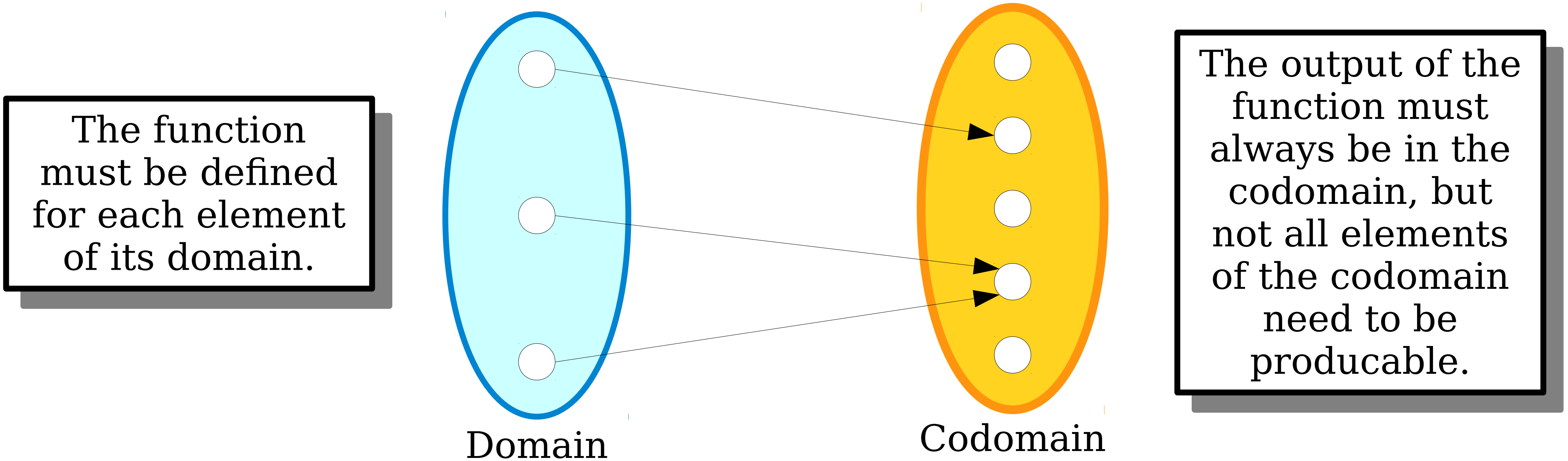
Cardinality

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.
- We write ***f* : *A* → *B*** to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.



Injective Functions

- A function $f : A \to B$ is called ***injective*** if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

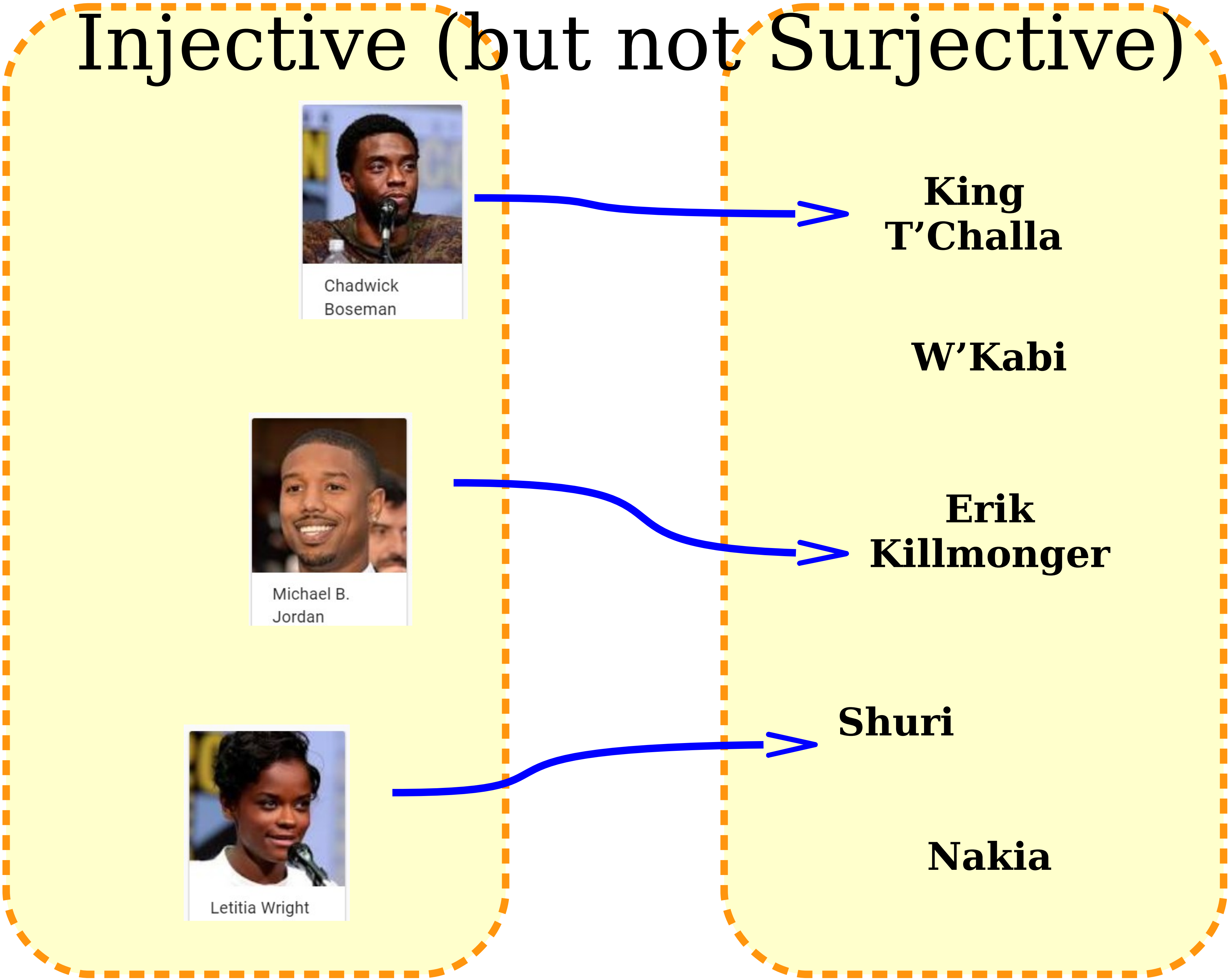
(*"Each 'question' has a different 'answer'"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

- ***Theorem:*** The composition of two injections is an injection.

Injective (but not Surjective)



Surjective Functions

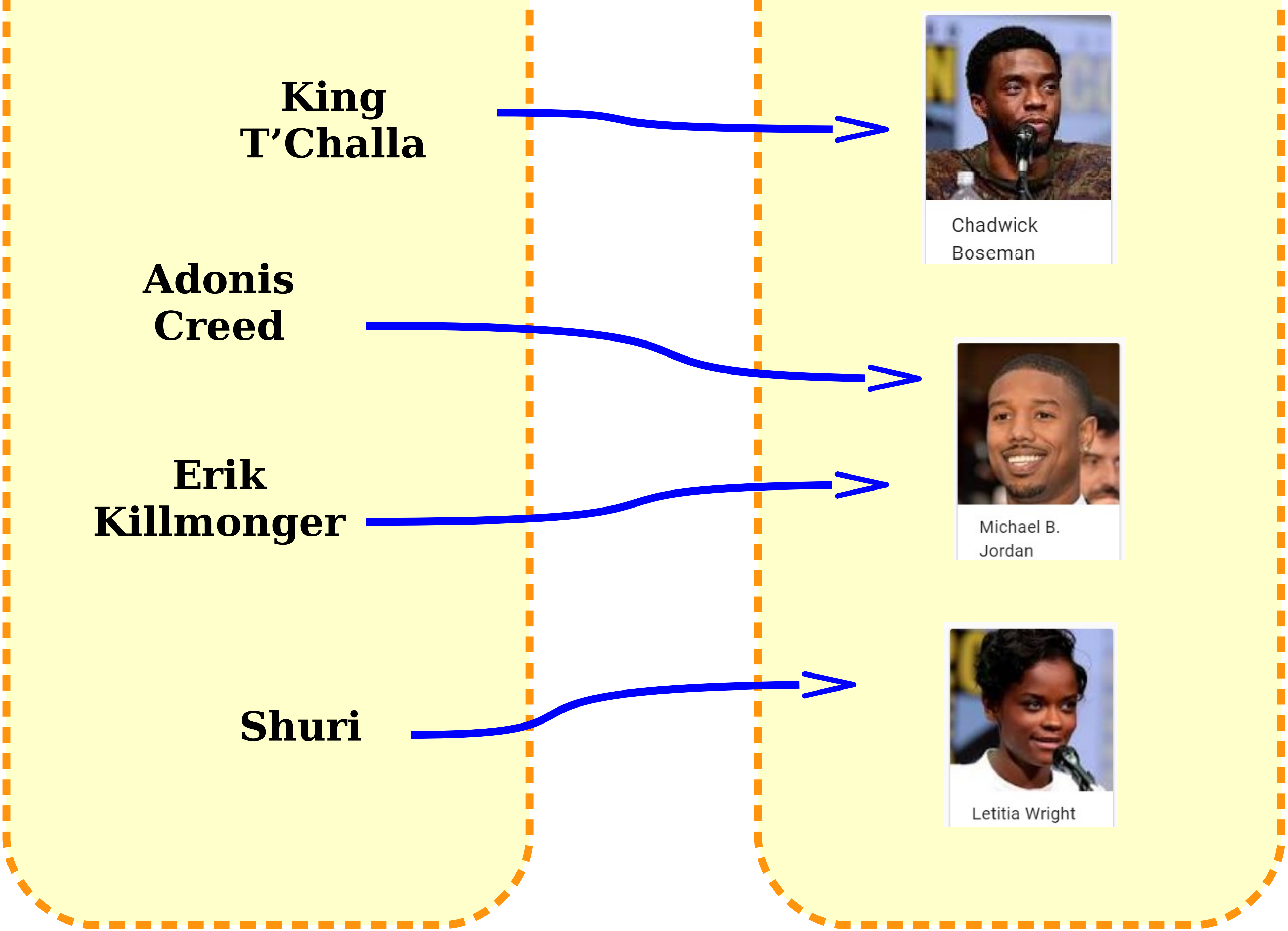
- A function $f : A \to B$ is called ***surjective*** if this FOL statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"There aren't any unused elements in the co-domain."*)

- ***Theorem:*** The composition of two surjections is a surjection.

Surjective (but not Injective)



Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- ***Theorem:*** The composition of two bijections is a bijection.
- ***Theorem(s):*** bijections are invertible and invertible functions are bijections.

New Stuff!

Cardinality Revisited

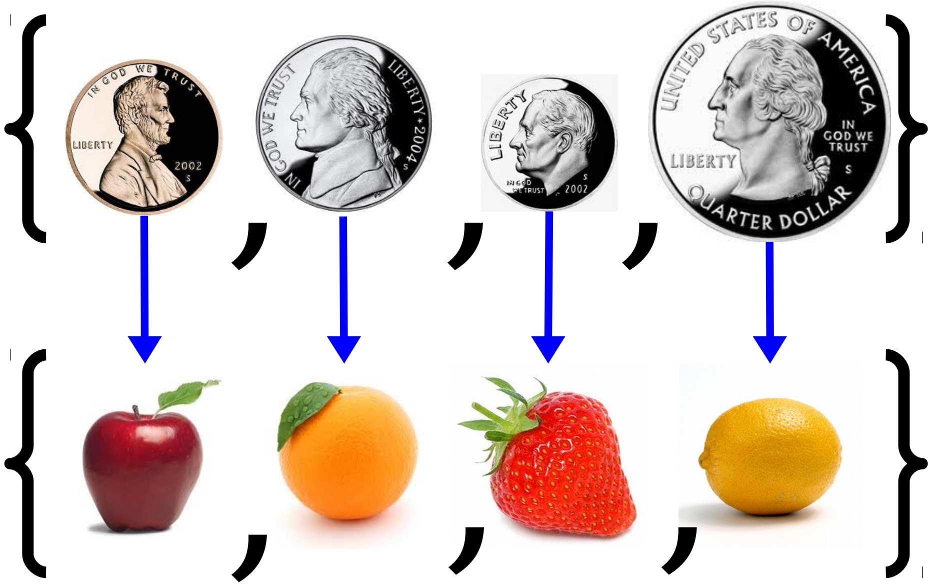
Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set S is basically its size, and is written $|S|$.
- For finite sets, cardinalities are natural numbers: $|\{1, 2, 3\}| = 3$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets: $|\mathbb{N}| = \aleph_0$

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

$|S| = |T|$ if there exists a *bijection* $f : S \to T$



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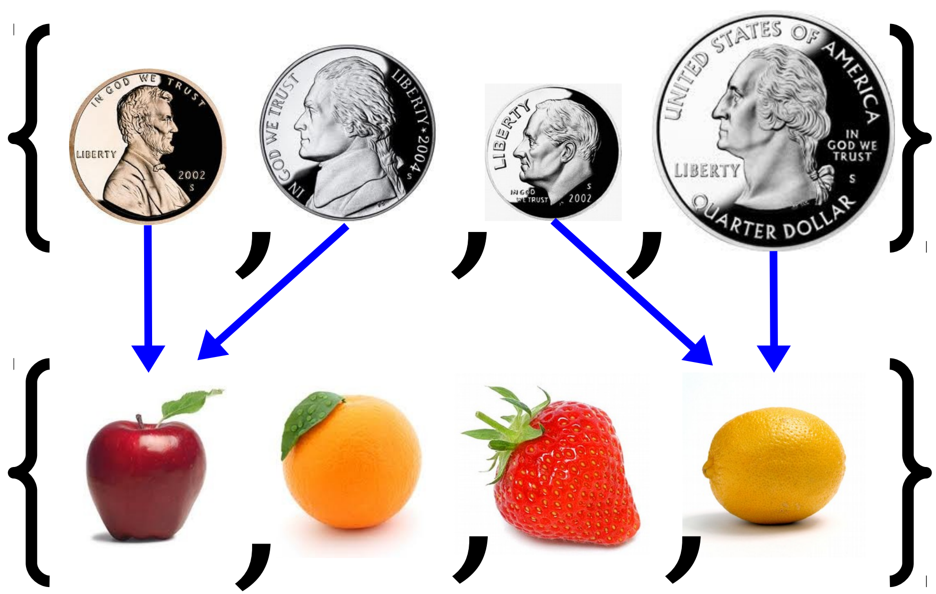

But remember that this doesn't mean that the existence of non-bijections *disproves* that two sets are the same size.

Fun with Cardinality

Terminology Refresher

- Let *a* and *b* be real numbers where $a \leq b$.
- The notation **[*a*, *b*]** denotes the set of all real numbers between *a* and *b*, inclusive.

$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

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Home on the Range

How does the number of real numbers in the rage 0 and 1 compare to the number of real numbers in the range 0 and 2?

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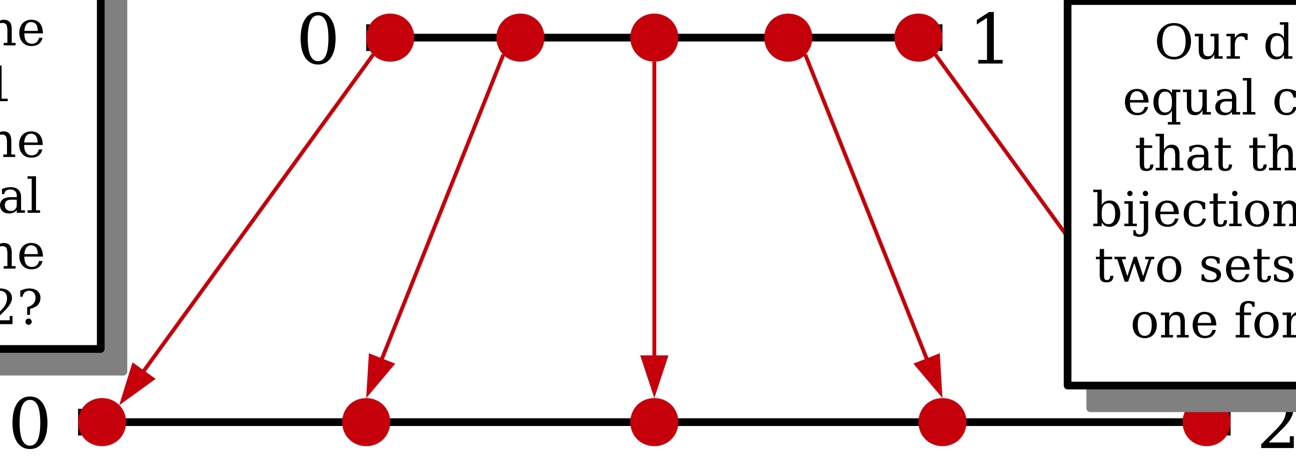

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$$f : [0, 1] \to [0, 2]$$

$$f(x) = 2x$$

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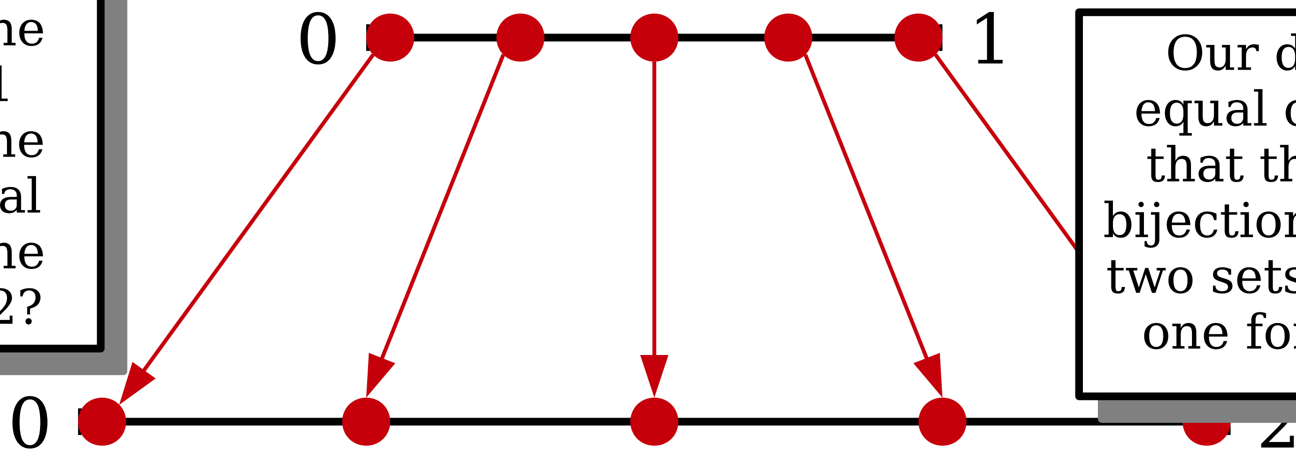

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YES!!!

EQUAL SIZE!!!

$$f : [0, 1] \to [0, 2]$$

$$f(x) = 2x$$

Theorem: $|[0, 1]| = |[0, 2]|$

Proof: Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$. We will prove that f is a bijection.

First, we will show that f is a well-defined function. Choose any $x \in [0, 1]$. This means that $0 \leq x \leq 1$, so we know that $0 \leq 2x \leq 2$. Consequently, we see that $0 \leq f(x) \leq 2$, so $f(x) \in [0, 2]$.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $2x_1 = 2x_2$, which in turn tells us that $x_1 = x_2$, as required.

Theorem: $|[0, 1]| = |[0, 2]|$

Proof: Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$.

How many of the following are proper ways of setting up the next part of this proof?

Choose any $x \in [0, 1]$. We will show there is a $y \in [0, 2]$ such that $f(x) = y$.

Pick any $y \in [0, 2]$. We will show there is an $x \in [0, 1]$ where $f(x) = y$.

Assume for the sake of contradiction that, for any $y \in [0, 2]$ and for any $x \in [0, 1]$, we have $f(x) \neq y$.

Finally, we will show that f is surjective.

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then a number between **0** and **3**.

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Finally, we will show that f is surjective. To do so, consider any $y \in [0, 2]$. We'll show that there is some $x \in [0, 1]$ where $f(x) = y$.

Let $x = {}^{y}/_{2}$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq {}^{y}/_{2} \leq 1$. We picked $x = {}^{y}/_{2}$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that $f(x) = 2x = 2({}^{y}/_{2}) = y$, so $f(x) = y$, as required. ■

Home on the Range

$$f : [0, 1] \to [0, 2]$$

$$f(x) = 2x$$

Home on the Range

$$f : [0, 1] \to [0, 3]$$

$$f(x) = 3x$$

Home on the Range

$$f : [0, 1] \to [0, 137]$$
$$f(x) = 137x$$

Home on the Range

(for any $k > 0$)

$$f : [0, 1] \rightarrow [0, k]$$

$$f(x) = kx$$

This means that ***cardinality*** (how many points there are) is a different idea than ***mass*** (how much those points weigh). Look into ***measure theory*** if you're curious to learn more!

Some Properties of Cardinality

Theorem: **For any set A, we have $|A| = |A|$.**

Proof: Consider any set A, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that f is a bijection.

First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of f that $x_1 = x_2$, as required.

Finally, we'll show that f is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$. Then $x \in A$ (since $y \in A$) and $f(x) = x = y$, as required. ■

Theorem: **If A, B, and C are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.**

Proof: Consider any sets A, B, and C where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from A to C.

Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \to C$.

Consider the function $g \circ f : A \to C$. Since g and f are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from A to C. Thus $|A| = |C|$, as required. ■

Practice exercise: Prove that if A and B are sets where $|A| = |B|$, then $|B| = |A|$.

Unequal Cardinalities

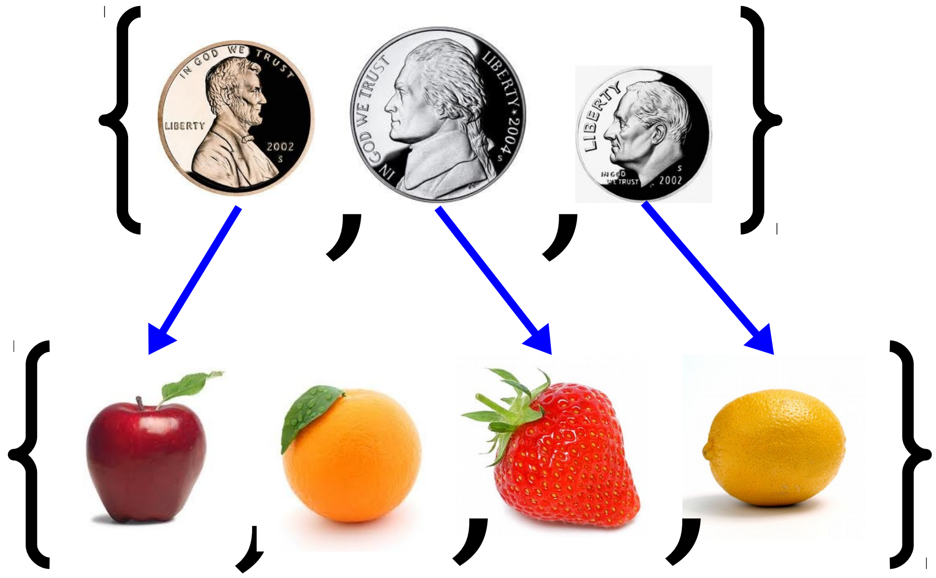
- Recall: $|A| = |B|$ if the following statement is true:

There exists a bijection $f : A \to B$

- What does it mean for $|A| \neq |B|$ to be true?

Every function $f : A \to B$ is not a bijection.

- This is a strong statement! To prove $|A| \neq |B|$, we need to show that *no possible function* from A to B can be injective and surjective.



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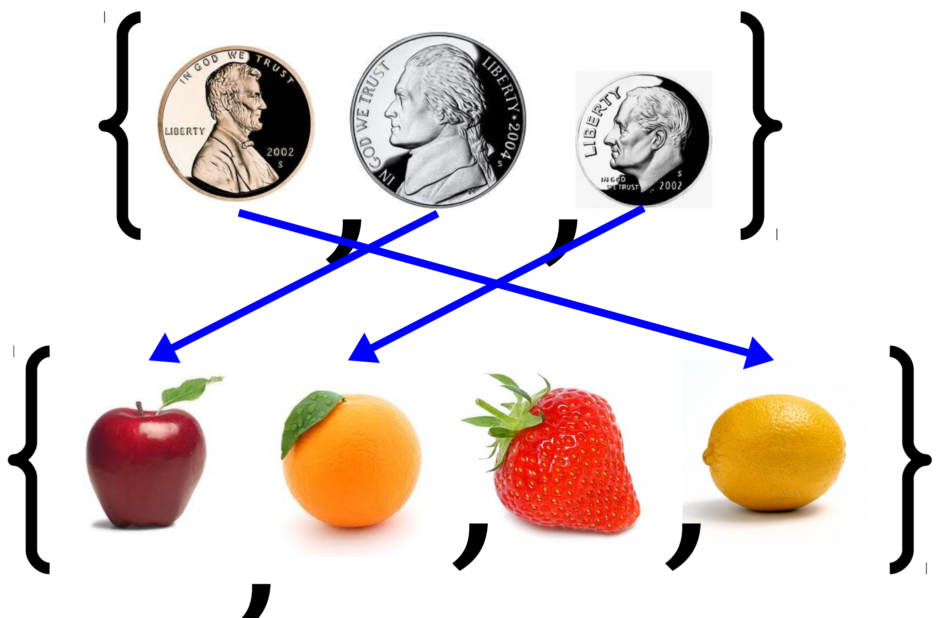

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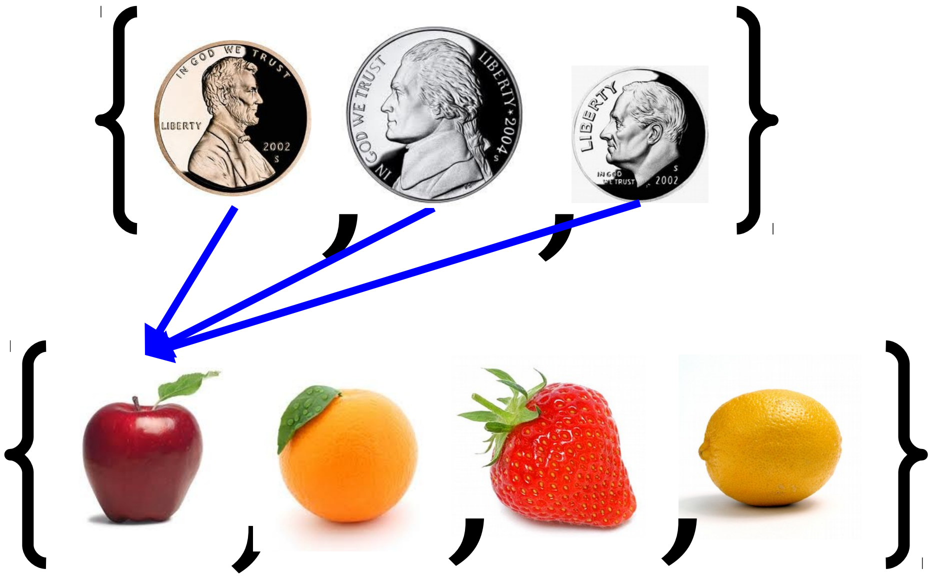

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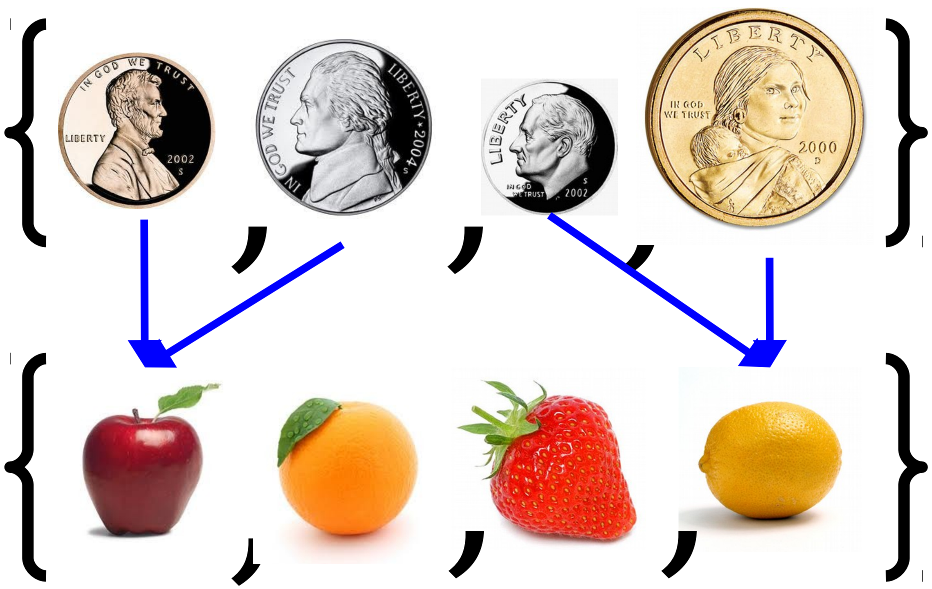

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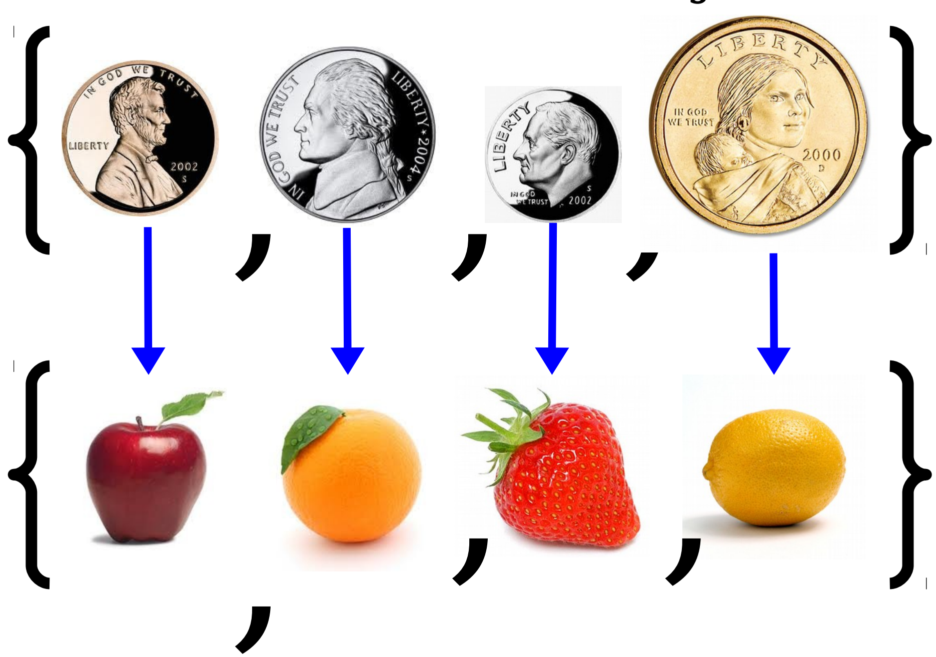

Cantor’s Theorem Revisited

Cantor's Theorem

- In our very first lecture, we sketched out a proof of ***Cantor's theorem***, which says that

If S is a set, then $|S| < |\wp(S)|$.

- That proof was visual and pretty hand-wavy. Let's see if we can go back and formalize it!

Where We're Going

- Today, we're going to formally prove the following result:

If S is a set, then $|S| \neq |\wp(S)|$.

- We've released an online Guide to Cantor's Theorem, which will go into *way* more depth than what we're going to see here.
- The goal for today will be to see how to start with our picture and turn it into something rigorous.
- On the next problem set, you'll explore the proof in more depth and see some other applications.

The Roadmap

- We're going to prove this statement:

  If S is a set, then $|S| \neq |\wp(S)|$.

- Here's how this will work:
  - Pick an arbitrary set S.
  - Pick an arbitrary function $f : S \to \wp(S)$.
  - Show that f is not surjective using a diagonal argument.
  - Conclude that there are no bijections from S to $\wp(S)$.
  - Conclude that $|S| \neq |\wp(S)|$.

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- Show that f is not surjective using a diagonal argument.
- Conclude that there are no bijections from S to $\wp(S)$.
- Conclude that $|S| \neq |\wp(S)|$.

This is a drawing of our function $f : S \to \wp(S)$.

$$x_0 \longleftrightarrow \{\ x_0,\ x_2,\ x_4,\ \dots\ \}$$

$$x_1 \longleftrightarrow \{\ x_3,\ x_5,\ \dots\ \}$$

$$x_2 \longleftrightarrow \{\ x_0,\ x_1,\ x_2,\ x_5,\ \dots\ \}$$

$$x_3 \longleftrightarrow \{\ x_1,\ x_4,\ \dots\ \}$$

$$x_4 \longleftrightarrow \{\ x_2,\ \dots\ \}$$

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$$\dots \longleftrightarrow \{\ \dots\ \}$$

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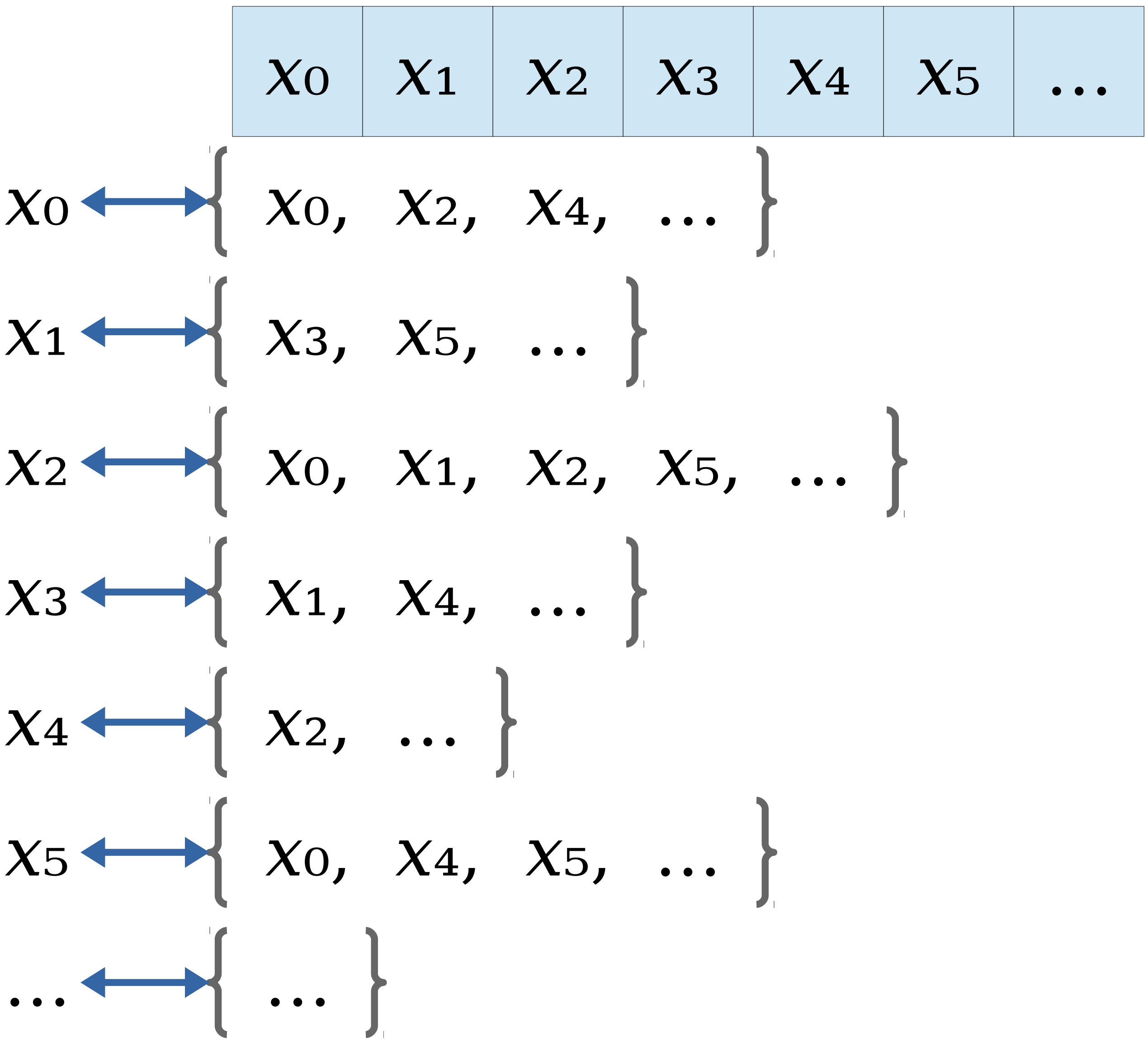

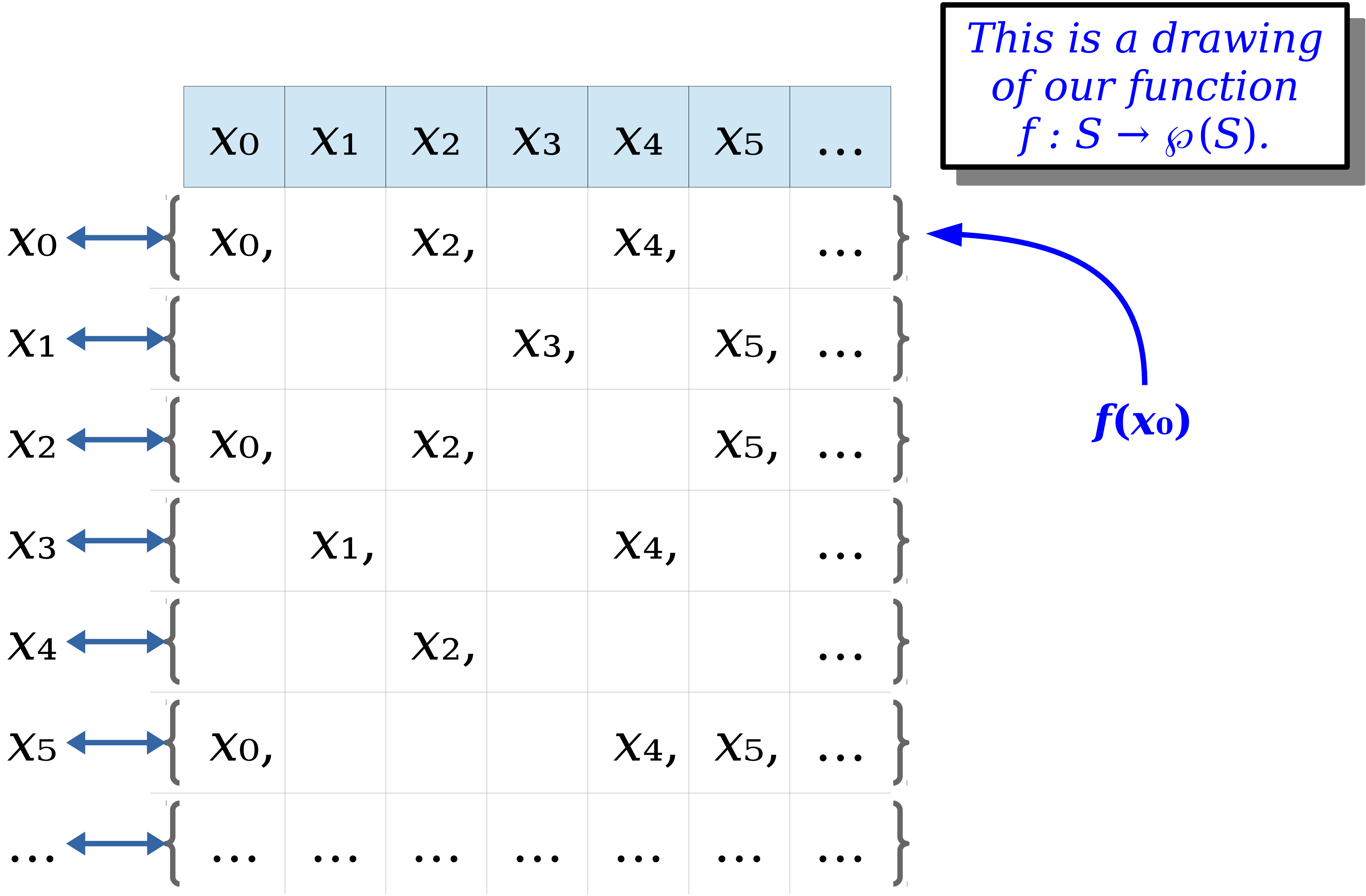

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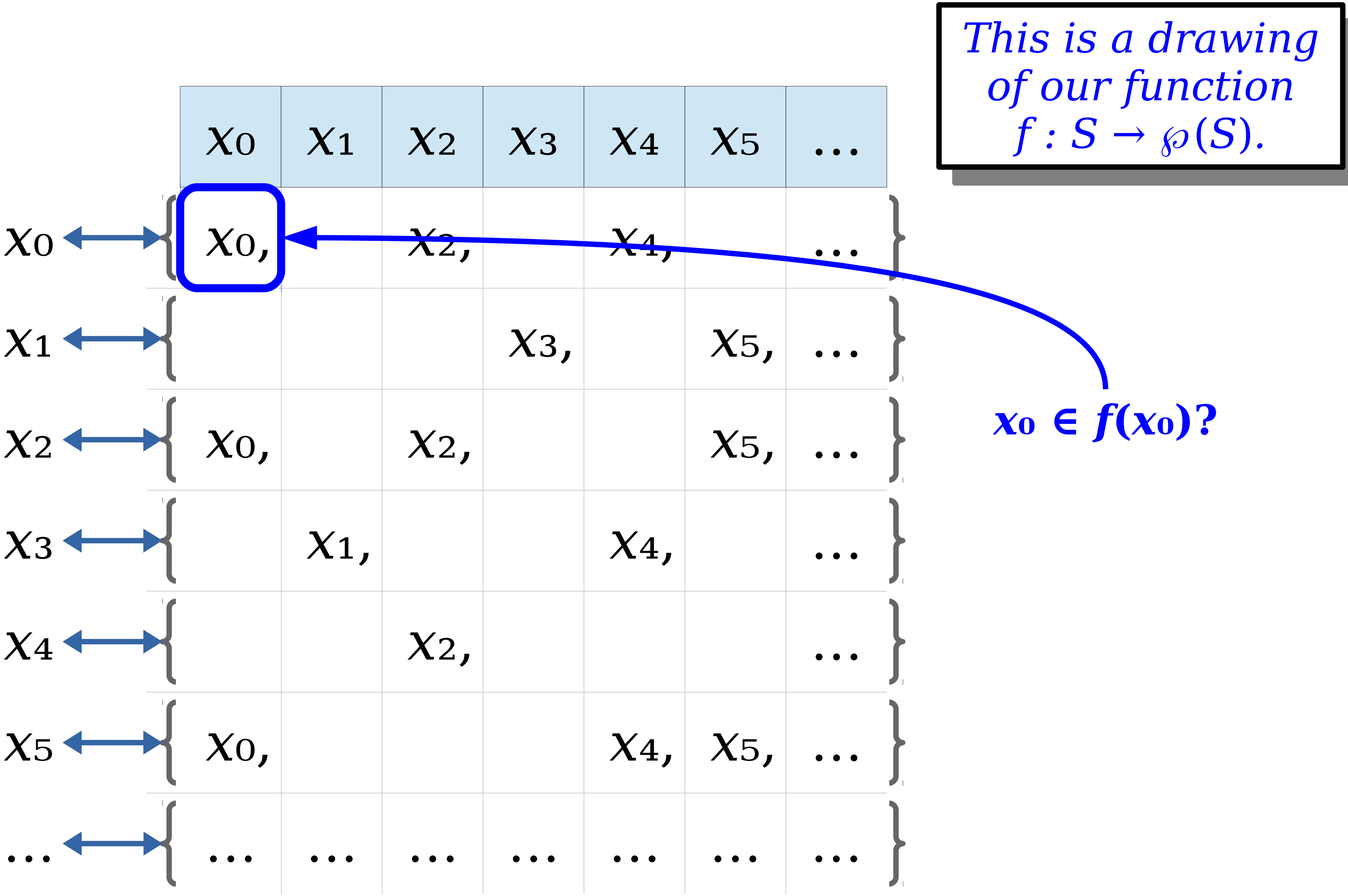
$f(x_0)$



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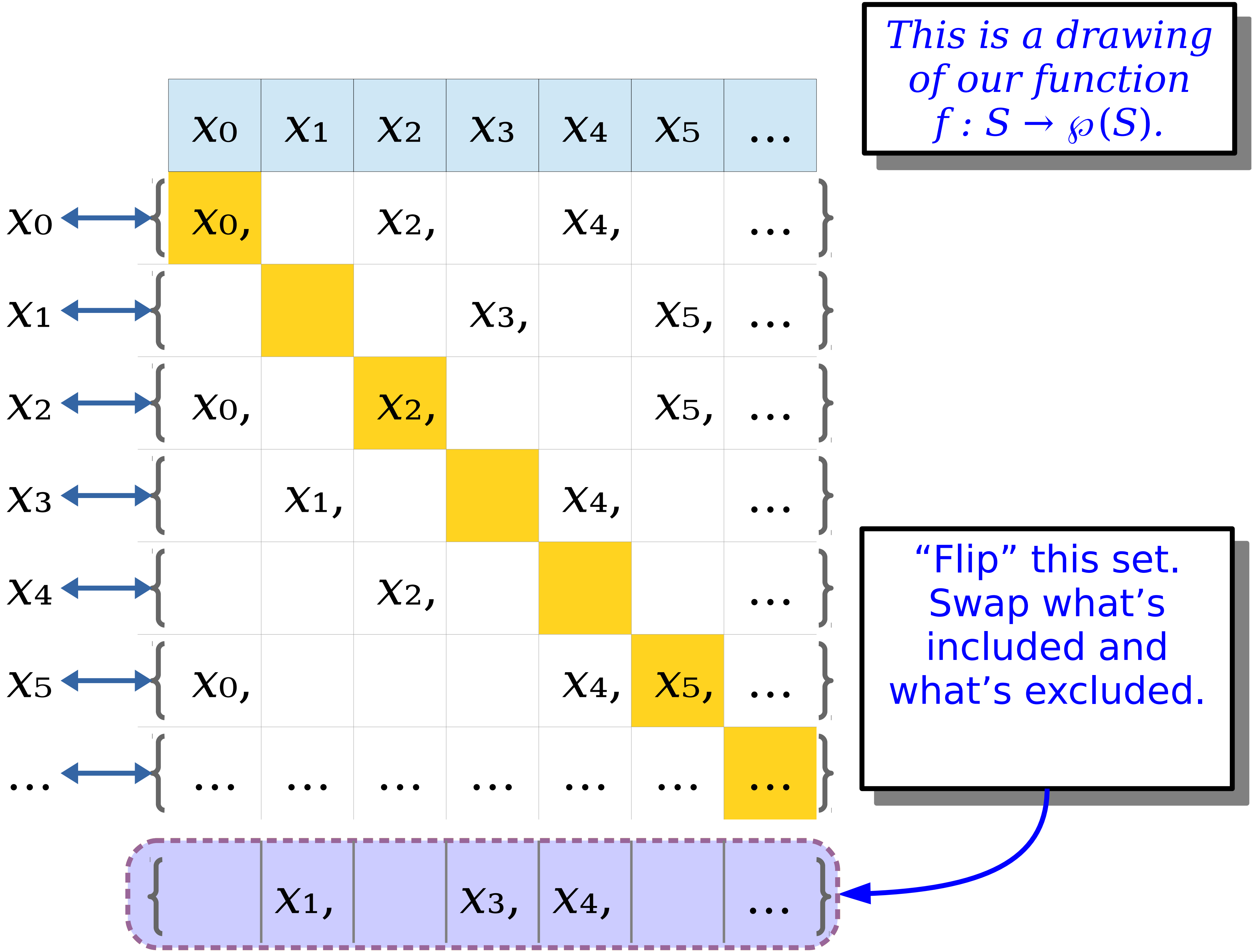
… {… … … … … … …}
$x_0 \in f(x_0)$?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x_3 { x_1, x_4, … }
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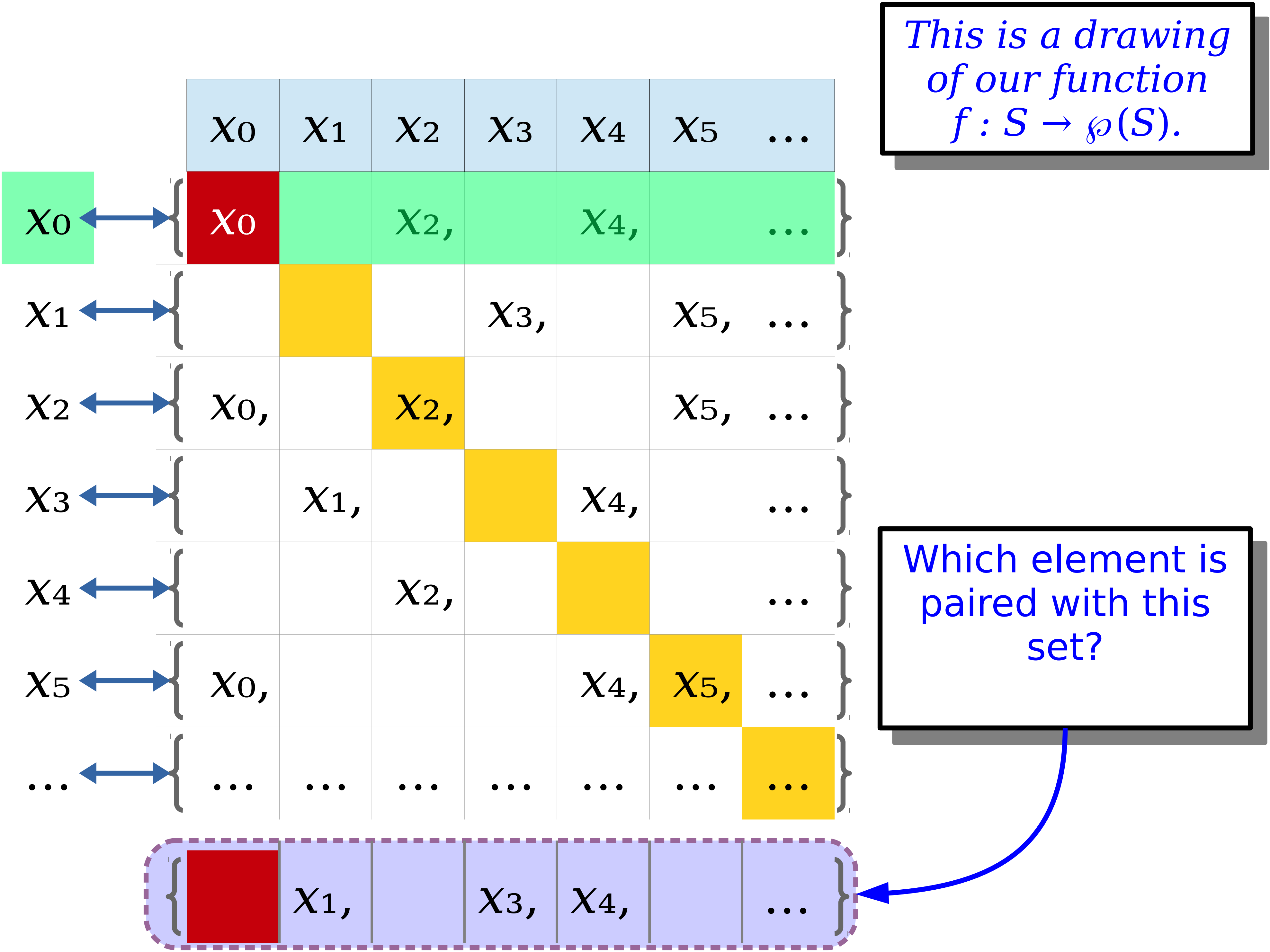
… { … … … … … … … }
“Flip” this set. Swap what’s included and what’s excluded.
{ x_1, x_3, x_4, … }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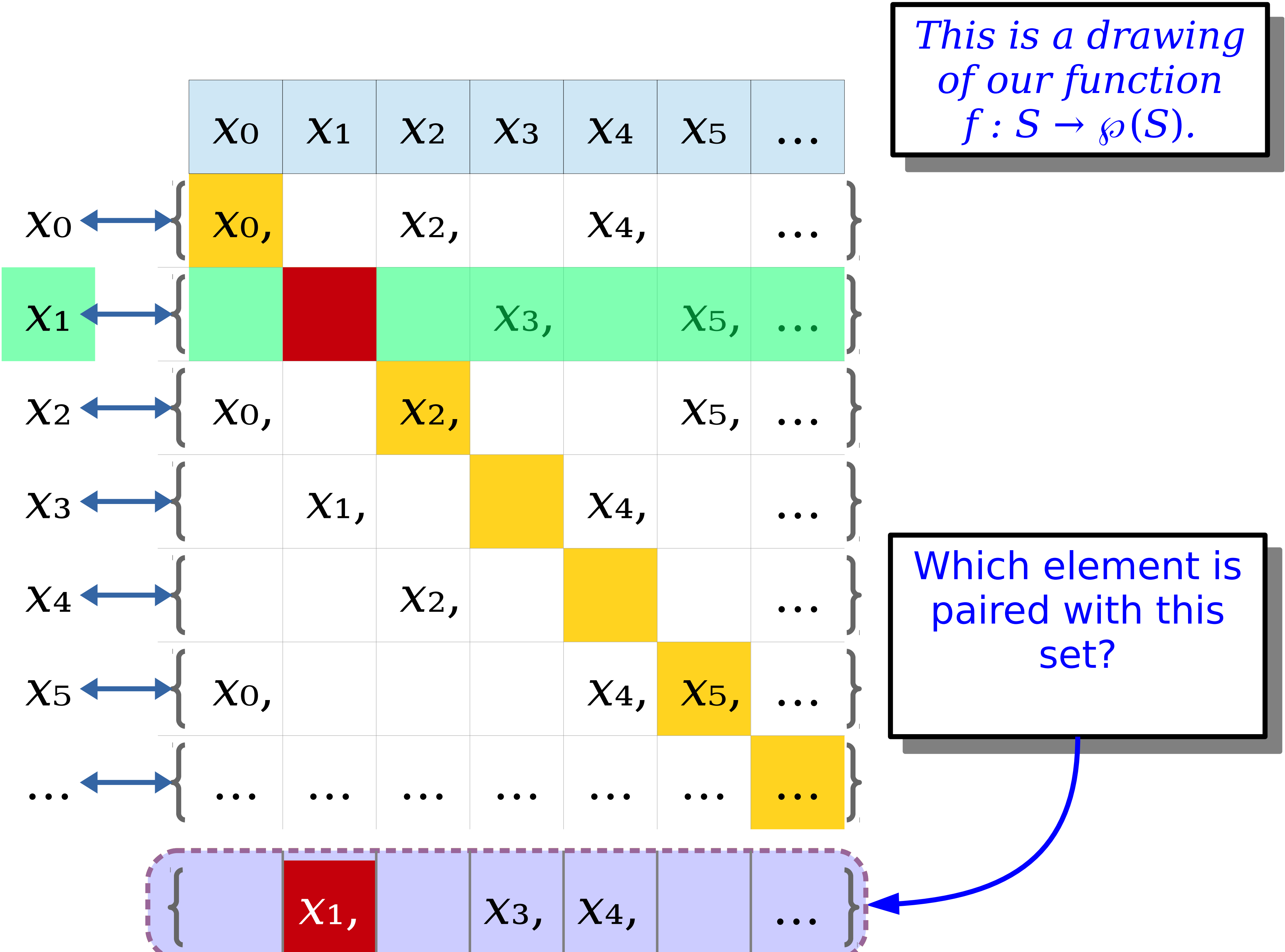
Which element is paired with this set?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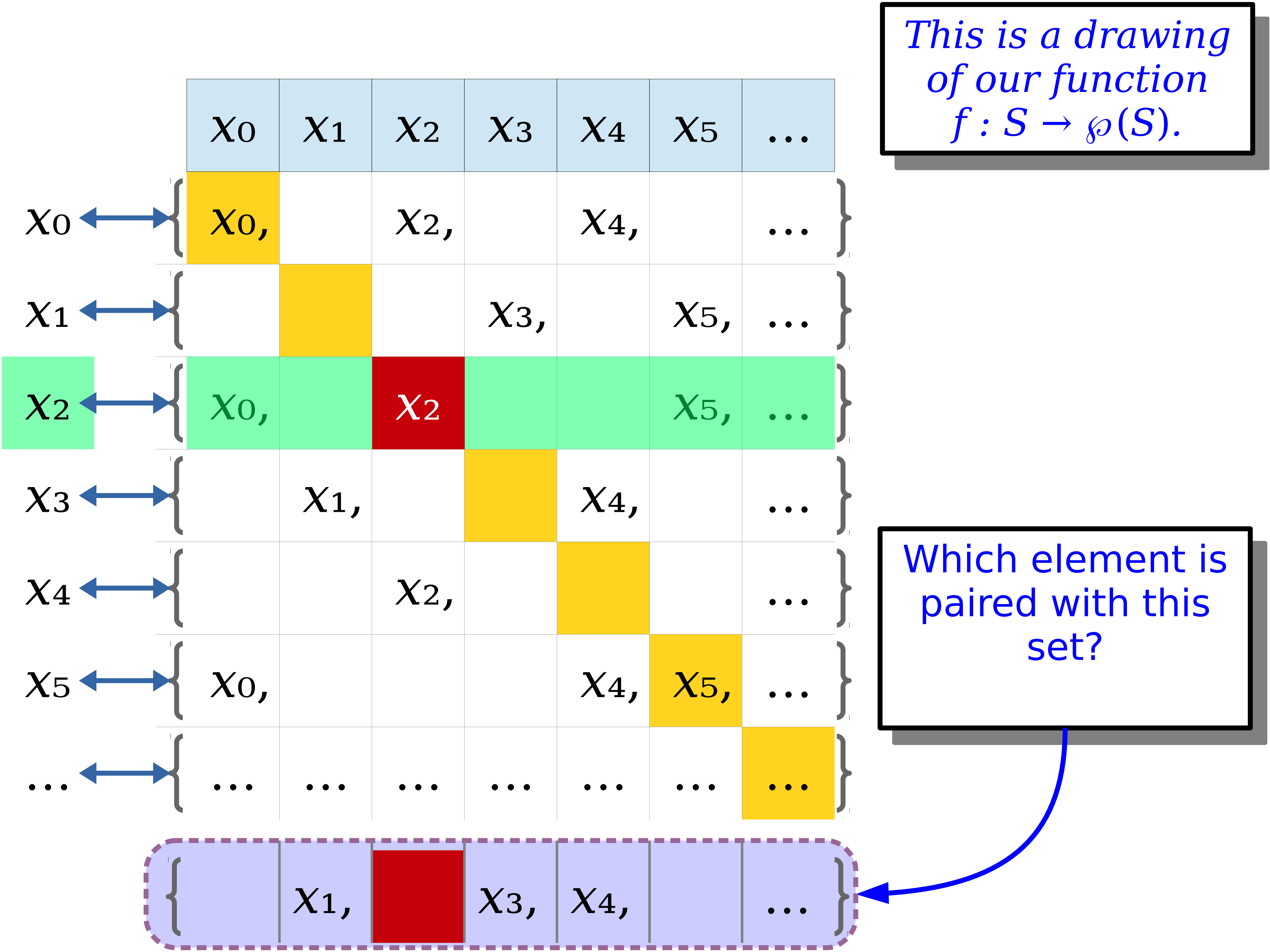
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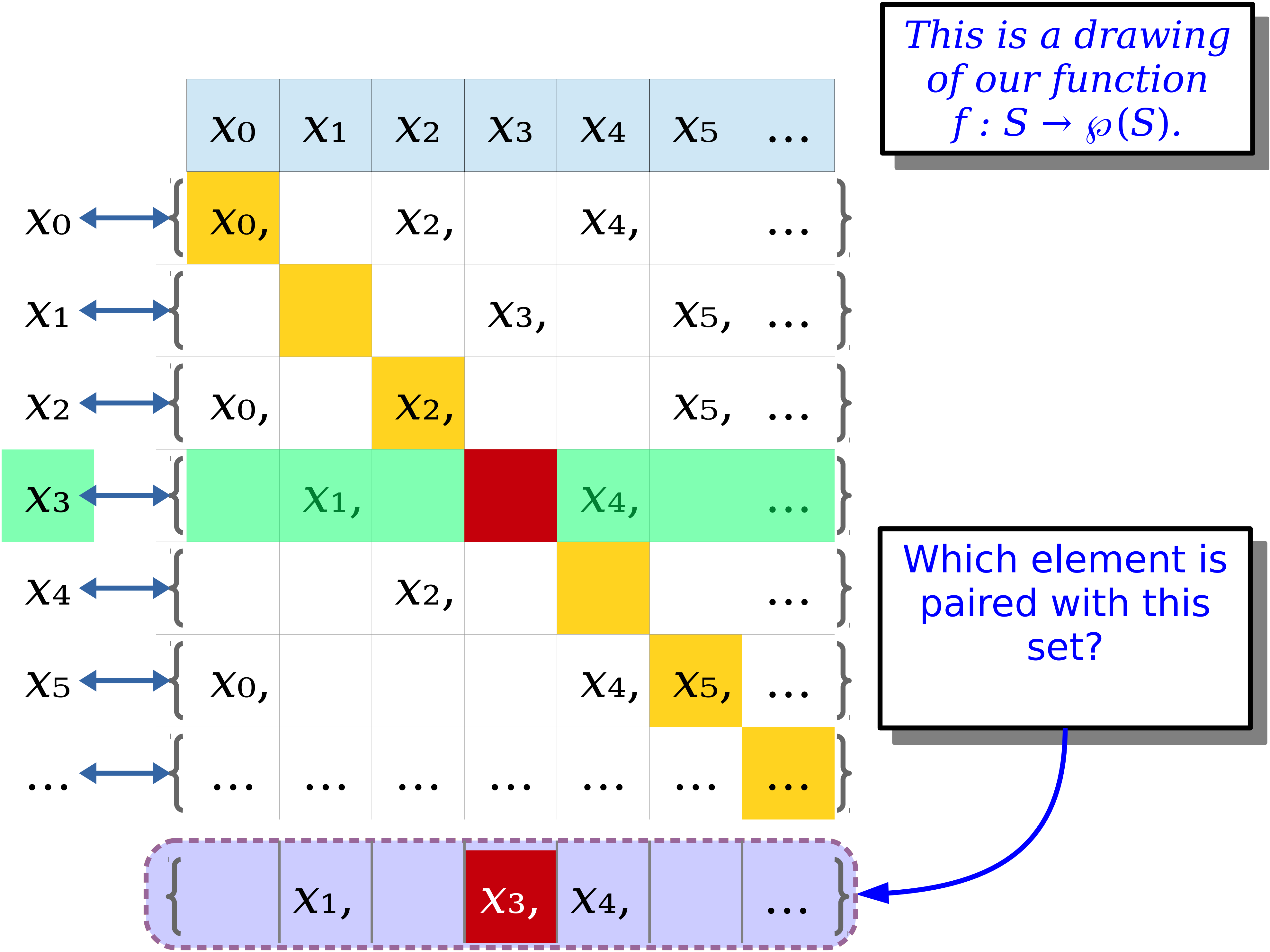
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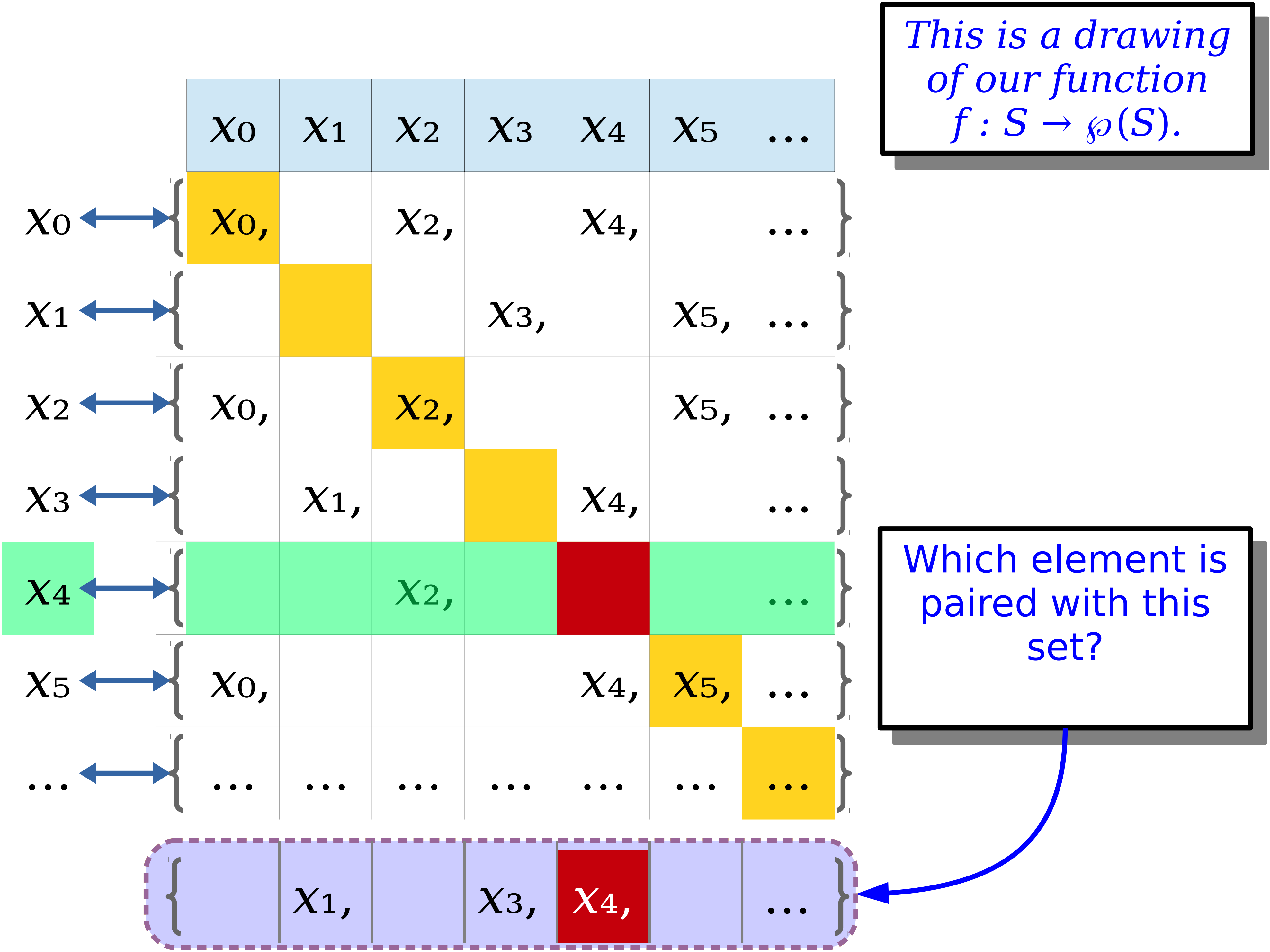
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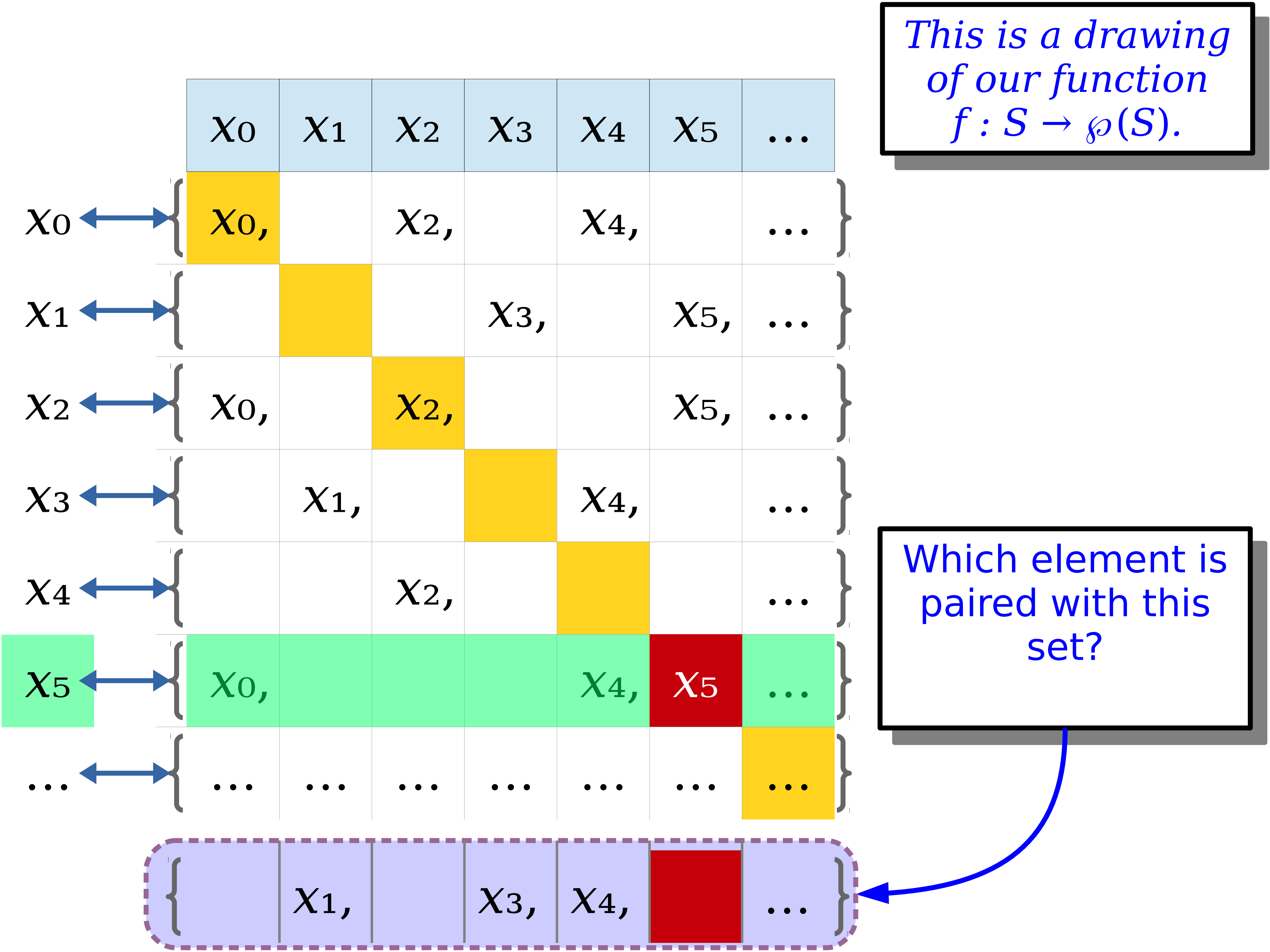

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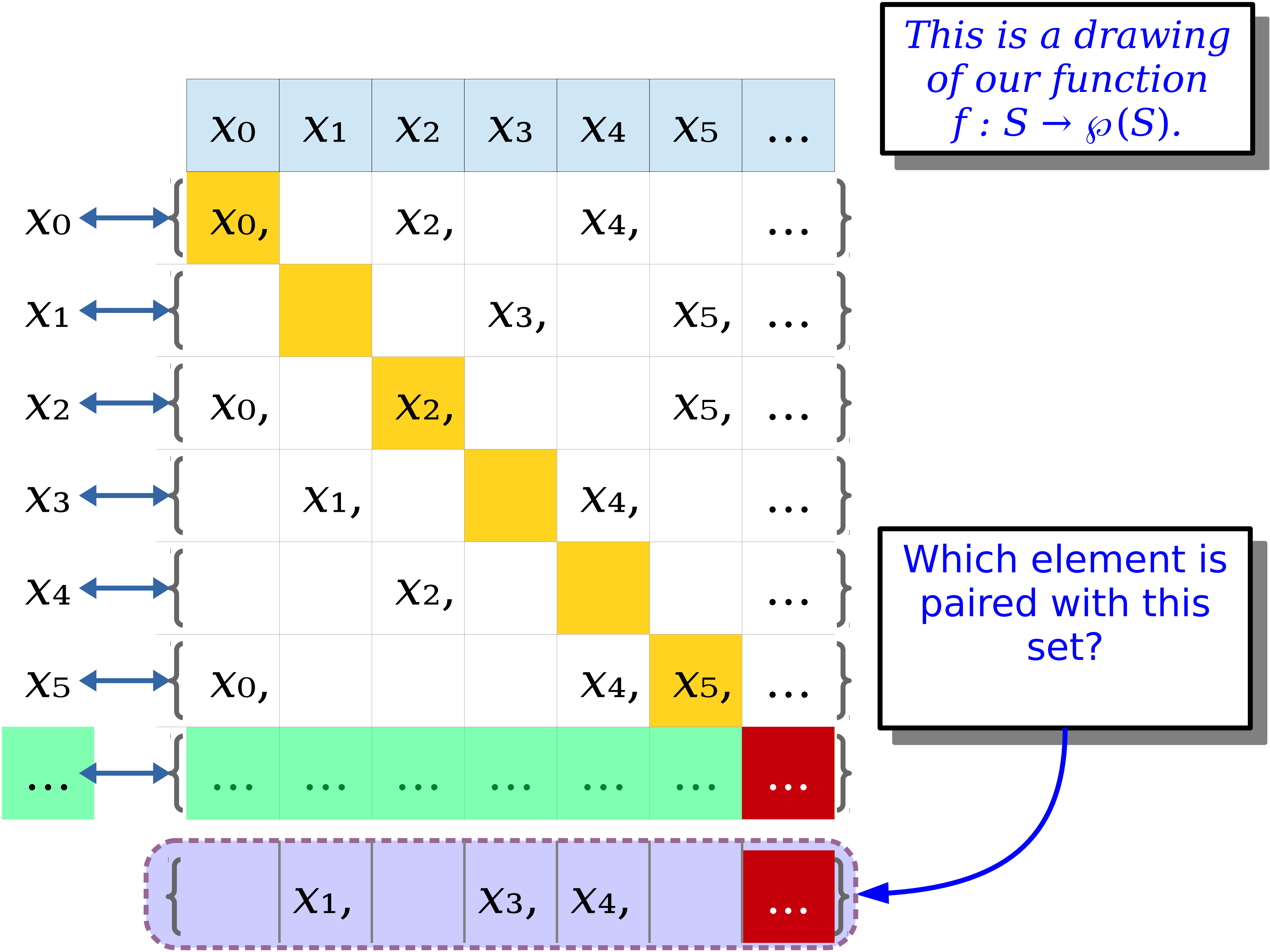
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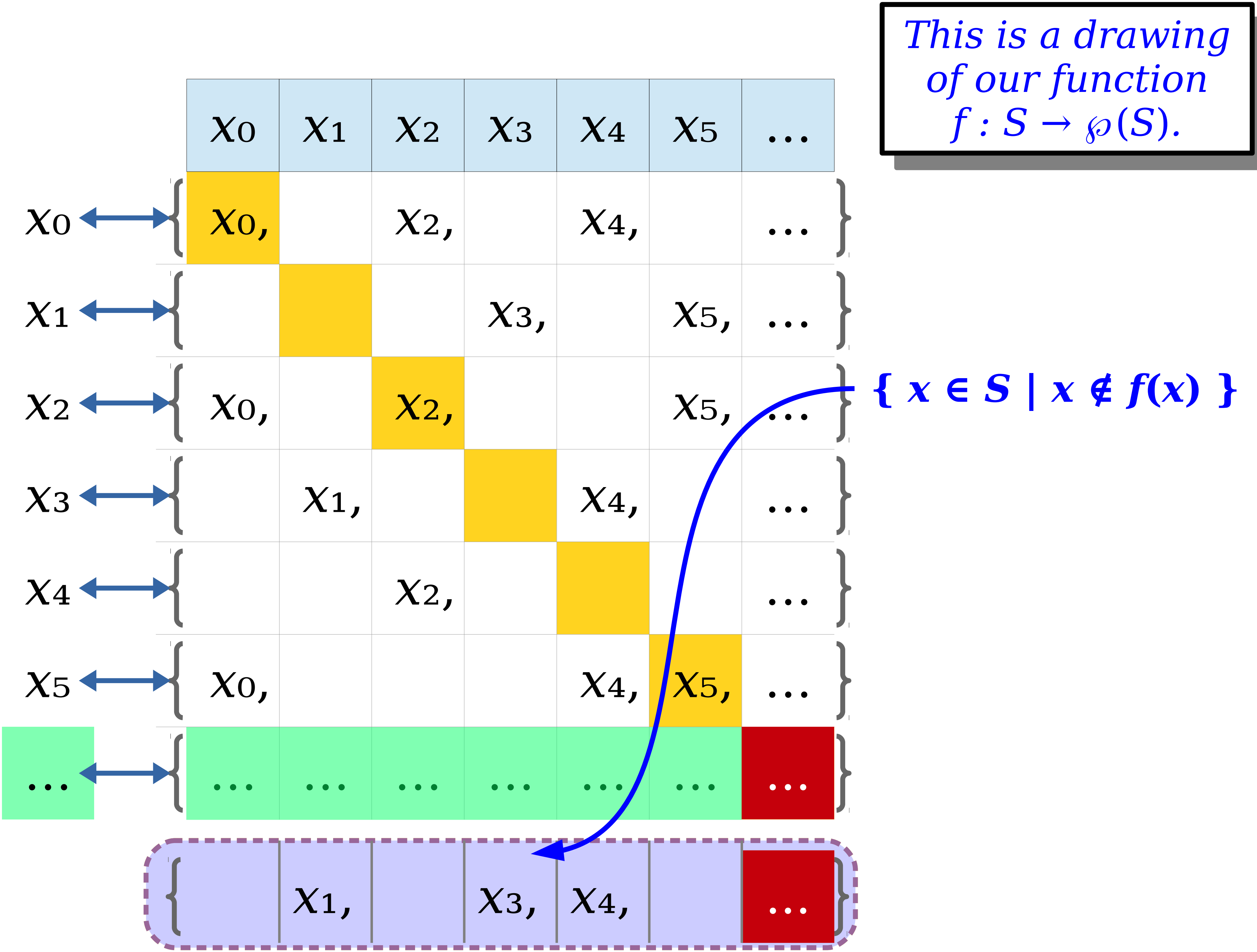
x_0 x_1 x_2 x_3 x_4 x_5 …
x_0 { x_0, x_2, x_4, … }
x_1 { x_3, x_5, … }
x_2 { x_0, x_2, x_5, … }
x_3 { x_1, x_4, … }
x_4 { x_2, … }
x_5 { x_0, x_4, x_5, … }
… { … … … … … … … }
{ x_1, x_3, x_4, … }
Which element is paired with this set?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... { ... ... }
$\{ x \in S \mid x \notin f(x) \}$
{ x_1, x_3, x_4, ... }

The Diagonal Set

- For any set S and function $f : S \to \wp(S)$, we can define a set D as follows:

$$\boldsymbol{D = \{\ x \in S \mid x \notin f(x)\ \}}$$

("The set of all elements x where x is not an element of the set f(x).")

- This is a formalization of the set we found in the previous picture.
- Using this choice of D, we can formally prove that no function $f : S \to \wp(S)$ is a bijection.

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The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Cardinality acts like an equivalence relation – but only because we can prove specific properties of how it behaves by relying on properties of function.
- Cantor's theorem can be formalized in terms of surjectivity.

Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
  - Building high-level structures out of lower-level ones!