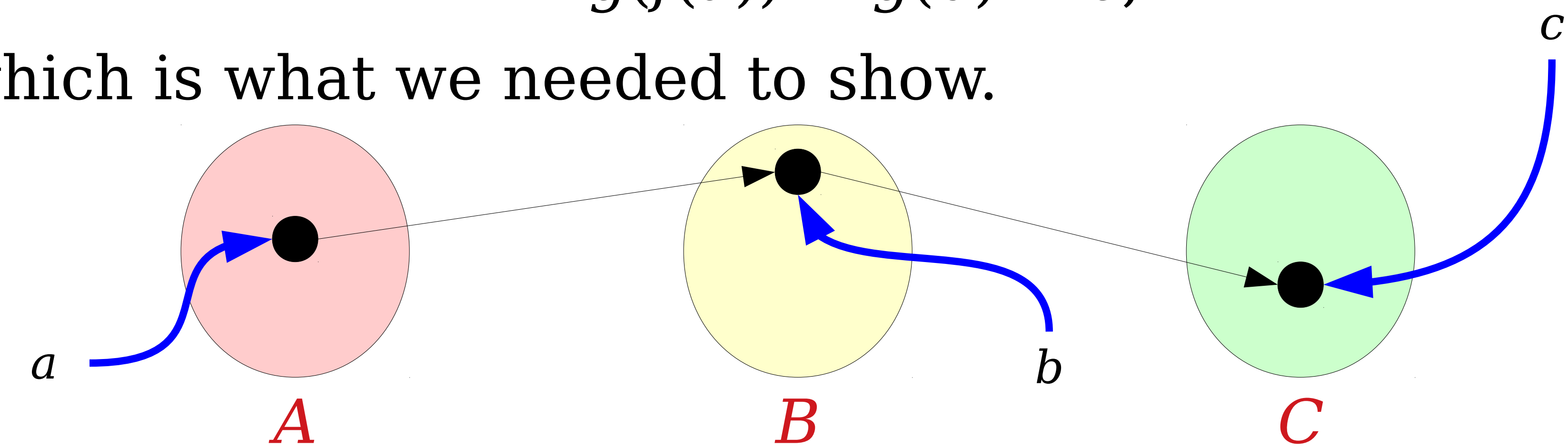

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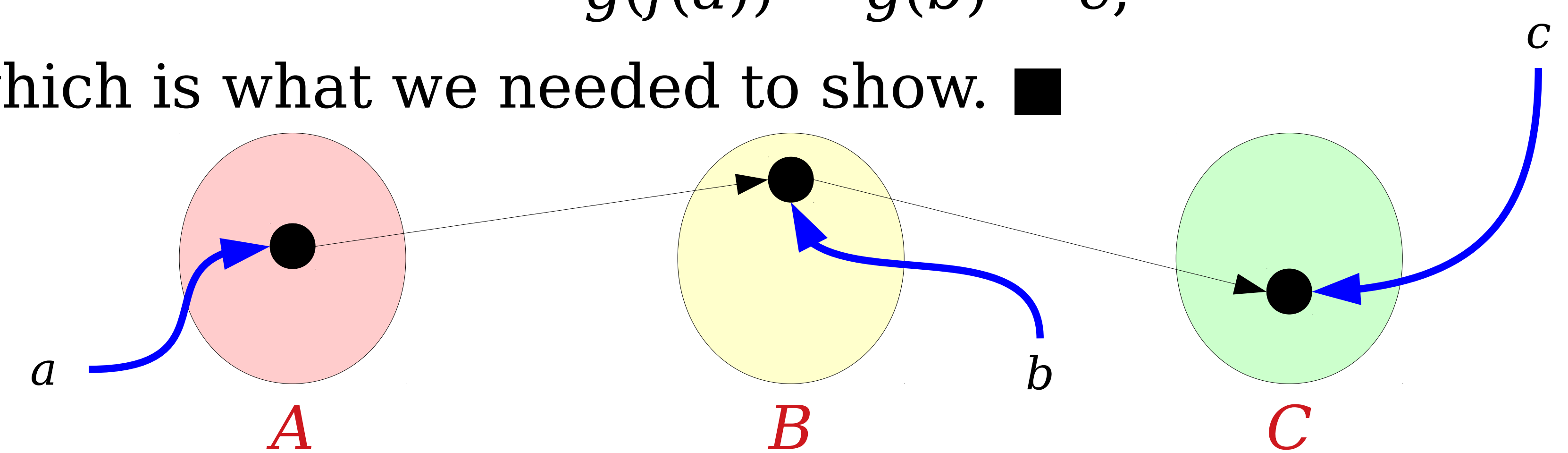

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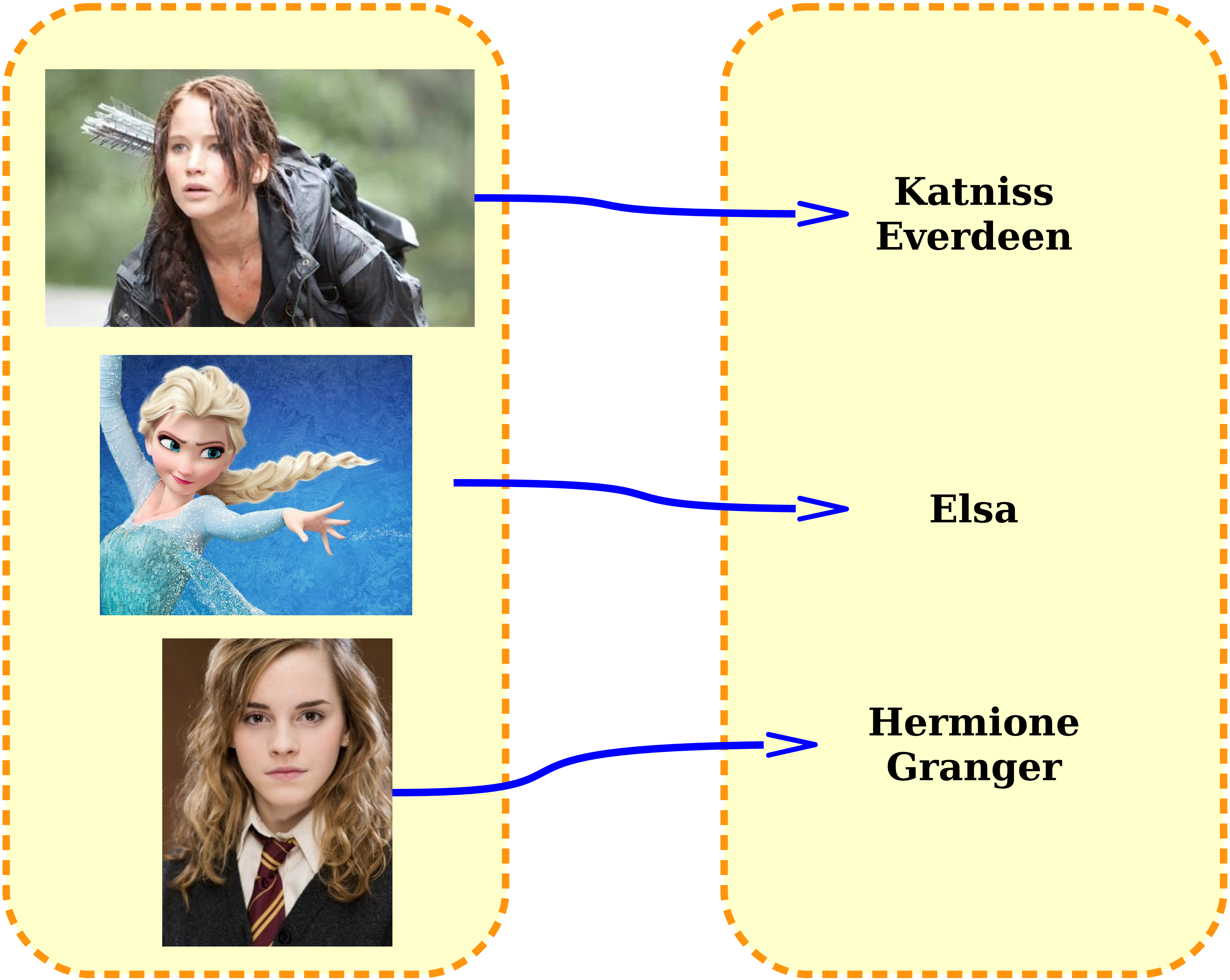
Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?


Katniss Everdeen
Elsa
Hermione Granger

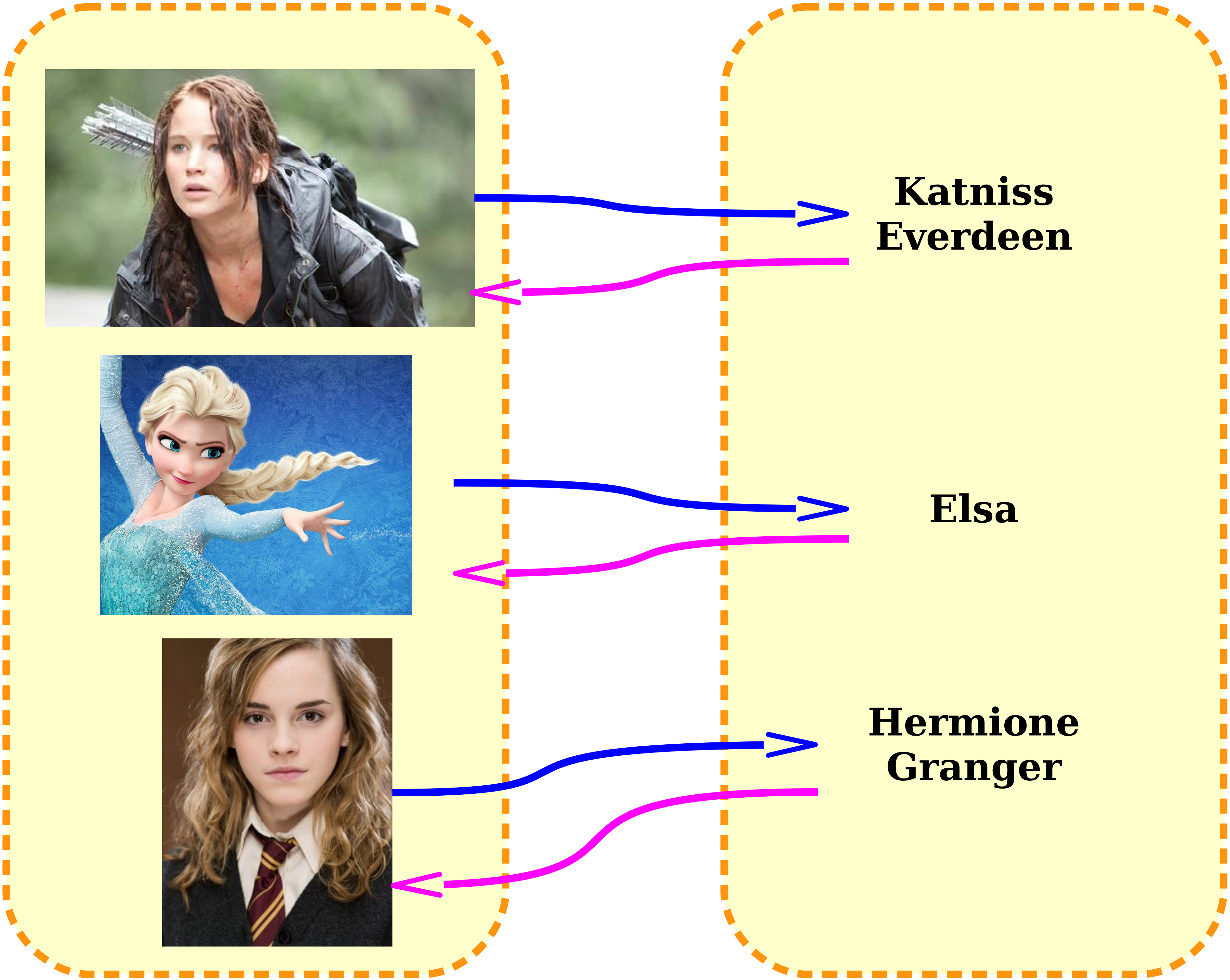
Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- Bijections are sometimes called ***one-to-one correspondences***.
  - Not to be confused with “one-to-one functions.”

Bijections and Composition

- Suppose that $f : A \to B$ and $g : B \to C$ are bijections.
- Is $g \circ f$ necessarily a bijection?
- ***Yes!***
    - Since both f and g are injective, we know that $g \circ f$ is injective.
    - Since both f and g are surjective, we know that $g \circ f$ is surjective.
    - Therefore, $g \circ f$ is a bijection.

Inverse Functions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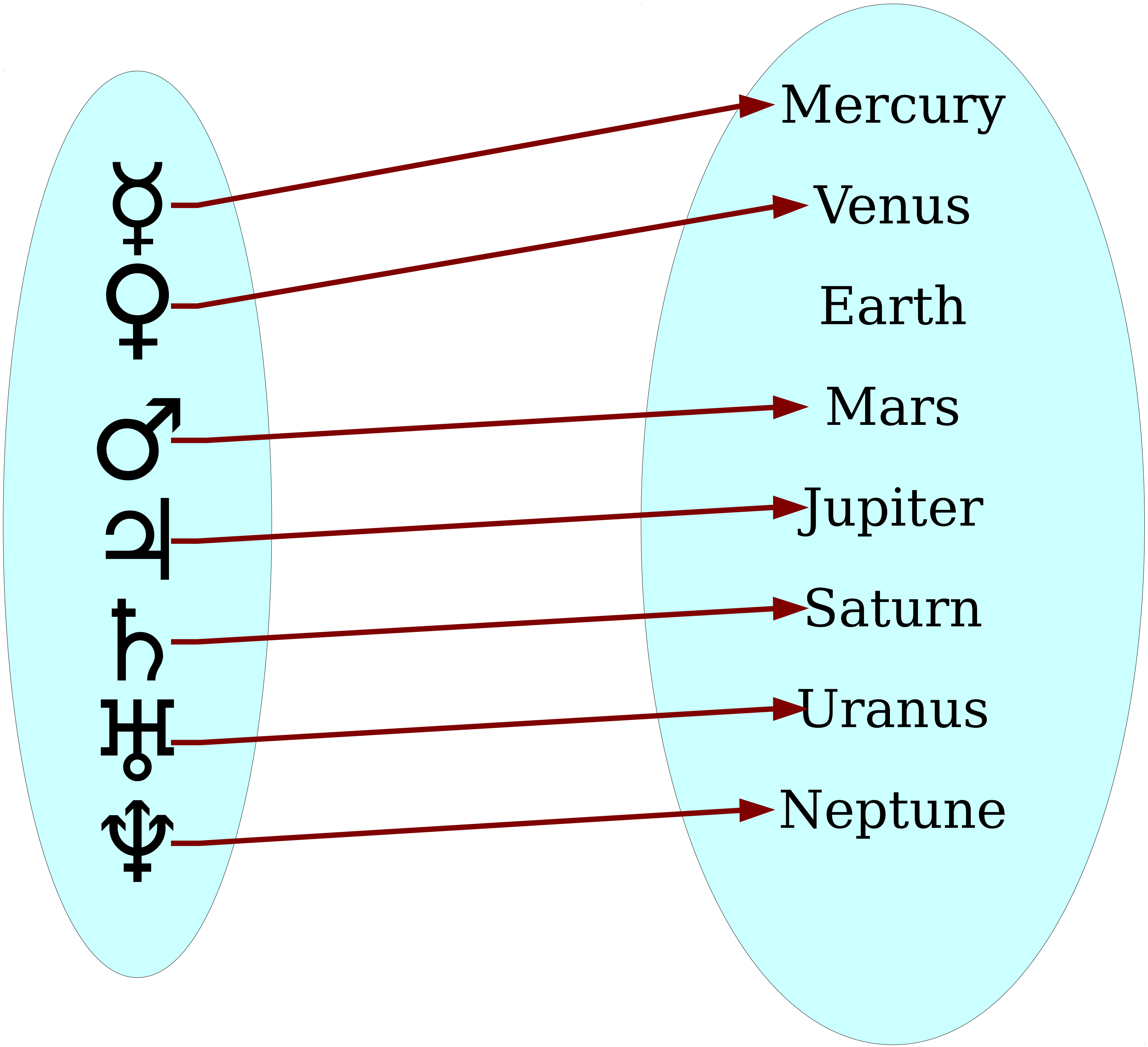
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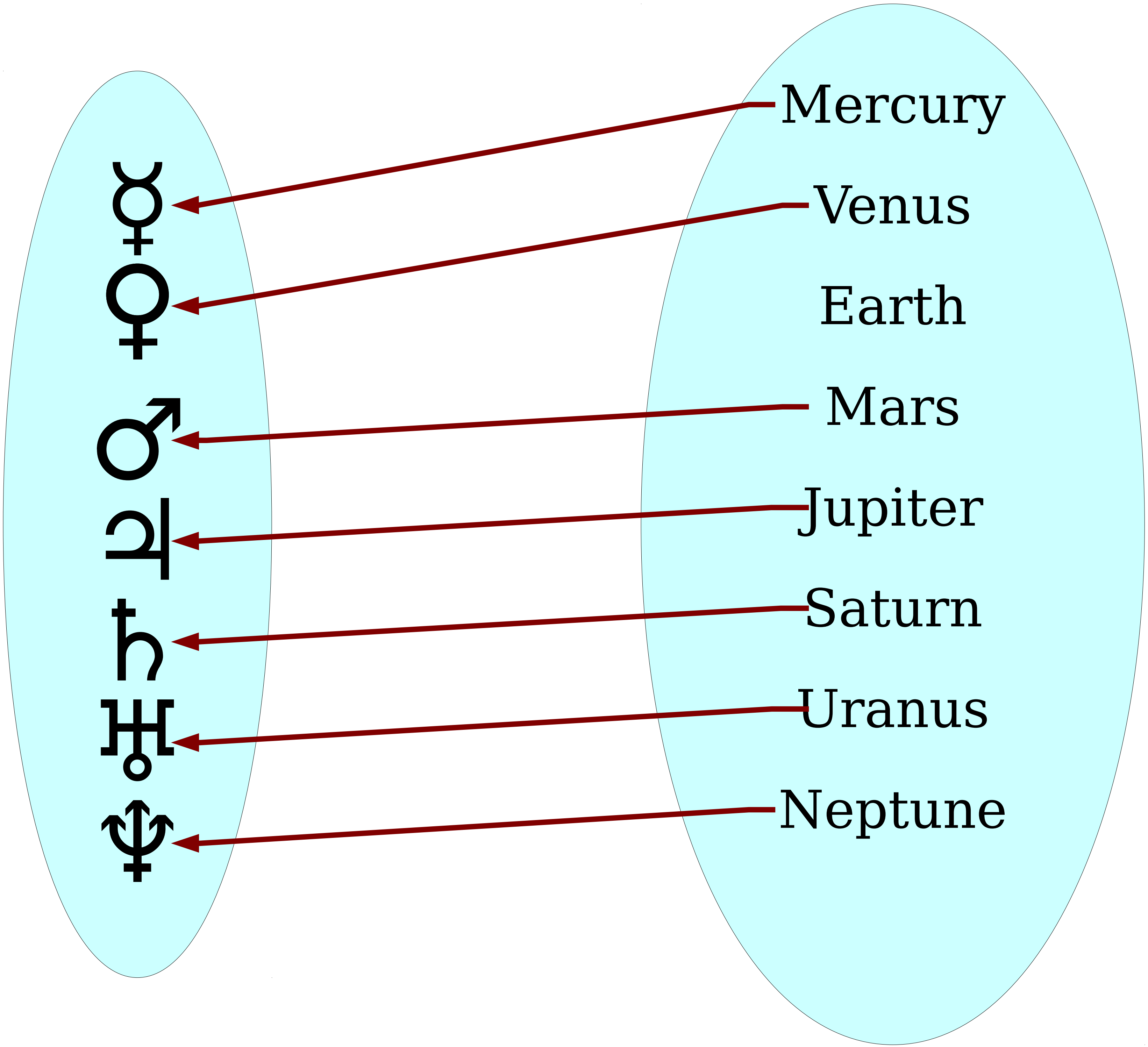
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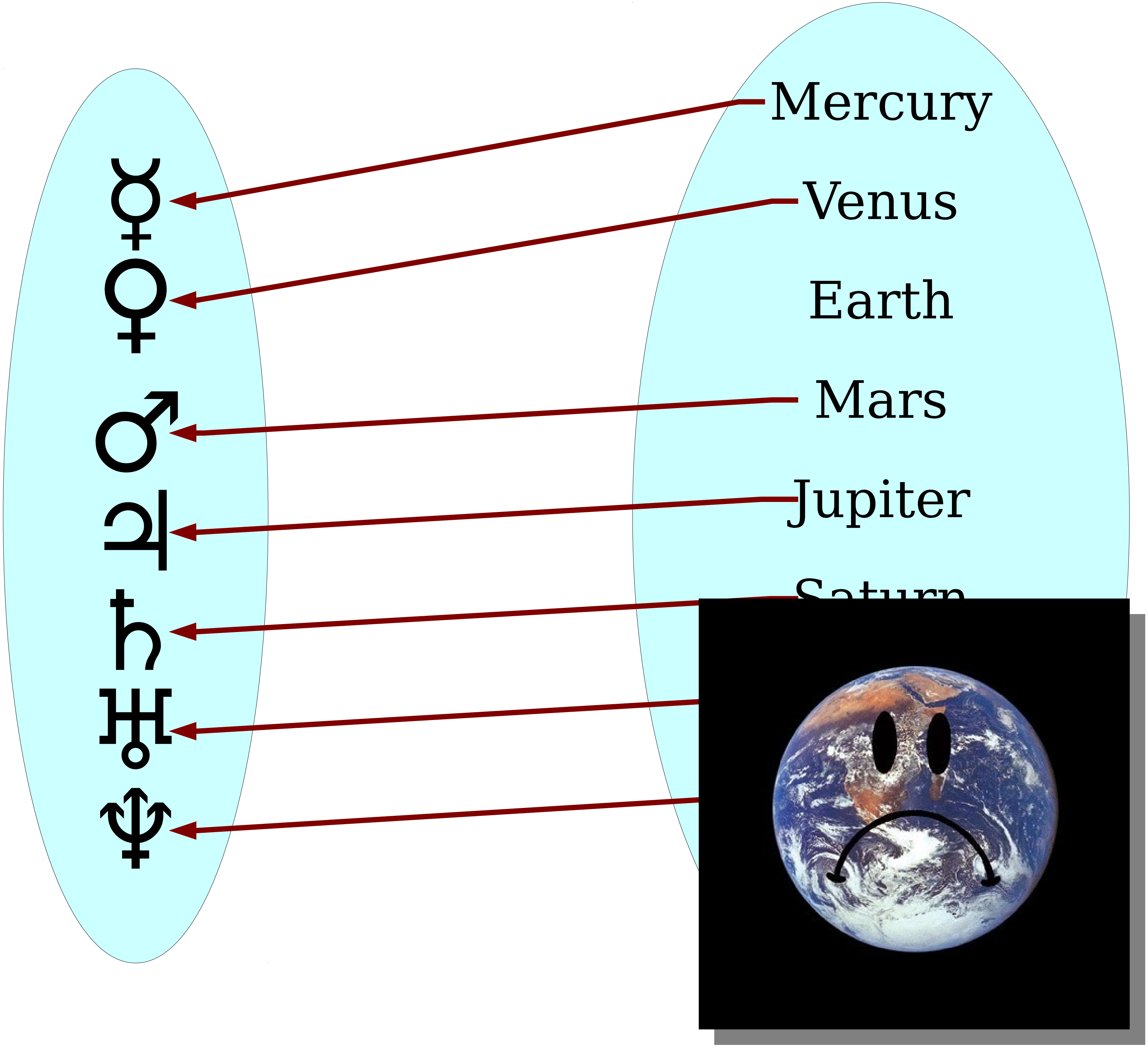

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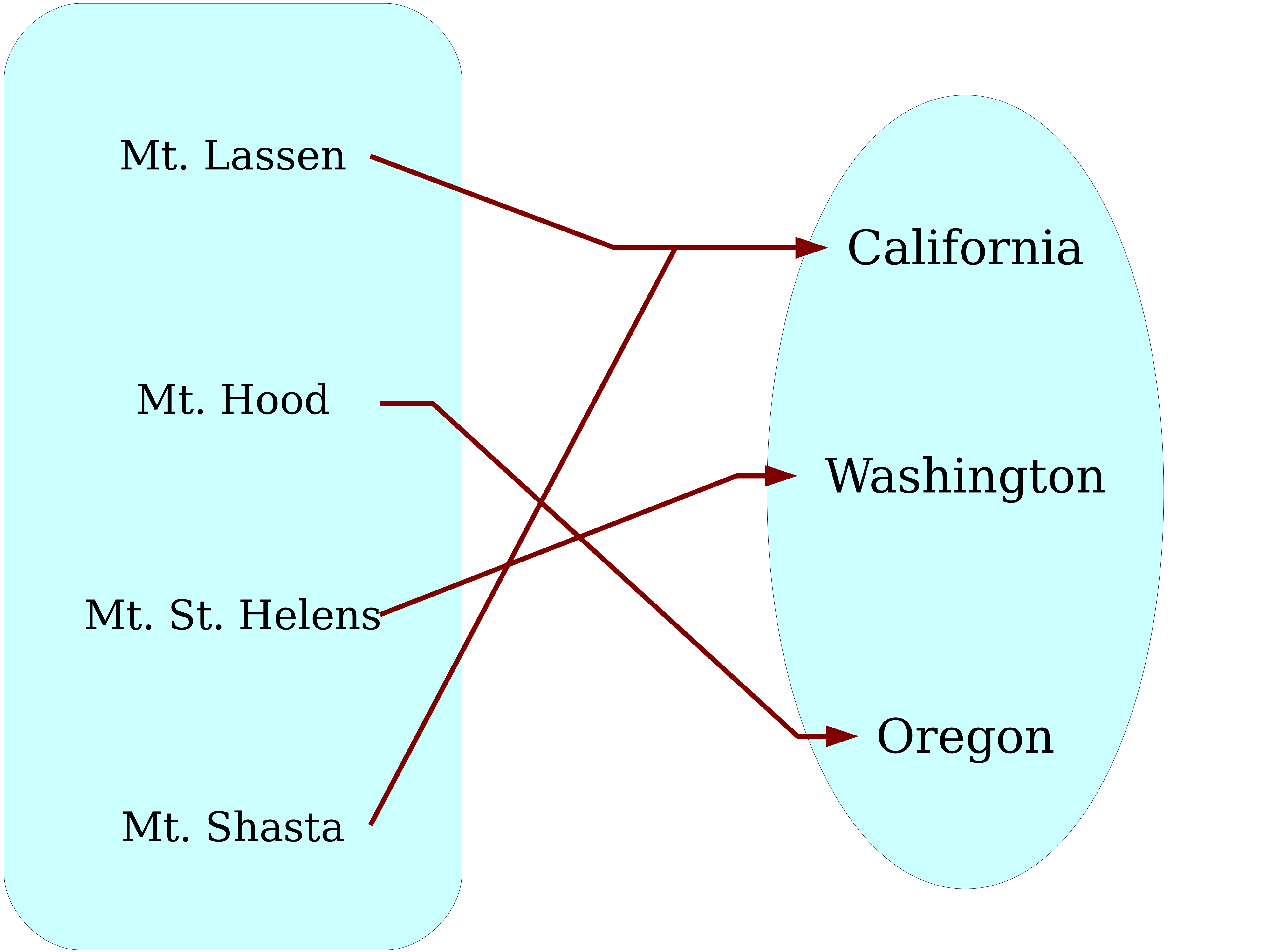
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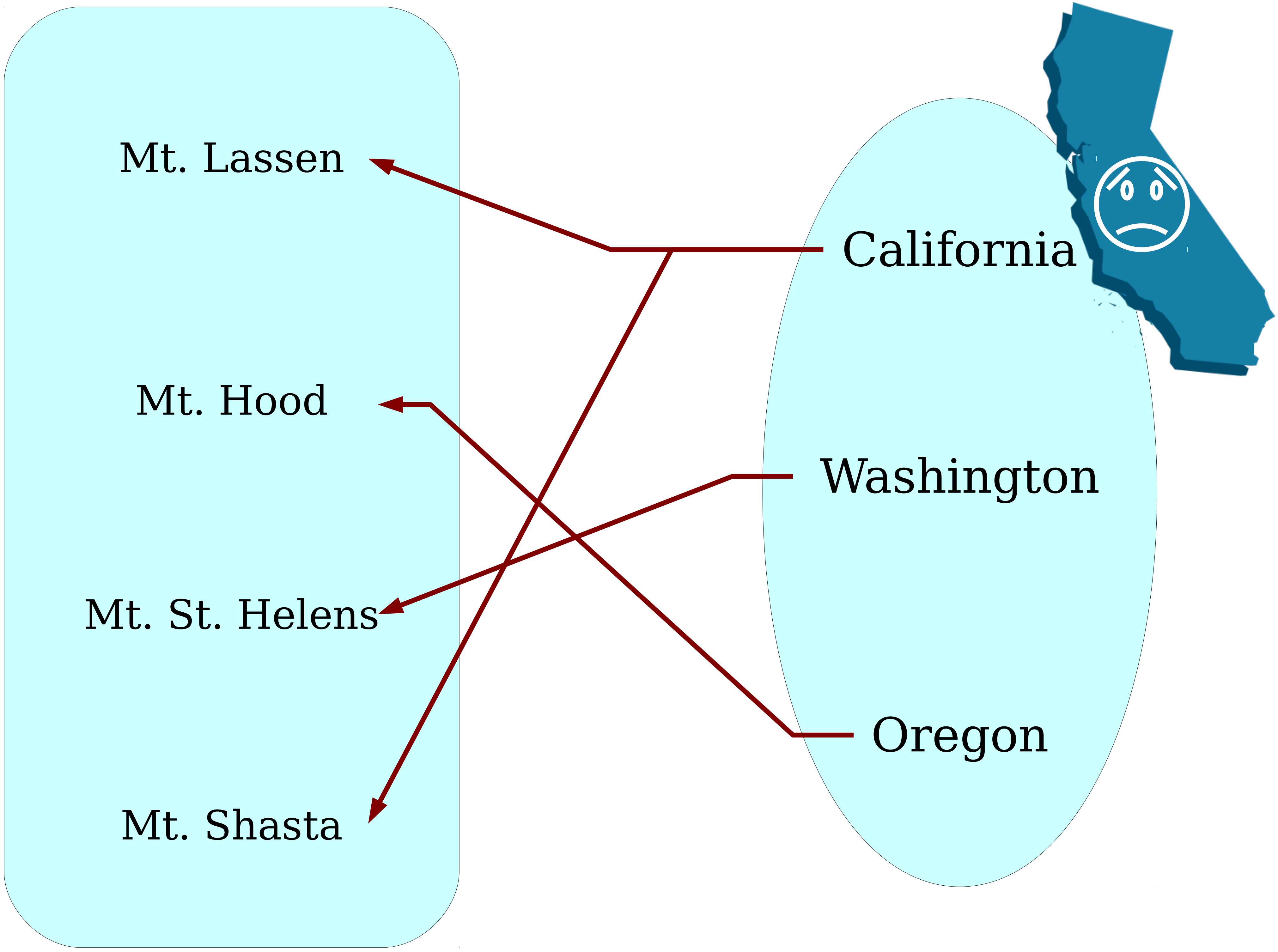

Mt. Lassen
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Inverse Functions

- In some cases, it's possible to "turn a function around."
- Let $f : A \to B$ be a function. A function $f^{-1} : B \to A$ is called an ***inverse of f*** if the following first-order logic statements are true about f and f^{-1}

$$\forall a \in A.\ (f^{-1}(f(a)) = a) \qquad \forall b \in B.\ (f(f^{-1}(b)) = b)$$

- In other words, if f maps a to b, then f^{-1} maps b back to a and vice-versa.
- Not all functions have inverses (we just saw a few examples of functions with no inverses).
- If f is a function that has an inverse, then we say that f is ***invertible***.

Inverse Functions

- ***Theorem:*** Let $f : A \to B$. Then f is invertible if and only if f is a bijection.
- These proofs are in the course reader. Feel free to check them out if you'd like!
- ***Really cool observation:*** Look at the formal definition of a function. Look at the rules for injectivity and surjectivity. Do you see why this result makes sense?

Where We Are

- We now know
  - what an injection, surjection, and bijection are;
  - that the composition of two injections, surjections, or bijections is also an injection, surjection, or bijection, respectively; and
  - that bijections are invertible and invertible functions are bijections.
- You might wonder why this all matters. Well, there's a good reason…

Problem Set Three

- The Problem Set Three checkpoint problem was due at 2:30PM today.
  - We’ll aim to get feedback to you by Wednesday.
  - Solutions are now available.
- The remaining problems are due on Friday at 2:30PM.
- As always, feel free to ask questions on Piazza or to stop by office hours with questions!
- PS2 solutions available tonight. We’ll get your work graded and returned by Wednesday.

Next Time

- ***Cardinality, Formally***
  - How do we rigorously define the idea that two sets have the same size?
- ***The Nature of Infinity***
  - It’s even weirder than you think!
- ***Cantor’s Theorem Revisited***
  - A formal proof of a major result!