Mathematical Vocabulary

> "You keep using that word. I do not think it means what you think it means."
> - Inigo Montoya, from *The Princess Bride*

Consider the humble **while** loop in most programming languages. Here's an example of a **while** loop in a piece of Java code:

```java
int x = 10;
while (x > 0) {
    x = x - 1;
    println(x);
}
```

There's something subtle in this loop. Notice that in the very last iteration of the loop, x will drop to zero, so the `println` call will print the value 0. This might seem strange, since the loop explicitly states that it runs while x is greater than 0.

If you've been programming for a while (pun not intended), you might not think much of this observation. "Of course," you might say, "that's just how **while** loops work. The condition is only checked at the top of the loop, so even if the condition becomes false in the middle of the loop, the loop keeps running." To many first-time programmers, though, this might seem completely counterintuitive. If the loop is really supposed to run *while* x is greater than 0, why doesn't it stop as soon as x becomes zero?

The reason this is interesting/tricky is that there's a distinction between the *informal* use of the word "while" in plain English and the *formal* use of the keyword **while** in software engineering. The dictionary definition of "while" can help you build a good intuition for how **while** loops work in actual code, but it doesn't completely capture the semantics of a **while** loop. For that, you need to learn exactly how **while** loops work and what they mean.

There are analogous concerns that arise in mathematics. Certain words and phrases, like "arbitrary," "by definition," and "without loss of generality," have very specific meanings in mathematical proofs that don't exactly match their English definitions. If you aren't aware of the specific concepts that these words and phrases imply in a mathematical proof, you can end up writing proofs that don't actually say what you think they say – just as you can easily write buggy code with a **while** loop if you don't fully understand some of the trickier semantics of how **while** loops work.

This handout contains a list of some common mathematical terms and phrases with very precise meanings. When you're writing proofs in this course, we recommend consulting this handout to make sure that everything you're saying means what you think it means.

Arbitrary

In mathematics, we commonly write statements like "let x be chosen arbitrarily" or "consider an arbitrary x" in the context of proving universal statements. If a variable x is declared to be chosen arbitrarily, it means that the reader of the proof should be able to supply any choice of x they like matching the criteria you've outlined and the proof should still work. For example, a statement of the form "consider an arbitrary natural number n" indicates to the reader that *any* choice of natural number n will work.

Be careful about using the word "arbitrary" in other contexts. For example, don't say something like this:

⚠   Choose an arbitrary $x = 137$.   ⚠

This statement is problematic because if x really is supposed to be chosen arbitrarily, we should have a lot of options to pick from, but here, x is defined to be 137. If you want to create a variable x whose value is 137, that's fine. Just say something like "let $x = 137$." Don't use the word "arbitrary," since that means something else.

Similarly, be careful about writing something like the following:

⚠   Since n is even, we know that $n = 2k$ for some arbitrary integer k.   ⚠

Here, we know that there is indeed some integer k where $n = 2k$, but it's not arbitrary. There's only one choice of k we can pick here that will work.

Assume

In mathematics, you are allowed to assume anything you'd like. You can assume that an integer n is even number, that a set S contains every natural number, that a player in a game behaves perfectly rationally, that there is a magic silver bear that rides on a narwhal, etc.. When you make an assumption, you're not arguing that something is true – you're just saying "hypothetically speaking, let's assume that this is true and see where it goes."

There are many places in mathematics where it's totally normal to make assumptions. If you're proving an implication of the form "If P, then Q," you typically would assume that P is true, then show under that assumption that Q must be true as well. In a proof by contradiction, you assume some statement is false in order to arrive at a contradiction. In a proof by induction, you assume the inductive hypothesis in the inductive step of the proof.

That said, you should be careful when making assumptions. If you're trying to prove that some result Q is true and you assume P in the course of doing so, you will ultimately need to justify why exactly P has to be true. Otherwise, your result (Q) is dangling in the air, held up only by your assumption (P), which may or may not be on solid ground.

Another weirdness with assumptions is that you're allowed to assume something that's patently false if you'd like. This is the norm in a proof by contradiction, where you assume something you know can't be right in order to derive a contradiction from it later on.

By Definition

When you're writing a proof, you'll at some point need to argue that some basic fact is true because it's "clearly" true. In some cases, you can do this by calling back to the definition of some term. For example, suppose that you want to prove that some number n is even. The definition of a number n being even is that there is some integer k such that $n = 2k$. Therefore, if you can show that there is an integer k such that $n = 2k$, then you can claim, *by definition*, that n is even. Claiming that something is true by definition means that if you were to actually look at the definition of the appropriate term or phrase, you would see that the statement is true because the definition says it is.

We often see people use the term "by definition" to claim something is true that, while true, isn't really true "by definition." For example, the following is an ***incorrect*** use of the phrase "by definition:"

⚠   By definition, we know that $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ ⚠

A good question here is "by definition *of what?*" Of the set $\{1, 2, 3\}$? Of the set $\{1, 2, 3, 4\}$? Or of the $\subseteq$ relation? While it's absolutely true that the set $\{1, 2, 3\}$ is a subset of the set $\{1, 2, 3, 4\}$, the justification "by definition" doesn't clearly articulate why.

In this case, "by definition" was *supposed* to refer to the definition of the $\subseteq$ relation. That definition says the following:

A set S is a subset of a set T, denoted $S \subseteq T$, if every element of S is also an element of T.

If you have this definition, you can look at the sets $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$ and, after checking that each element of the first set is an element of the second, can conclude that indeed that the first set is a subset of the second. However, the reader of the proof still has to put in some extra work to confirm that this is the case. A better way to justify why $\{1, 2, 3\}$ is a subset of $\{1, 2, 3, 4\}$ would be to say something like this:

Because every element of $\{1, 2, 3\}$ is an element of $\{1, 2, 3, 4\}$,
by definition we see that $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$.

If

The word "if" in mathematics is overloaded (it has several different meanings that differ by context) and probably causes more confusion than most other terms.

The first context for the word "if" arises in the context of implications. Many statements that we'd like to prove are implications. For example:

If n is an even natural number, then n^2 is an even natural number.
If $A \subseteq B$ and $B \subseteq A$, then $A = B$

In this context, "if" sets up a one-directional implication. The statement "If P, then Q" means that if P is true, then Q is true. The implication doesn't necessarily flow backwards – it's quite possible that Q can be true even if P isn't true.

You sometimes see "if" used in statements like these:

The number p^2 is rational if p is rational.
A graph is four-colorable if it's planar.

These statements also set up implications, though the implications flow the opposite direction. The statement "P if Q" is equivalent to "If Q, then P," meaning that the second part of the statement implies the first. Be careful when reading statements with "if" in them to make sure that you understand what statement is being articulated!

The confusing part about "if" is that we also use the word "if" in definitions, which behave differently than implications. For example, here are some mathematical definitions:

We say that a number n is even if there is an integer k such that $n = 2k$.
We'll call a graph a planar graph if it can be drawn in the plane with no edges crossing.
A set S is called countably infinite if $|S| = |\mathbb{N}|$.

These terms introduce new mathematical definitions by using the word "if." This can be a bit confusing because, unlike implications, definitions flow in both directions. For example, if n is even, then I can write it as $2k$ for some integer k; independently, if I find a number of the form $2k$, then I know that it's even. Even though as written the definition seems to say "if $n = 2k$, then n is even" as if it's an implication, it's really saying that the term "n is even" means exactly "there is an integer k where $n = 2k$." Knowing one of these automatically tells you the other.

So how do you differentiate between these two cases? One simple test you can use is to ask whether you're working with a brand-new term or whether you're working with a term you've already seen before. If some new concept or notation is being introduced for the very first time, chances are you're looking at a definition, so "if" means "is defined to mean." If you see existing terms being linked together, chances are you're looking at an implication. Not sure which one you're looking at? Just ask!

Iff

No, that's not a typo. The word "iff" (if you can even call it a word) is a mathematical shorthand meaning "if and only if." Unlike "if," which sets up a one-directional implication, "iff" sets up a two-way implication. For example, the statement

n is an integer iff n^2 is an integer

means "if n is an integer, then n^2 is an integer, and if n^2 is an integer, then n is an integer."

When reading mathematics, be careful to pay attention to whether you're reading "if" or "iff," since they mean very different things. Also, when writing mathematics, be careful not to write "iff" when you mean "if." We find that a lot of first-time proofwriters end up using "iff" where they mean "if," often because "iff" looks a lot cooler. It's fine to use "iff" in your proofs – just be careful to make sure that it's the right word for the job!

Vacuously True

A *vacuous truth* is a statement that is true because it doesn't apply to anything. There are two main situations in which you see vacuous truths. First, you'll often encounter statements like

"Every X has property Y"

when there are no objects of type X. In this case, you can claim that the above statement is vacuously true, because there are no X's. For example, the statements "every unicorn is pink" and "every set S where $|S| = |\wp(S)|$ contains 137" are vacuously true because there are no unicorns (sorry) and there are no sets S where $|S| = |\wp(S)|$ (thanks, Cantor).

The other case where vacuous truths arise is in statements like

"If P is true, then Q is true"

when the statement P is never true. These statements are also called vacuously true. For example, the statement "if $2 + 2 = 5$, then everyone gets free ice cream" is vacuously true because $2 + 2 \neq 5$, and the statement "if the American flag in 2014 had 54 stars on it, then the universe is on fire" is vacuously true because the American flag in 2014 had only 50 stars on it.

One of the more common classes of mistakes we see in proofs – especially in the base cases of inductive proofs (we'll cover them in a few weeks) – is when a proof incorrectly claims that a statement is vacuously true. For example, consider this statement:

> "Every set with no elements has cardinality 0."

This statement is indeed a true statement: there's just one set with no elements, the empty set, and its cardinality is indeed zero. However, it's not a *vacuously true* statement because there is an empty set. If a proof were to claim that this statement is vacuously true, the proof would be wrong – it's true, but not vacuously.

We also see a lot of proofs that try to use vacuous truth in places where it doesn't apply. For example, consider this statement:

> "There is a pink unicorn."

This statement is not vacuously true, nor is it true at all. Instead, this statement is false. Why is this?

Here are two different ways you can see this. First, there's the "common sense" intuition for this statement. The statement claims that if you searched far and wide, eventually you'd find a pink unicorn. This isn't true, though: because there are no unicorns, there certainly aren't any pink unicorns. Therefore, saying that there is a unicorn when there aren't actually any unicorns would be incorrect.

Here's another way to think about this. Let's ask a related question: what is the negation of the above statement? To determine this, let's look closely at the original statement. The statement "there is a pink unicorn" is an existential statement, since it's essentially this statement:

> "There is a unicorn that is pink."

We can negate this statement using the standard technique for negating existential statements. That gives us this statement:

> "Every unicorn is not pink."

This statement, interestingly enough is vacuously true – there aren't any unicorns, so it's true that every unicorn isn't pink. Since the negation of the statement "there is a pink unicorn" is *true*, it must be the case that the statement "there is a pink unicorn" must be *false*.

More generally, *existential statements cannot be vacuously true*. Vacuous truth only applies to implications and to universal statements.

Without Loss of Generality

Suppose you come to a step in a proof where you have two numbers p and q and you know for a fact that exactly one of them is even and exactly one of them is odd. You could write the remainder of the proof like this:

> We now consider two cases.
>
> > *Case 1: p is even and q is odd.* [*some argument*]
> >
> > *Case 2: p is odd and q is even.* [*some argument*]
>
> In both cases we see that [*something holds*], so [*we draw some conclusion*]

Depending on the structure of what you're proving, it's quite possible that you need to have totally separate arguments for the two cases. However, in many cases, you'll find that the argument you're making is *exactly* the same, but with the values of p and q interchanged. When that happens, you can avoid writing out the cases and duplicating your argument by using the magical and wonderful phrase "without loss of generality." For example:

> **Without loss of generality**, assume that p is odd and that q is even. Then
> [*some argument*], so [*we draw some conclusion*]

The phrase "without loss of generality" is a shorthand for "there are several different cases that we need to consider, but they're basically all the same and so we're going to make a simplifying assumption that doesn't miss any cases." You're welcome to use it to collapse structurally identical cases together if you'd like. However, be careful when using it! You can only use "without loss of generality" if all the cases really are symmetric and if you don't have any special information that would distinguish the objects under consideration.