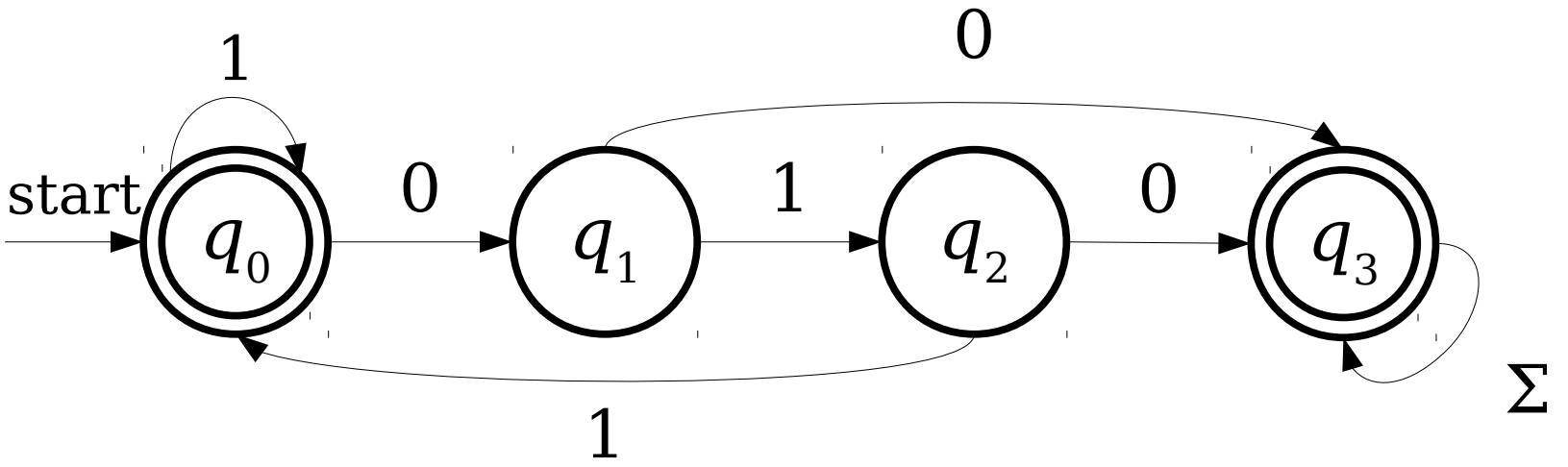
Finite Automata

Part Three

Recap from Last Time

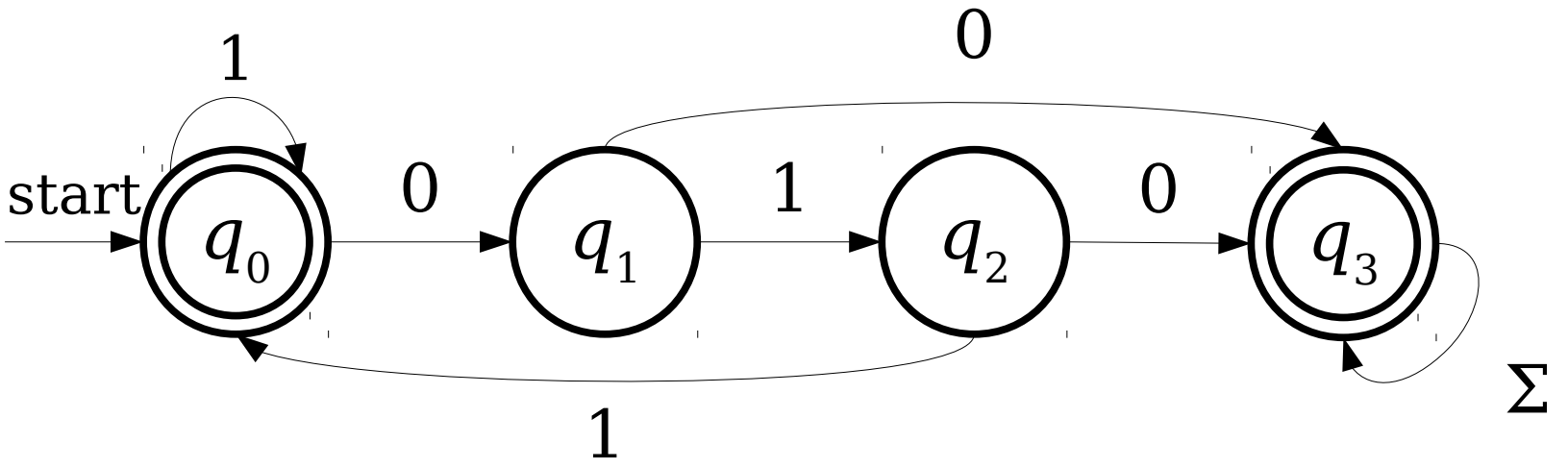
Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

These stars indicate accepting states.

Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

Since this is the first row, it's the start state.

If D is a DFA, the ***language of D***, denoted $\mathscr{L}(D)$, is $\{ w \in \Sigma^* \mid D \text{ accepts } w \}$.

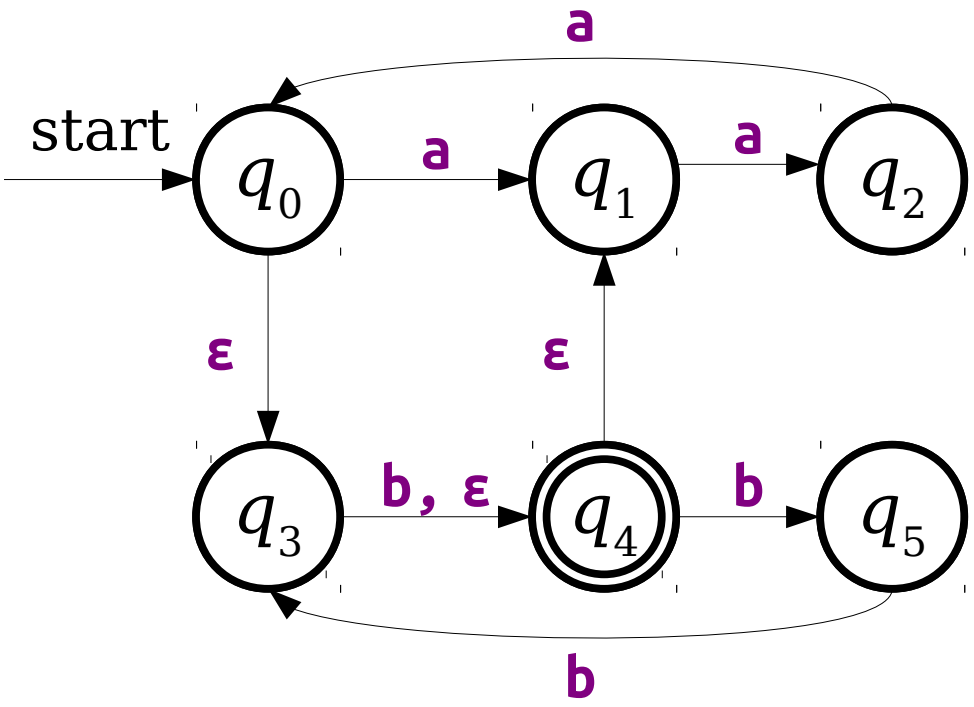
A language L is called a ***regular language*** if there exists a DFA D such that $\mathscr{L}(D) = L$.

NFAs

- An ***NFA*** is a
  - ***N***ondeterministic
  - ***F***inite
  - ***A***utomaton
- Can have missing transitions or multiple transitions defined on the same input symbol.
- Accepts if *any possible series of choices* leads to an accepting state.

ε-Transitions

- NFAs have a special type of transition called the **ε-transition**.
- An NFA may follow any number of ε-transitions at any time without consuming any input.



Massive Parallelism

- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.
- The NFA accepts if *any* of the states that are active at the end are accepting states. It rejects otherwise.

Just how powerful *are* NFAs?

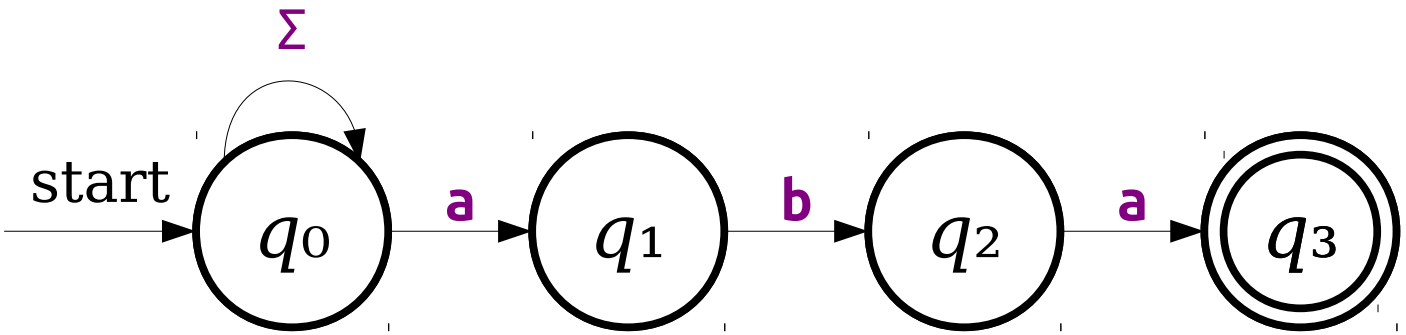
New Stuff!

NFAs and DFAs

- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question:*** Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is ***yes***!

Thought Experiment:

How would you simulate an NFA in software?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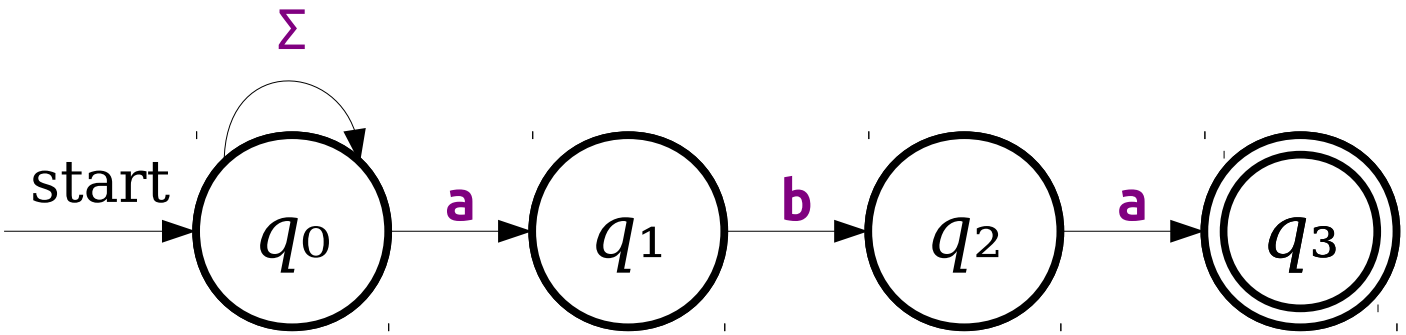
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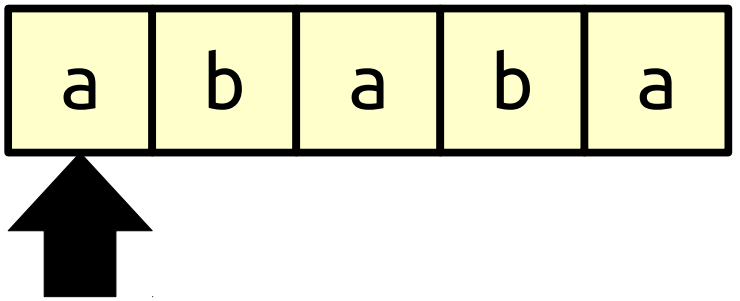
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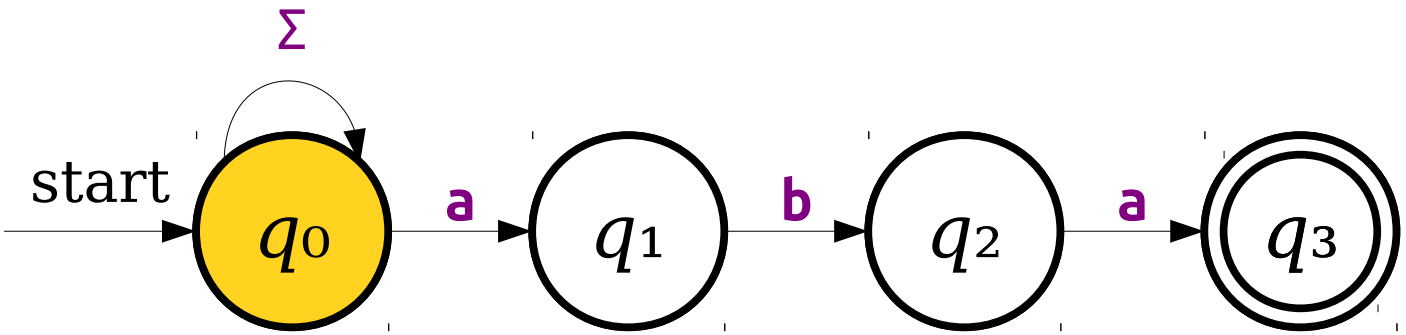
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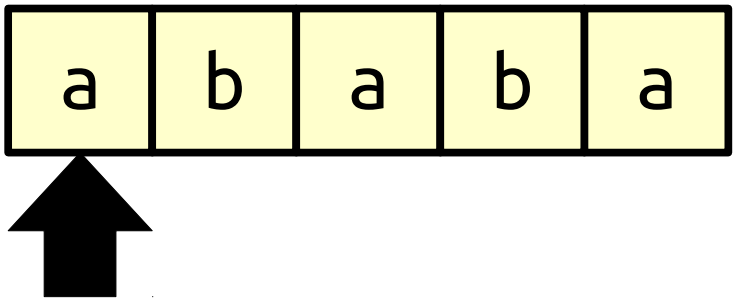
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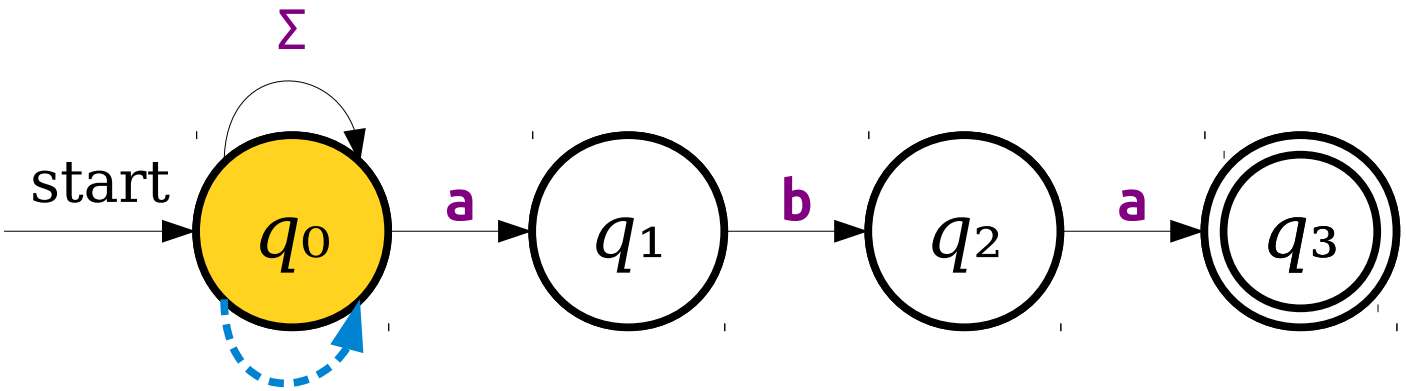
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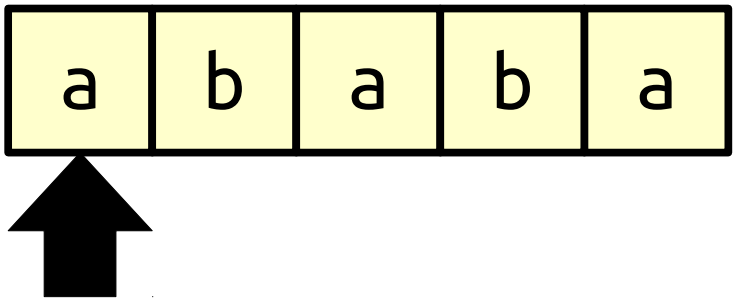
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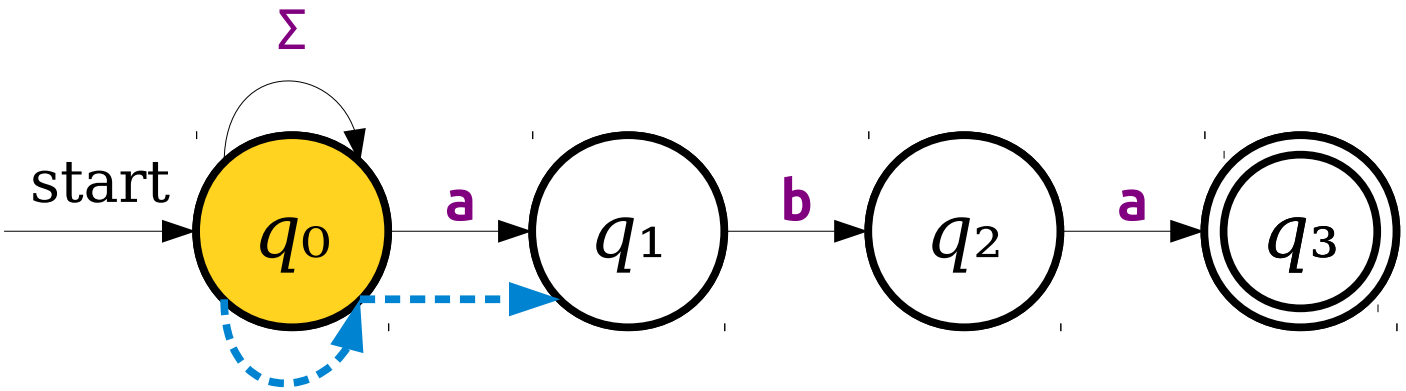
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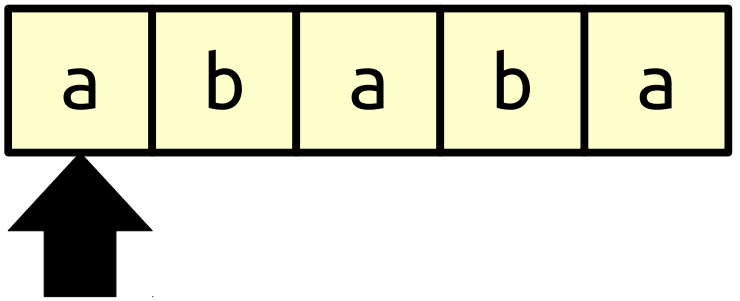
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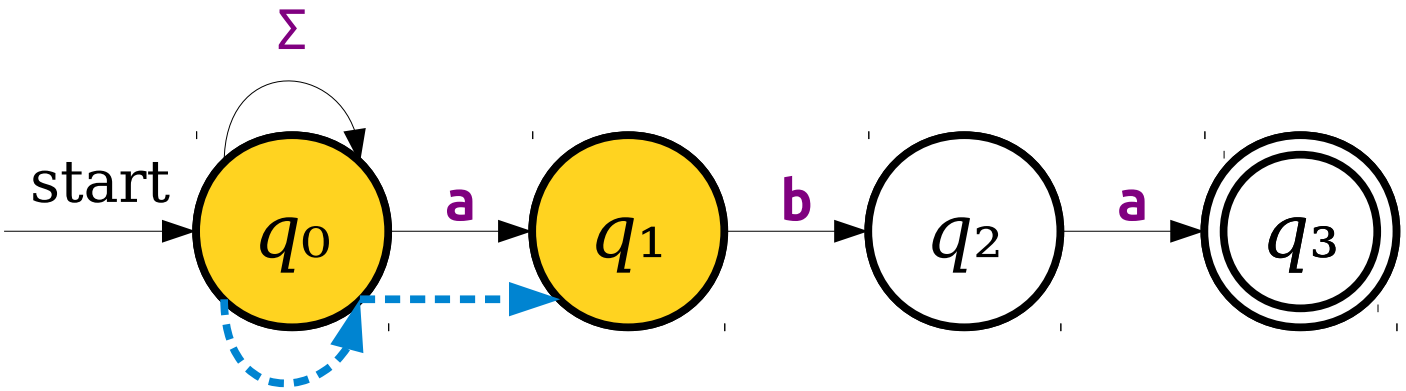
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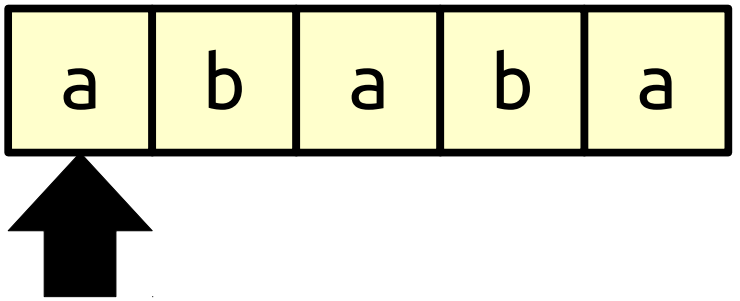
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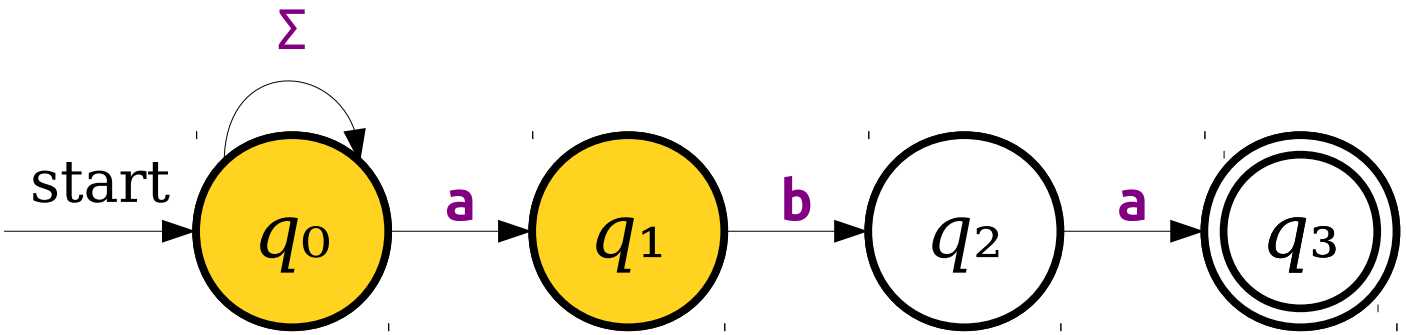
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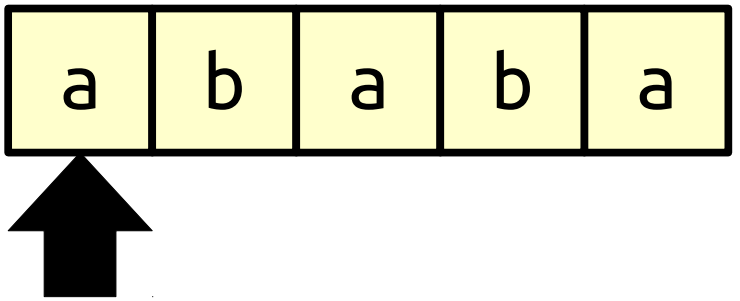
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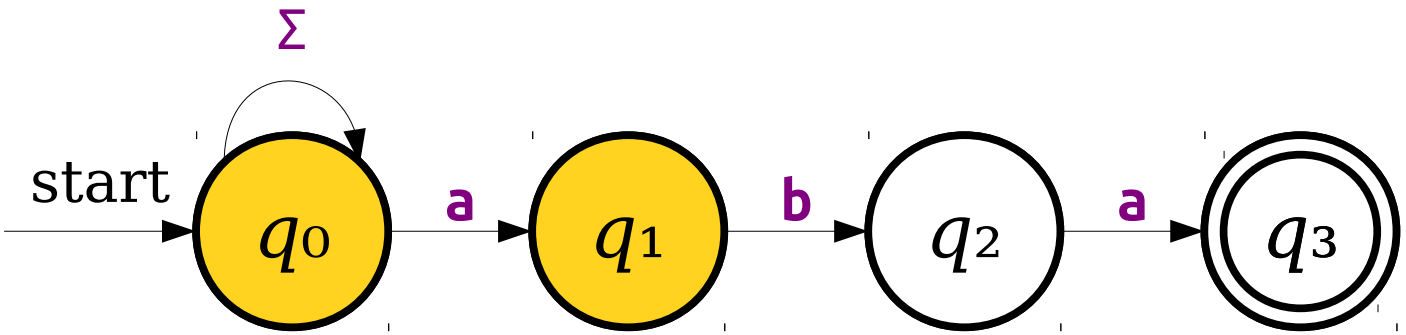
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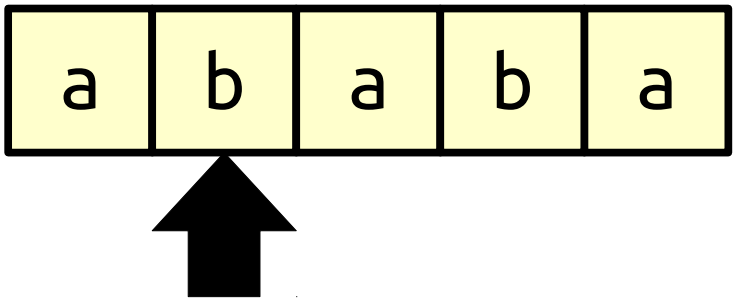
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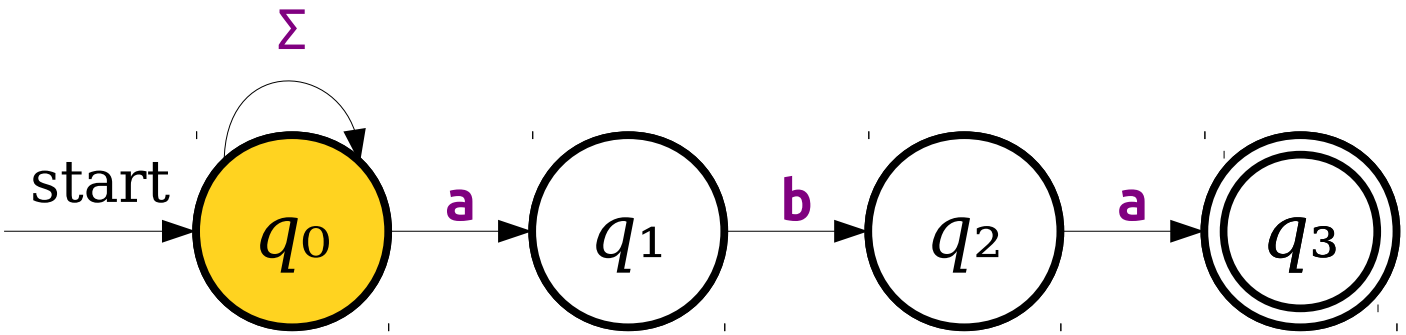
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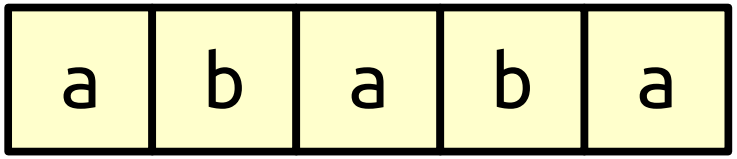
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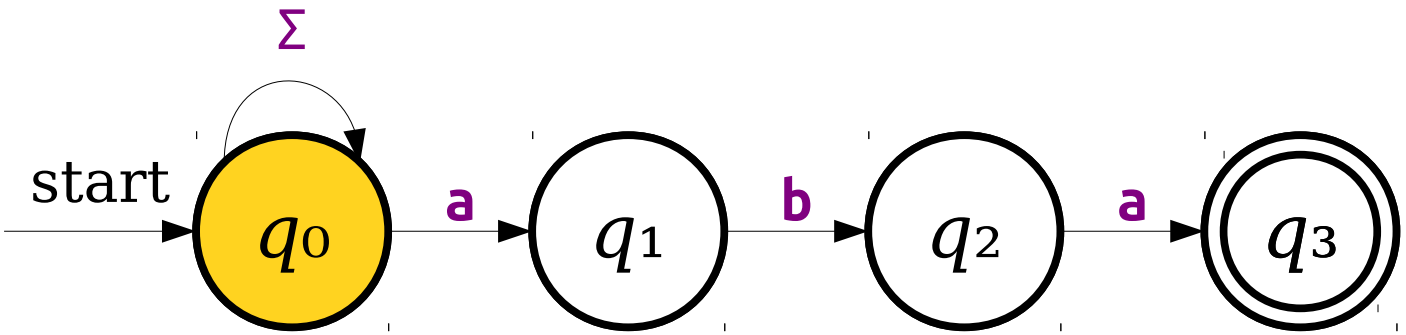
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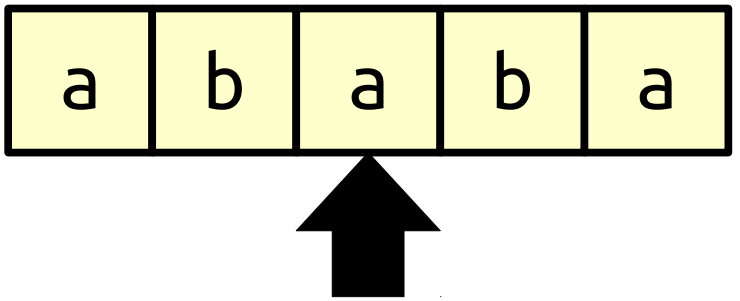
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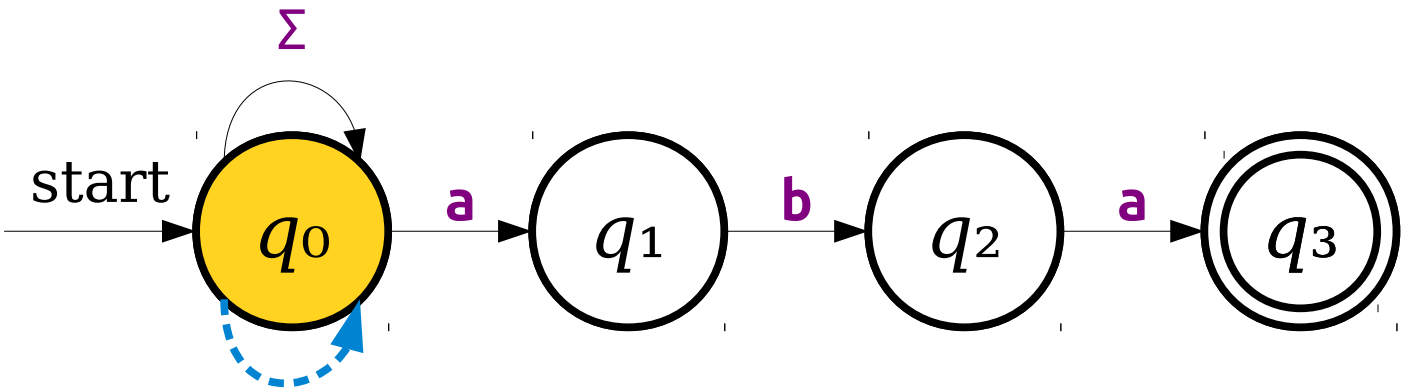
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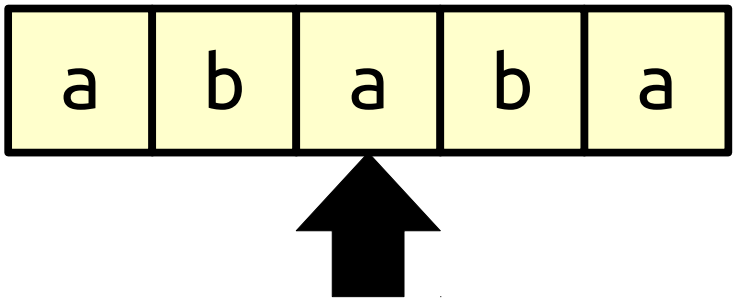
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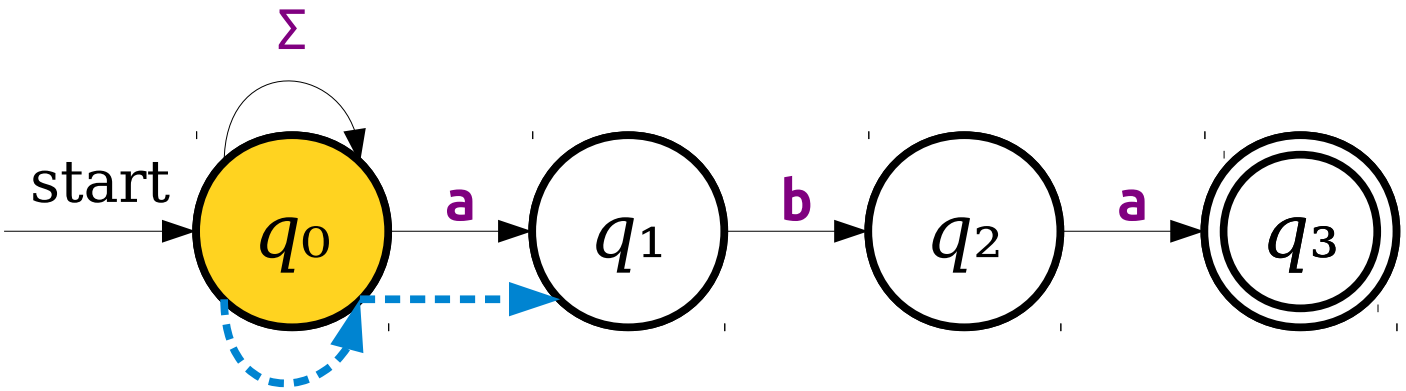
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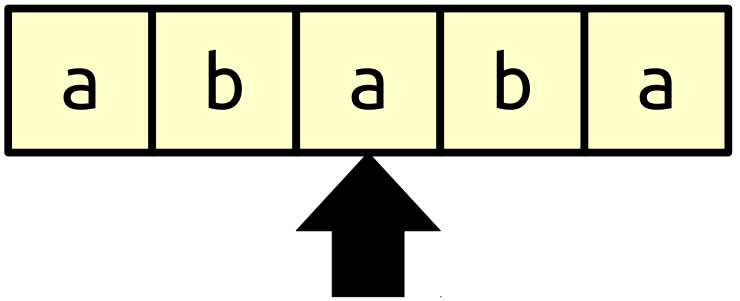
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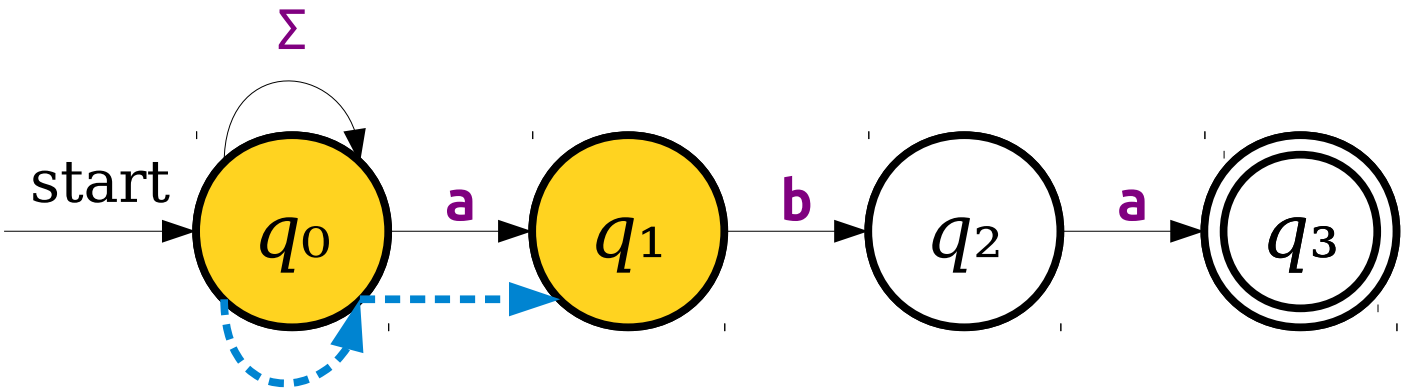
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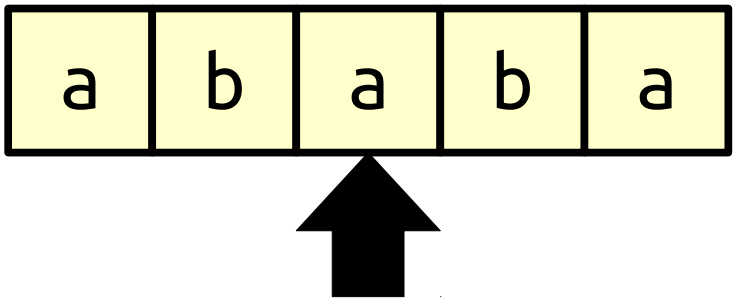
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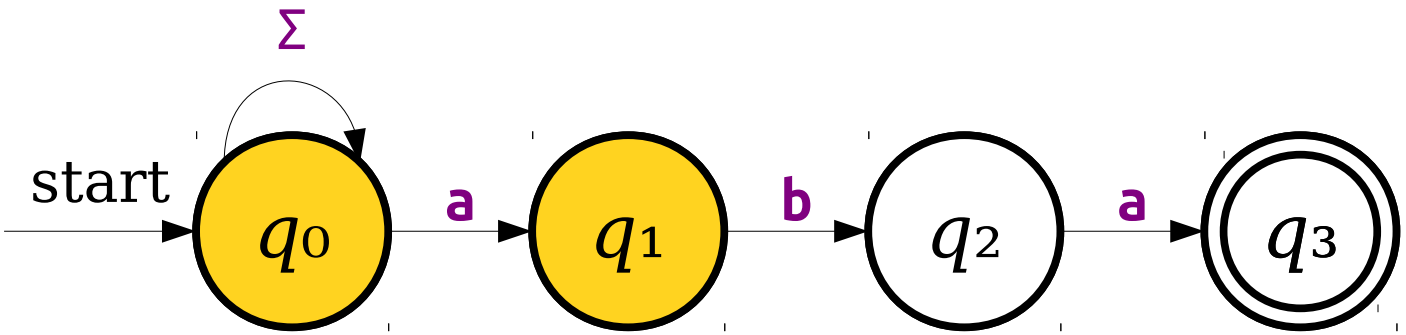
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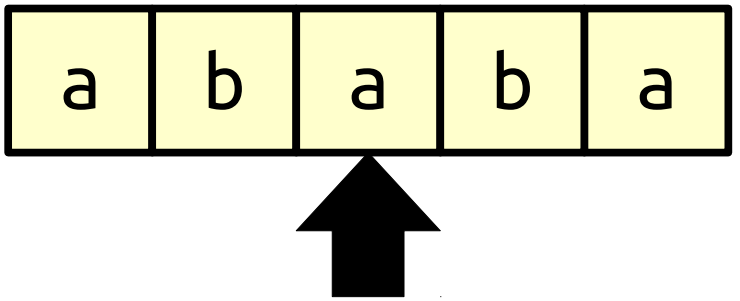
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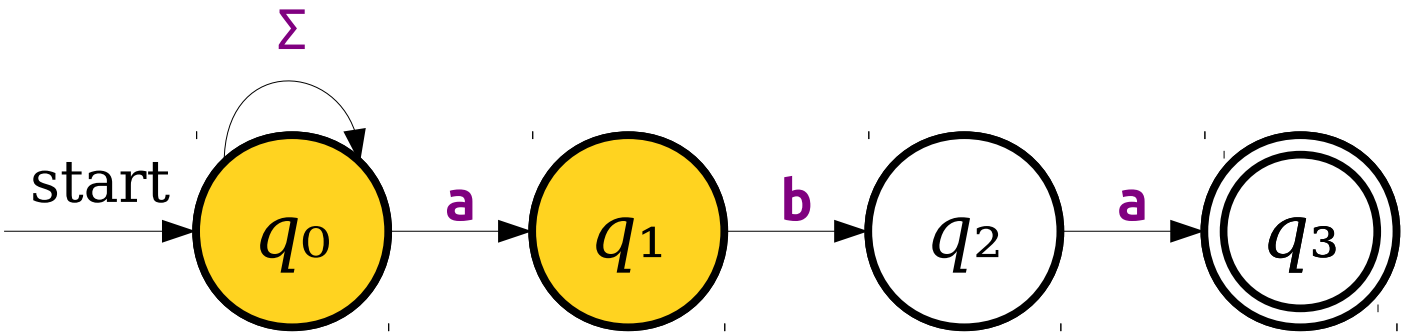
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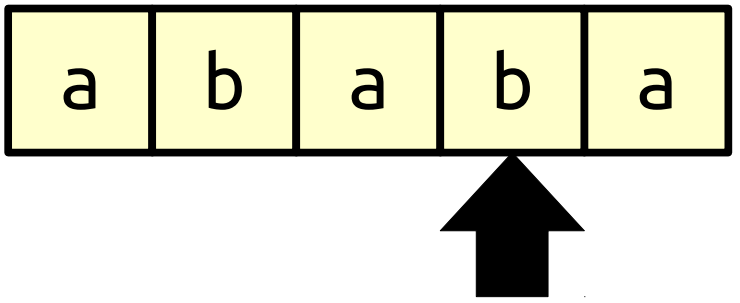
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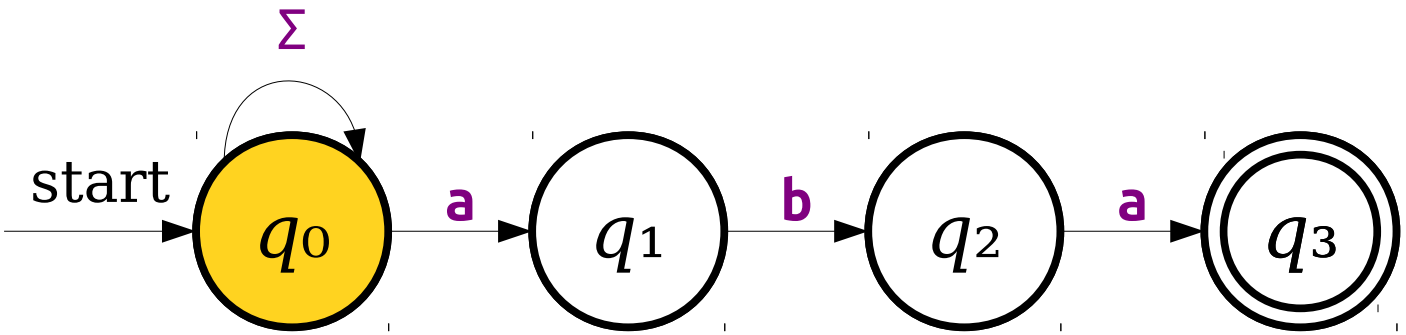
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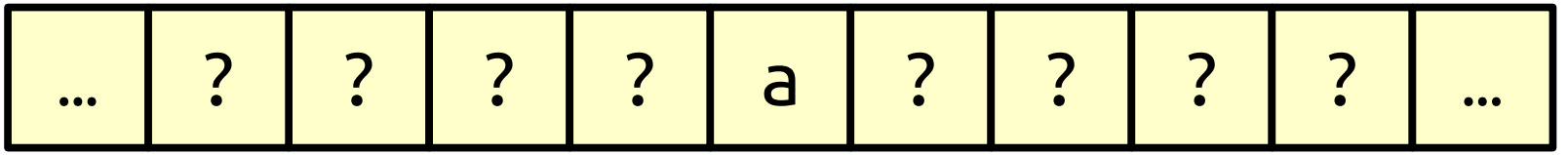
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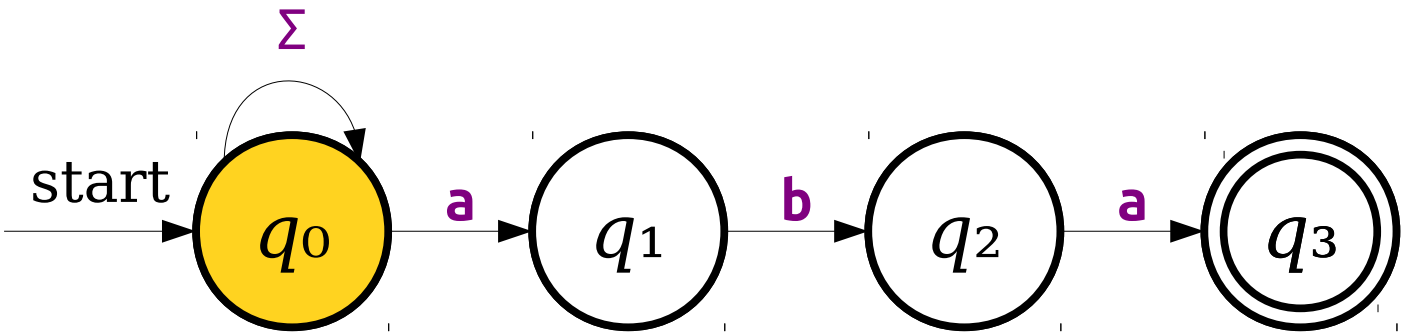
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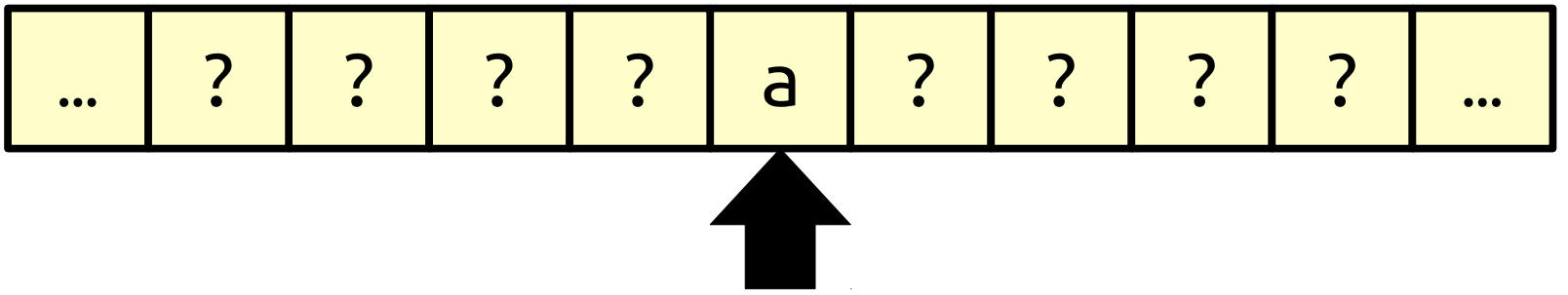
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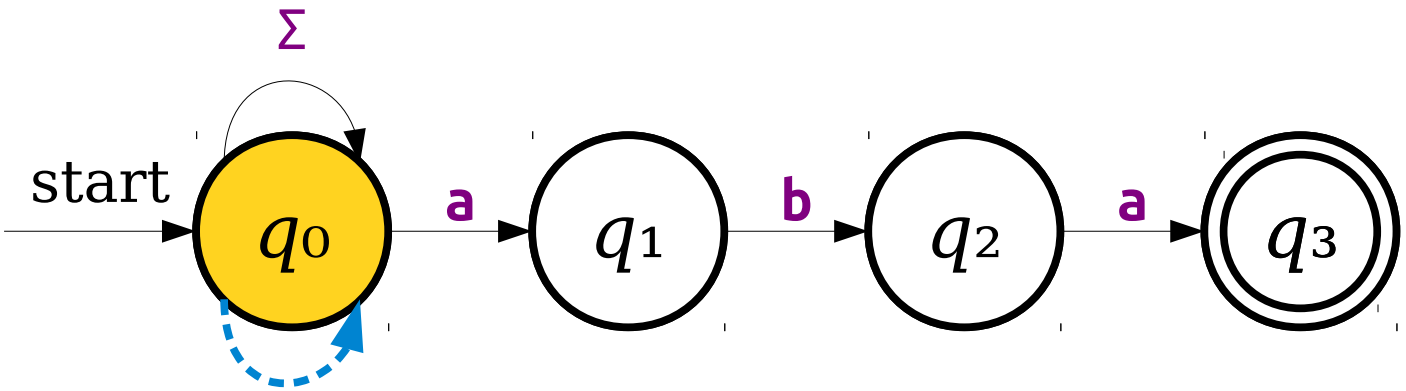
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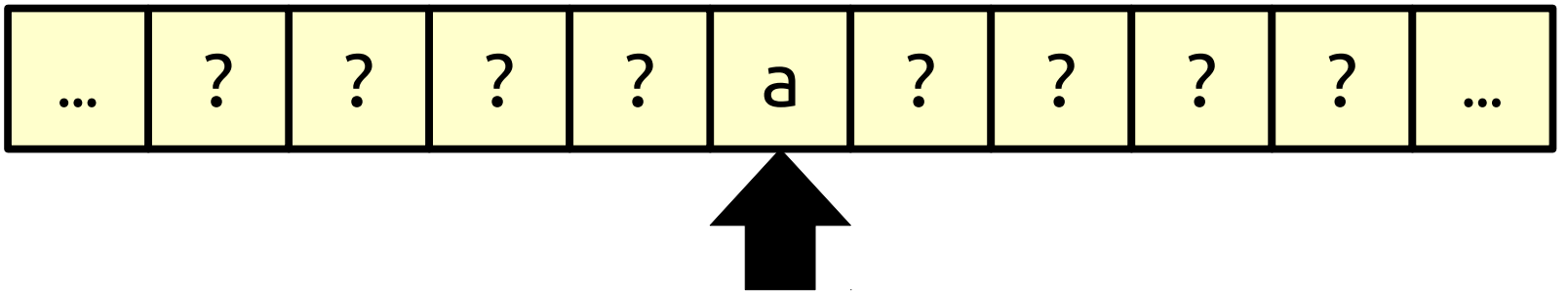
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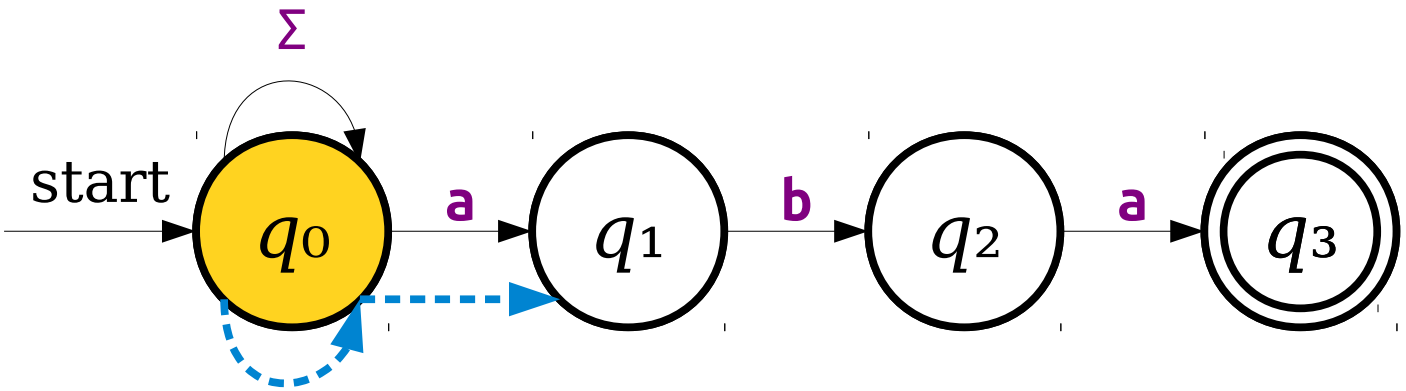
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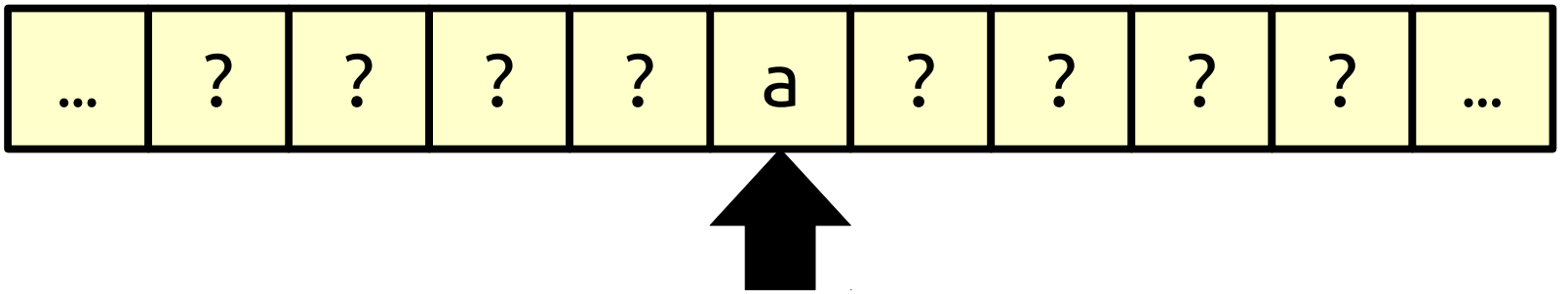
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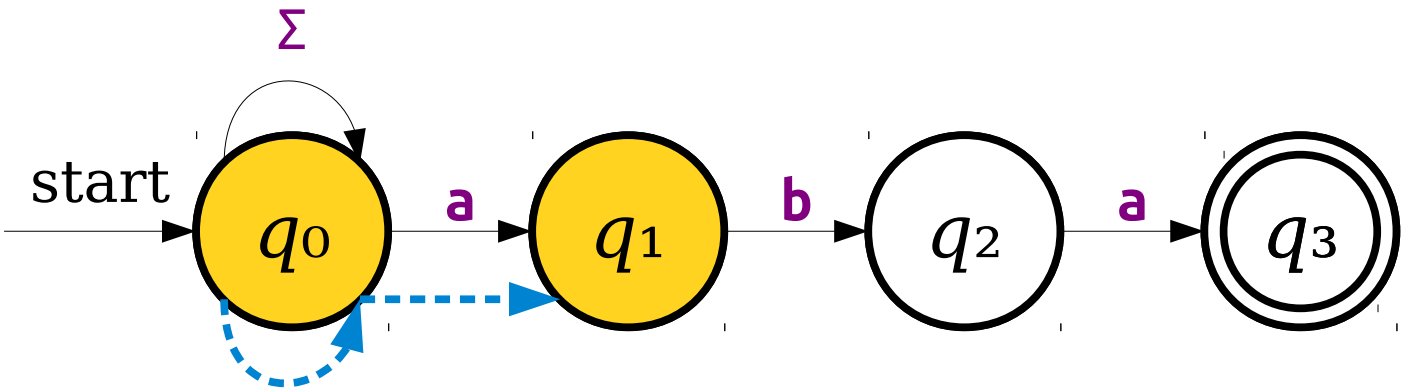
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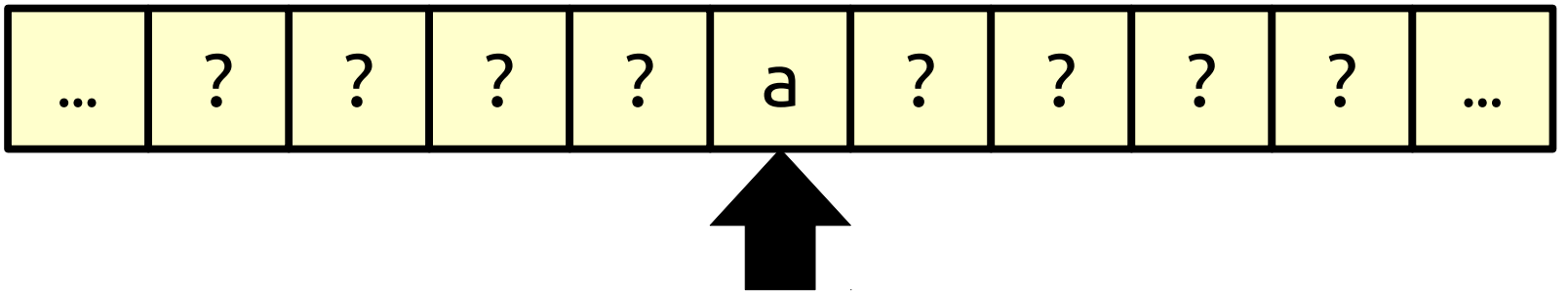
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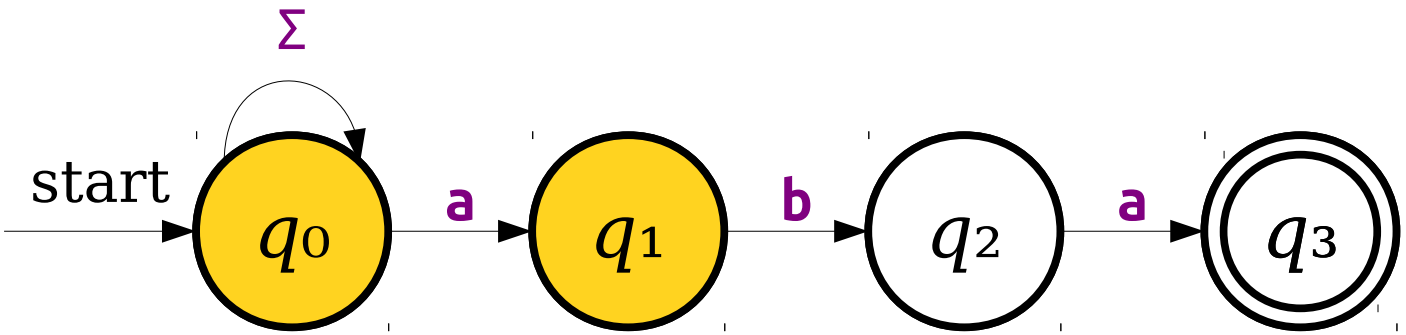
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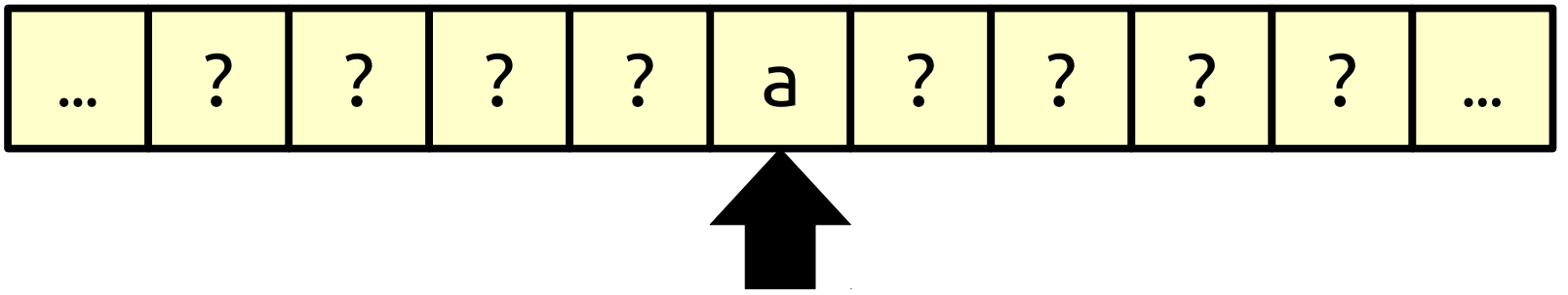
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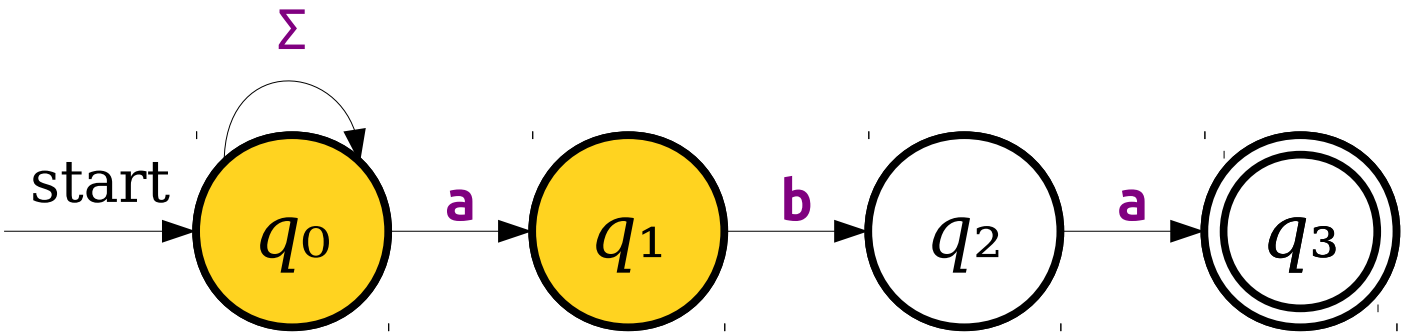
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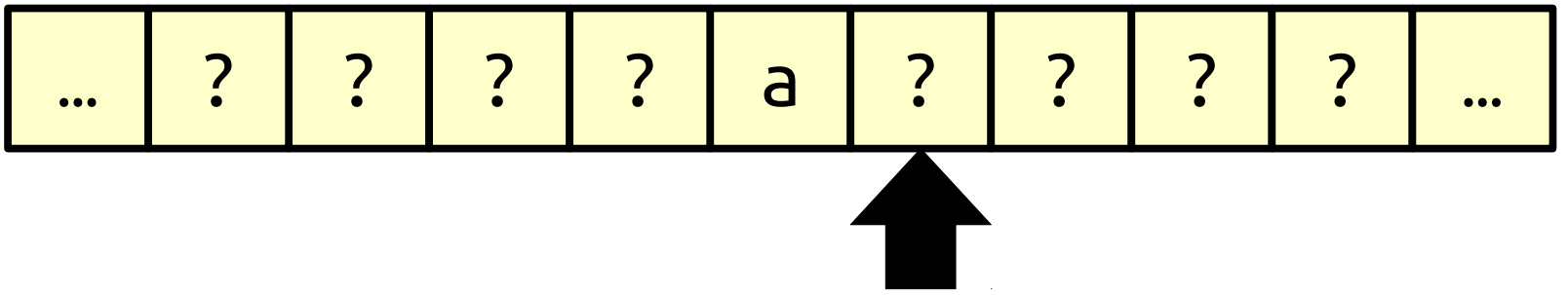
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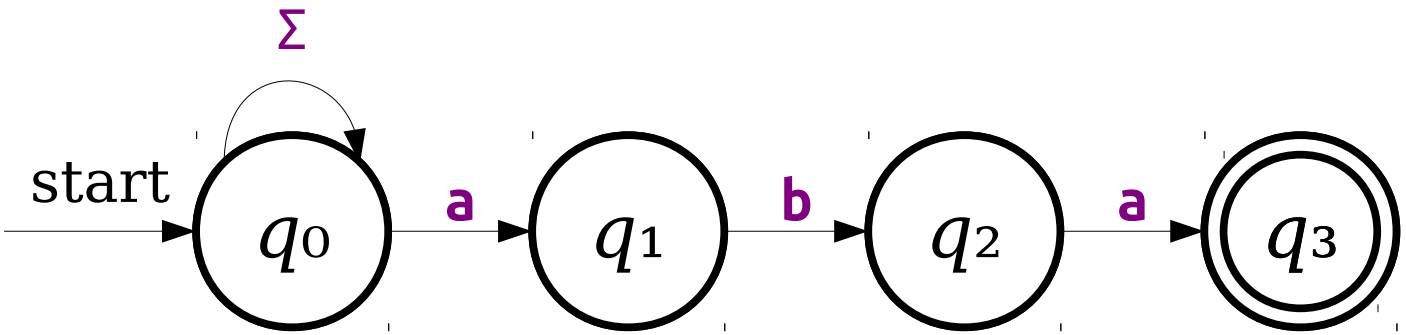
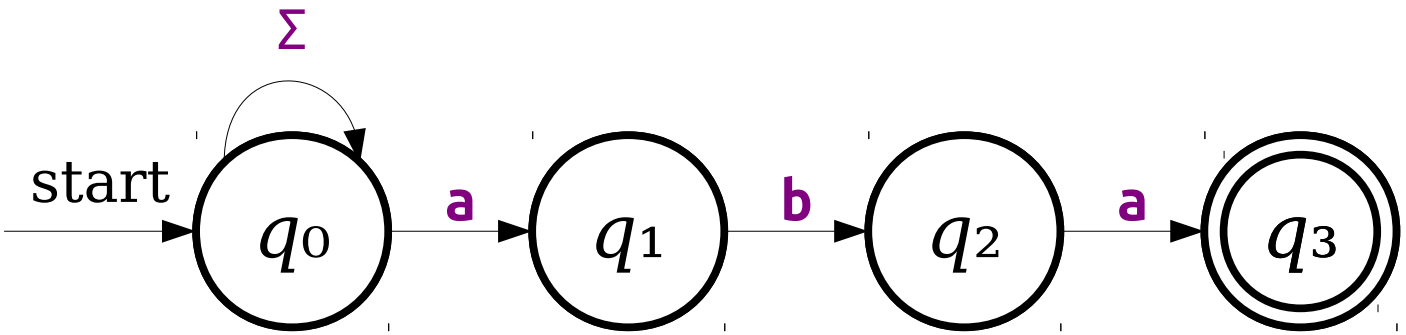


	a
$\{q_0\}$	$\{q_0, q_1\}$



	a	b
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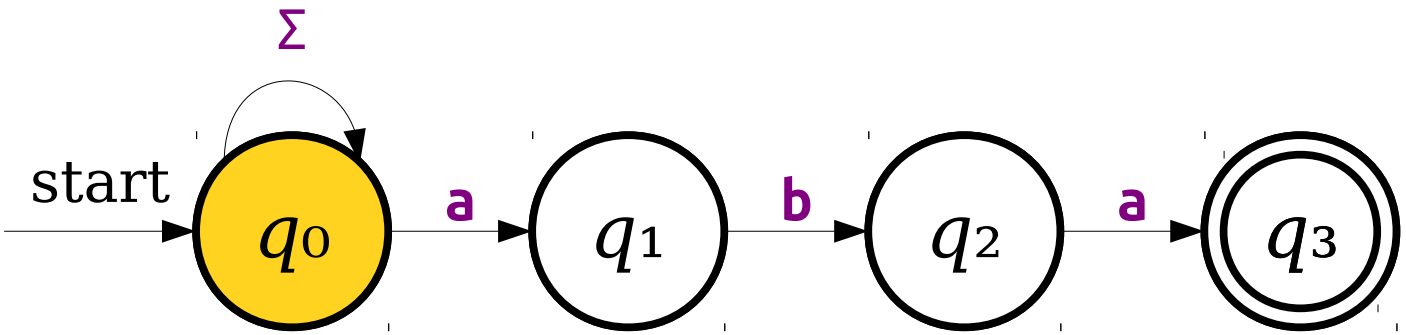

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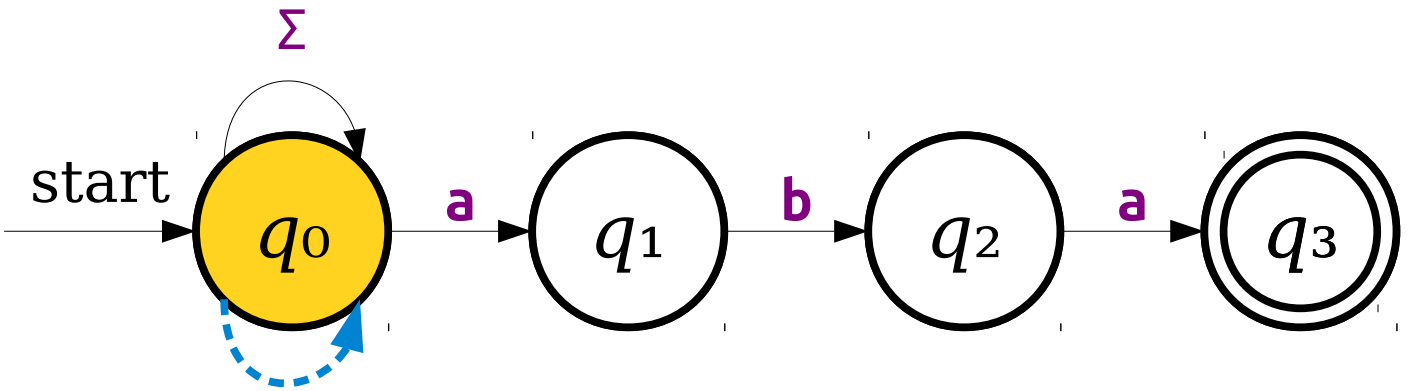

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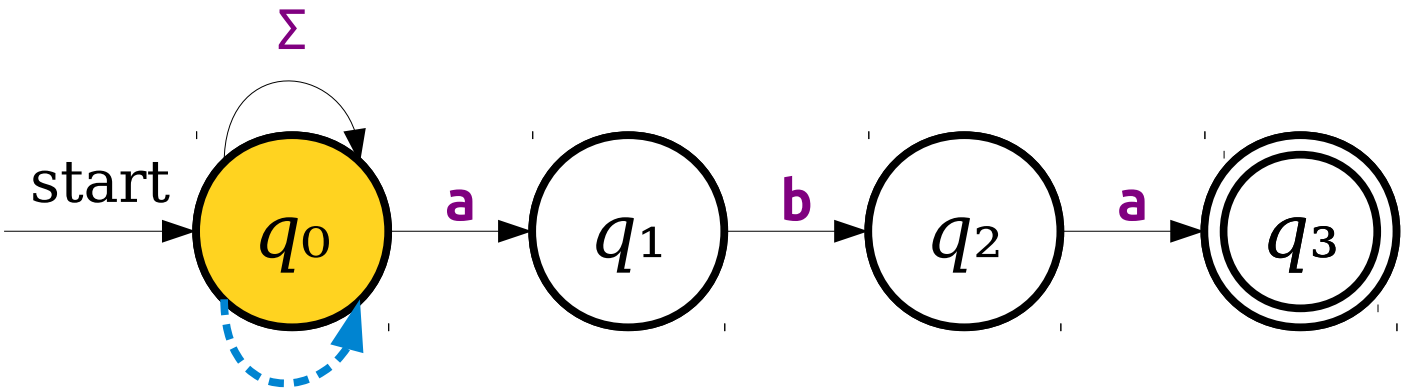

	a	b
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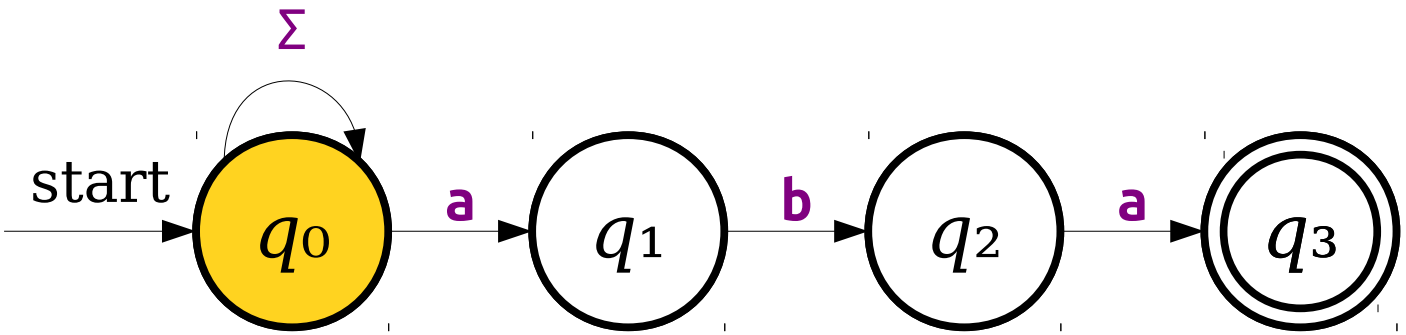

	a	b
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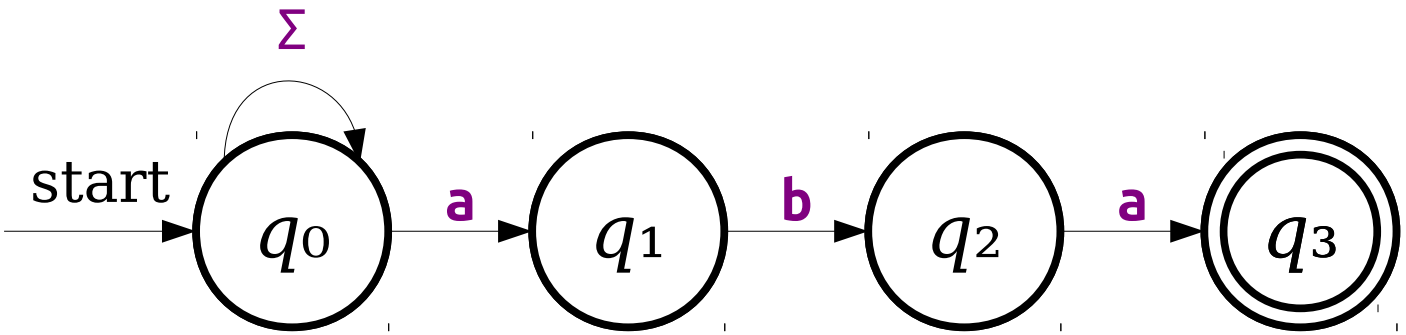

	a	b
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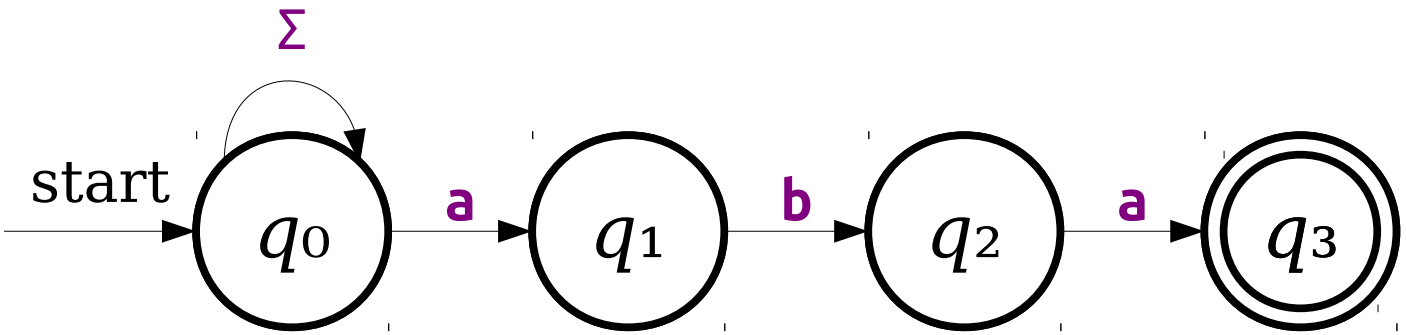

	a	b
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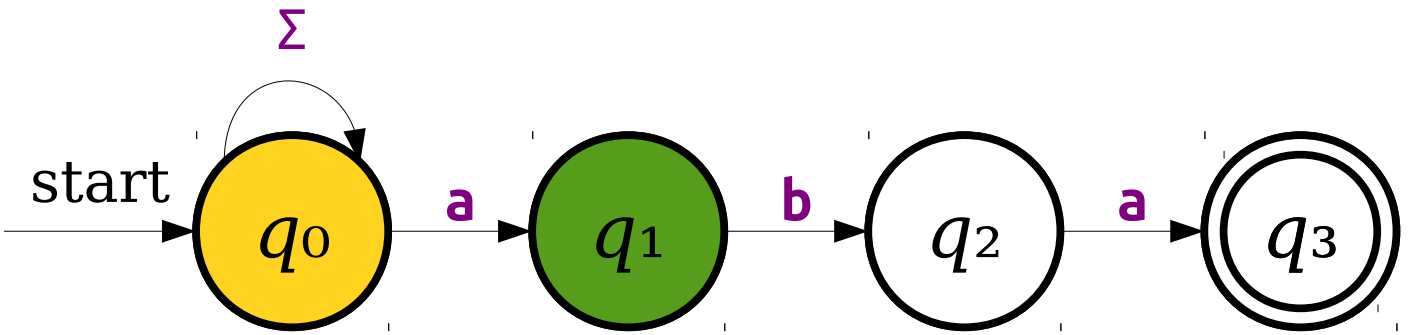

	a	b
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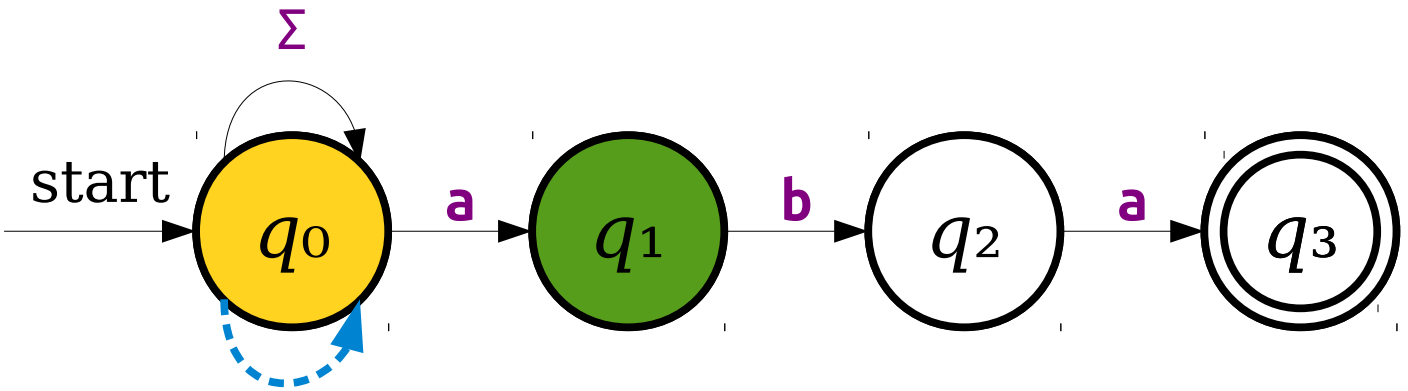

	a	b
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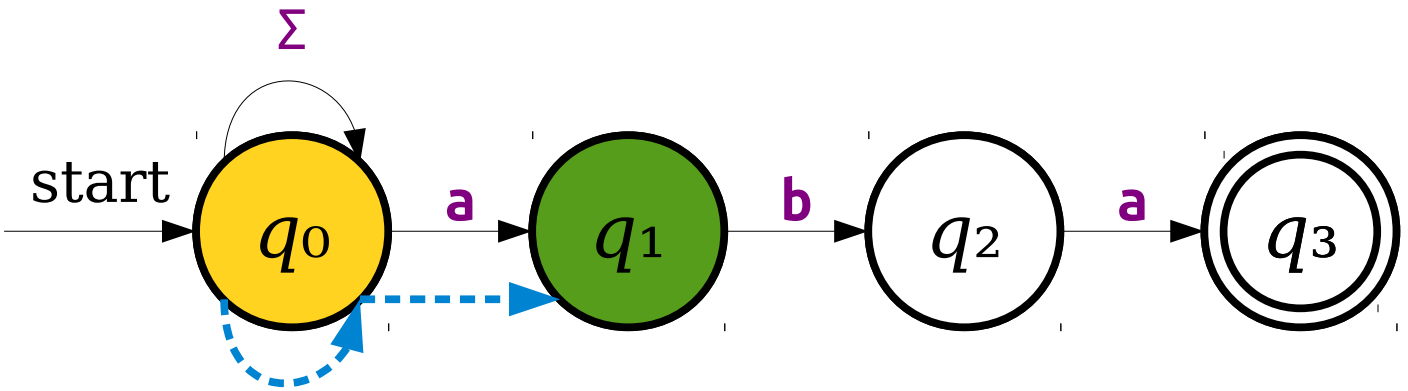

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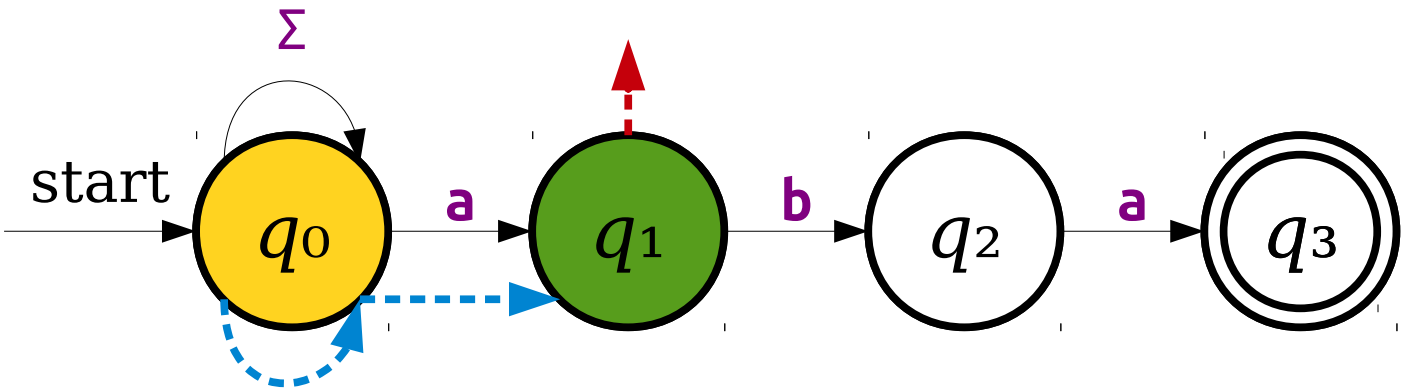
$\{q_0, q_1\}$		



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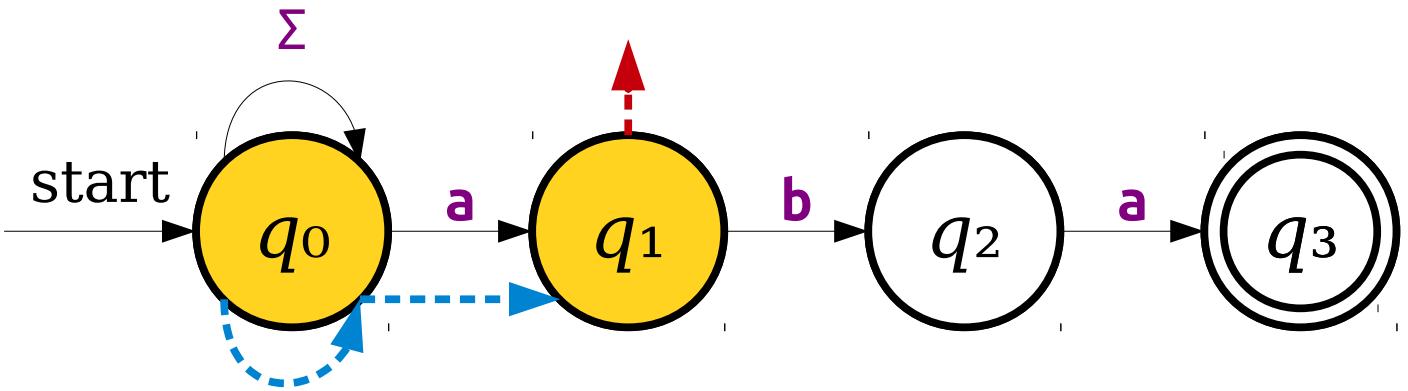

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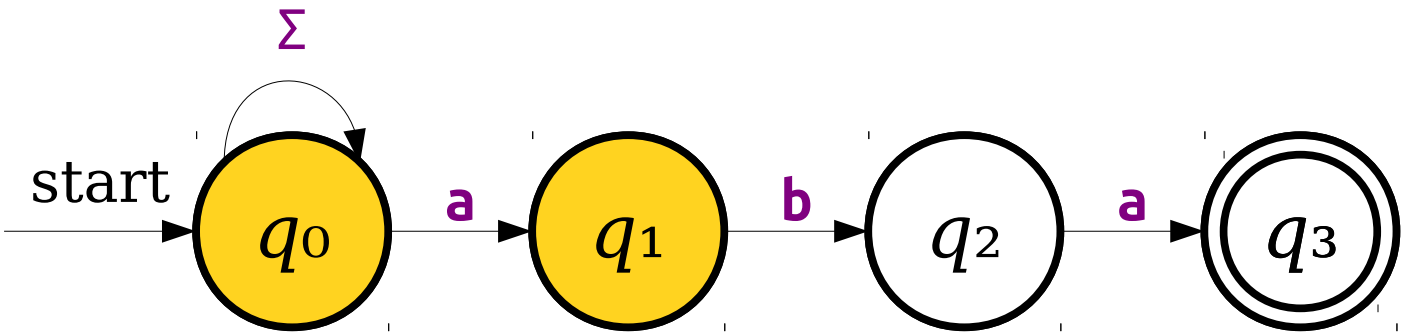

	a	b
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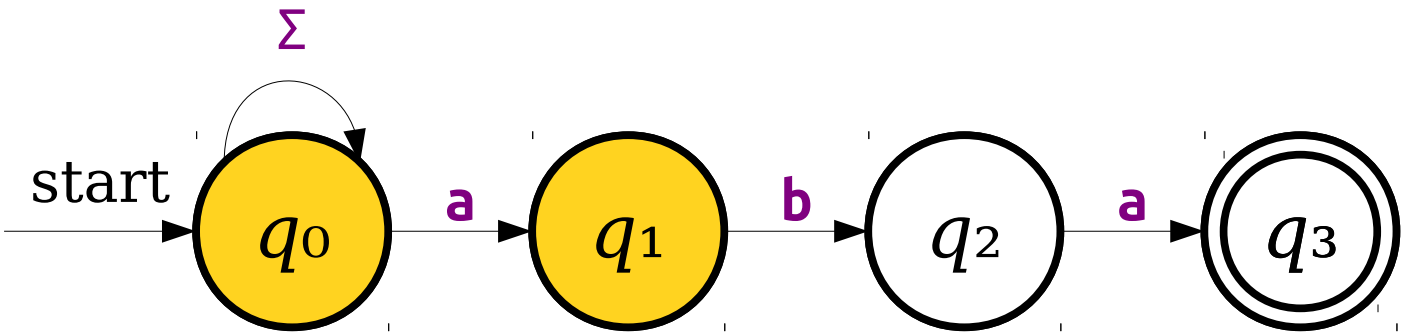

	a	b
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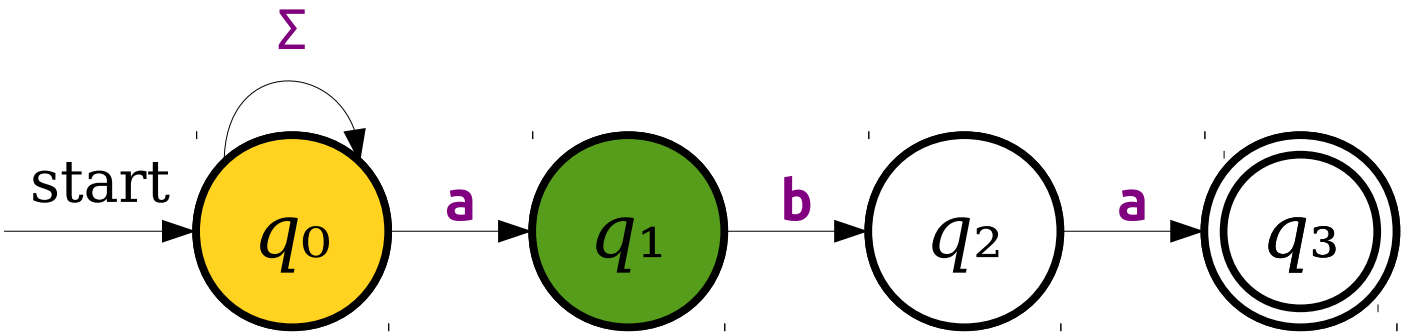
$\{q_0, q_1\}$	$\{q_0, q_1\}$	



	a	b
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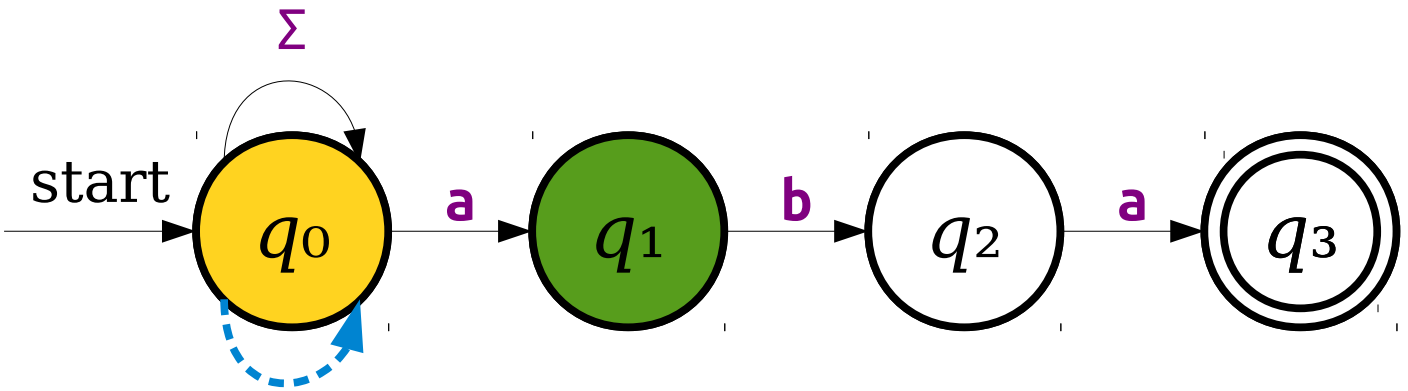

	a	b
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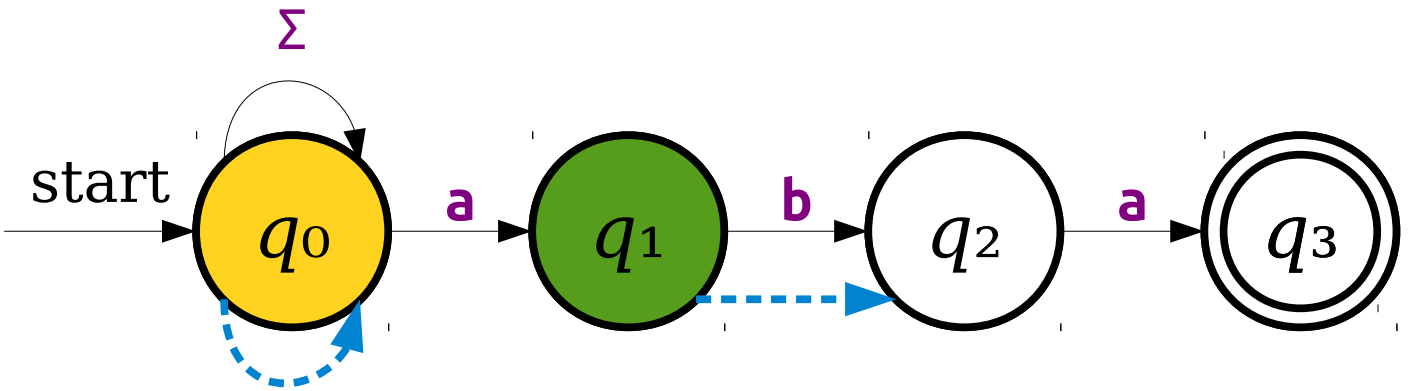
$\{q_0, q_1\}$	$\{q_0, q_1\}$	



	a	b
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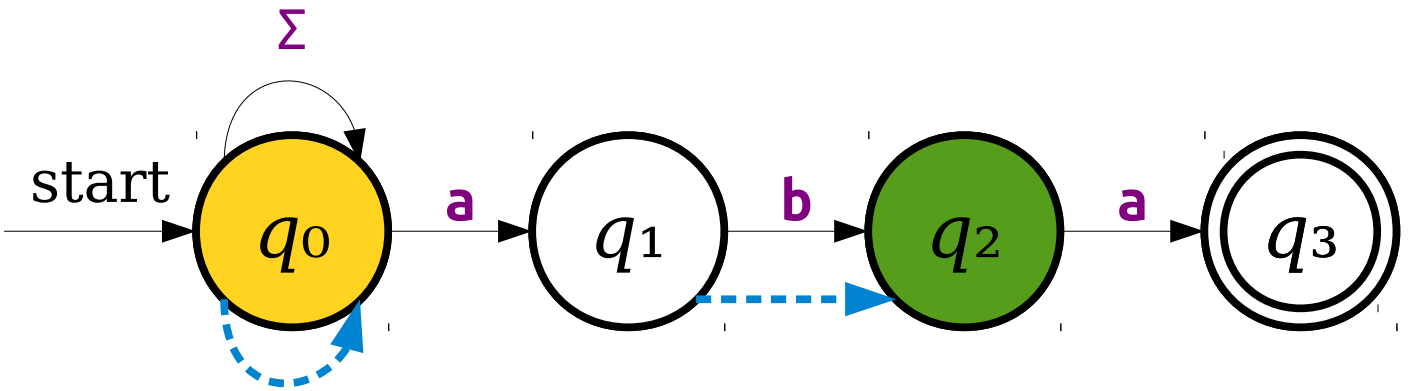

	a	b
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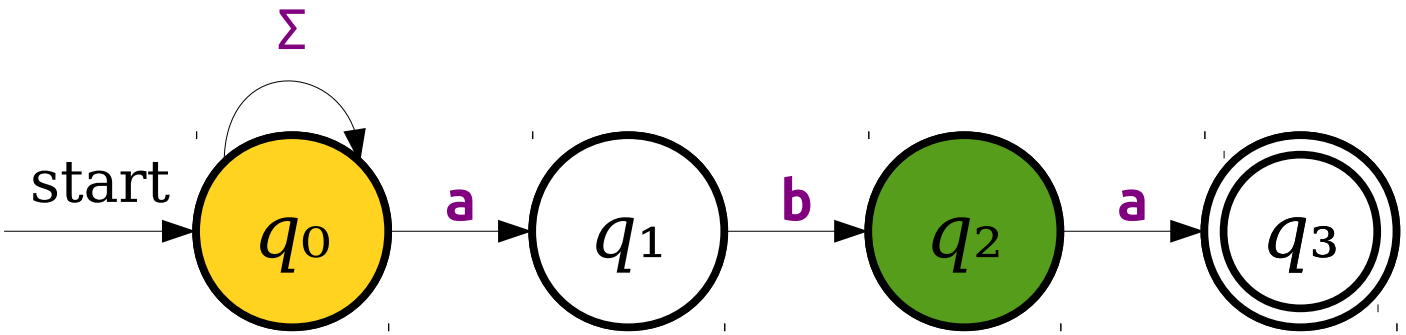

	a	b
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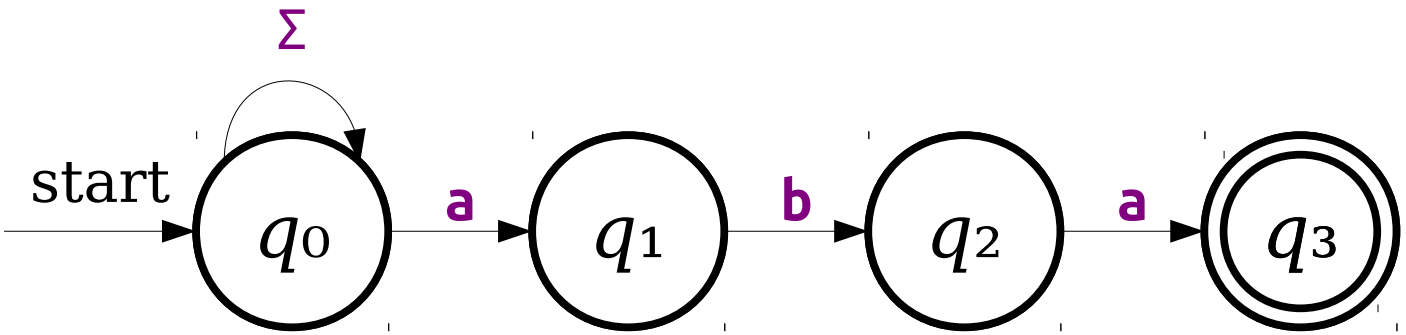

	a	b
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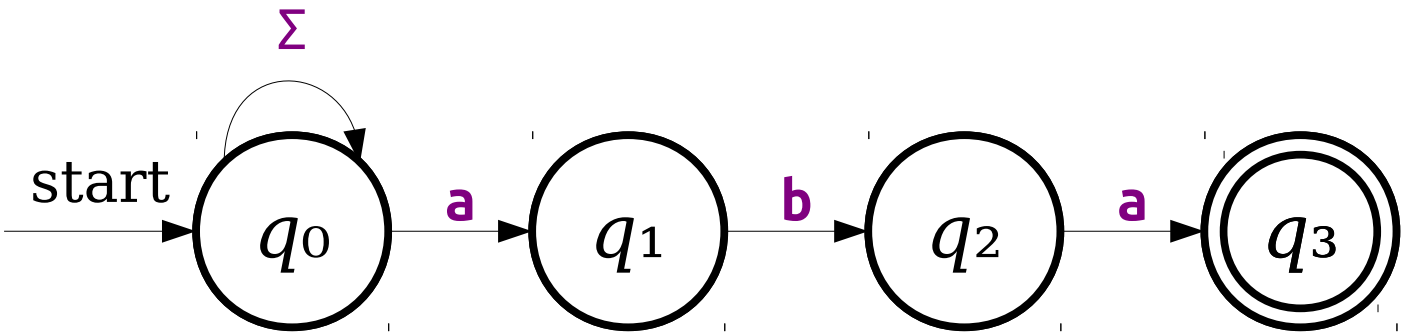

	a	b
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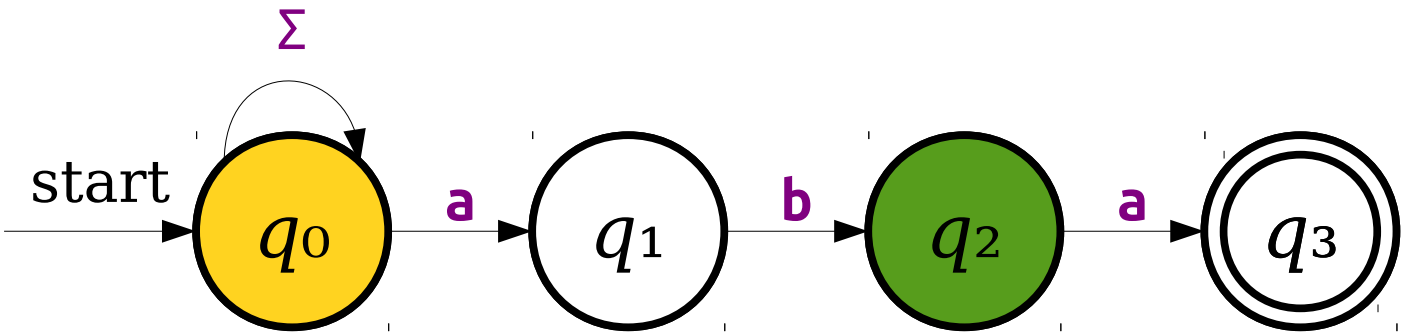

	a	b
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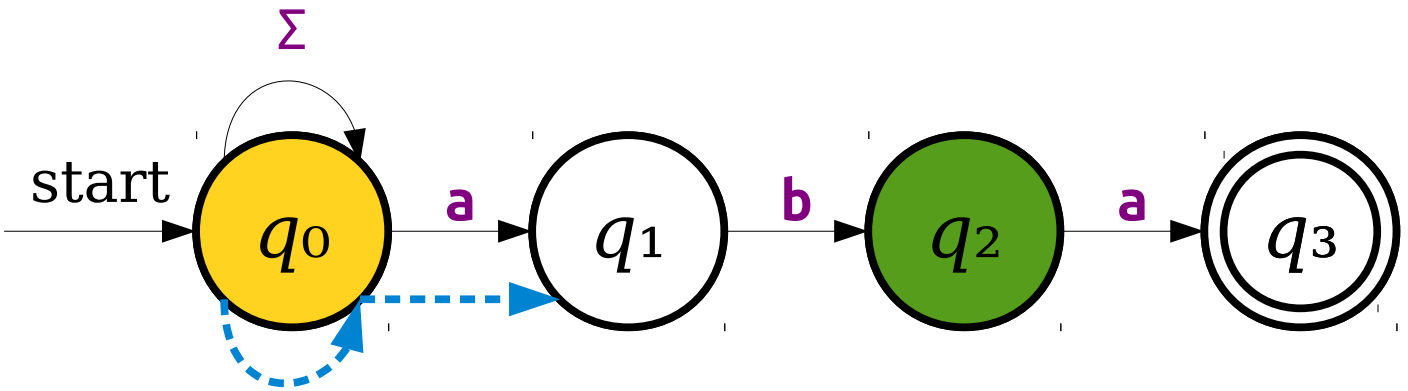

	a	b
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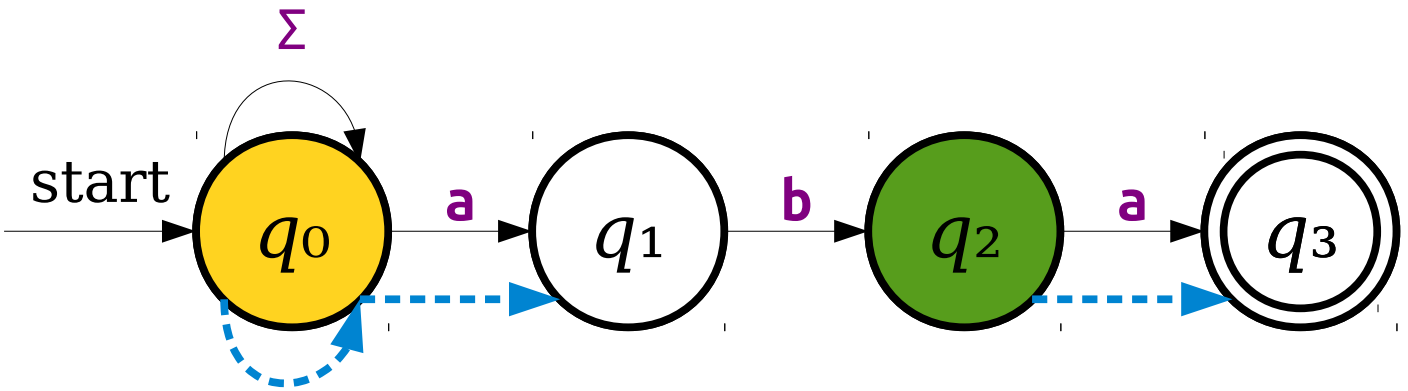
$\{q_0, q_2\}$		



	a	b
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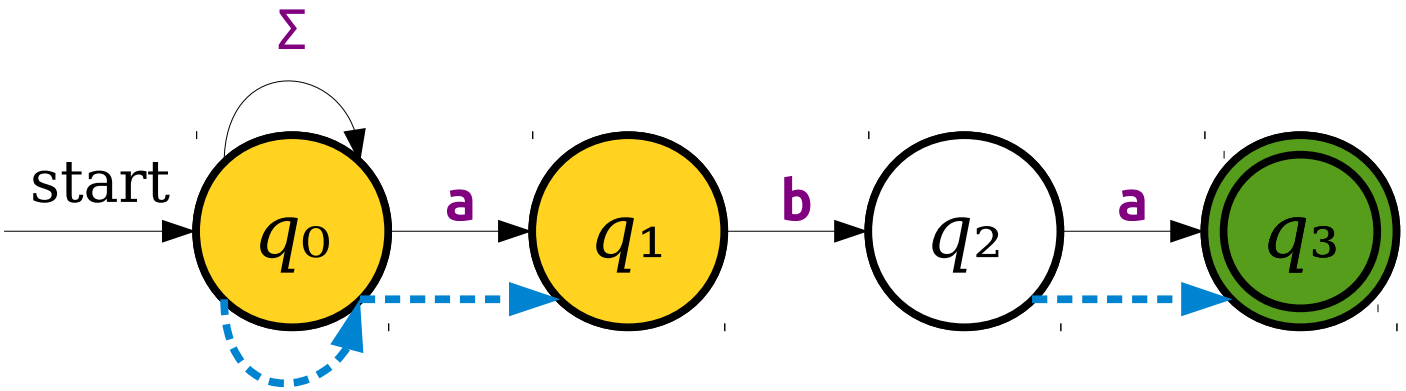

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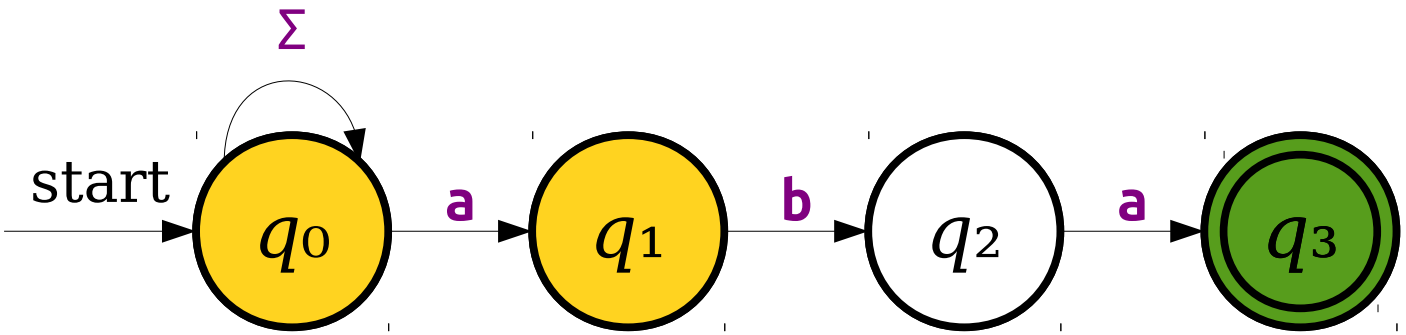

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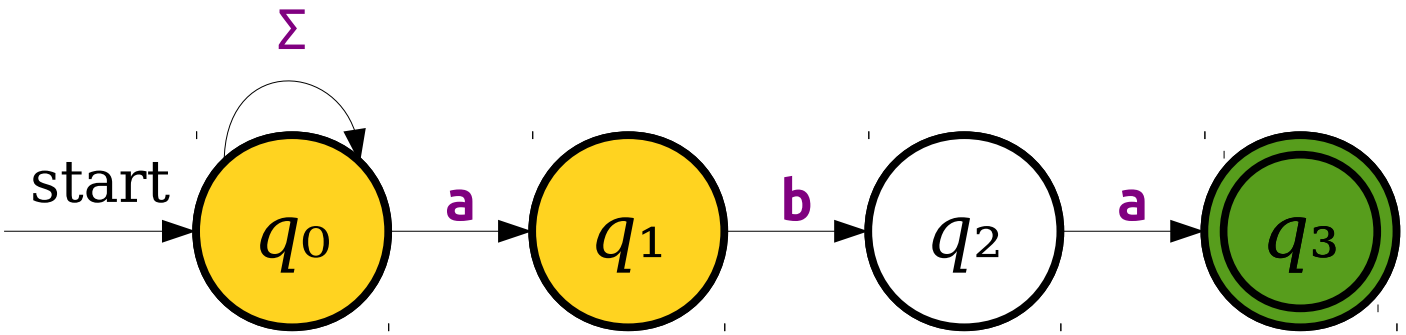

	a	b
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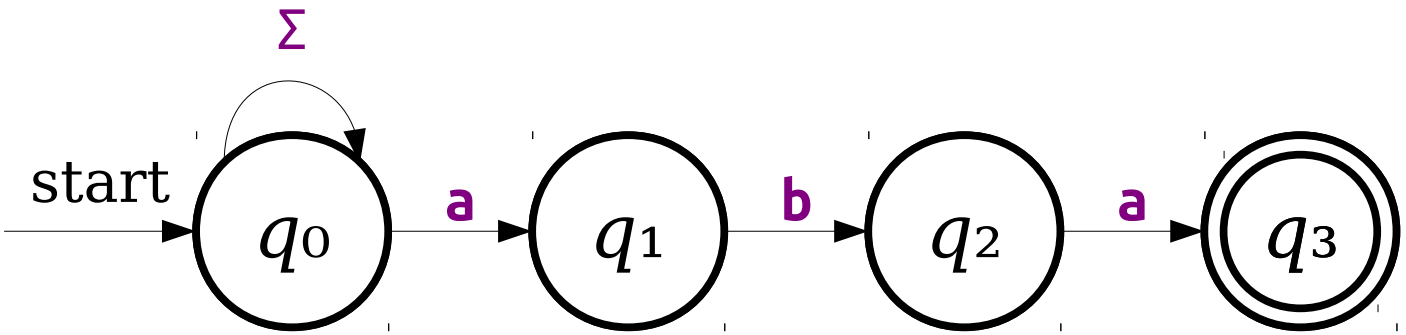

	a	b
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$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
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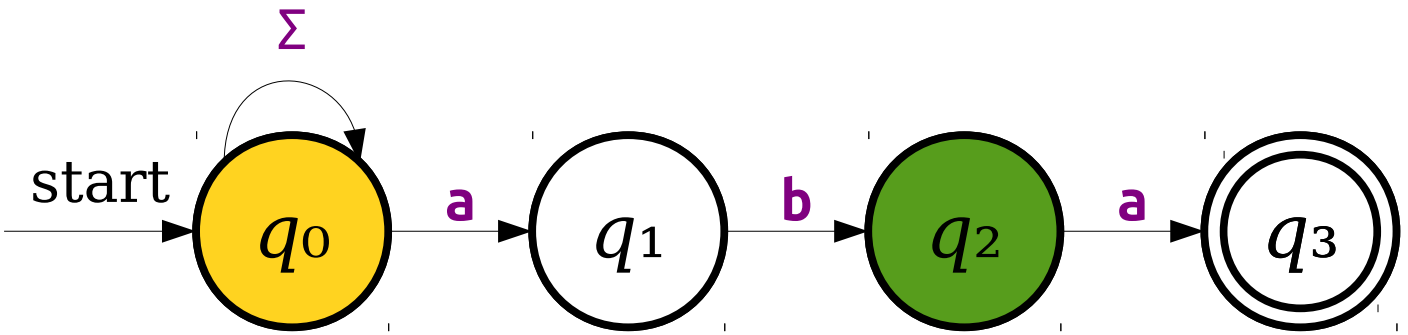

	a	b
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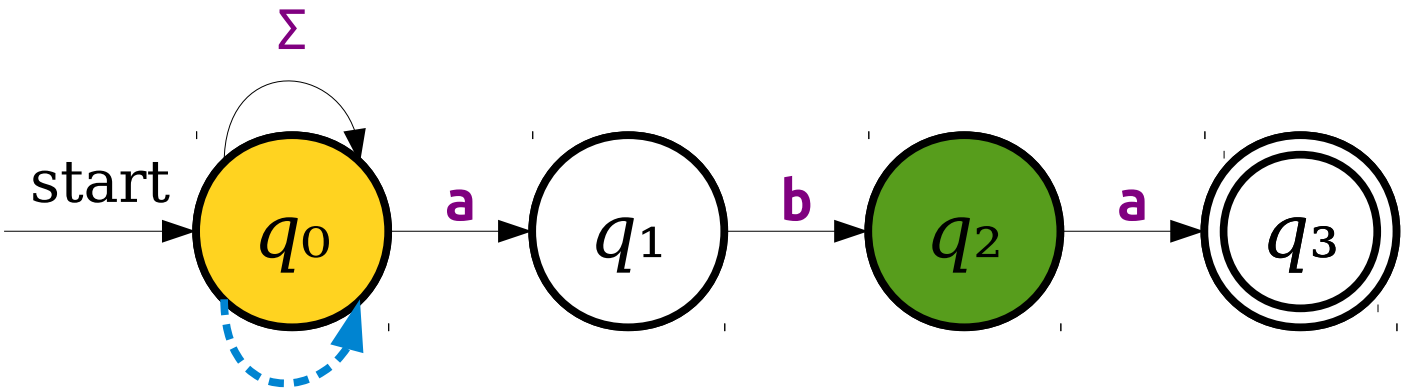

	a	b
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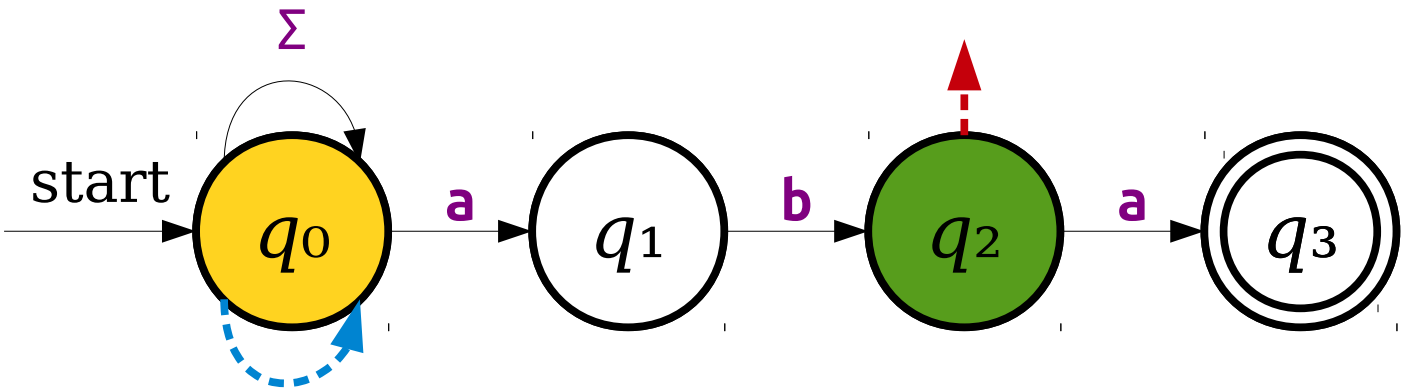

	a	b
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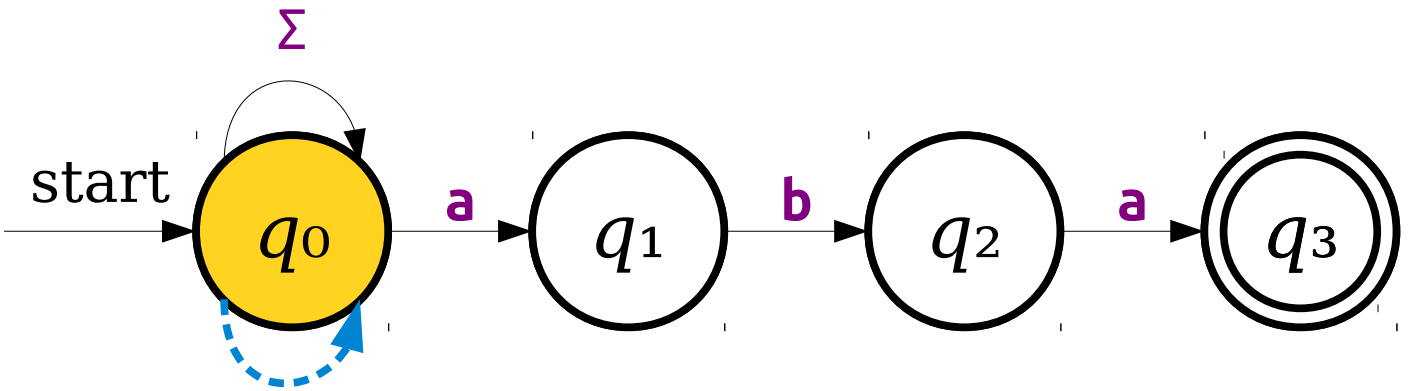

	a	b
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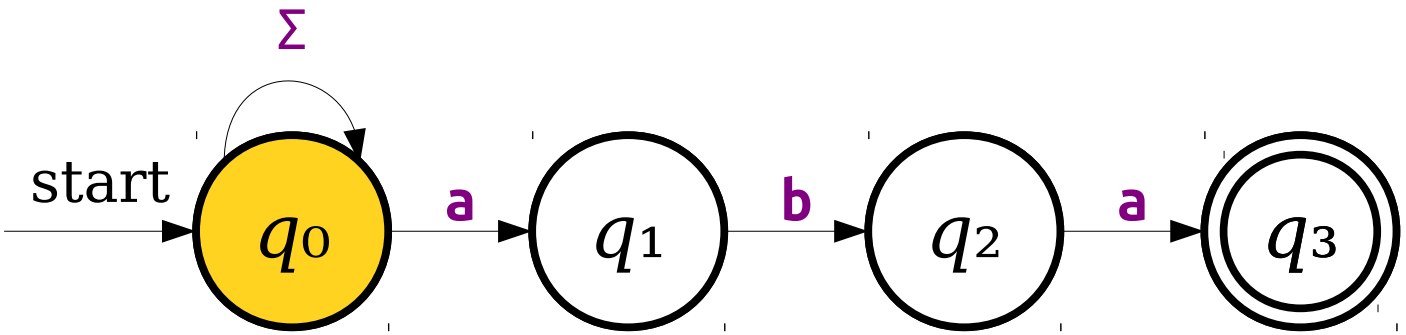

	a	b
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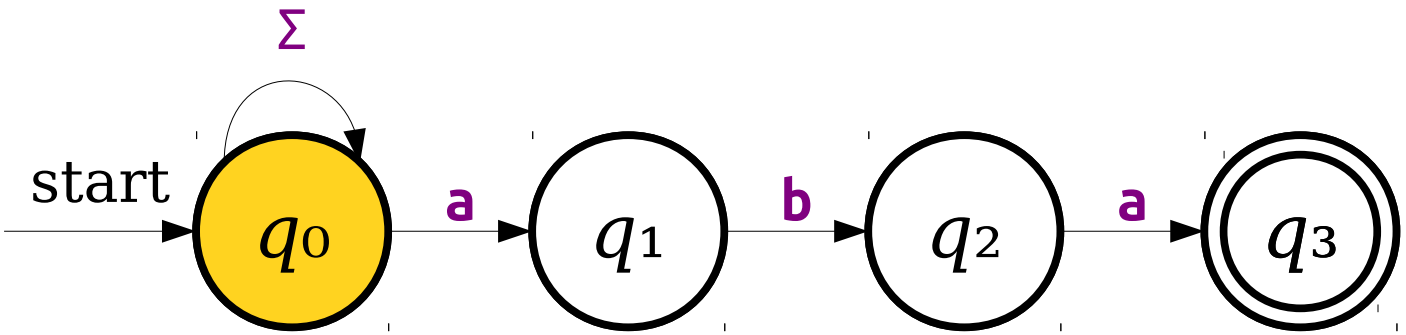

	a	b
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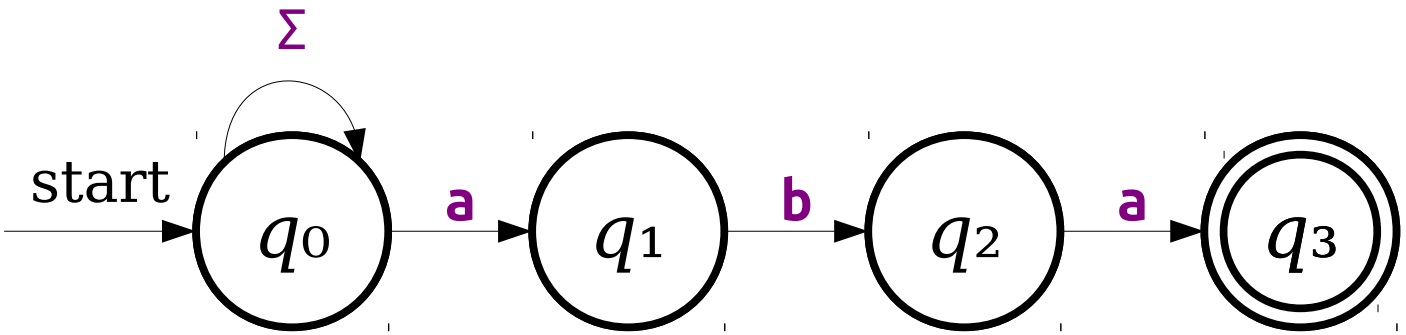

	a	b
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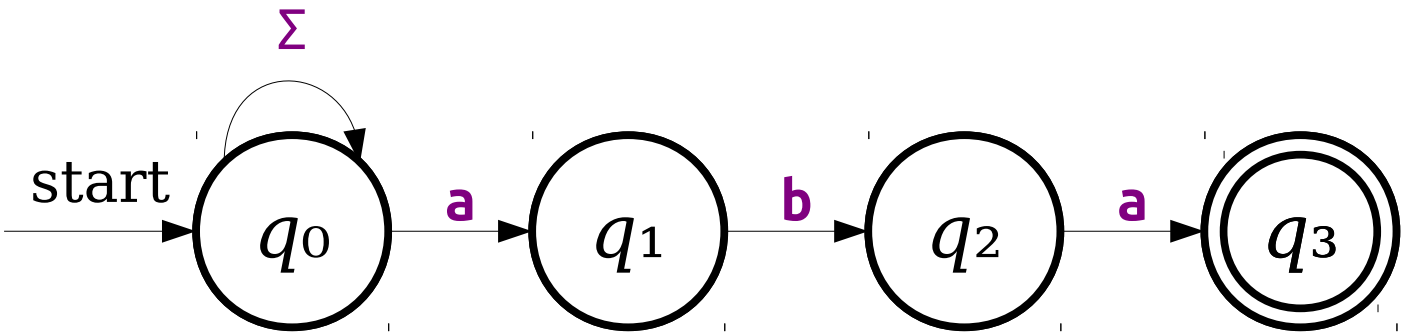

	a	b
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$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
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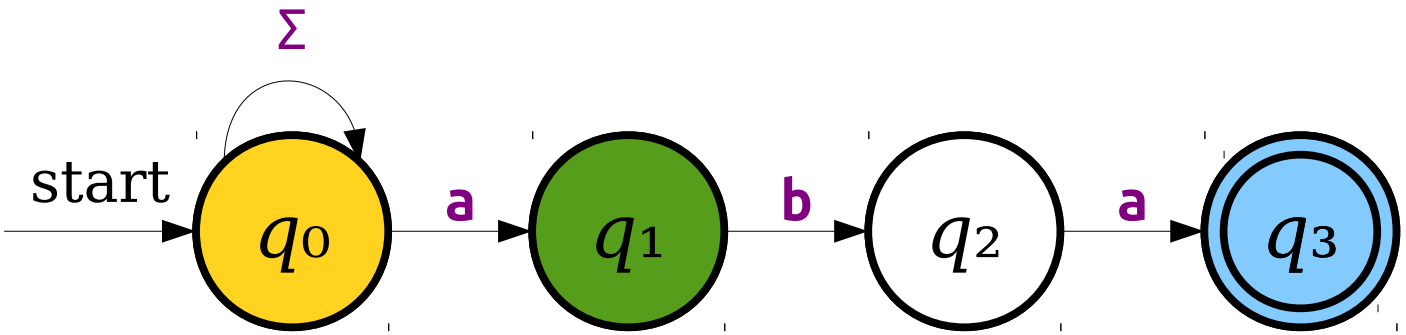

	a	b
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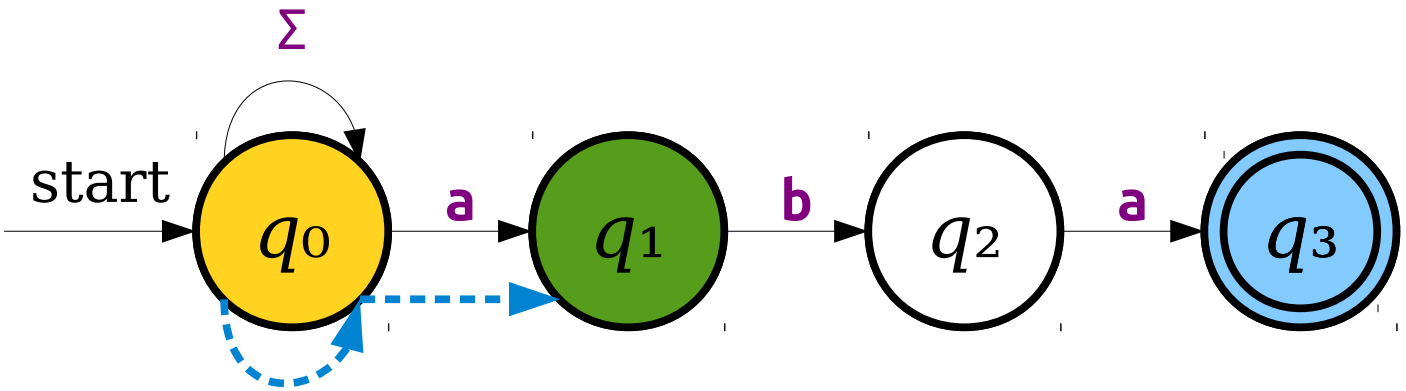
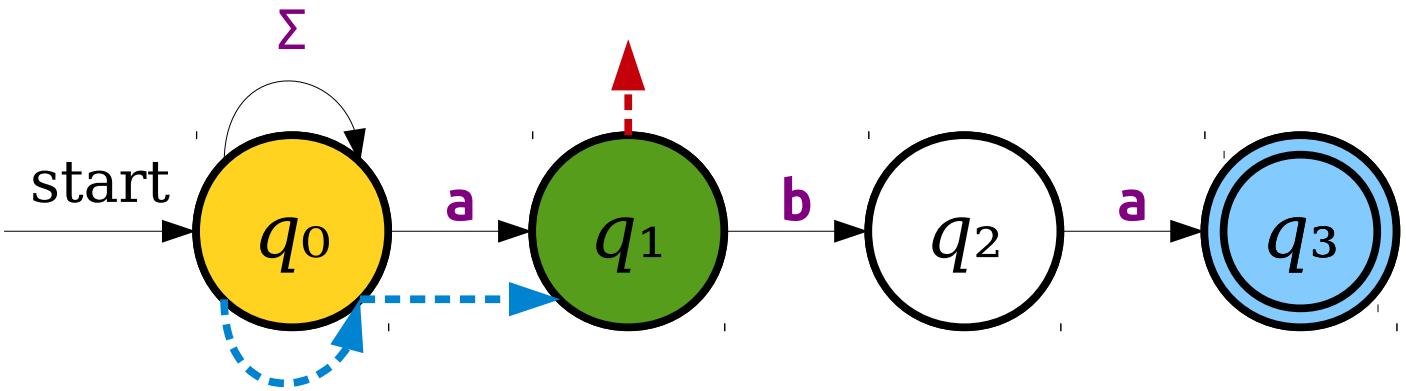


	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$\{q_0, q_1, q_3\}$		



	a	b
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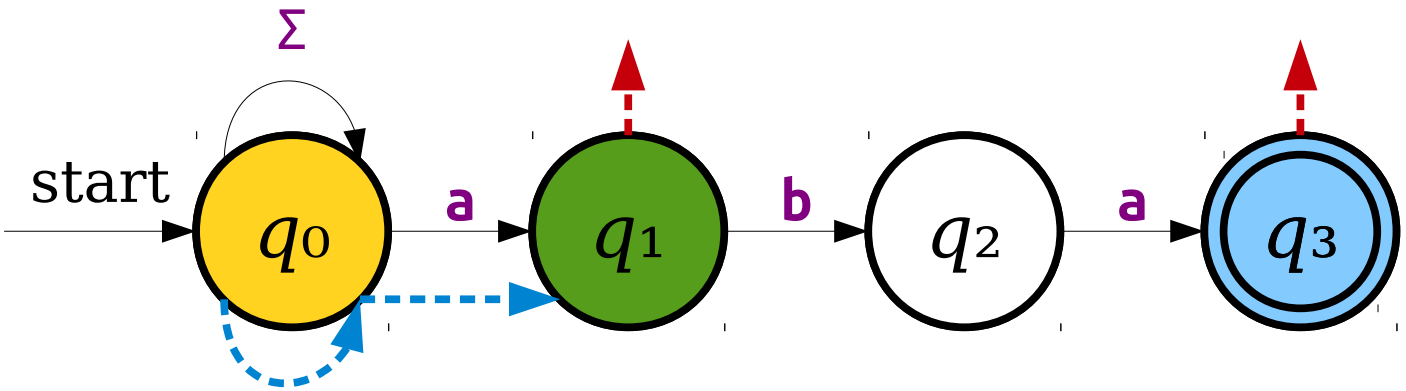
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$\{q_0, q_1, q_3\}$		



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
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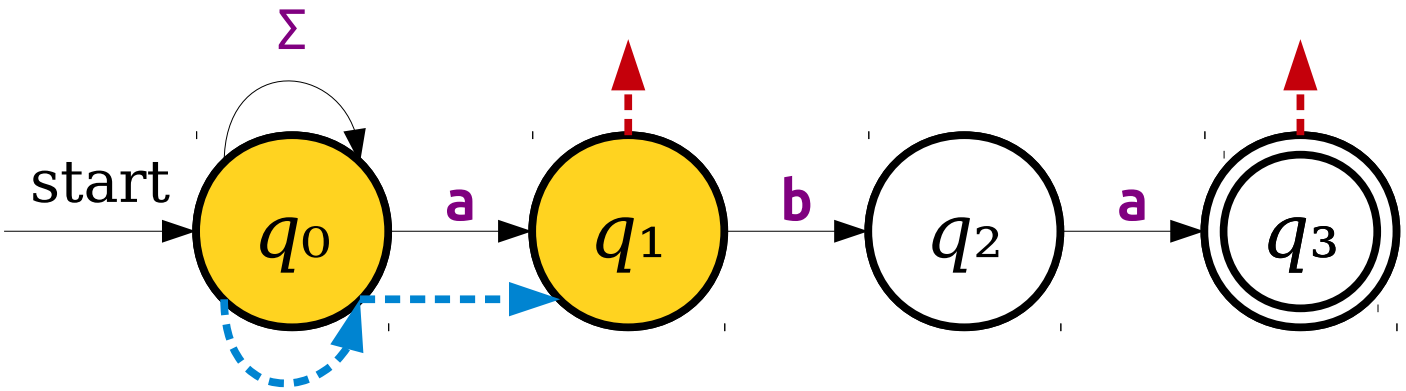
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$\{q_0, q_1, q_3\}$		



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
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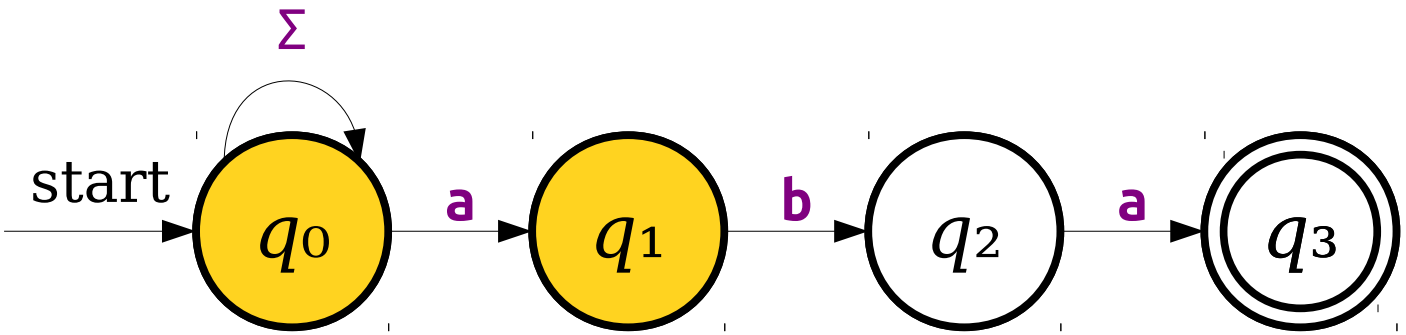
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$\{q_0, q_1, q_3\}$		



	a	b
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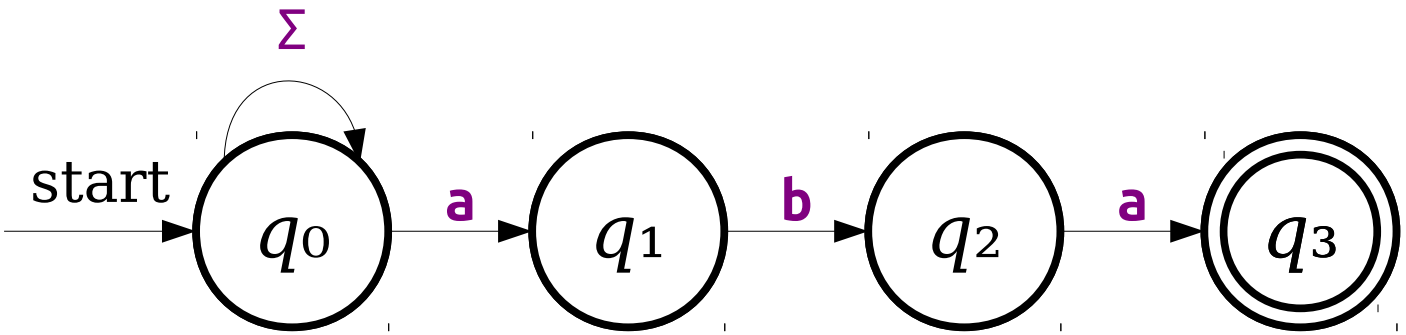
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	



	a	b
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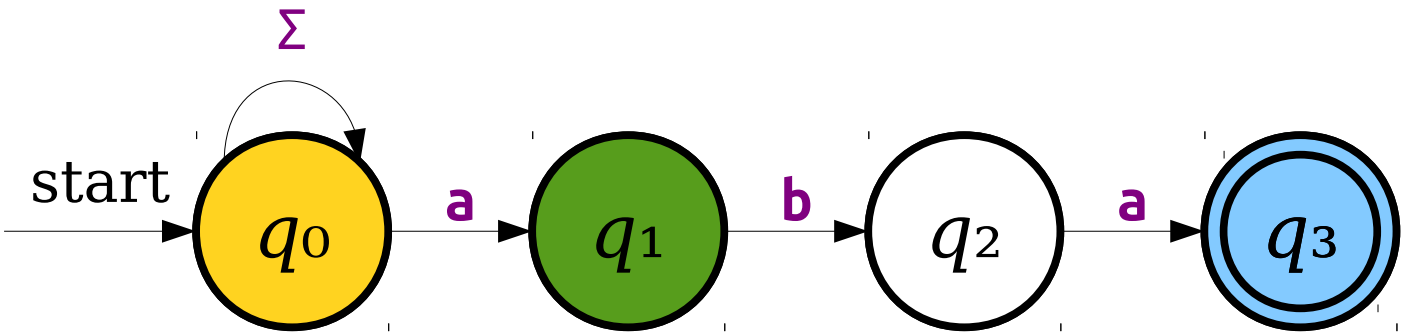
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	



	a	b
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$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
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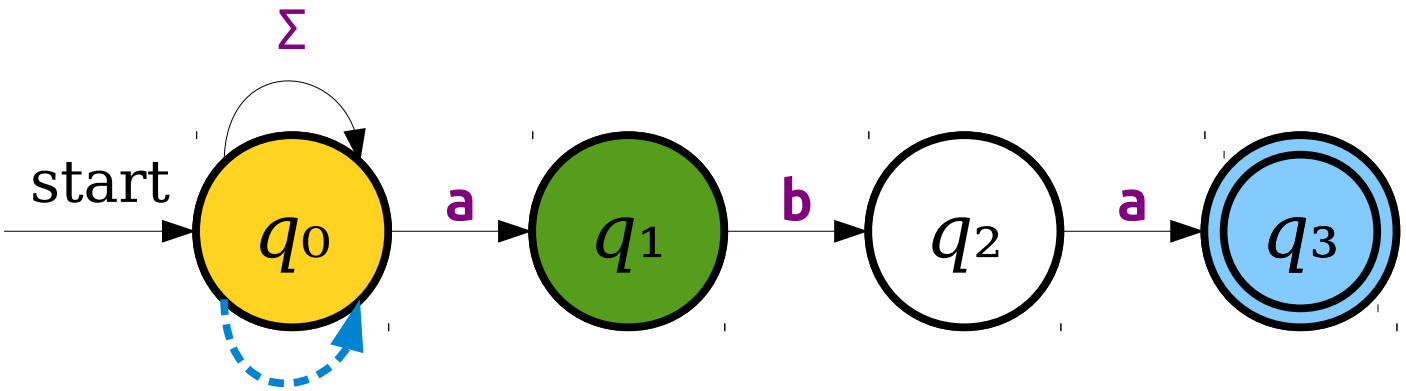

	a	b
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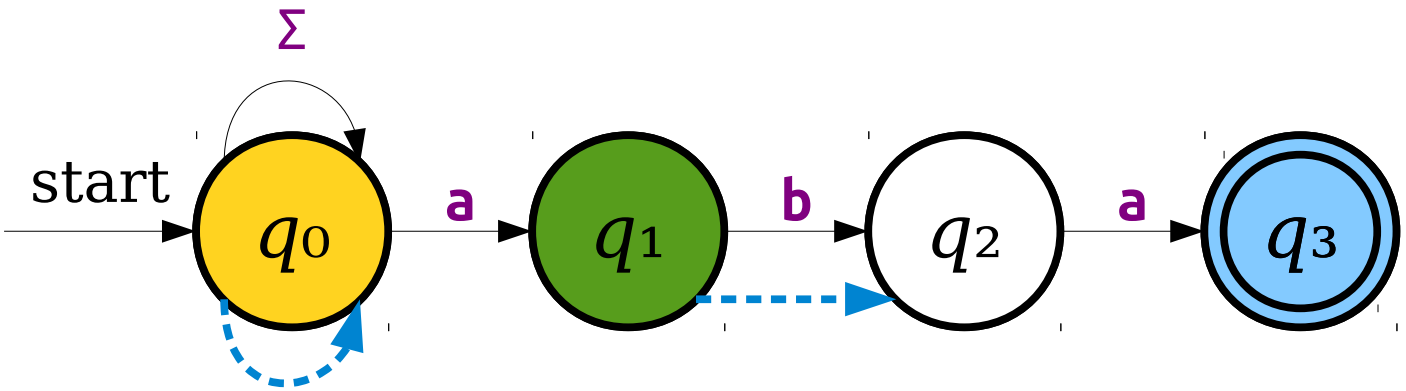
$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	



	a	b
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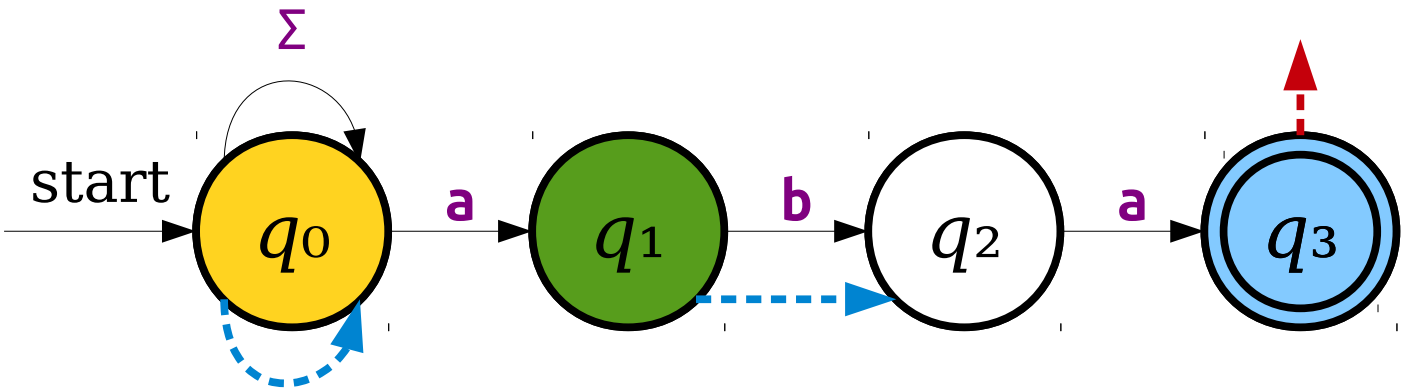
$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	



	a	b
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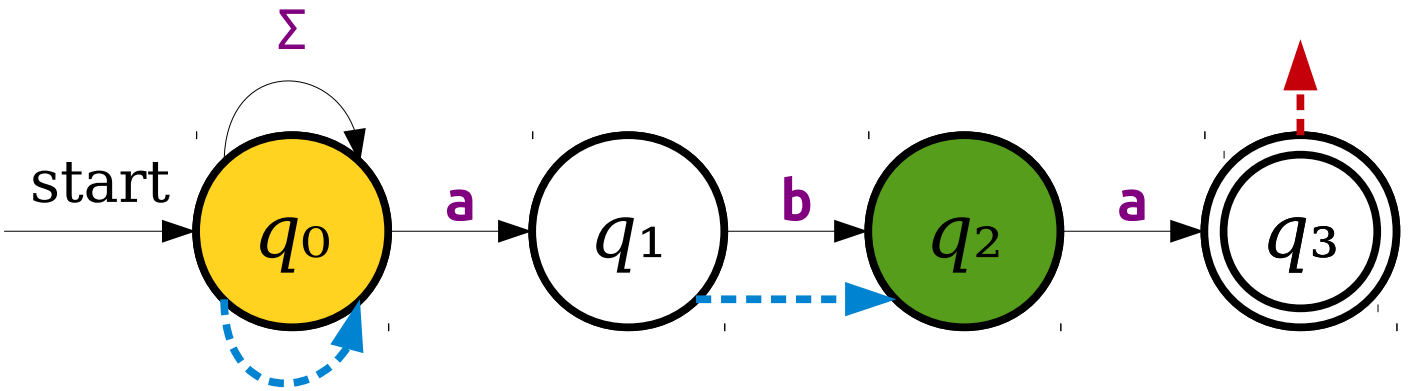
$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	



	a	b
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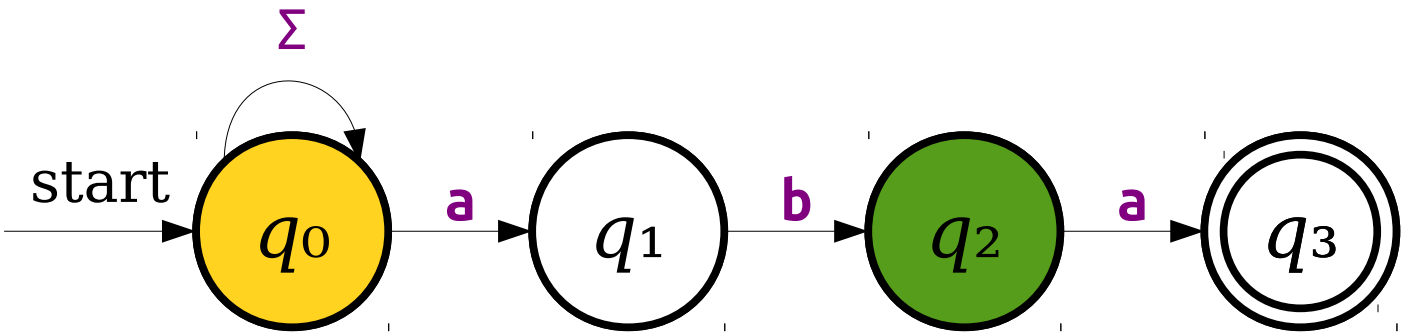
$\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	



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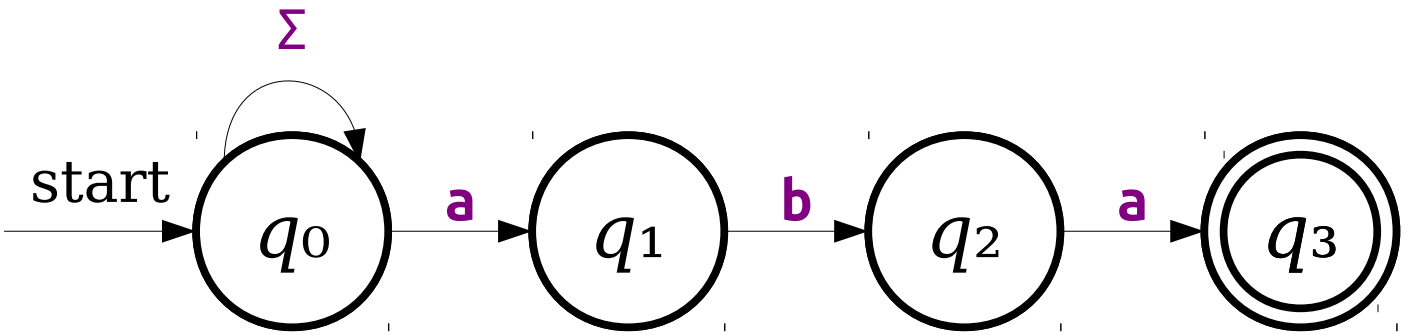

	a	b
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$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
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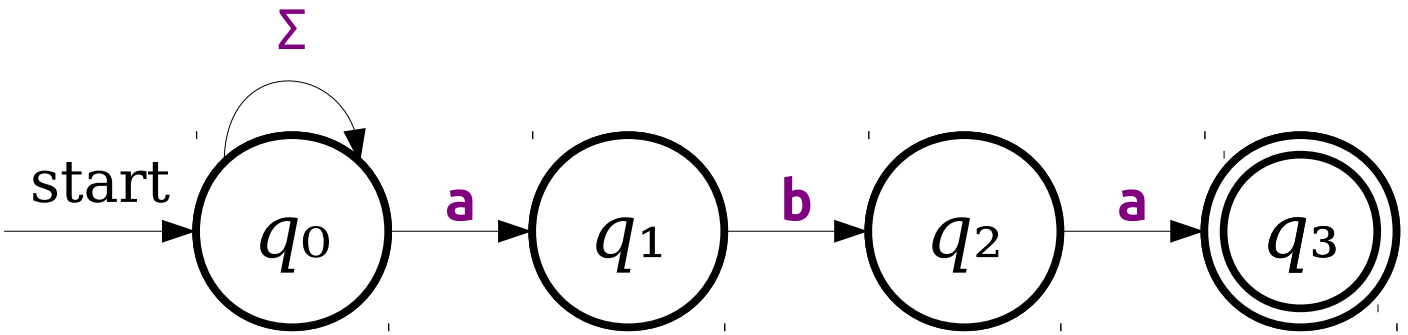

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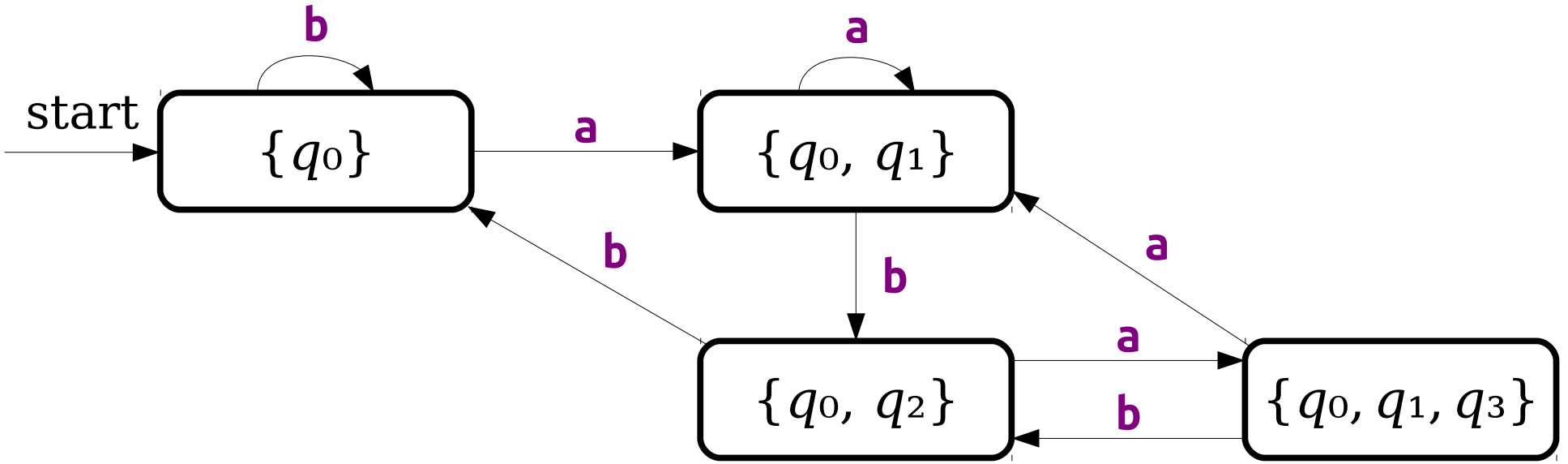

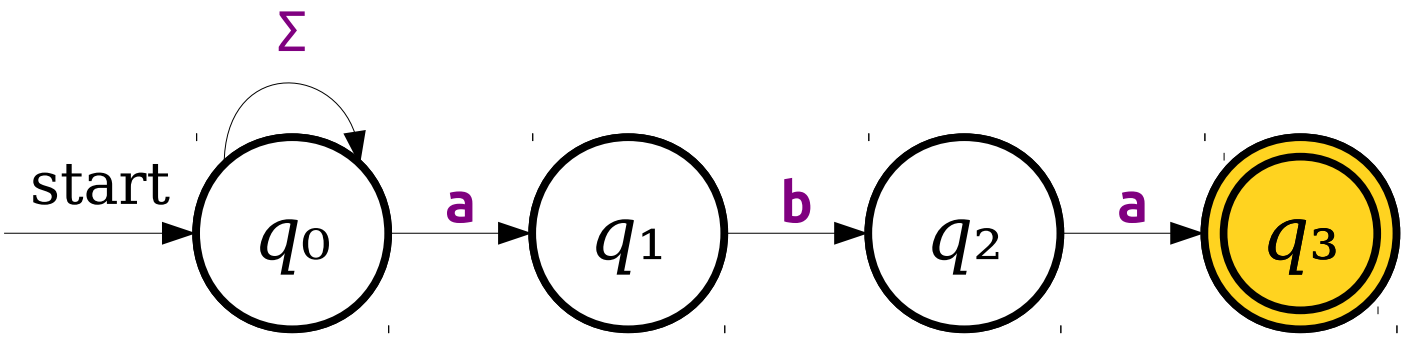

	a	b
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$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
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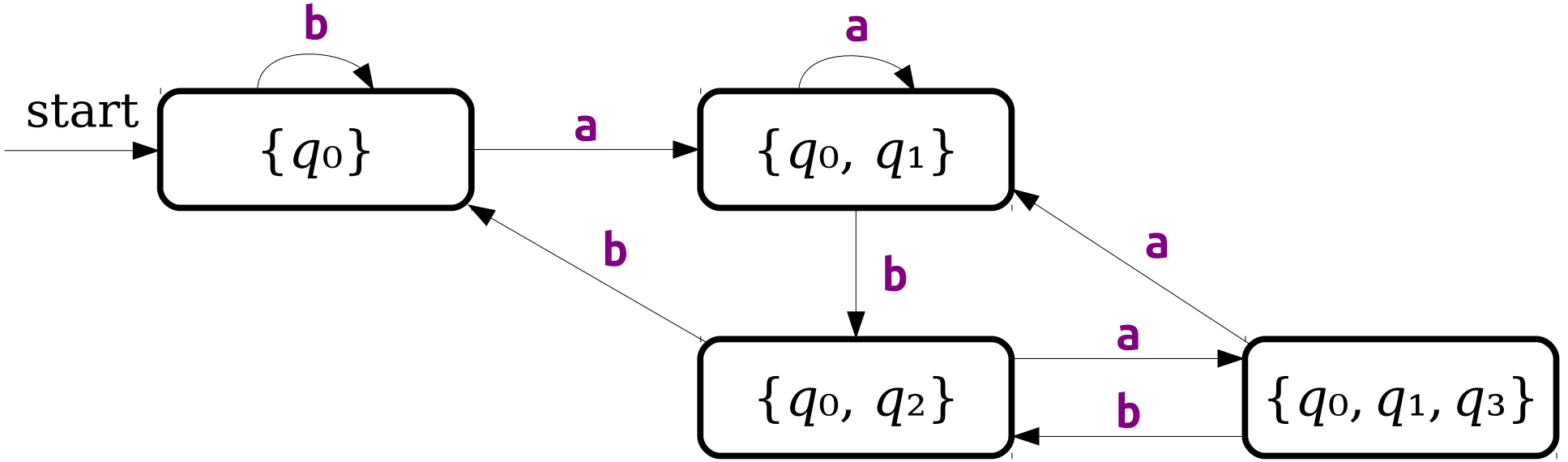

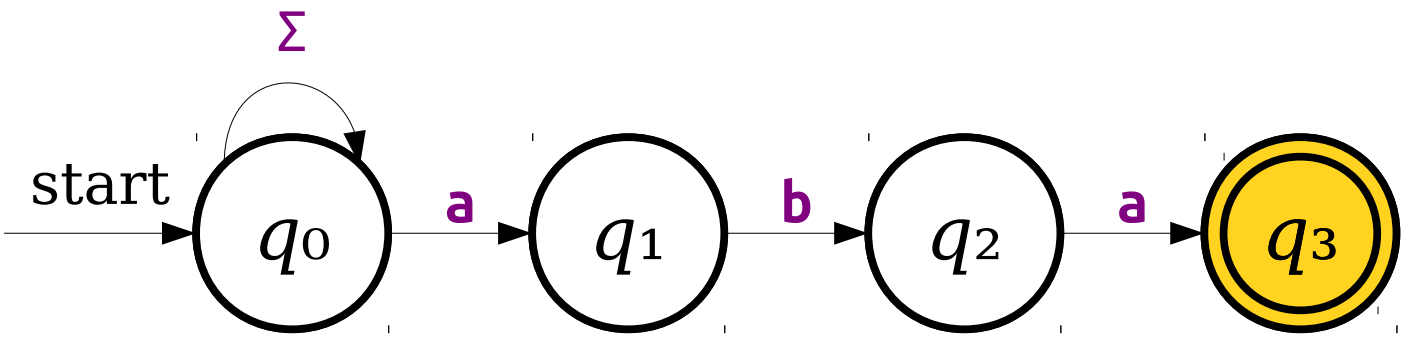

	a	b
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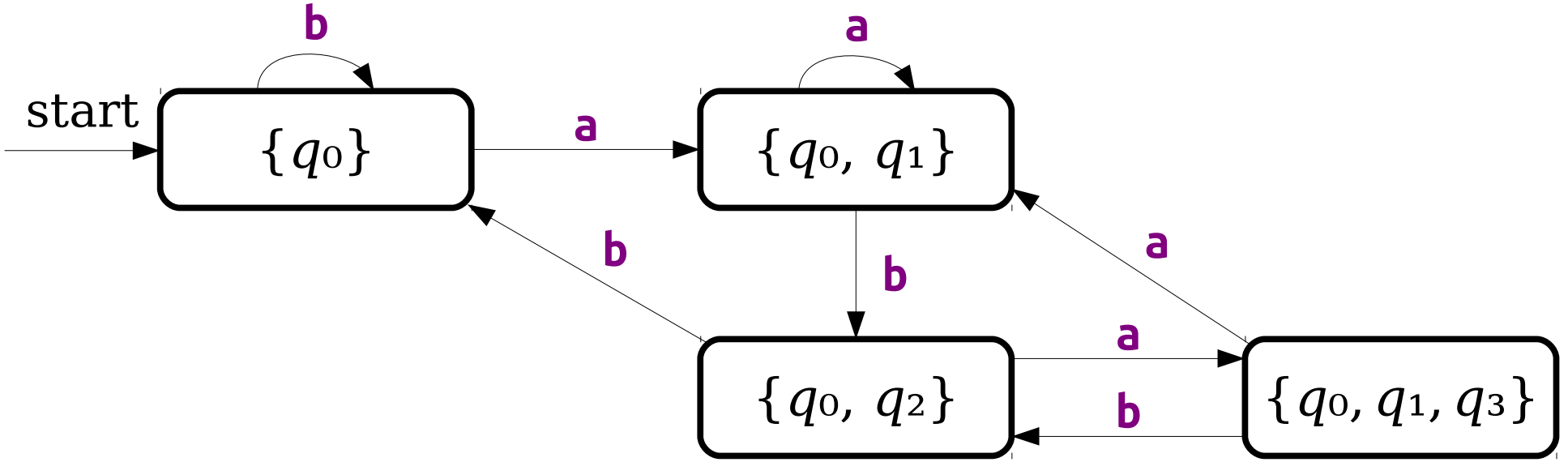
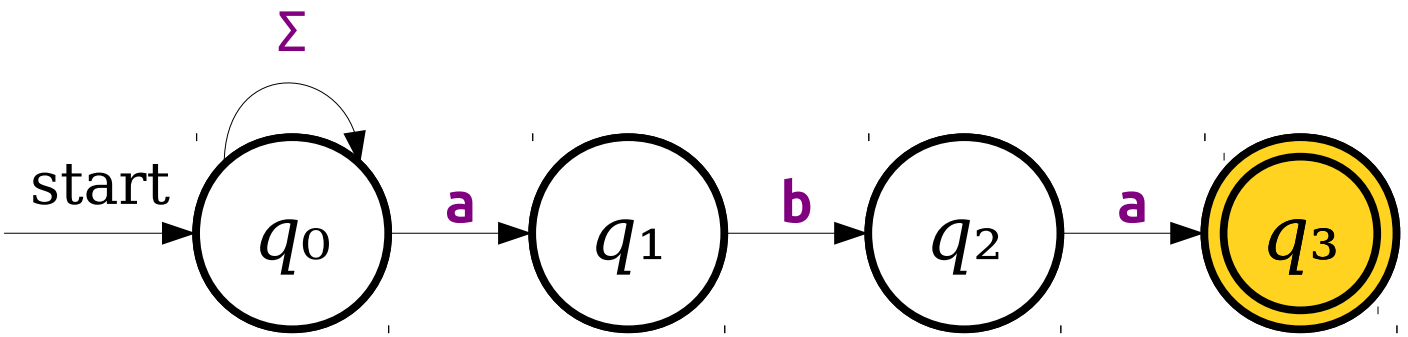
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$*\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$





	a	b
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$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
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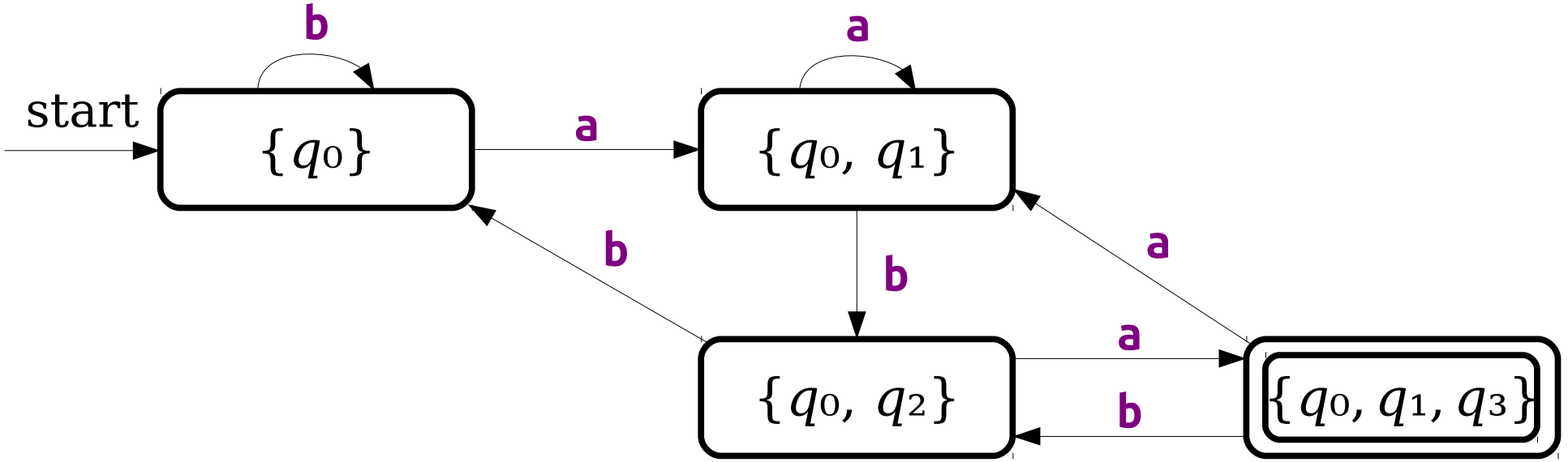

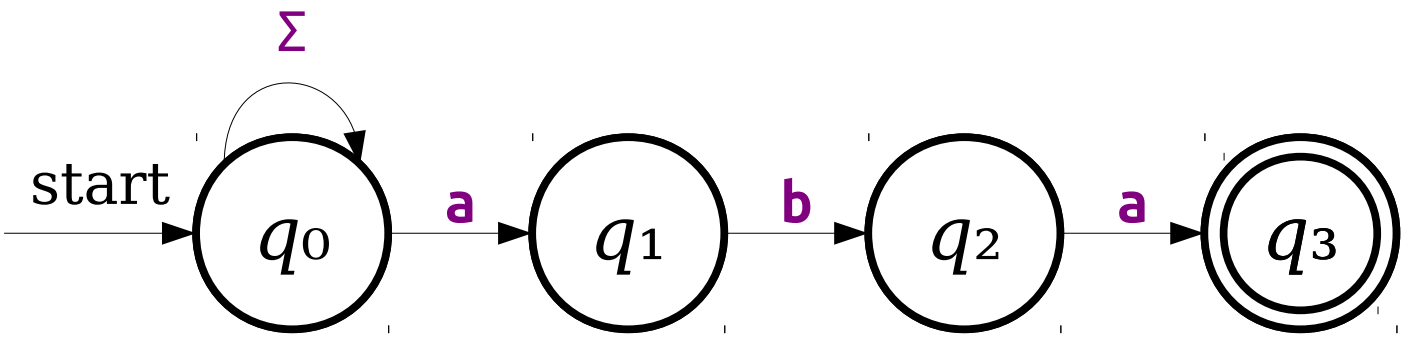

	a	b
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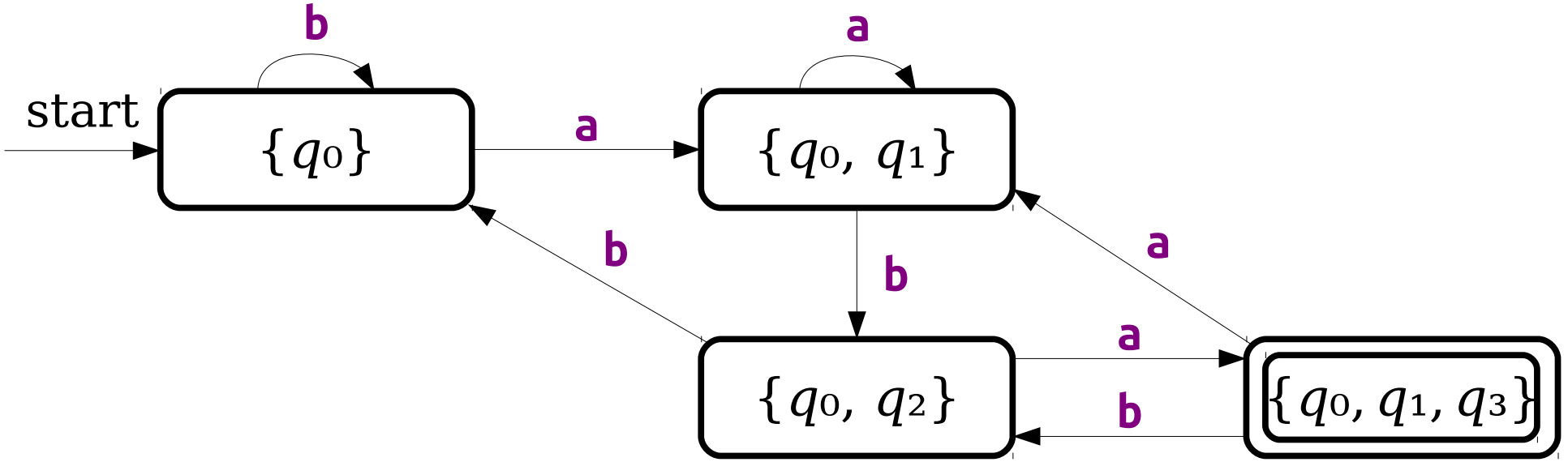
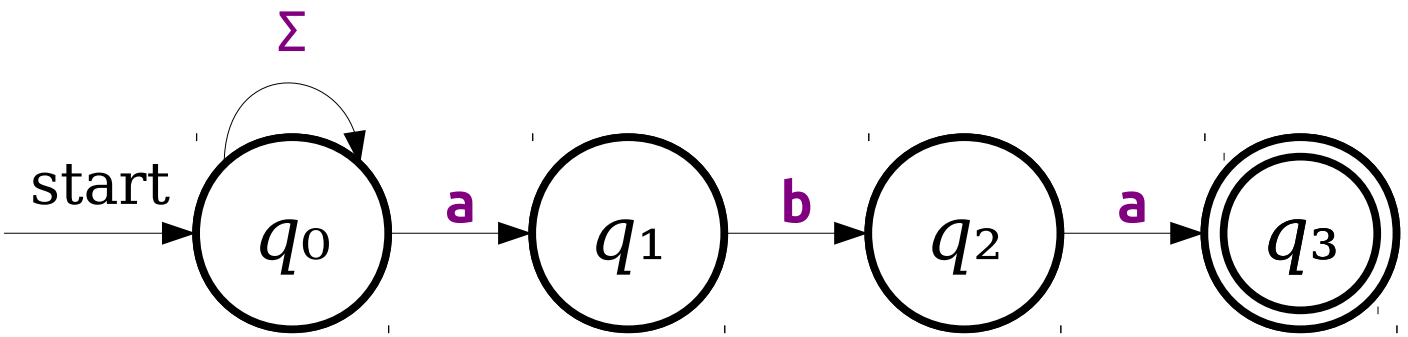
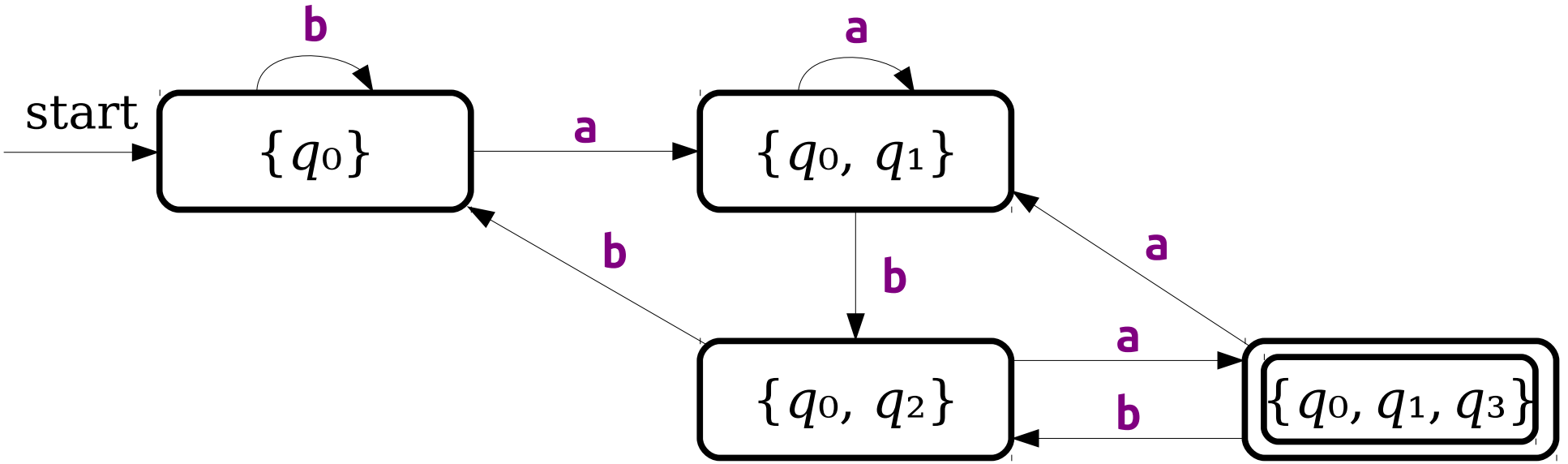
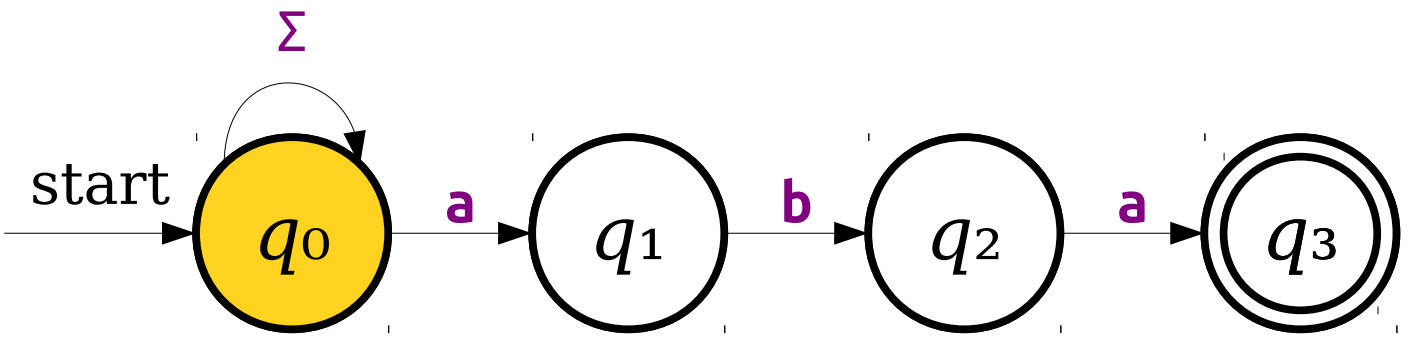


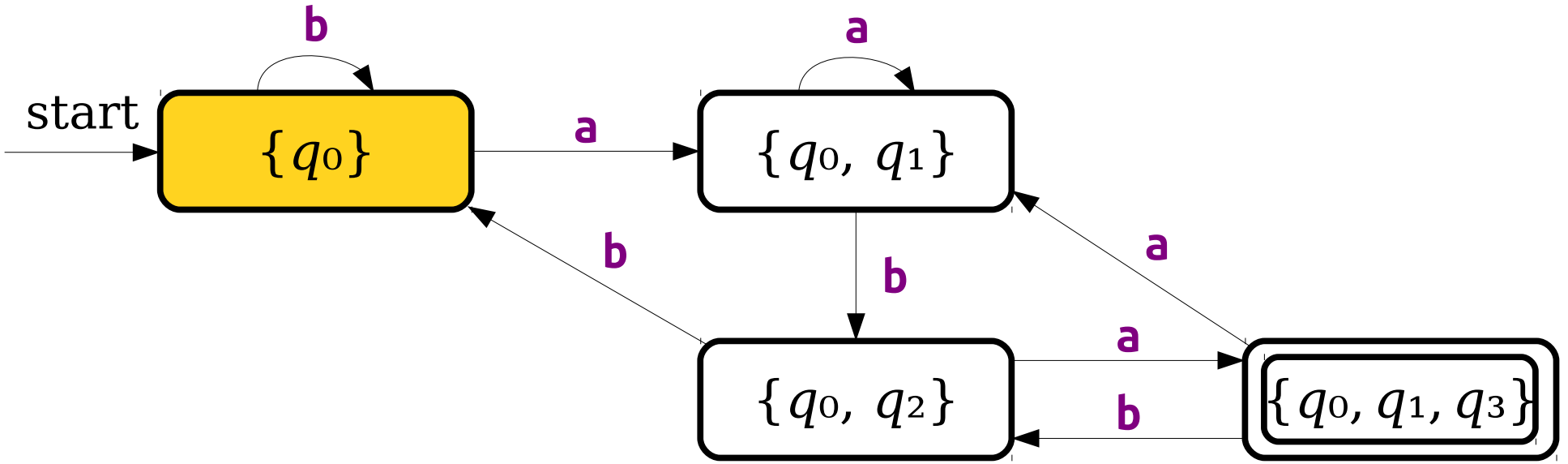


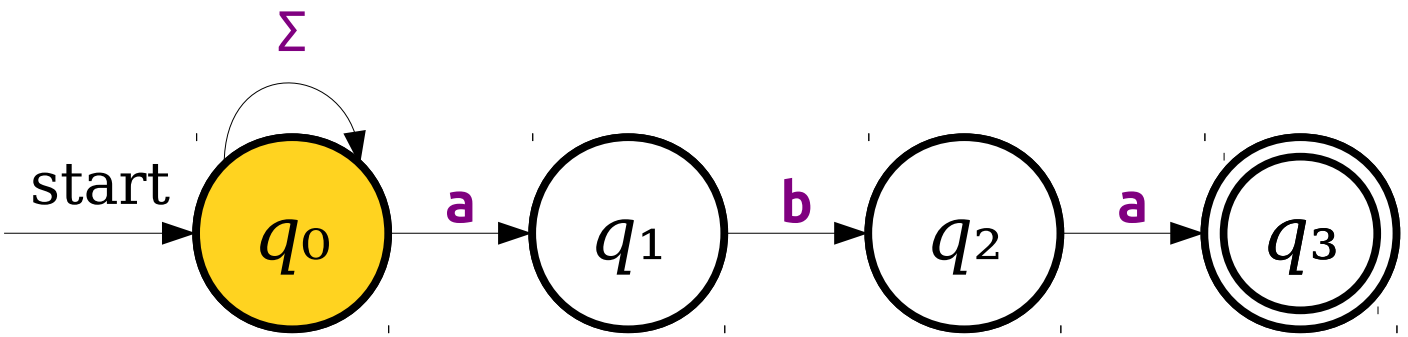
a	b	a	a	b	a





a	b	a	a	b	a




Σ
start
q_0
a
q_1
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q_2
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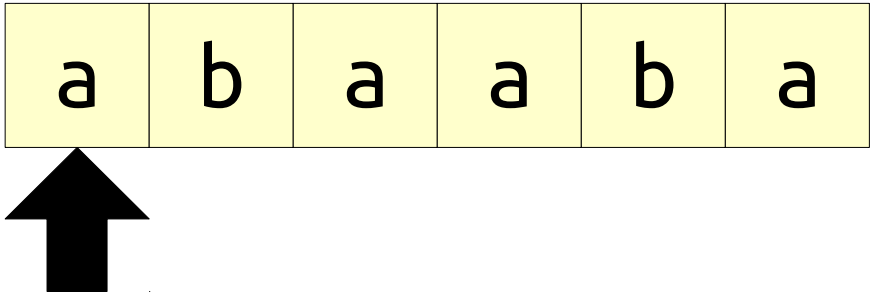
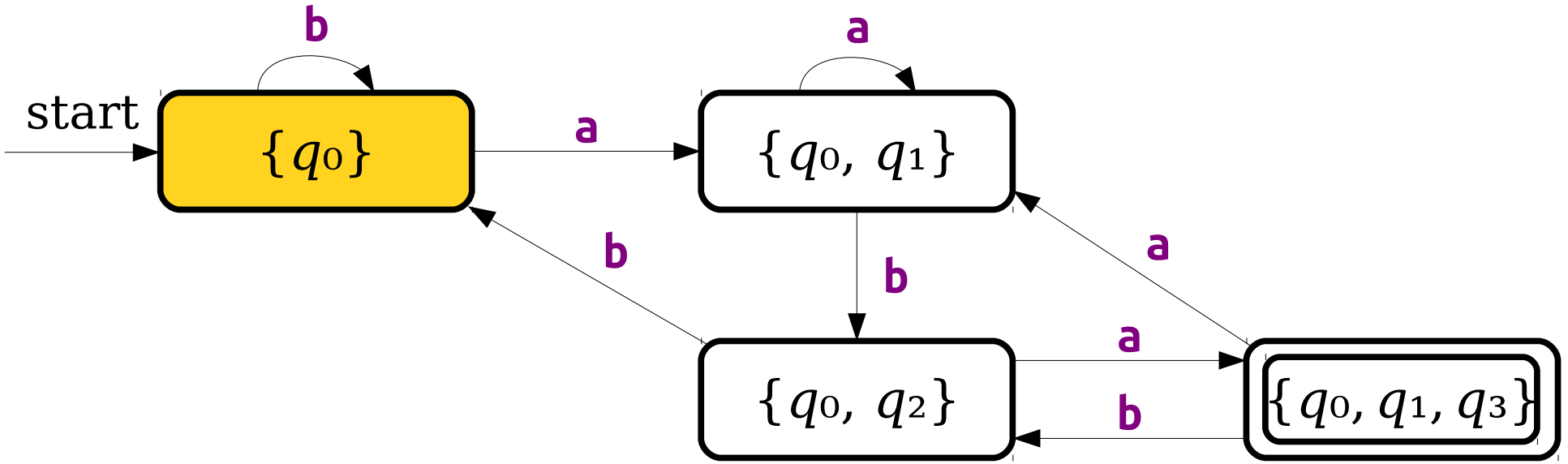
q_3


a
b
a
a
b
a


b
a
start
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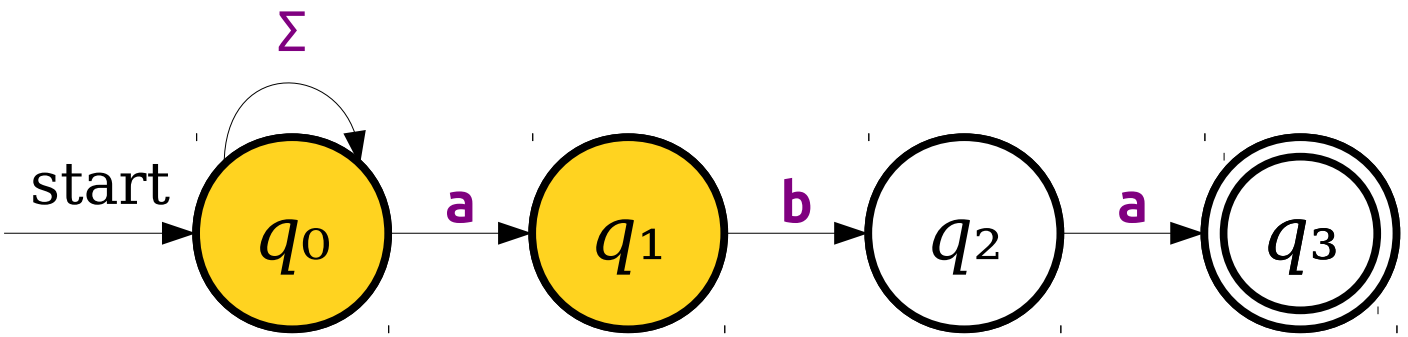
a
$\{q_0, q_1\}$
b
b
a
a
$\{q_0, q_2\}$
b
$\{q_0, q_1, q_3\}$


Σ
start
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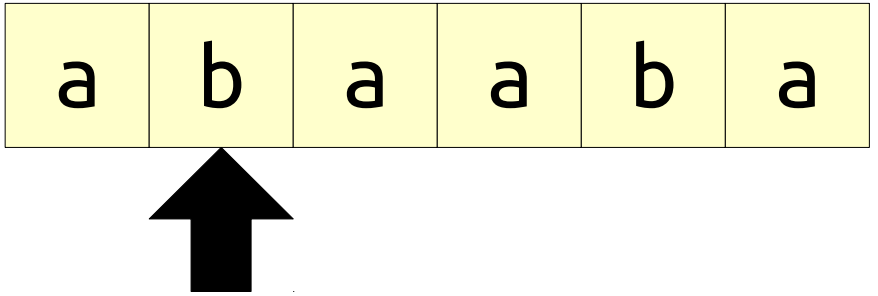
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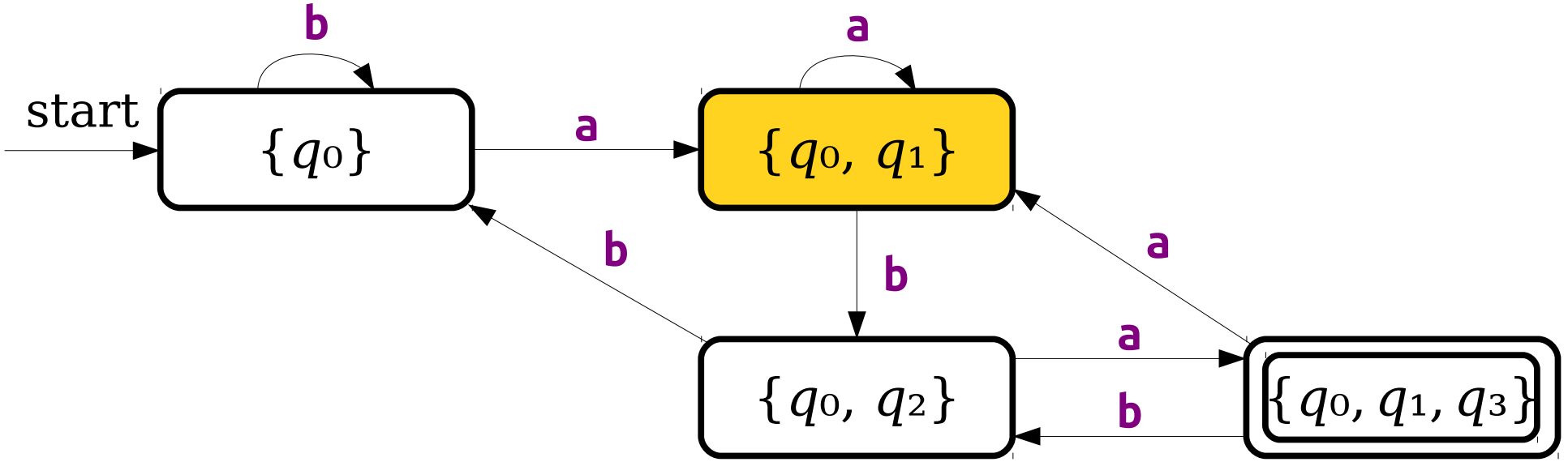
b
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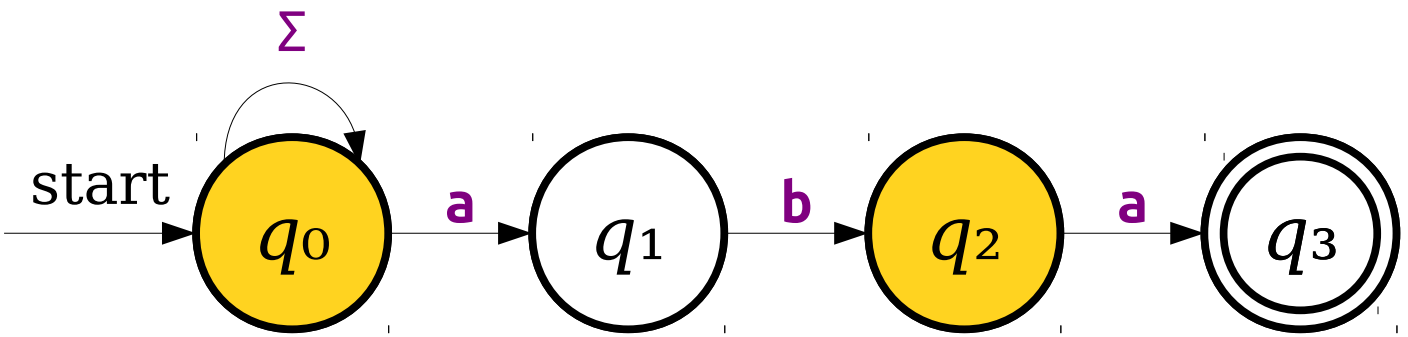
b
$\{q_0, q_1, q_3\}$


Σ
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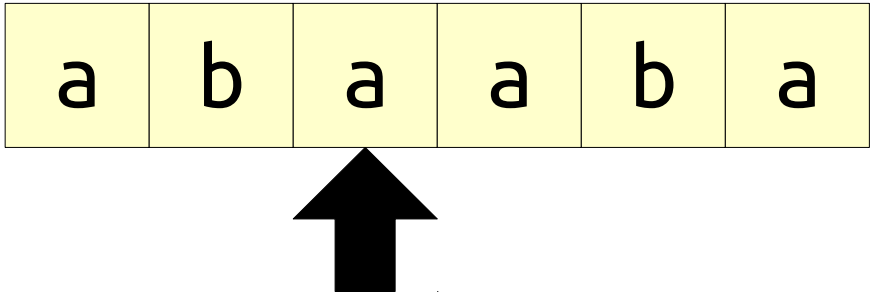
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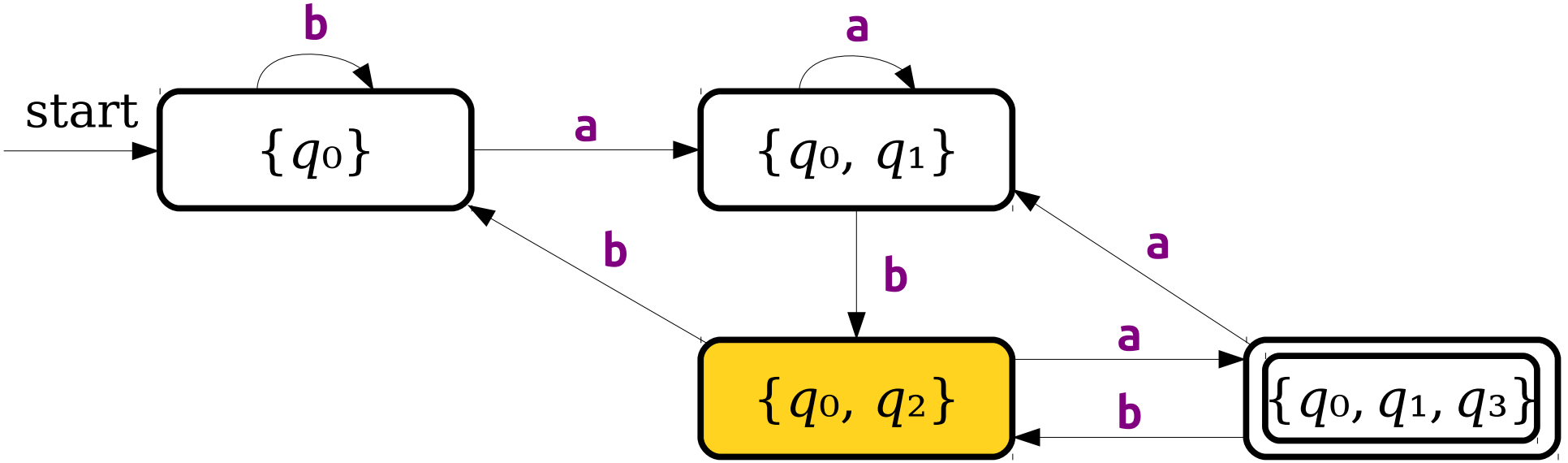
b
a
start
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a
$\{q_0, q_2\}$
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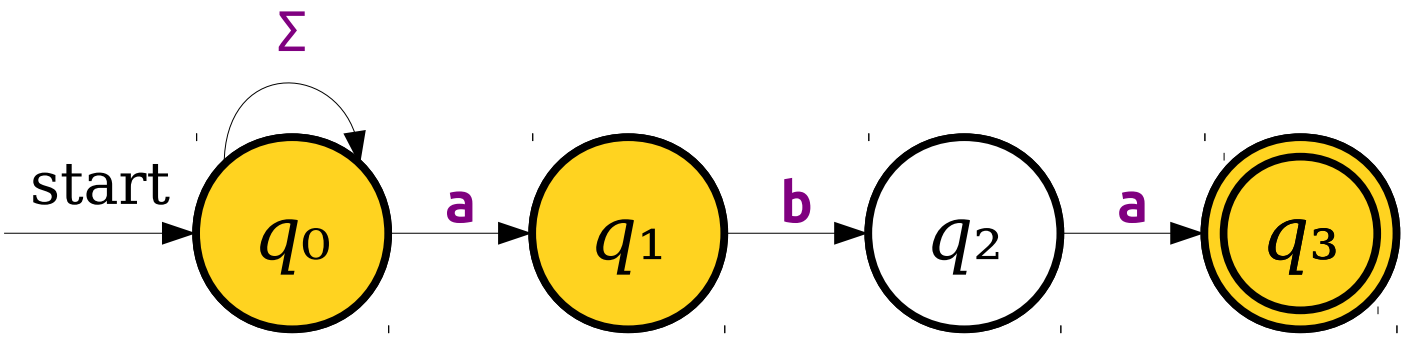
Σ
start
q_0
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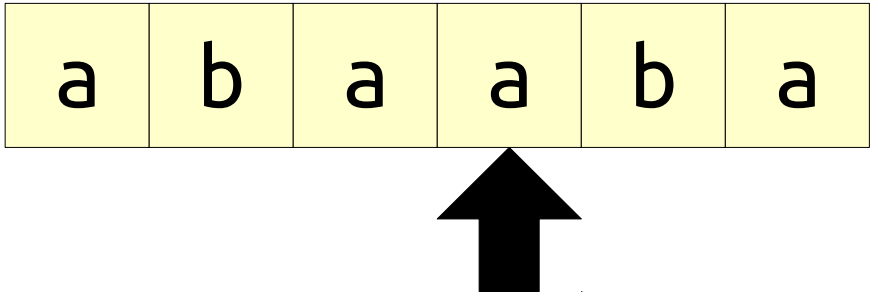
q_3


a
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a
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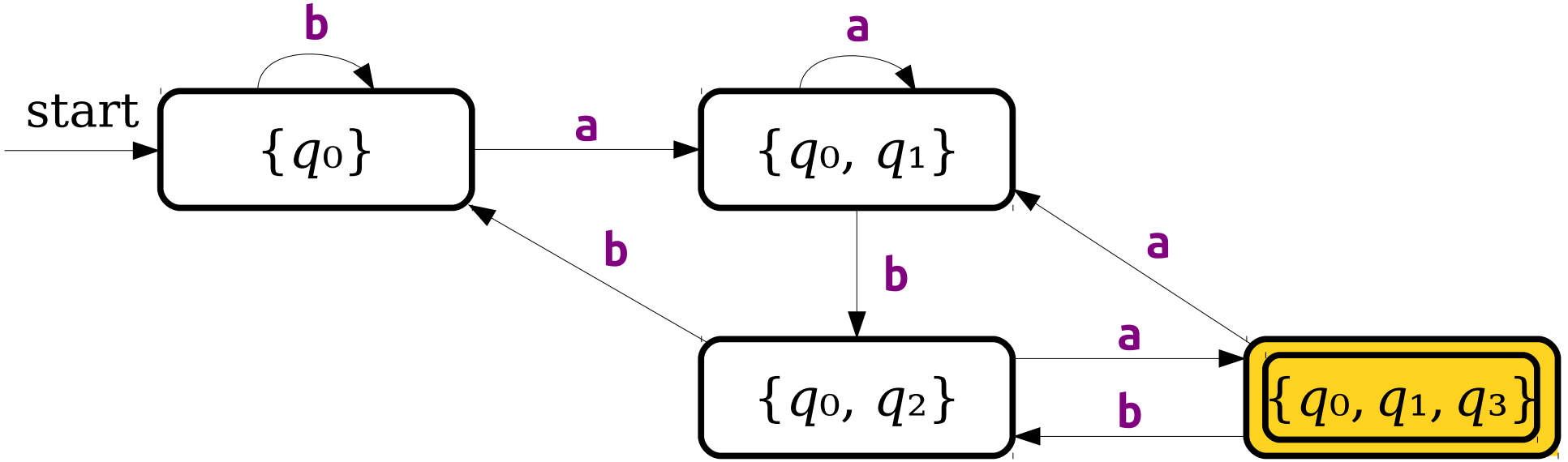
b
a


b
a
start
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a
$\{q_0, q_1\}$
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b
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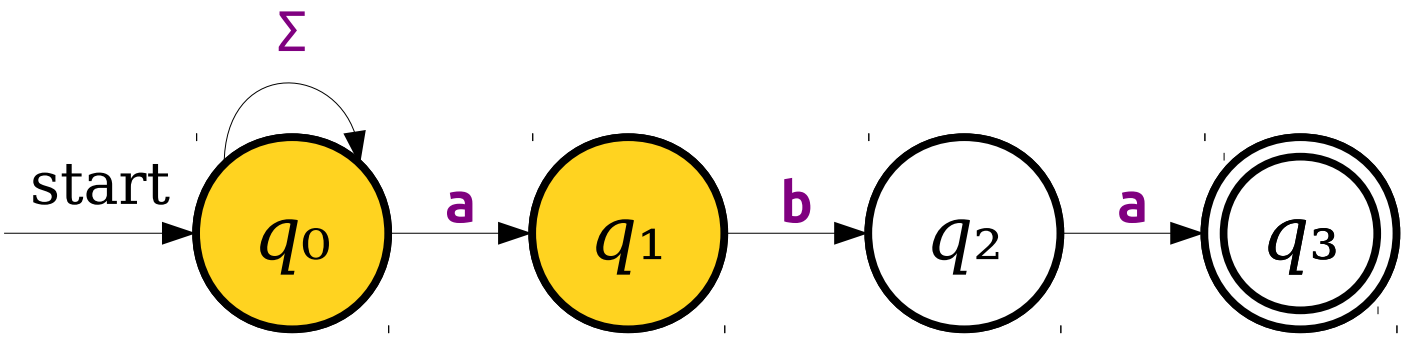
$\{q_0, q_2\}$
b
$\{q_0, q_1, q_3\}$


Σ
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q_1
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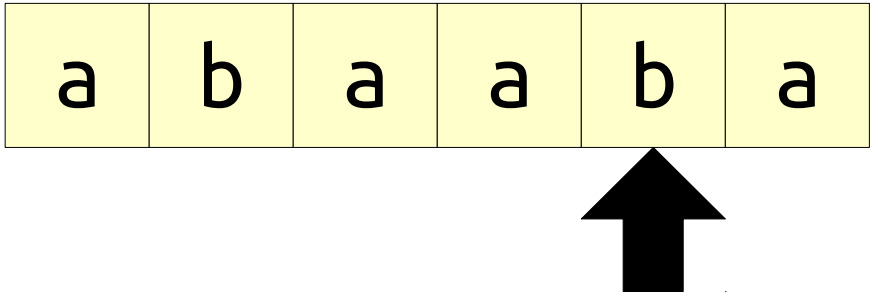
q_3


a
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b
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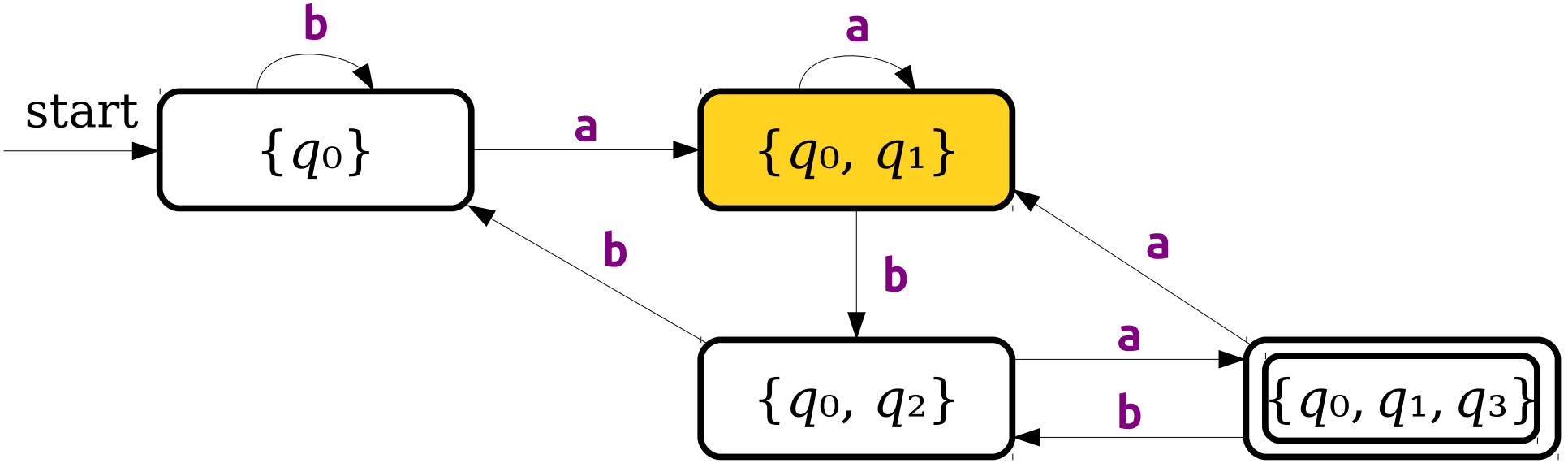
b
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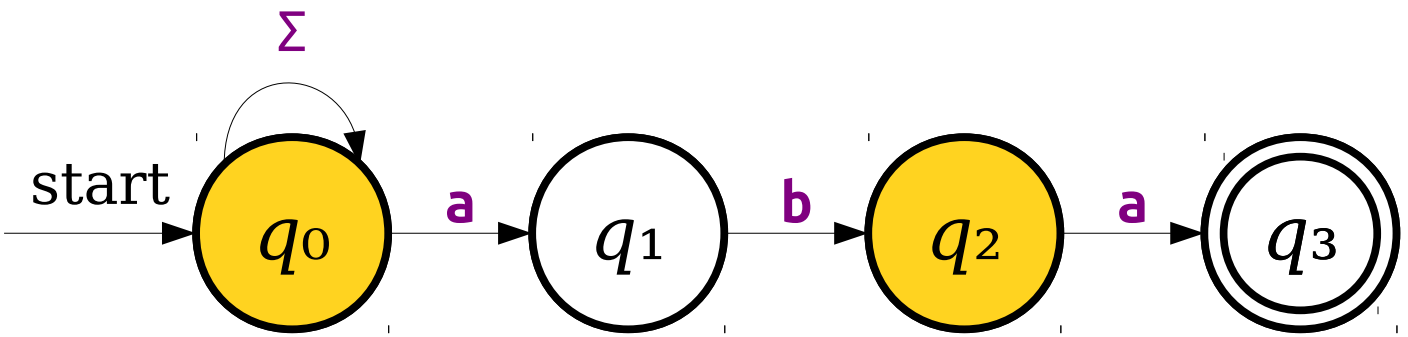
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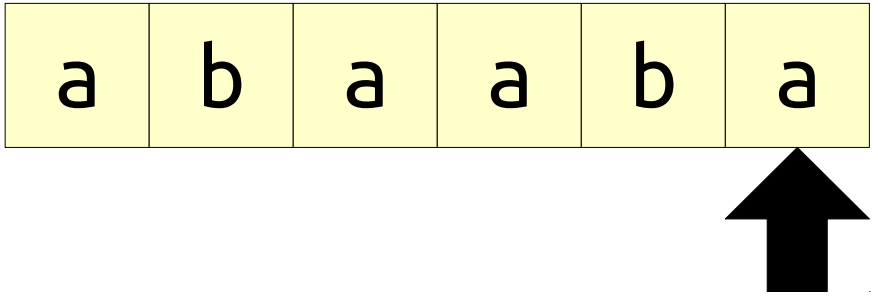
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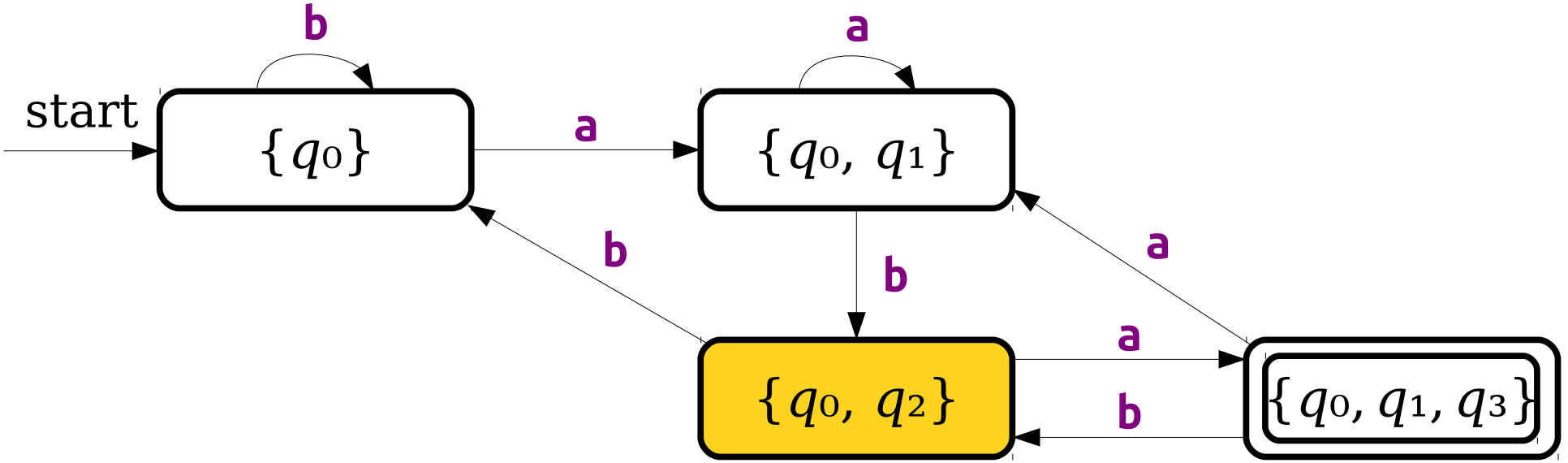
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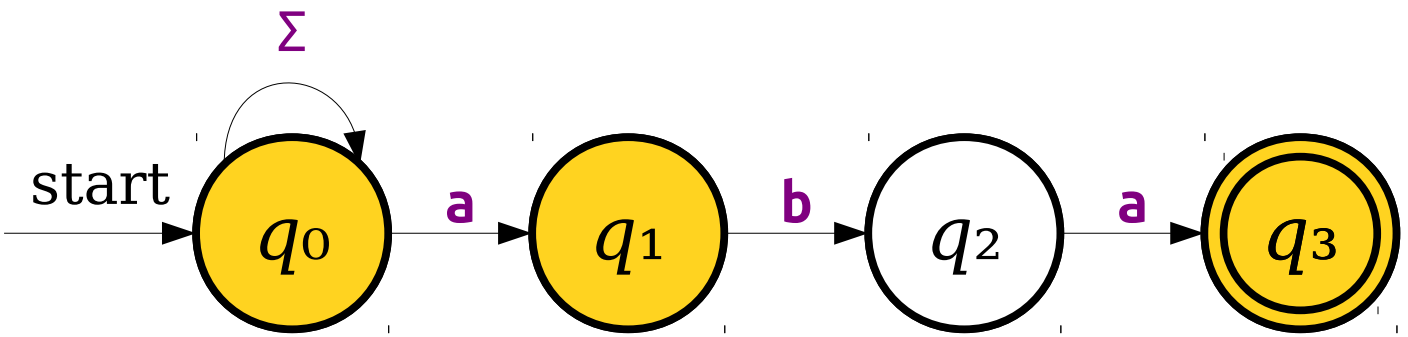
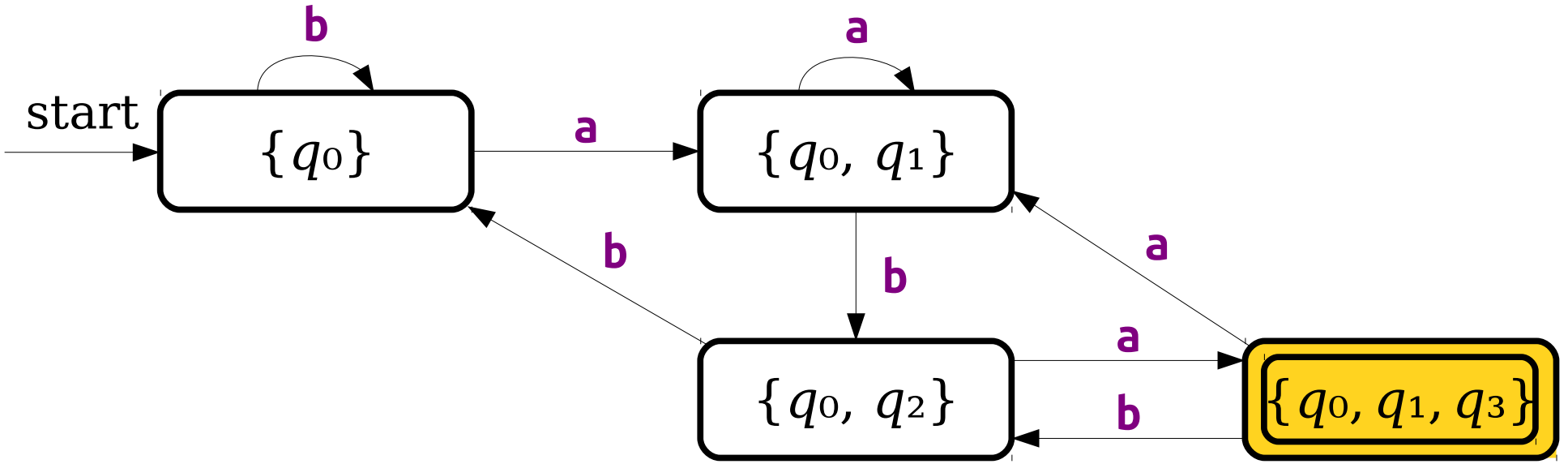


a	b	a	a	b	a



The Subset Construction

- This procedure for turning an NFA for a language L into a DFA for a language L is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.
- There's an online ***Guide to the Subset Construction*** with a more elaborate example involving ε-transitions and cases where the NFA dies; check that for more details.

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size n, but no DFAs of size less than 2^n?

A language L is called a ***regular language*** if there exists a DFA D such that $\mathscr{L}(D) = L$.

An Important Result

Theorem: A language L is regular if and only if there is some NFA N such that $\mathscr{L}(N) = L$.

Proof Sketch: Pick a language L. First, assume L is regular. That means there's a DFA D where $\mathscr{L}(D) = L$. Every DFA is "basically" an NFA, so there's an NFA (D) whose language is L.

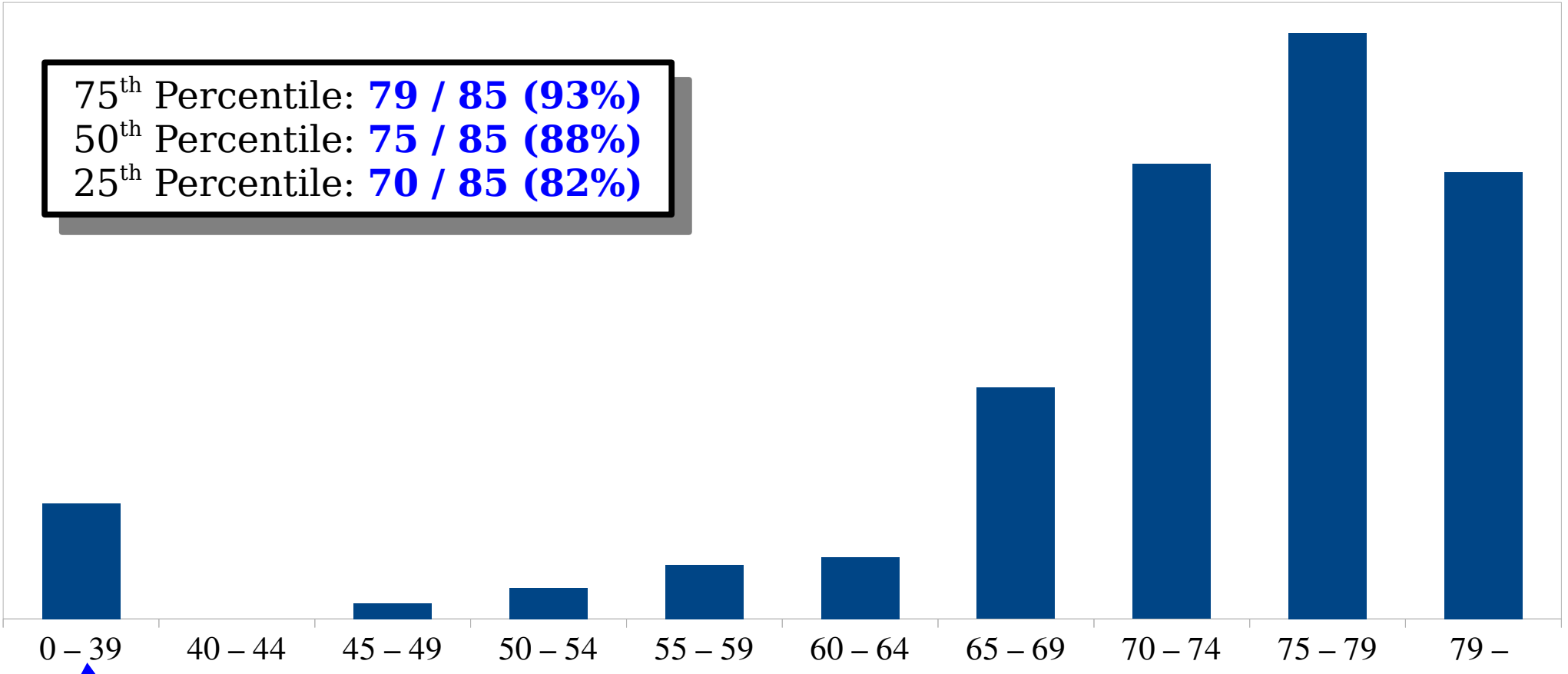
Next, assume there's an NFA N such that $\mathscr{L}(N) = L$. Using the subset construction, we can build a DFA D where $\mathscr{L}(N) = \mathscr{L}(D)$. Then we have that $\mathscr{L}(D) = L$, so L is regular. ■-*ish*

Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

Time-Out for Announcements!

Problem Set Four Grades



Many of these grades are because folks forgot to list partners – please check to make sure you're getting credit for the work you're doing, and let us know if your partner forgot to add you.

Problem Set Six

- Problem Set Five was due at 2:30PM today.
  - Want to use late days? One late day will extend this deadline to 2:30PM Saturday, and a second will extend it to 2:30PM Sunday.
- Problem Set Six goes out today. It's due next Friday at 2:30PM.
  - Play around with DFAs, NFAs, language transformations, and their properties!
  - Explore how all the discrete math topics we've talked about so far come into play!

DFA/NFA Editor

- We have an online DFA/NFA editor you'll use to answer and submit some of the questions for PS6.
- This tool will let you design and test your automata on a number of different inputs.
- You can also use it to explore on your own!
- One quick note: unlike the previous coding questions, we will only run the autograder once the problem set comes due. As a result, ***make sure to test your solutions thoroughly before submitting!***
  - Think about edge cases. What are some small strings that might break things? Some longer strings?
  - Pretend you haven't looked at your automata and just saw the language itself. What would be cases you'd expect would be really tricky?

Looking for a Partner?

- I’ve heard from many of you that you’re now looking for a problem set partner.
- Don’t forget that Piazza has a lovely “Search for Teammates” feature that you can use to do this.
- It’s like speed dating for theory!

Midterm Practice Problems

- If you'd like to get a jump on studying for the second midterm, feel free to work through the four practice exams we've posted to the course website.
- There's also Extra Practice Problems 2 to work through.
- We'll be holding a practice midterm exam ***next Wednesday*** evening from ***7PM – 10PM***, location TBA. It'll use an exam that's not yet posted to the course website.

Your Questions

“How can you "differentiate" yourself as a programmer? Especially, at Stanford since you are one out of so many.”

My first question is why you'd want to differentiate yourself as a programmer – that's not something you necessarily need to do at this point. I'd focus a lot more on skill acquisition and on finding what makes you happy before worrying about this. There aren't many times where you need to "stand out" of the crowd as a programmer, and most of them will arise because you're competent, talented, and easy to work with.

Your personal identity doesn't have to be tied to your coding skills. You're a whole person and these skills are just a part of that.

“Why do you like the number 137 so much?”

It's the reciprocal of the fine structure constant, rounded to the nearest integer. It's also a great "nothing-up-my-sleeve" number.

Back to CS103!

Properties of Regular Languages

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

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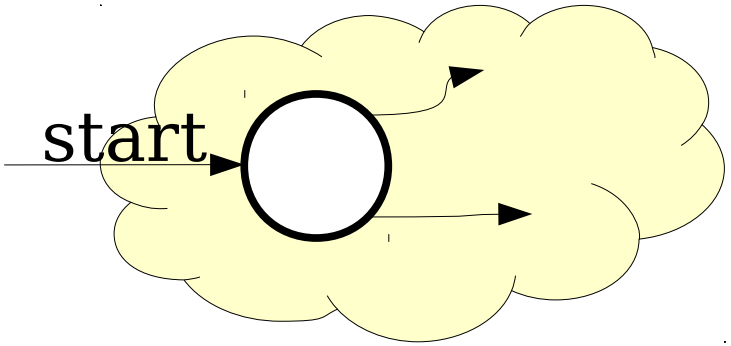
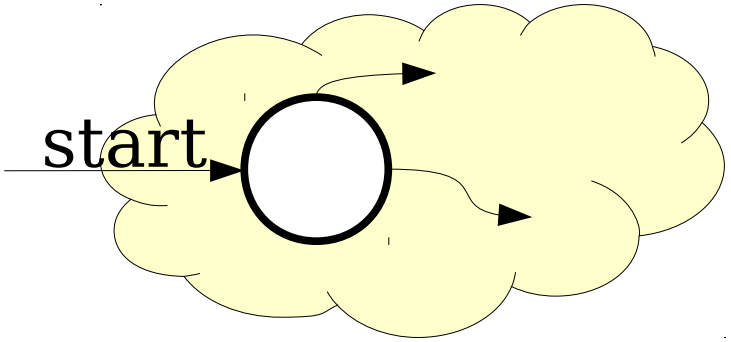


Machine for L_1



Machine for L_2

The Union of Two Languages

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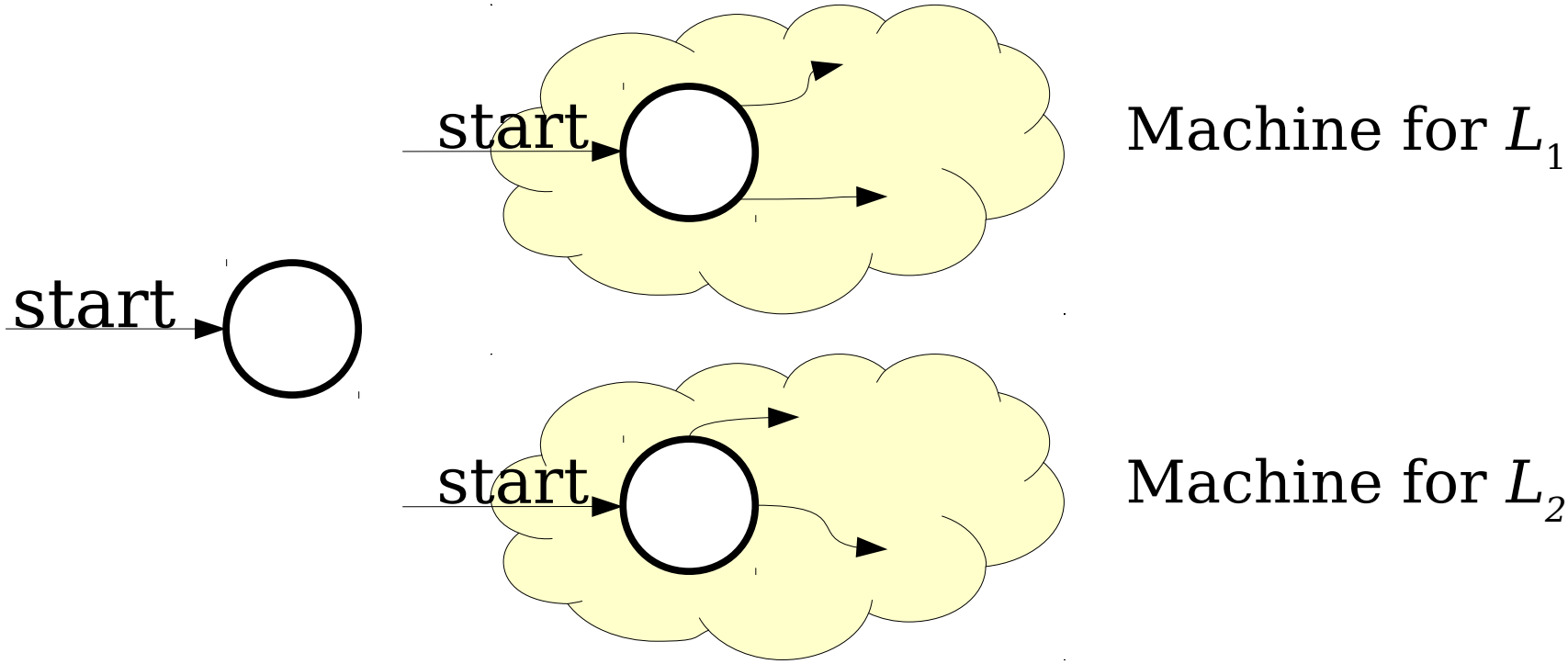

The Union of Two Languages

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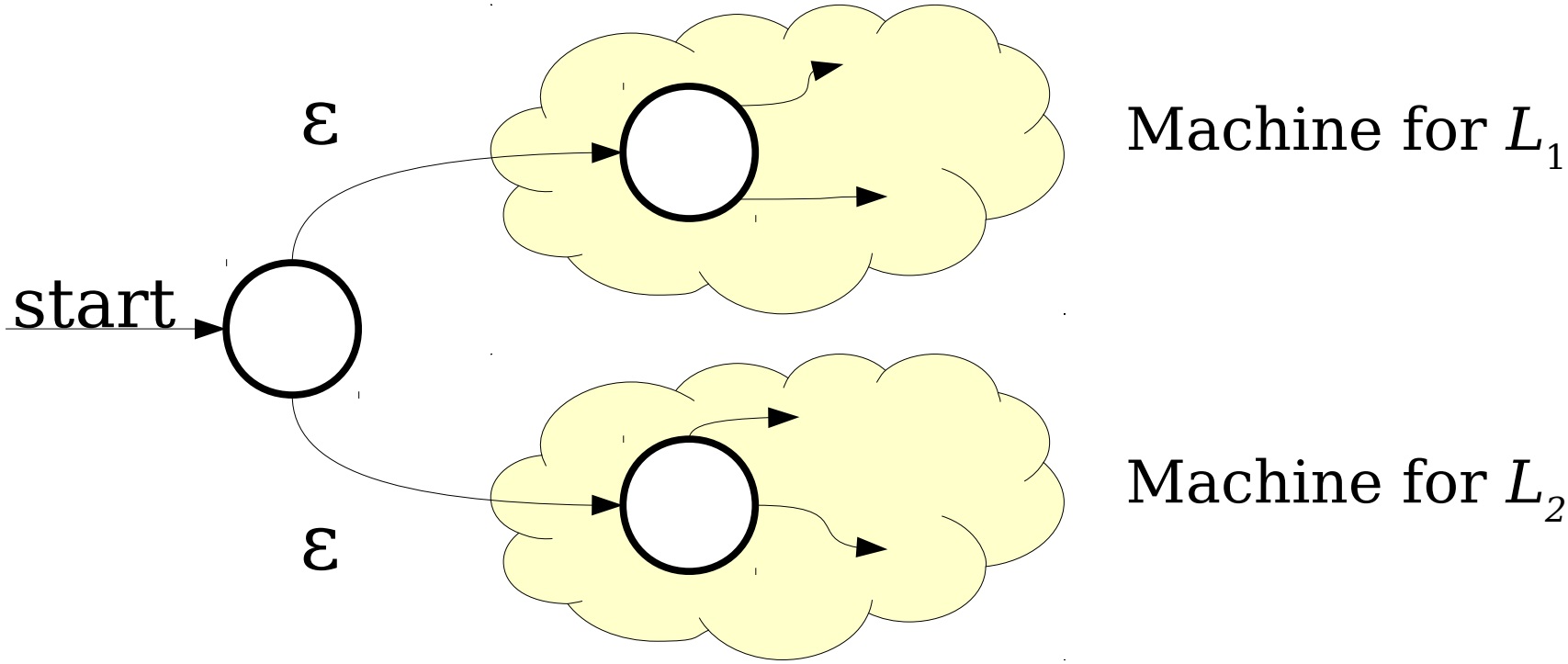

The Union of Two Languages

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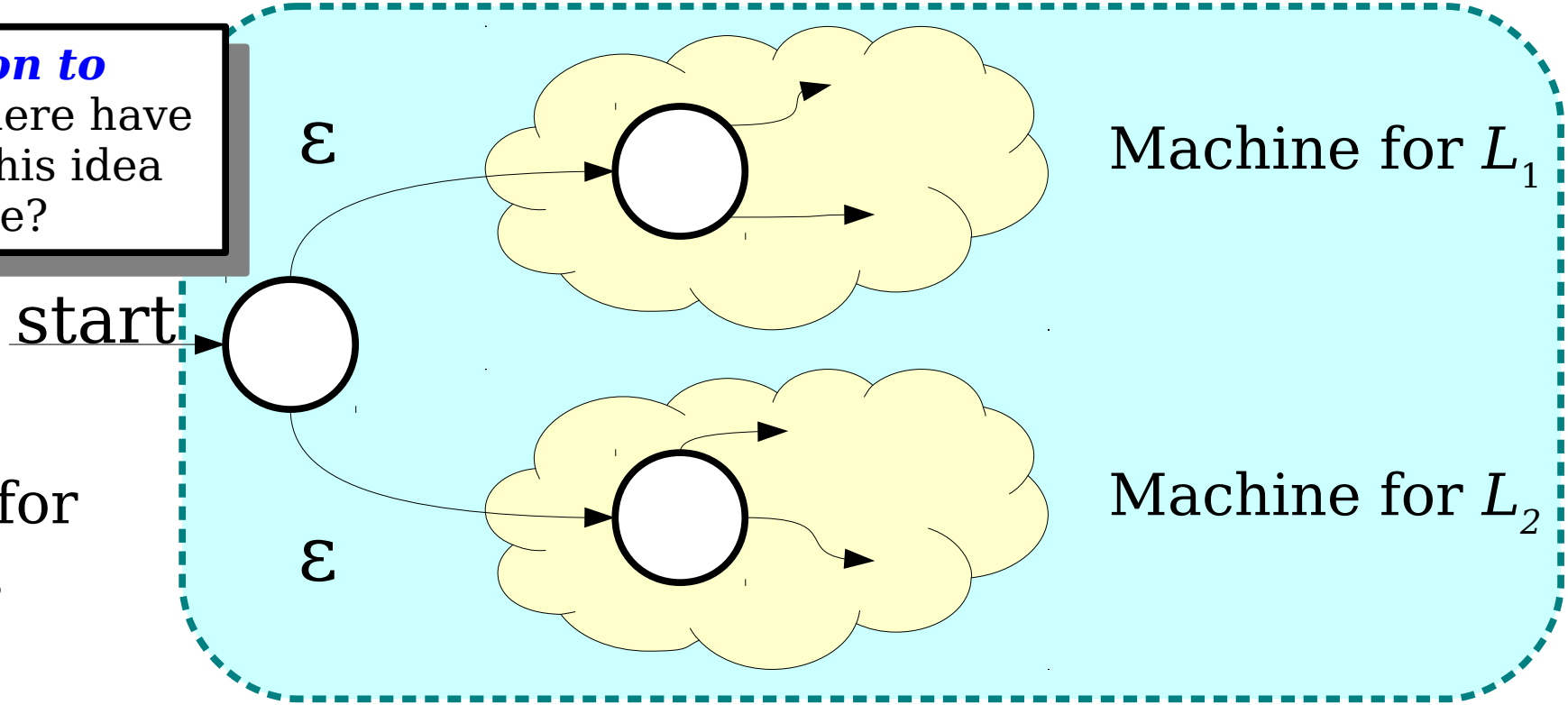
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

Question to ponder: where have you seen this idea before?



Machine for $L_1 \cup L_2$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?

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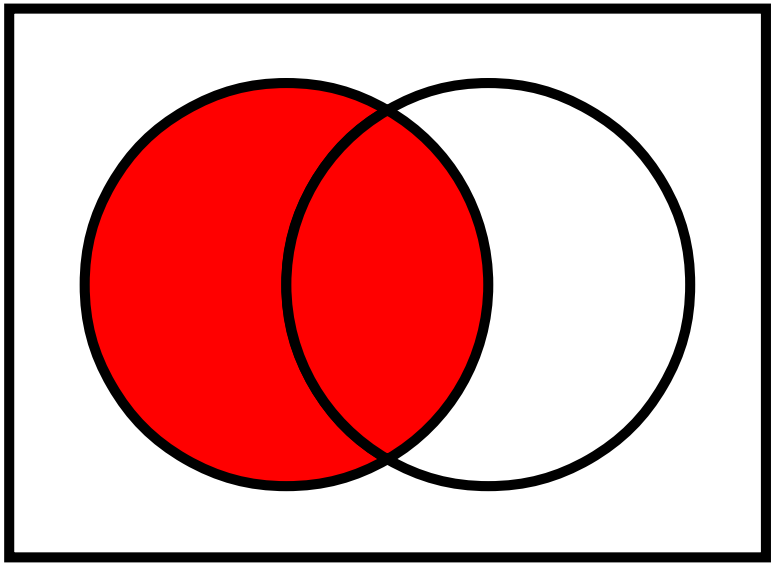
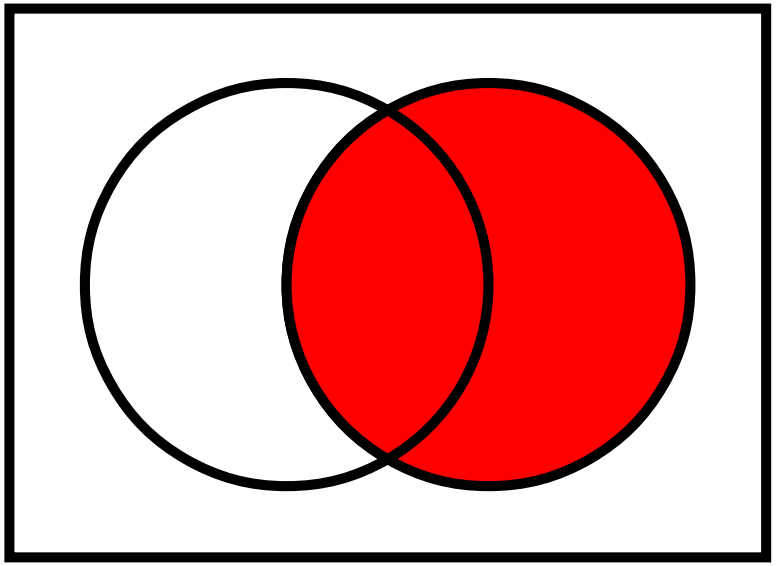


L_1



L_2

The Intersection of Two Languages

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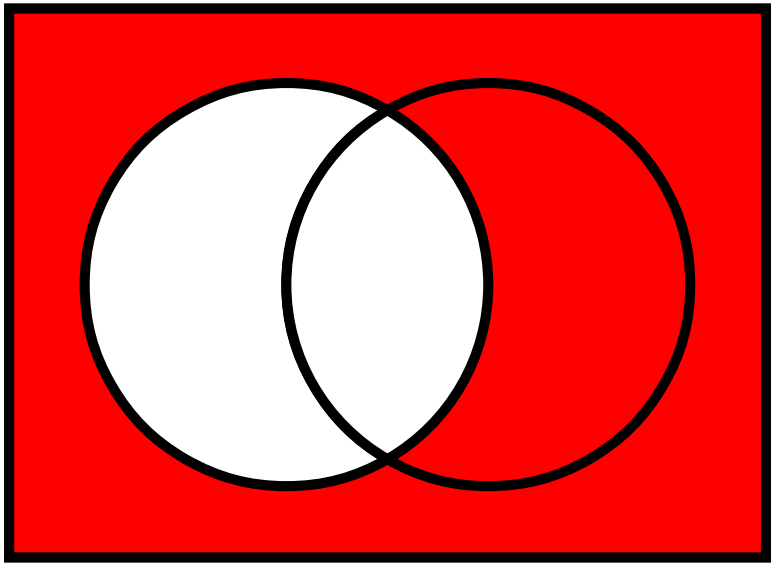
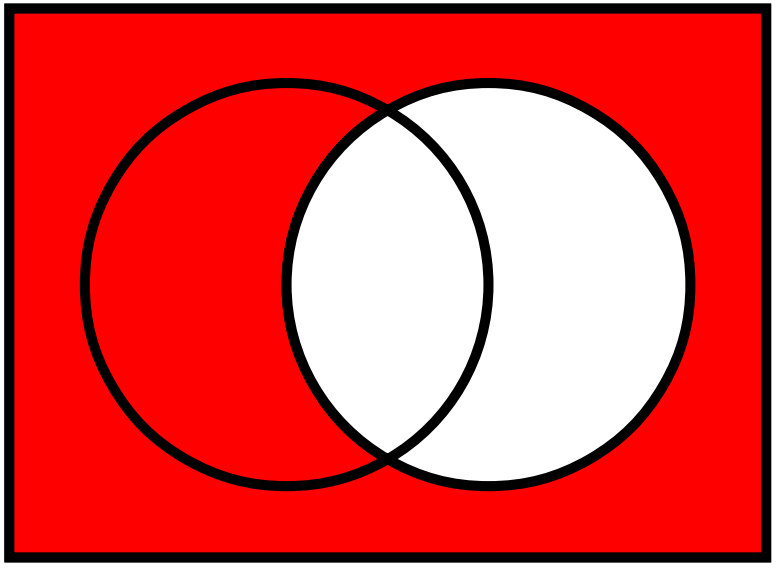


$\overline{L}_1$



$\overline{L}_2$

The Intersection of Two Languages

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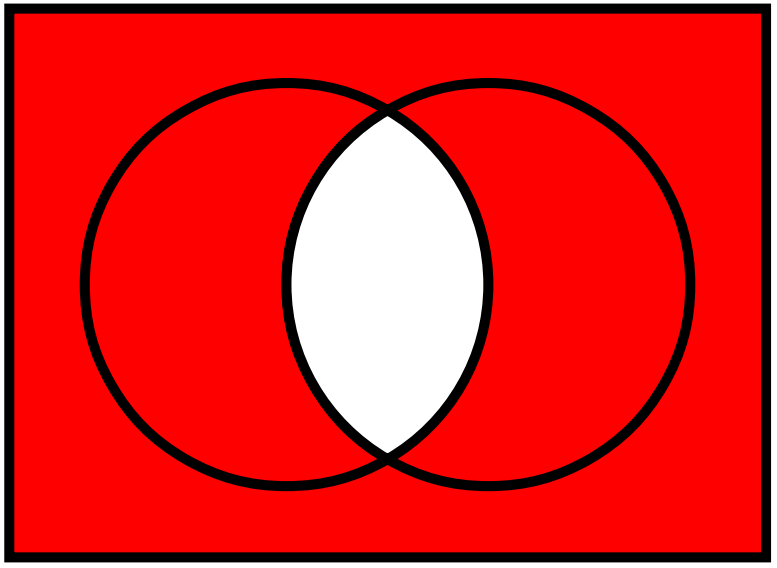
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?



$$\overline{L}_1 \cup \overline{L}_2$$

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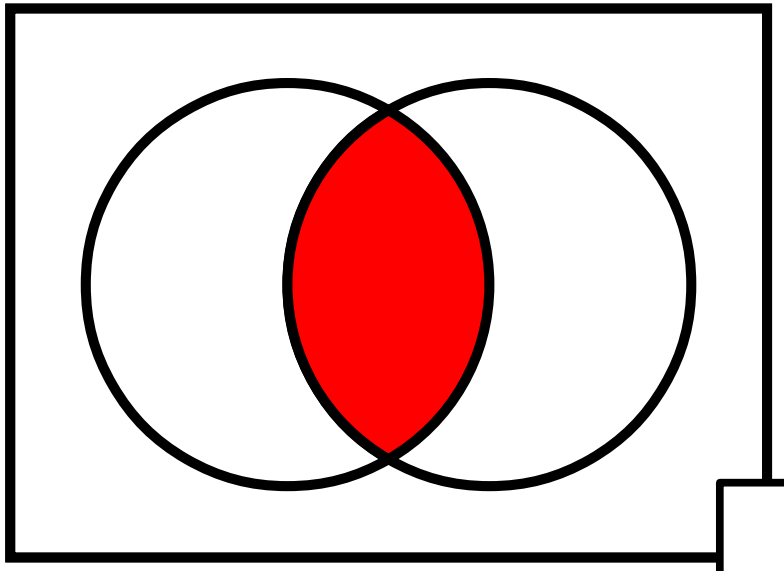

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of w and x, denoted ***wx***, is the string formed by tacking all the characters of x onto the end of w.
- Example: if w = `quo` and x = `kka`, the concatenation wx = `quokka`.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

$$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

$$wxy = w(xy) = (wx)y$$

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

Concatenation Example

- Let Σ = { `a`, `b`, …, `z`, `A`, `B`, …, `Z` } and consider these languages over Σ:
  - ***Noun*** = { `Puppy`, `Rainbow`, `Whale`, … }
  - ***Verb*** = { `Hugs`, `Juggles`, `Loves`, … }
  - ***The*** = { `The` }
- The language ***TheNounVerbTheNoun*** is
  - { `ThePuppyHugsTheWhale`, `TheWhaleLovesTheRainbow`, `TheRainbowJugglesTheRainbow`, … }

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

- Two views of L_1L_2:
  - The set of all strings that can be made by concatenating a string in L_1 with a string in L_2.
  - The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.

This is closely related to, but different than, the Cartesian product.

Question to ponder: In what ways are concatenations similar to Cartesian products? In what ways are they different?

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?

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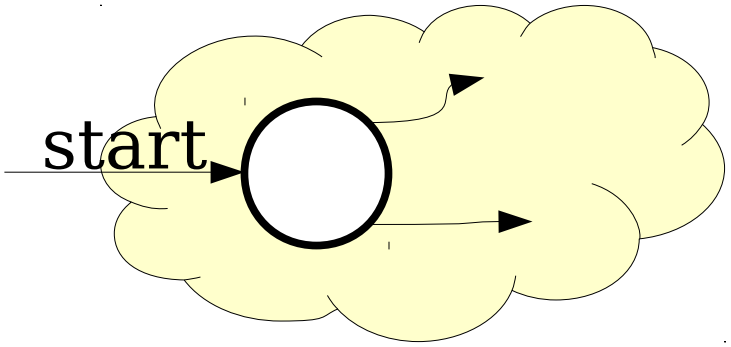
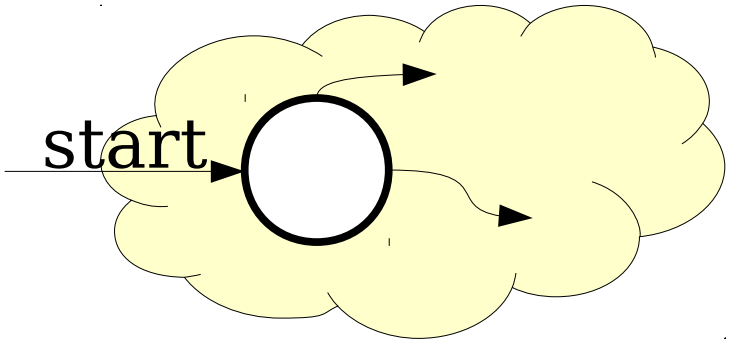


Machine for L_1



Machine for L_2

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
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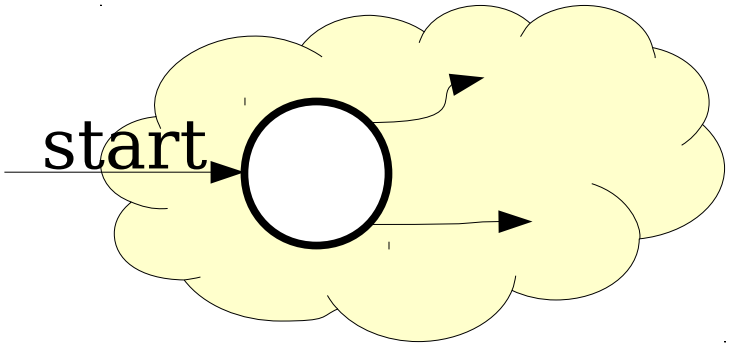

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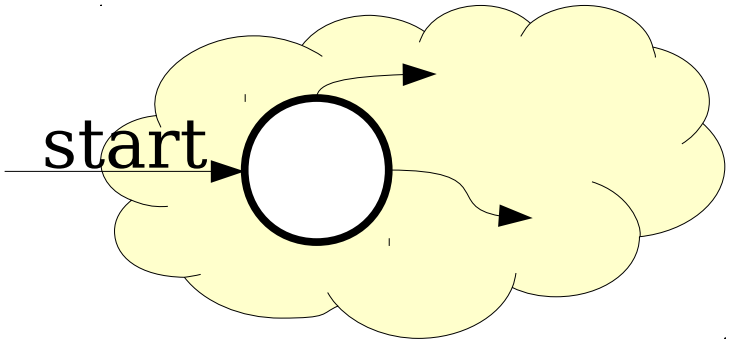

Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

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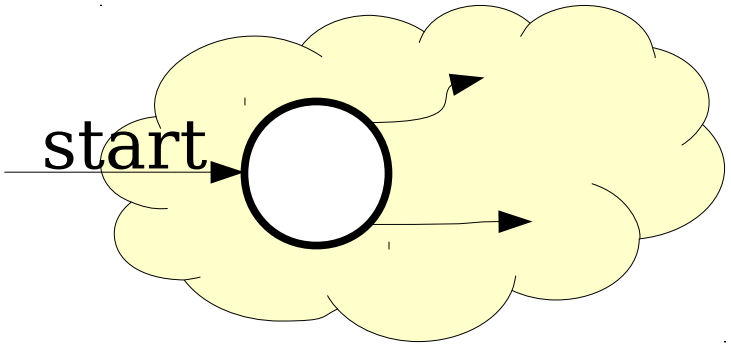

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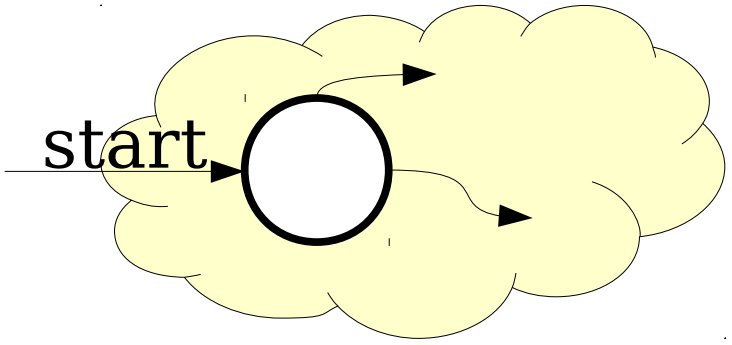

Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

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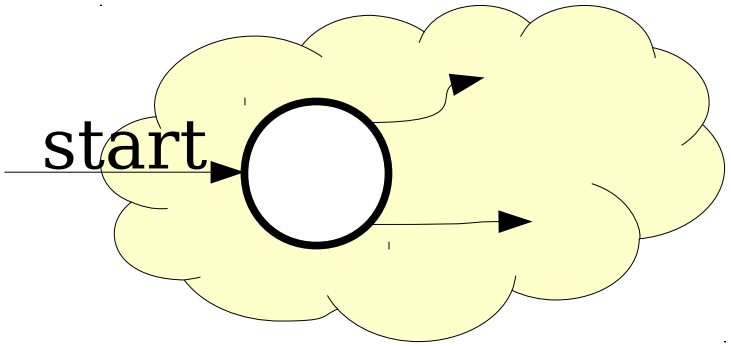

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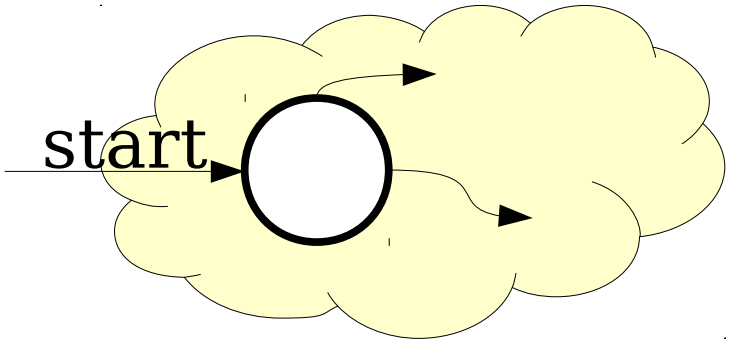


Machine for L_2

b	o	o	k

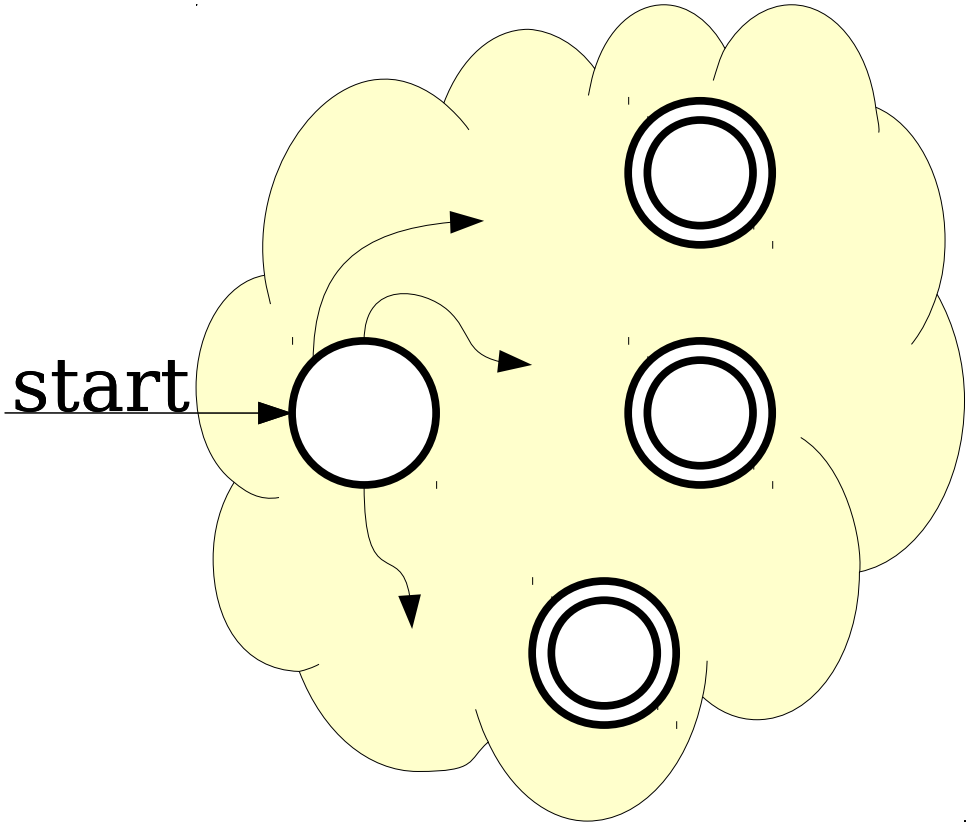
k	e	e	p	e	r

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea***:
  - Run a DFA/NFA for L_1 on w.
  - Whenever it reaches an accepting state, optionally hand the rest of w to a DFA/NFA for L_2.
  - If the automaton for L_2 accepts the rest, $w \in L_1L_2$.
  - If the automaton for L_2 rejects the remainder, the split was incorrect.

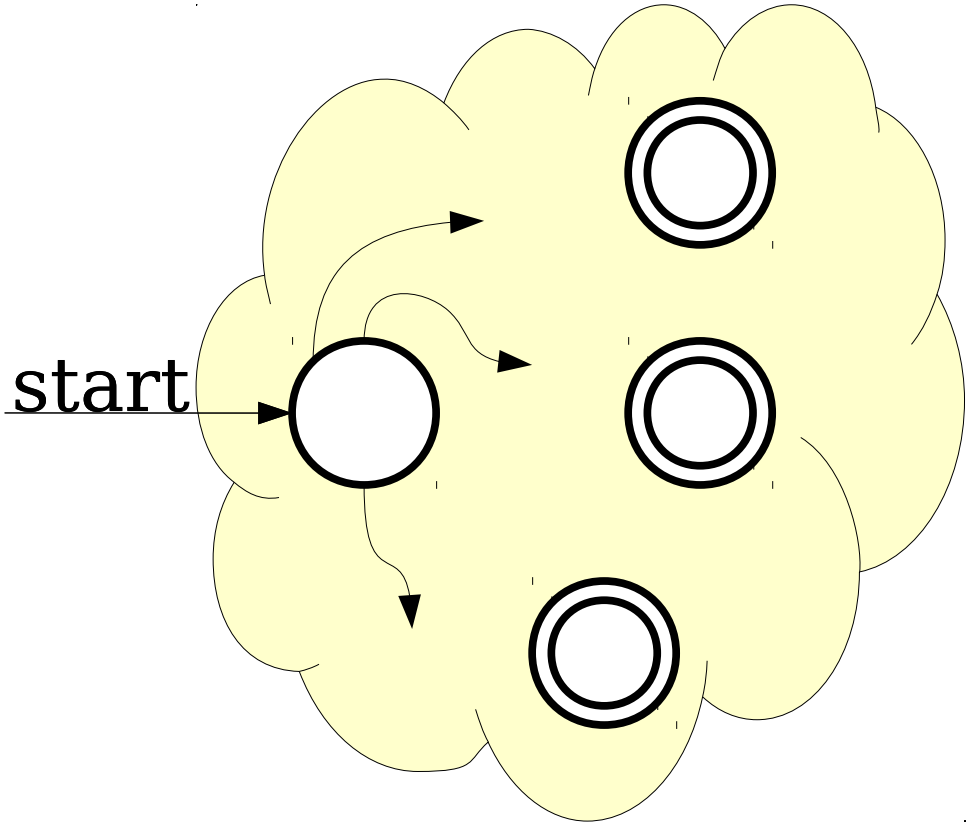
Concatenating Regular Languages

Concatenating Regular Languages

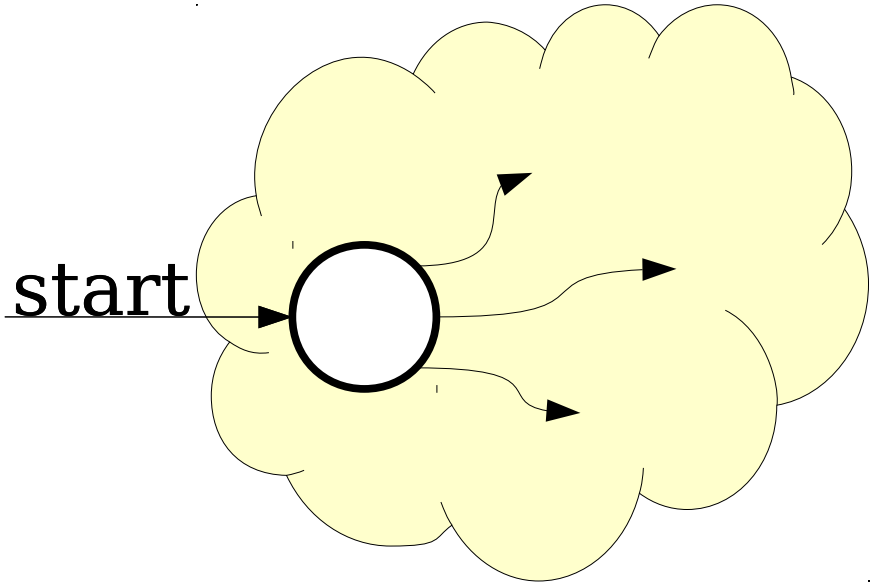


Machine for L_1

Concatenating Regular Languages

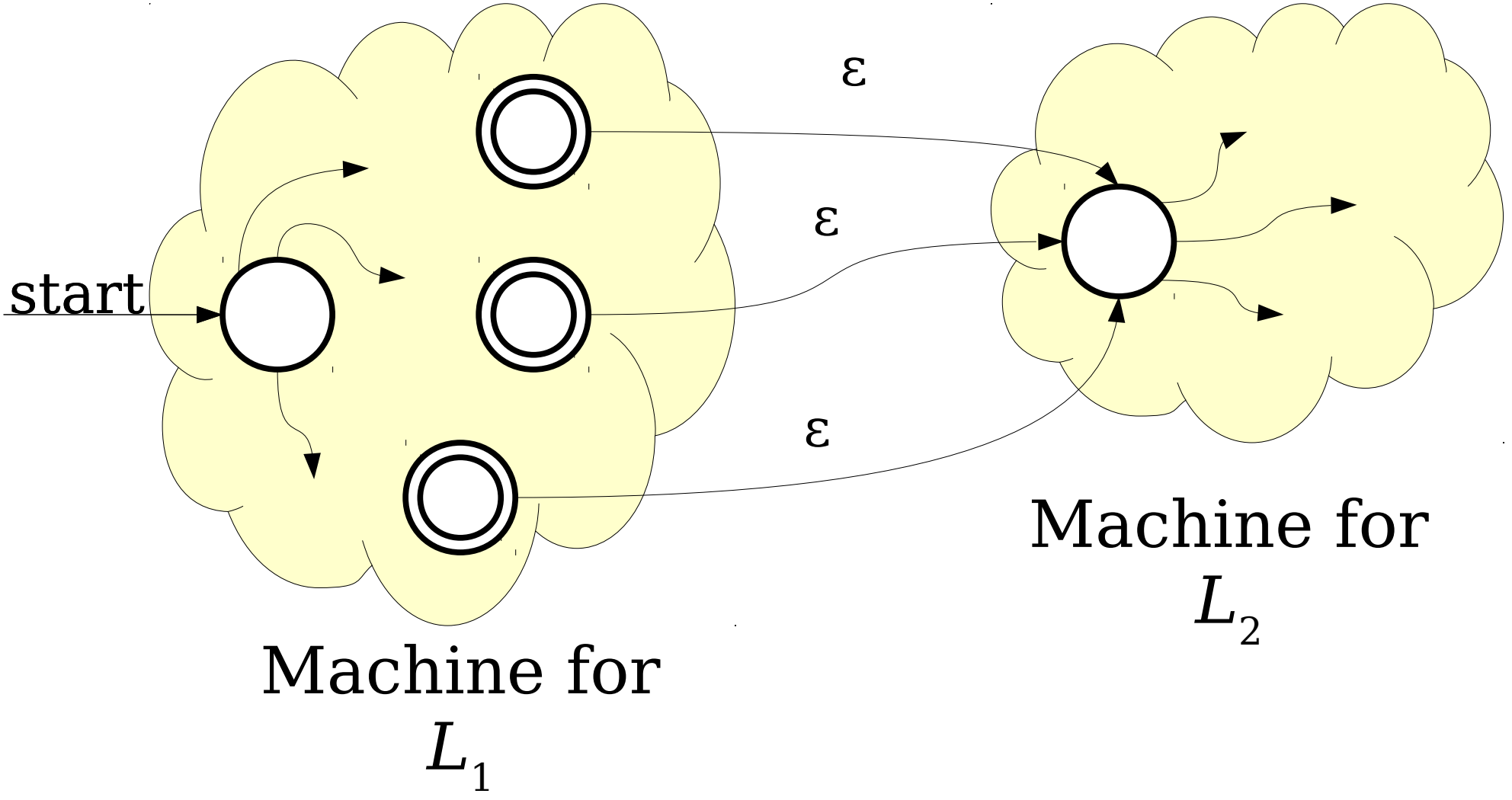


Machine for L_1

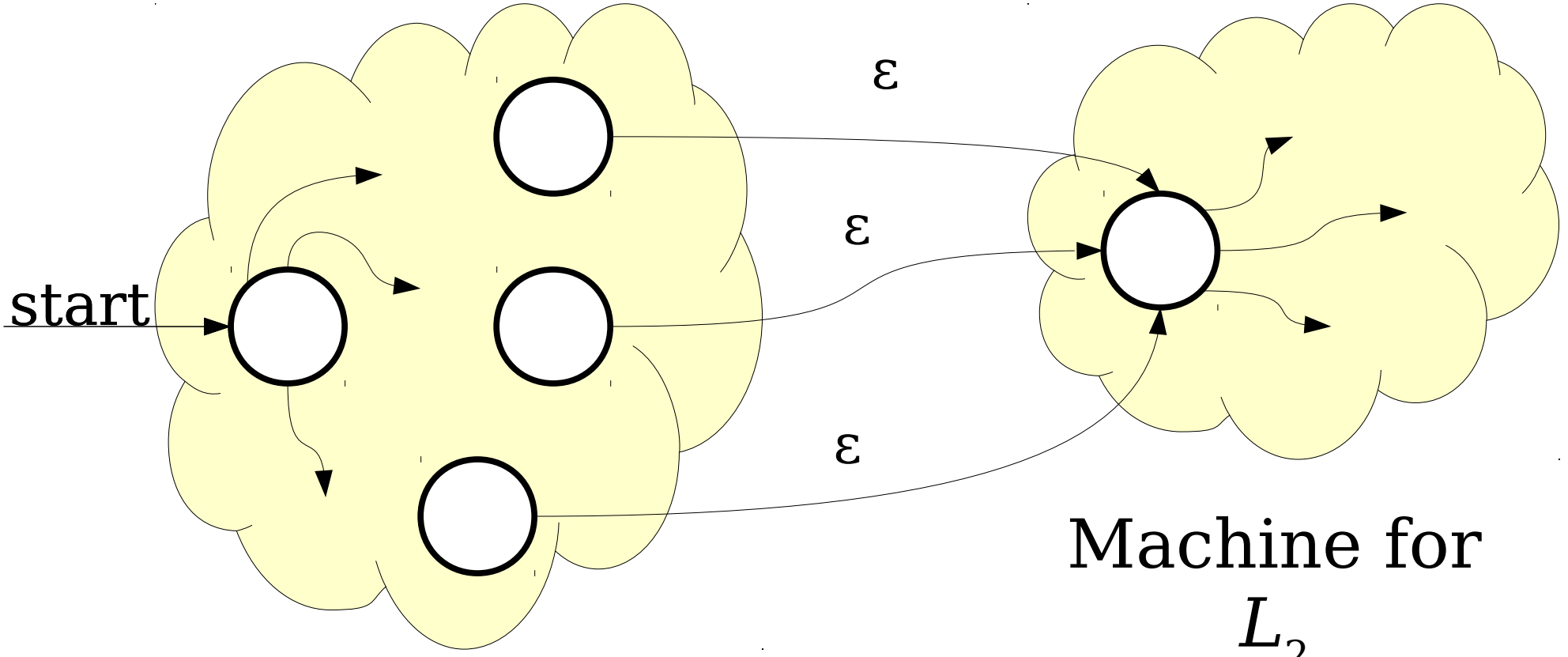


Machine for L_2

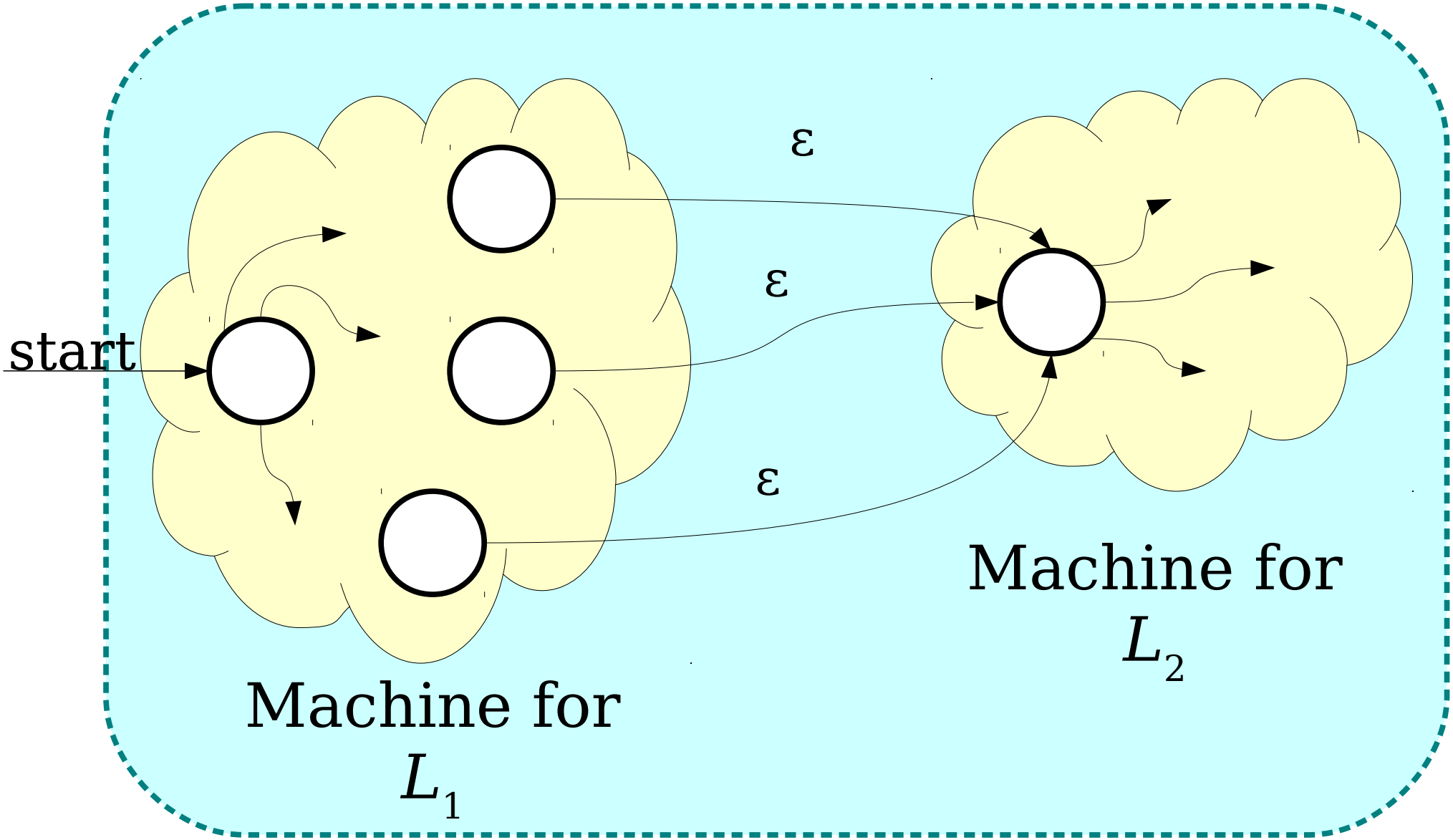
Concatenating Regular Languages



Concatenating Regular Languages



Concatenating Regular Languages



Machine for $L_1 L_2$

Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**,
aabaab, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**,
baabb, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \varnothing$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\varnothing^0$?

The Kleene Star

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, L^* is the language all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question to ponder:*** What is $\emptyset^*$?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

...

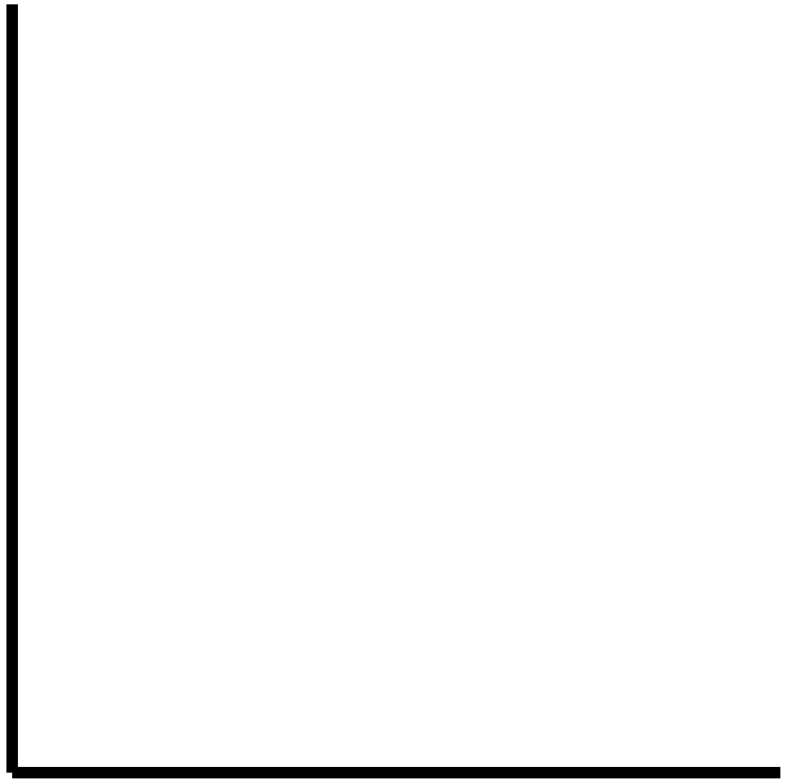
}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

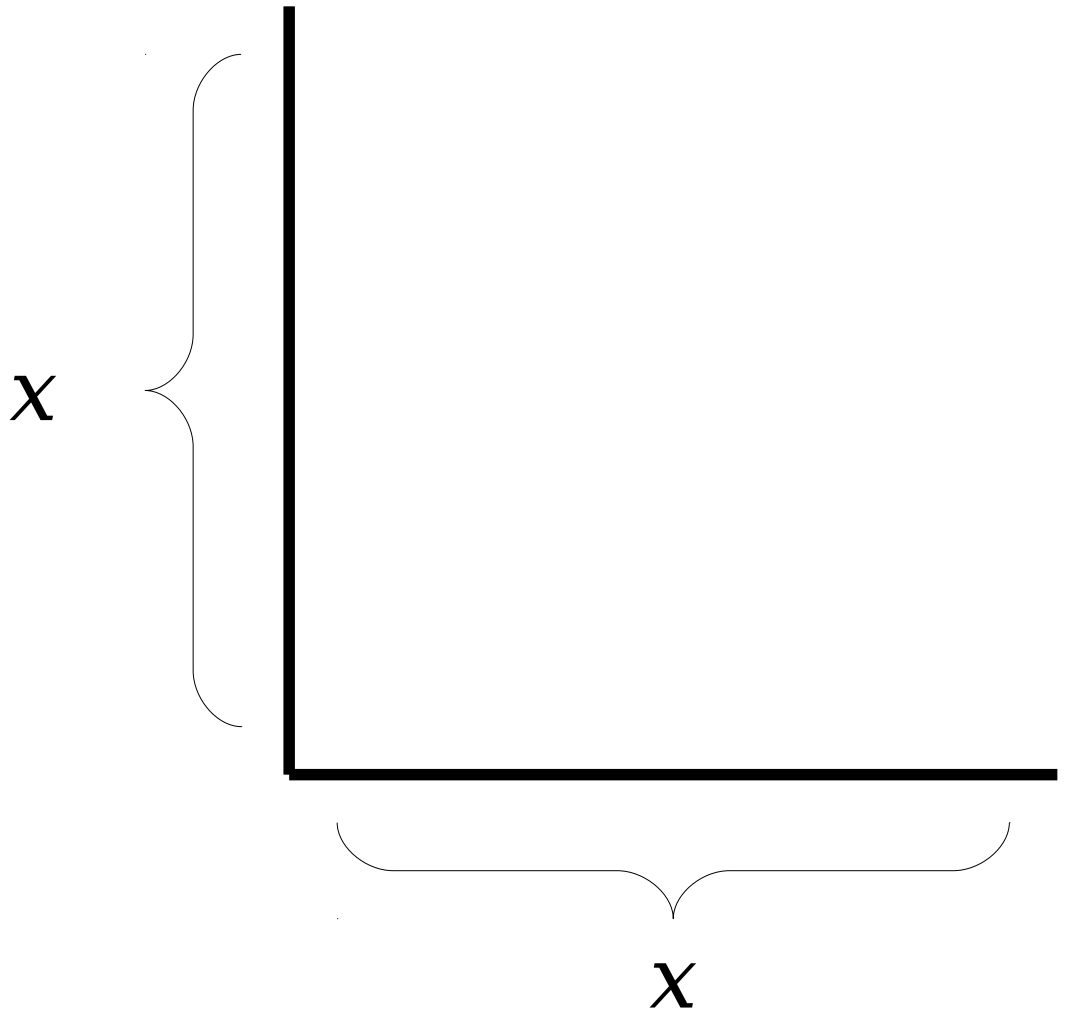
Reasoning about Infinity

- If L is regular, is L^* necessarily regular?
- ⚠ **A Bad Line of Reasoning:** ⚠
  - $L^0 = \{ \varepsilon \}$ is regular.
  - $L^1 = L$ is regular.
  - $L^2 = LL$ is regular
  - $L^3 = L(LL)$ is regular
  - …
  - Regular languages are closed under union.
  - So the union of all these languages is regular.

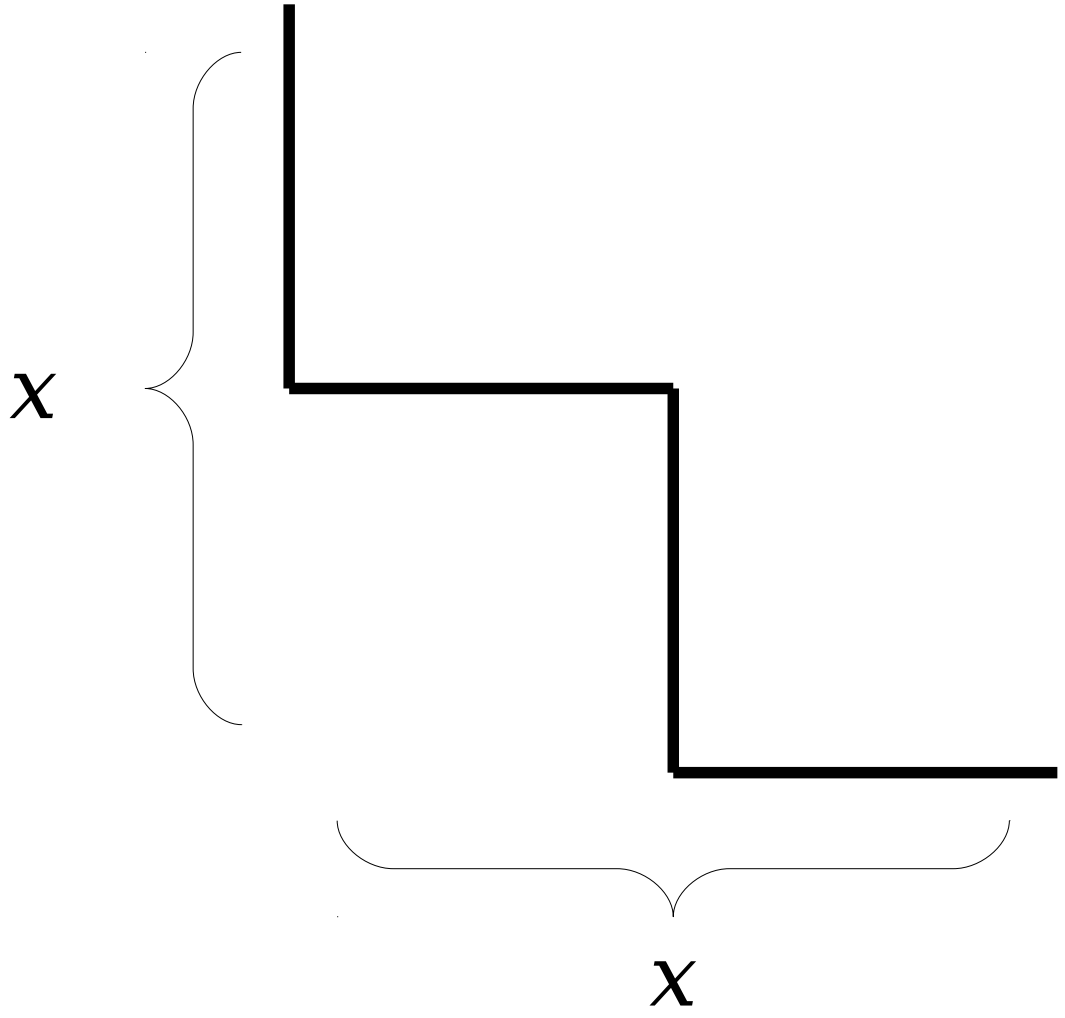
Reasoning about Infinity



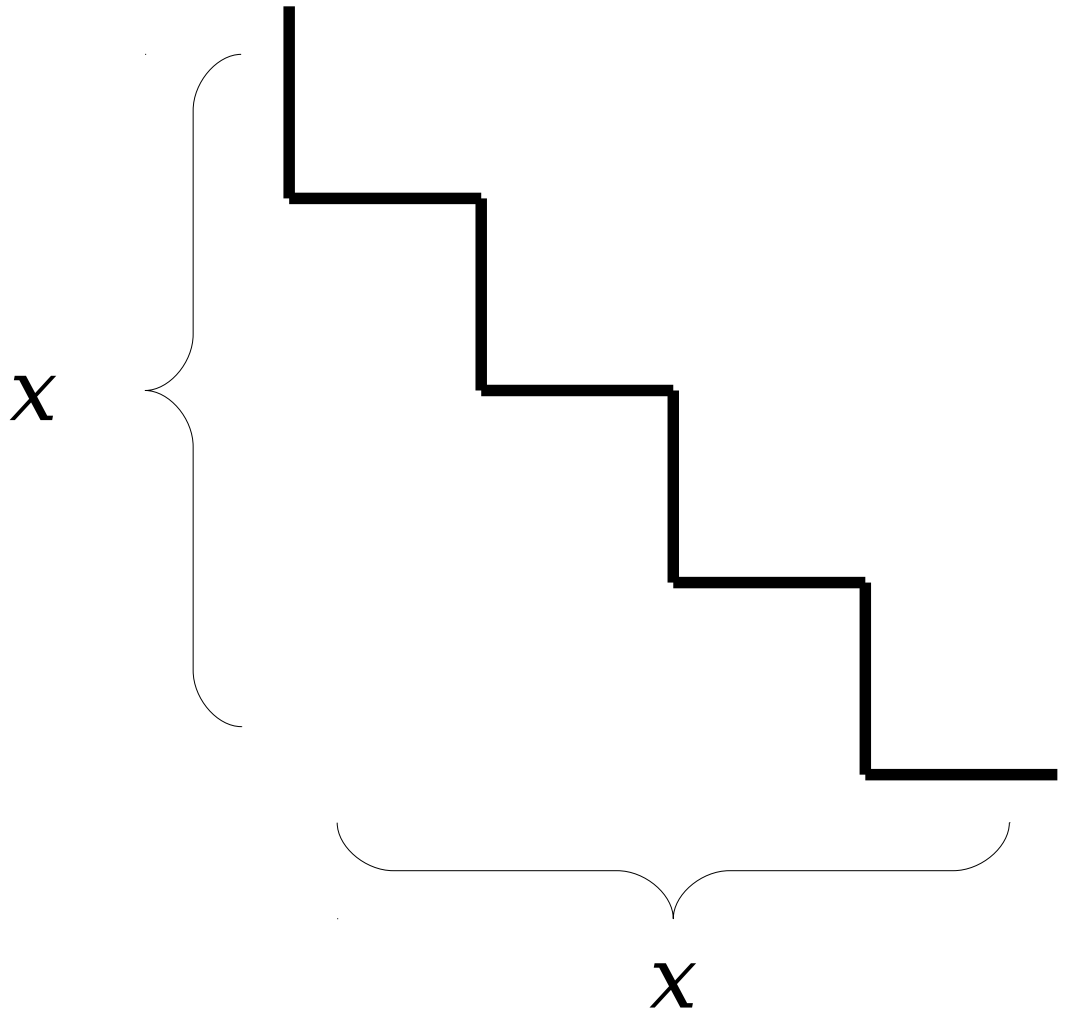
Reasoning about Infinity



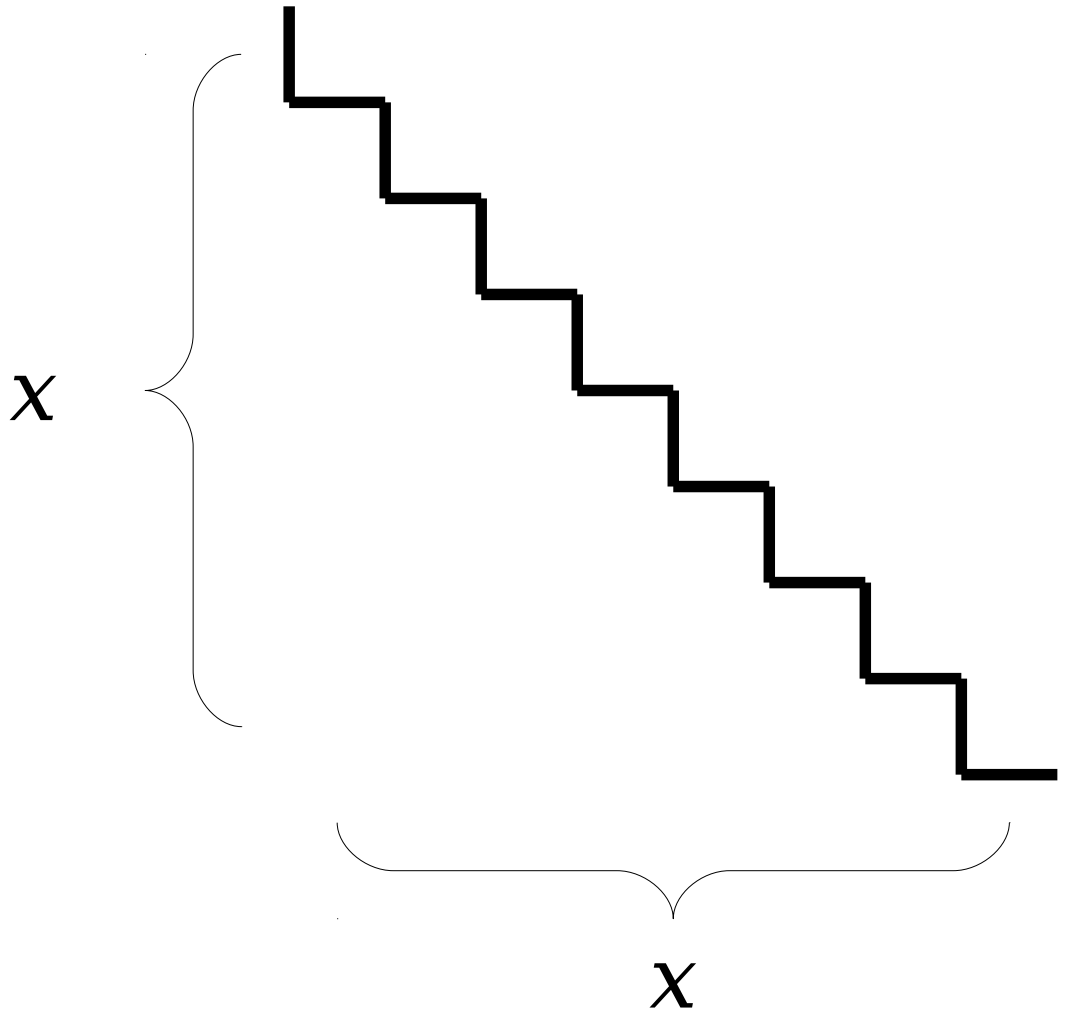
Reasoning about Infinity



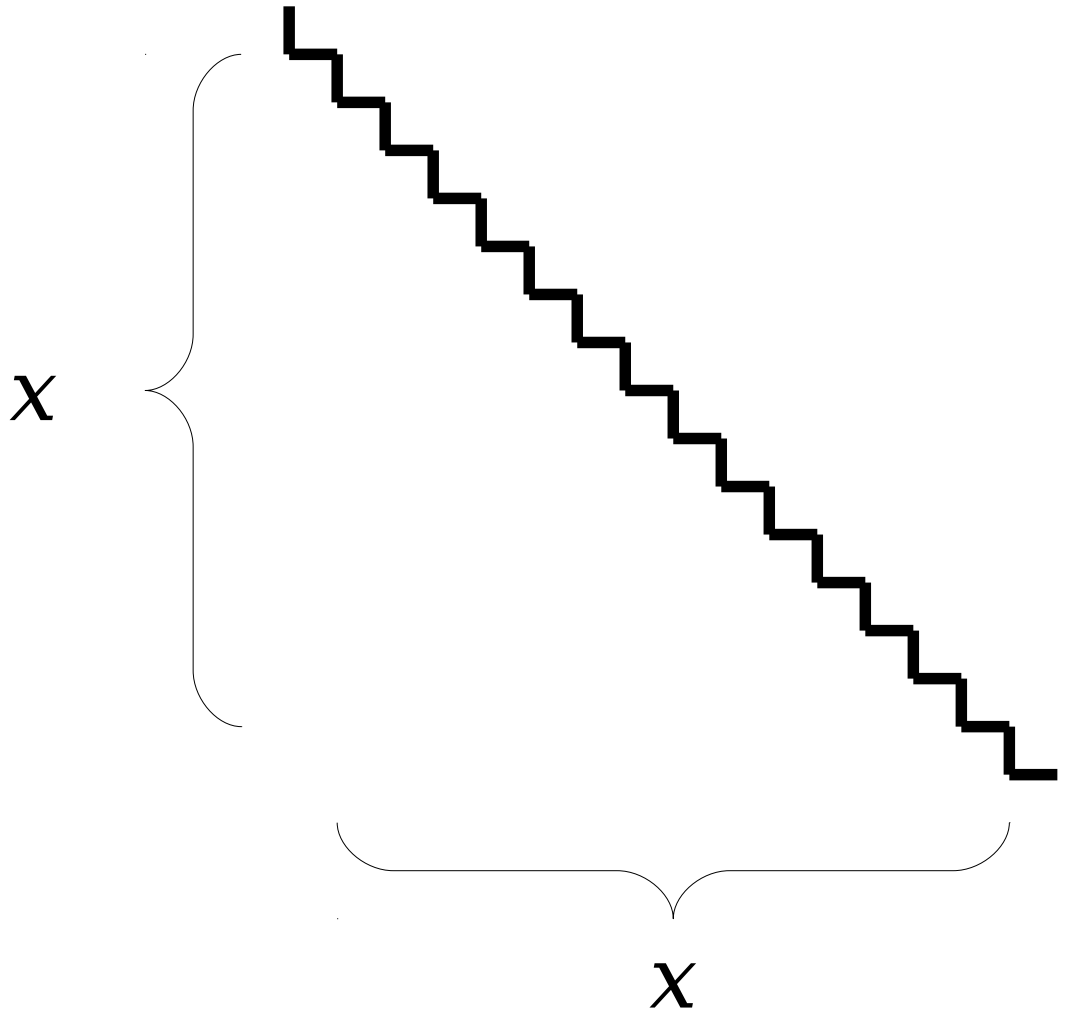
Reasoning about Infinity



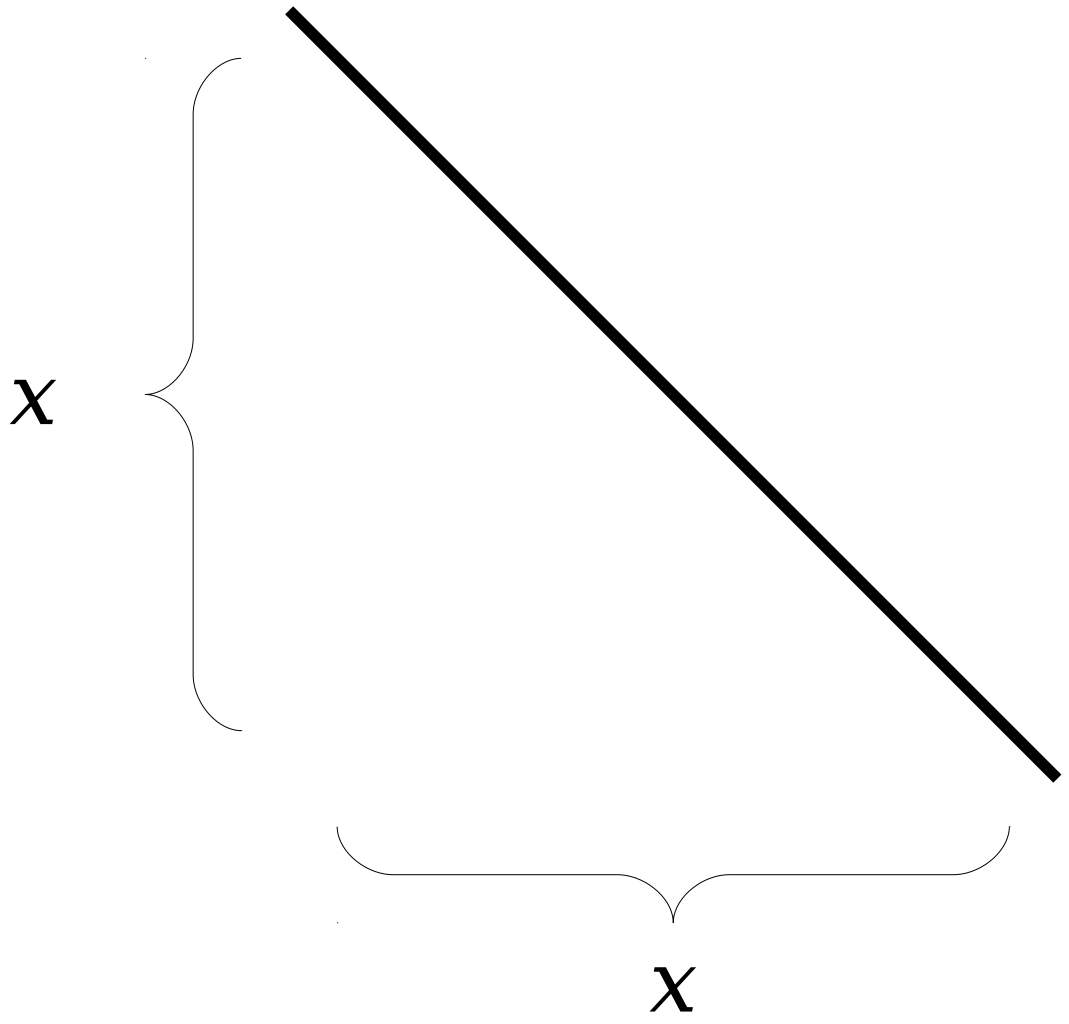
Reasoning about Infinity



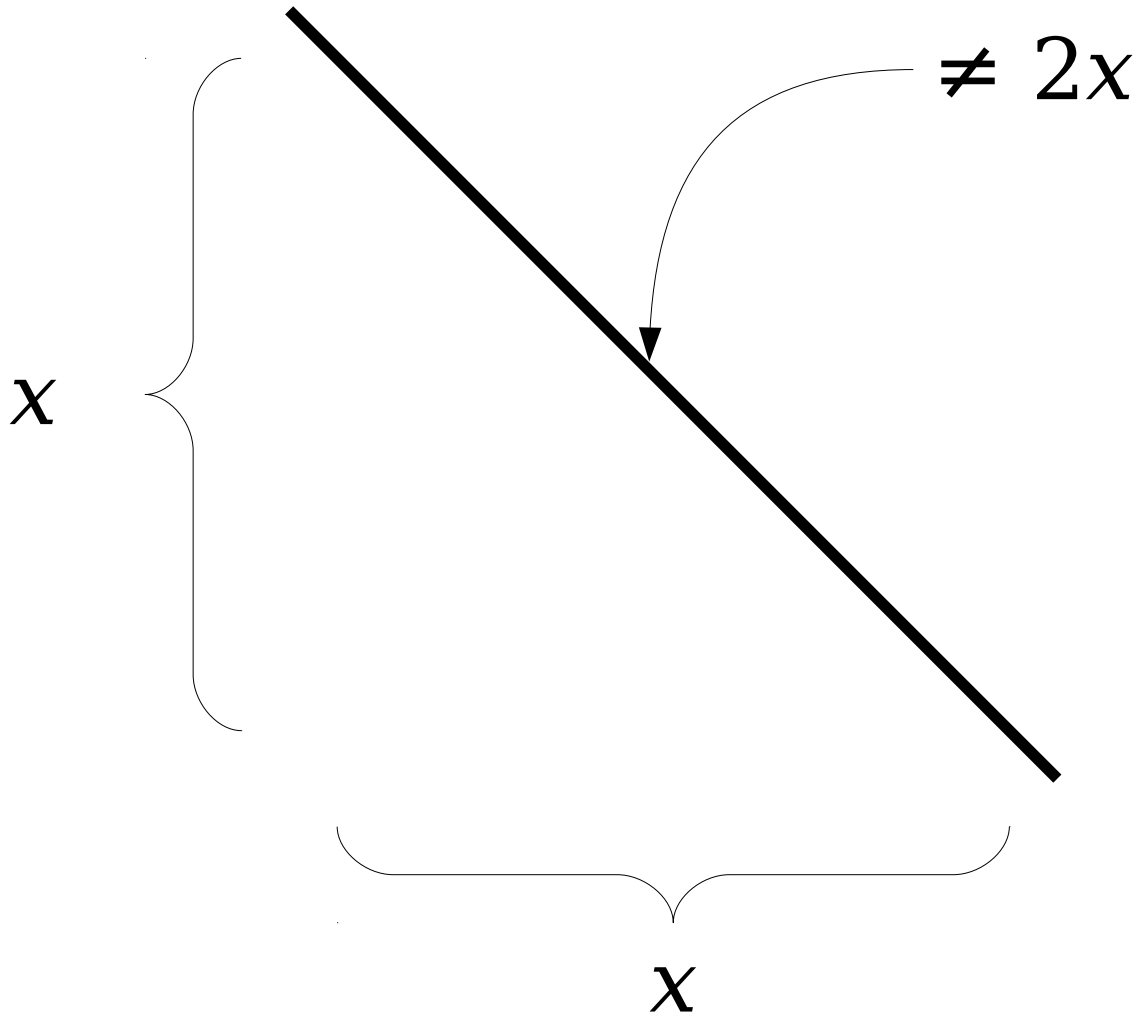
Reasoning about Infinity



Reasoning about Infinity



Reasoning about Infinity



Reasoning about Infinity

$$0.9 < 1$$

Reasoning about Infinity

$$0.99 < 1$$

Reasoning about Infinity

$$0.999 < 1$$

Reasoning about Infinity

$$0.9999 < 1$$

Reasoning about Infinity

$$0.99999 < 1$$

Reasoning about Infinity

$$0.9999\overline{9} \not< 1$$

Reasoning about Infinity

0 is finite

Reasoning about Infinity

1 is finite

Reasoning about Infinity

2 is finite

Reasoning about Infinity

3 is finite

Reasoning about Infinity

4 is finite

Reasoning about Infinity

∞ is finite

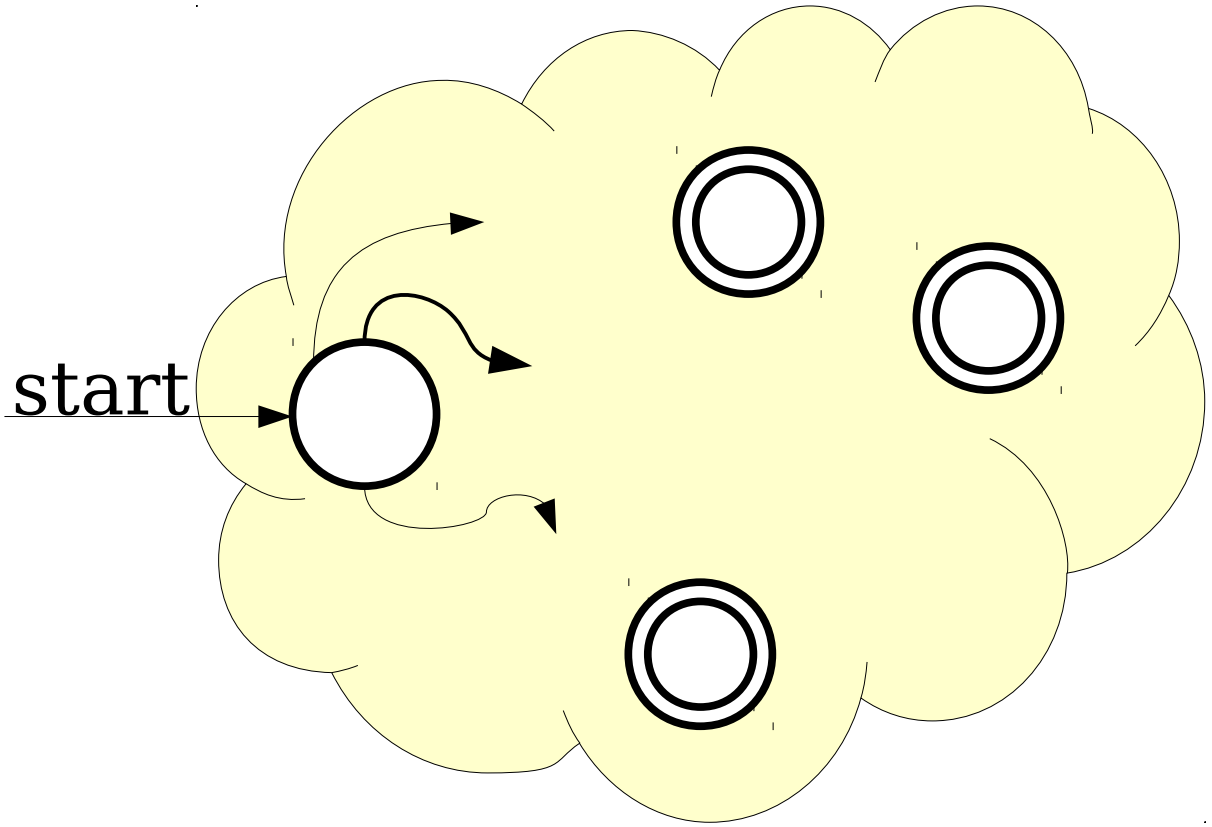
^ not

Reasoning About the Infinite

- If a series of finite objects all have some property, the “limit” of that process *does not* necessarily have that property.
- In general, it is not safe to conclude that some property that always holds in the finite case must hold in the infinite case.
  - (This is why calculus is interesting).
- So our earlier argument ($L^* = L^0 \cup L^1 \cup \ldots$) isn’t going to work.
- We need a different line of reasoning.

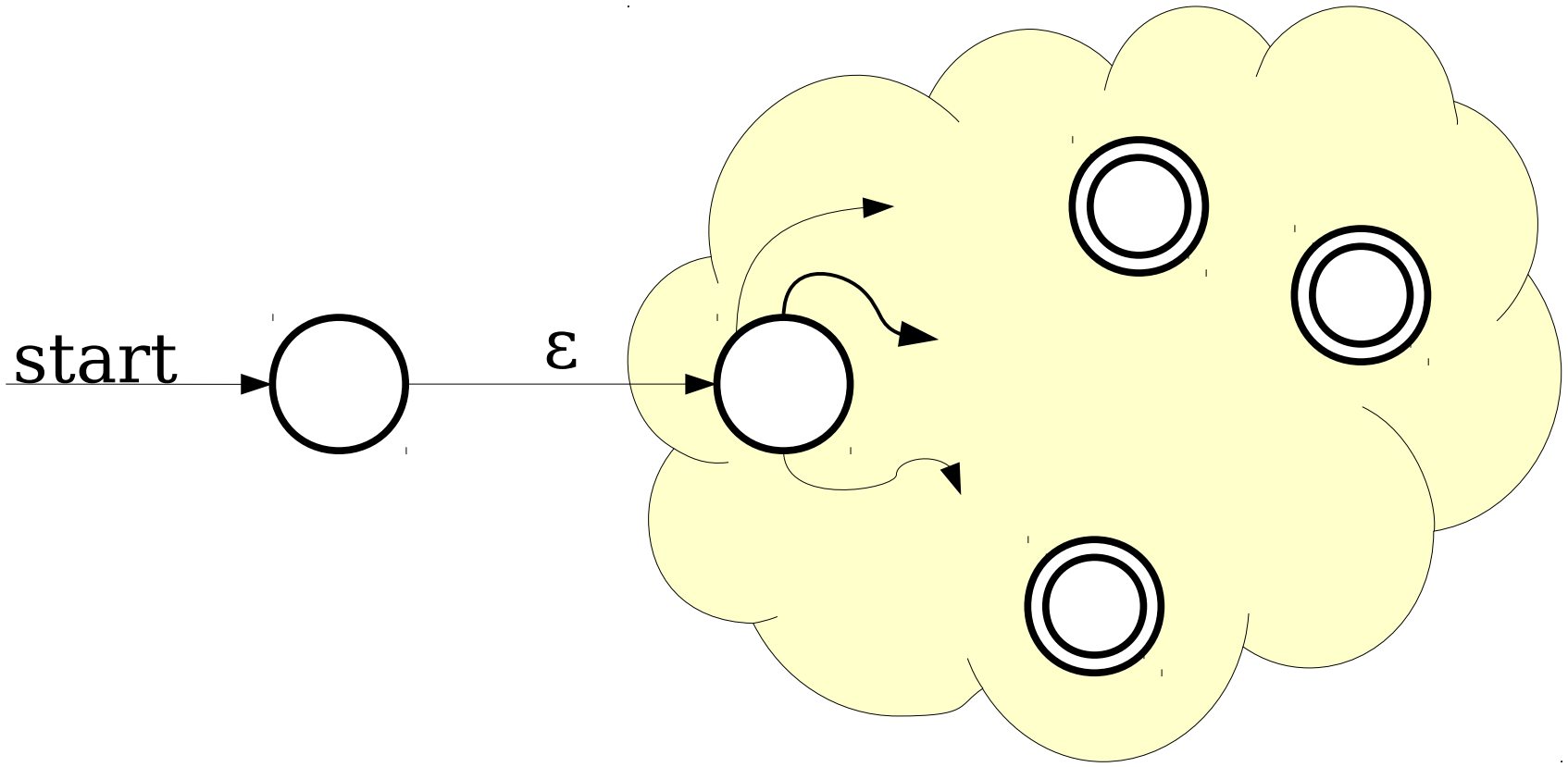
Idea: Can we directly convert an NFA for language L to an NFA for language L*?

The Kleene Star

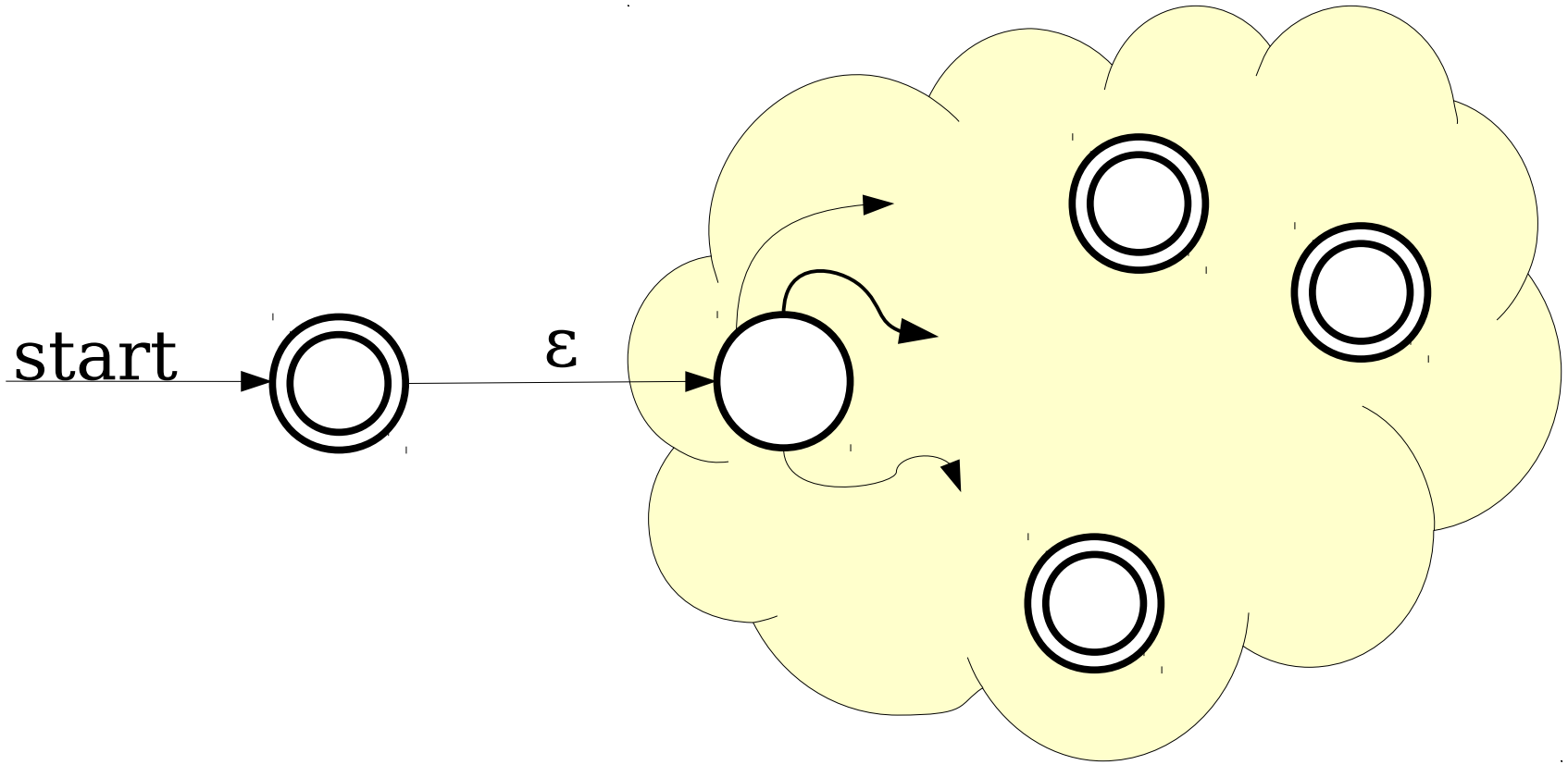


Machine for L

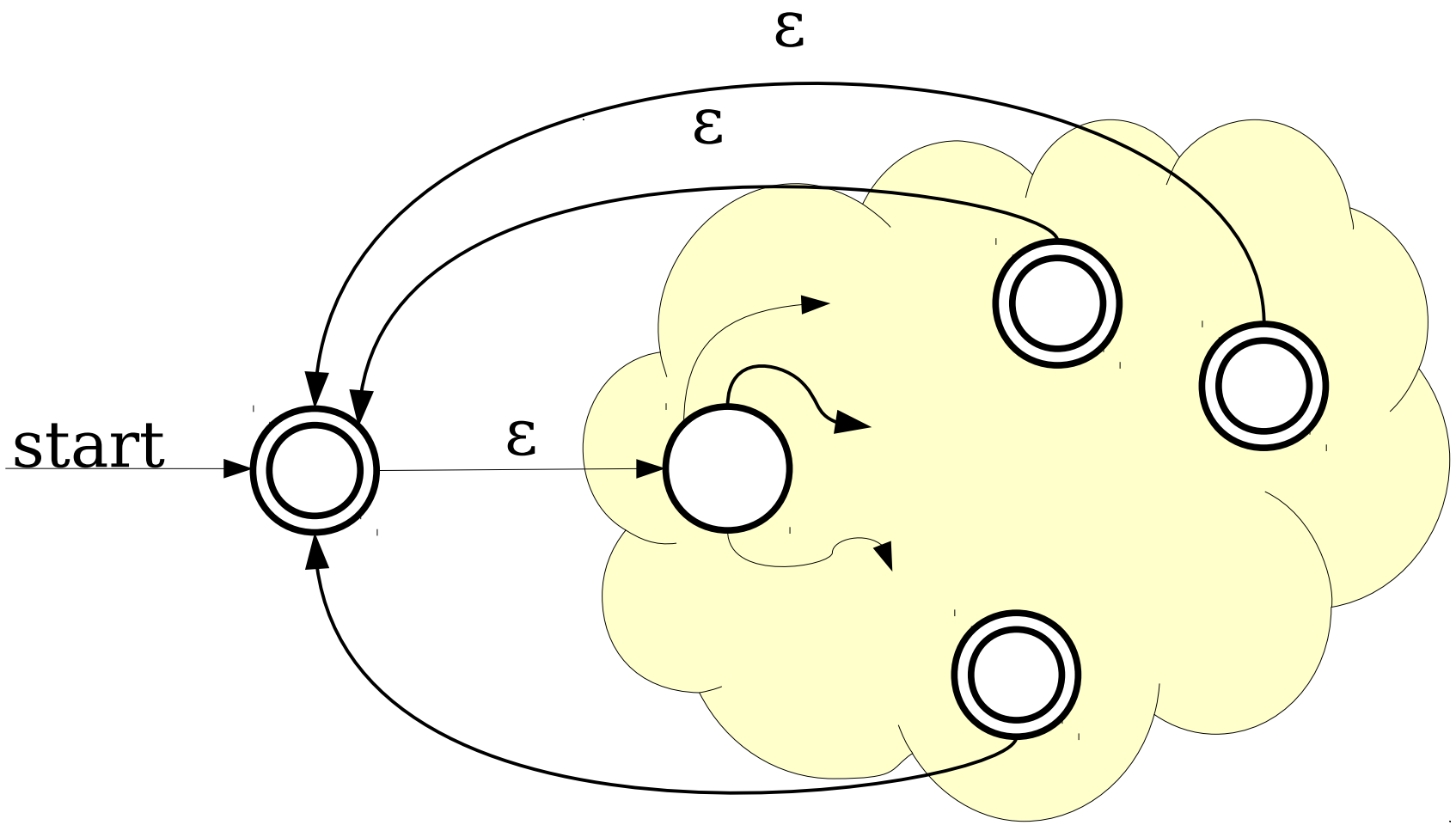
The Kleene Star



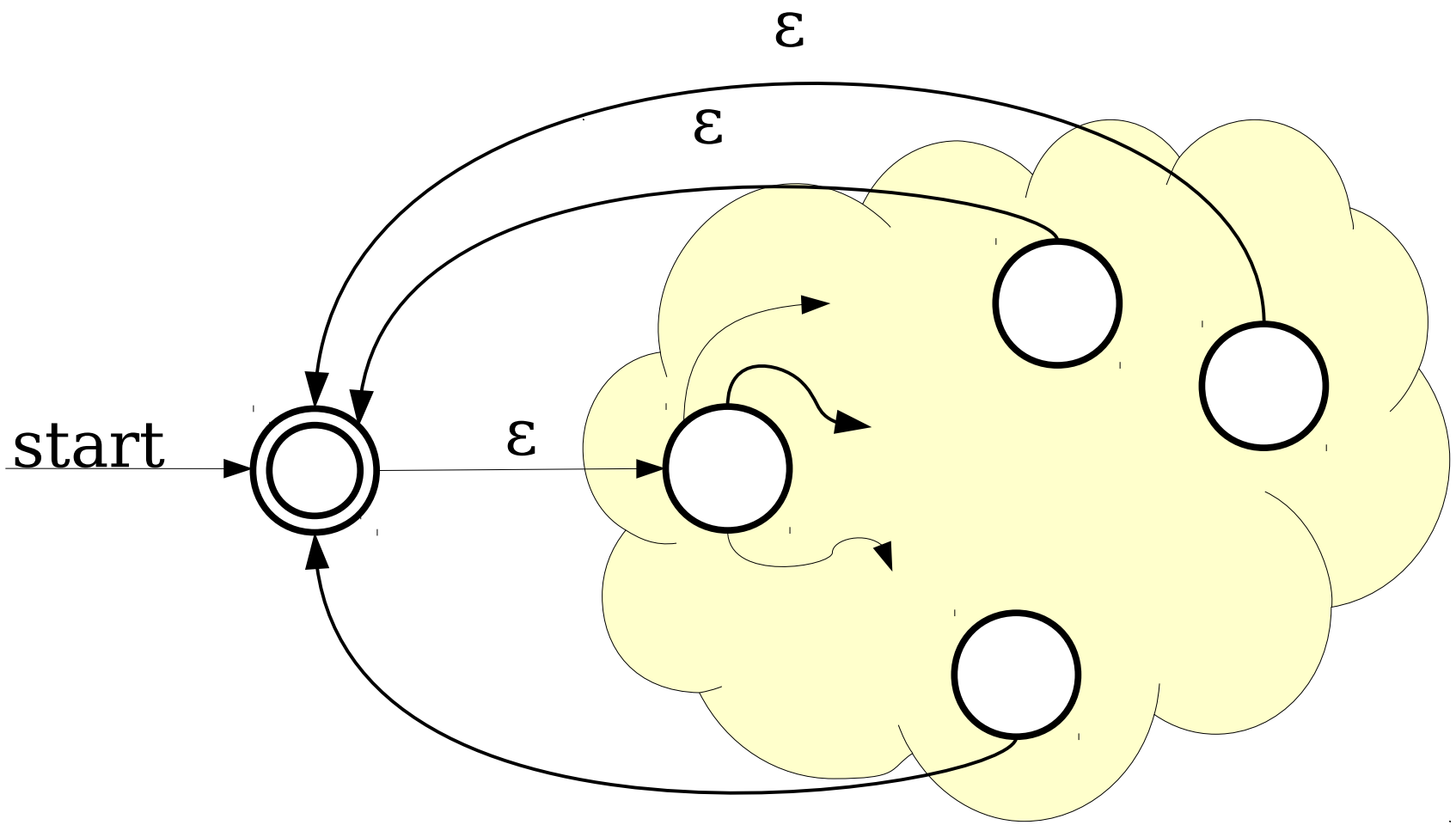
The Kleene Star



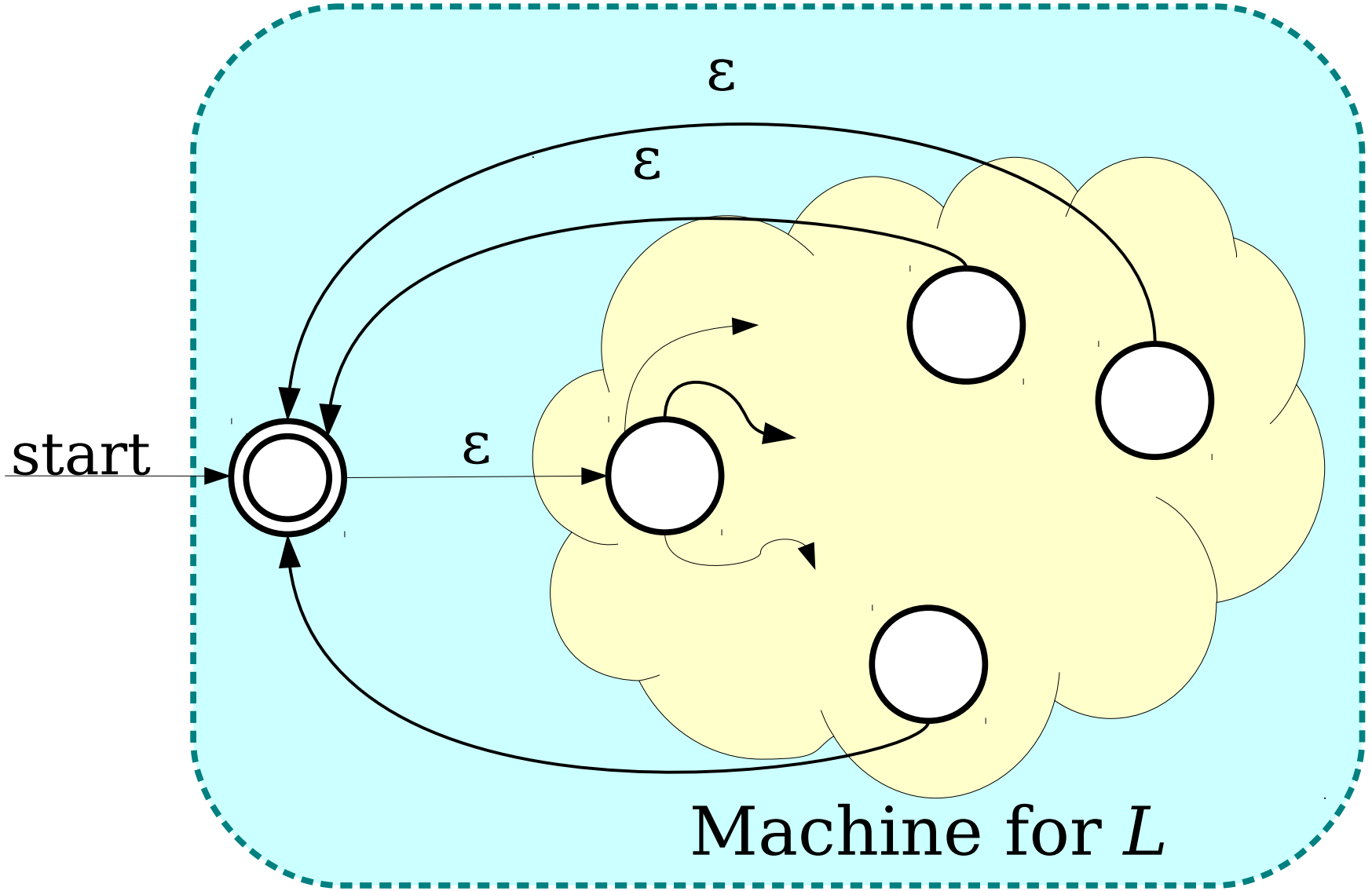
The Kleene Star



The Kleene Star

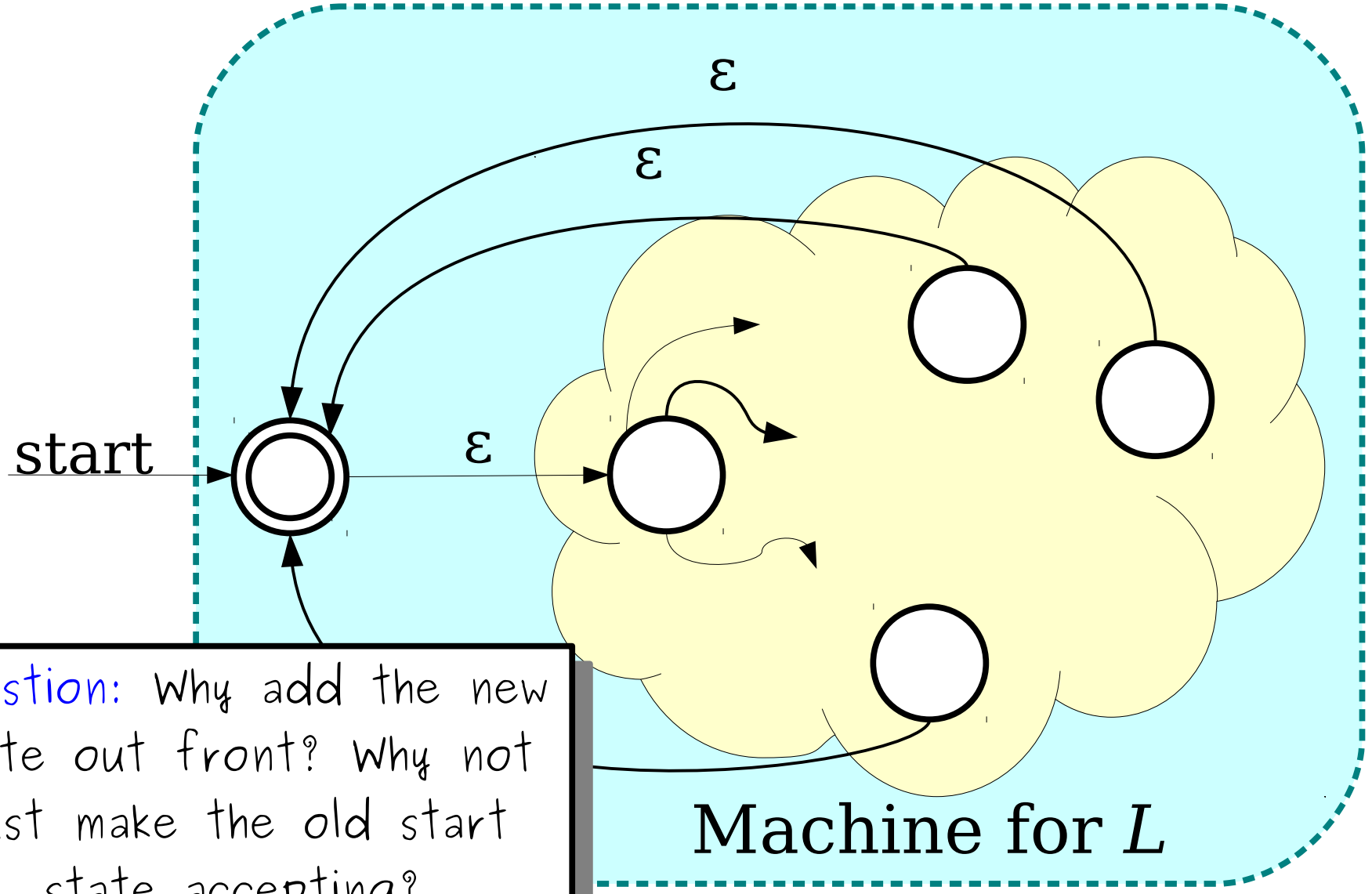


The Kleene Star



Machine for L^*

The Kleene Star



Question: Why add the new state out front? Why not just make the old start state accepting?

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L}_1$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!