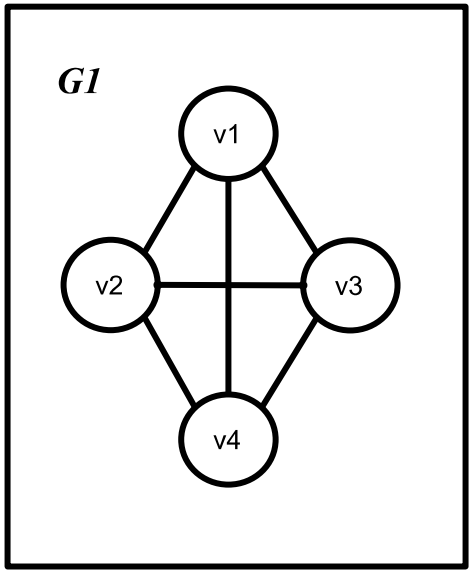
Verifiers and RE

Get ready to answer
some questions in rapid-fire style!
(about 10 seconds per question,
but I won’t close the recorded poll for a
while so don’t stress about that)

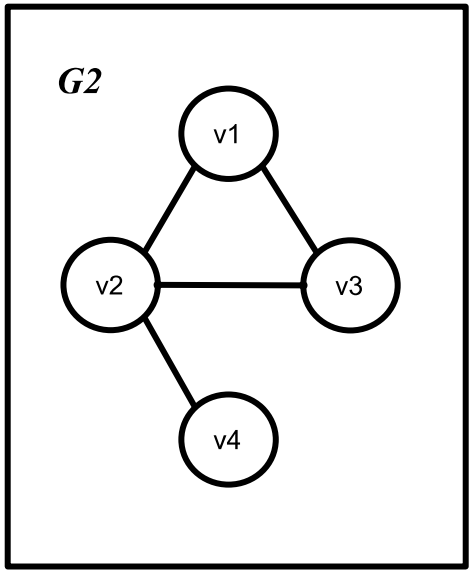
Definition:

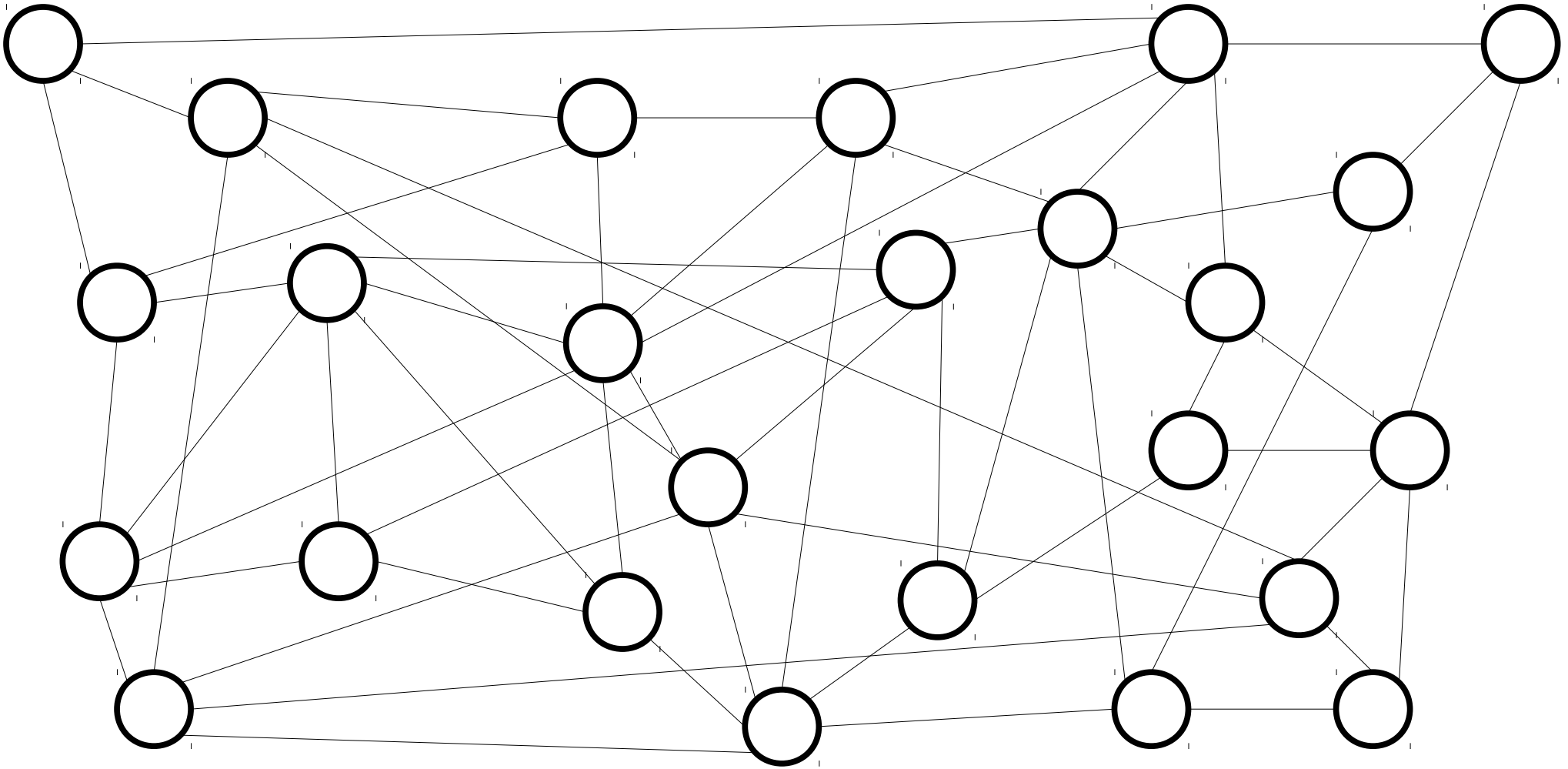
A ***k*-Clique** is a set of k vertices of a graph that are all adjacent to each other (all possible edges between those k vertices are present in the graph).

has a 4-Clique:



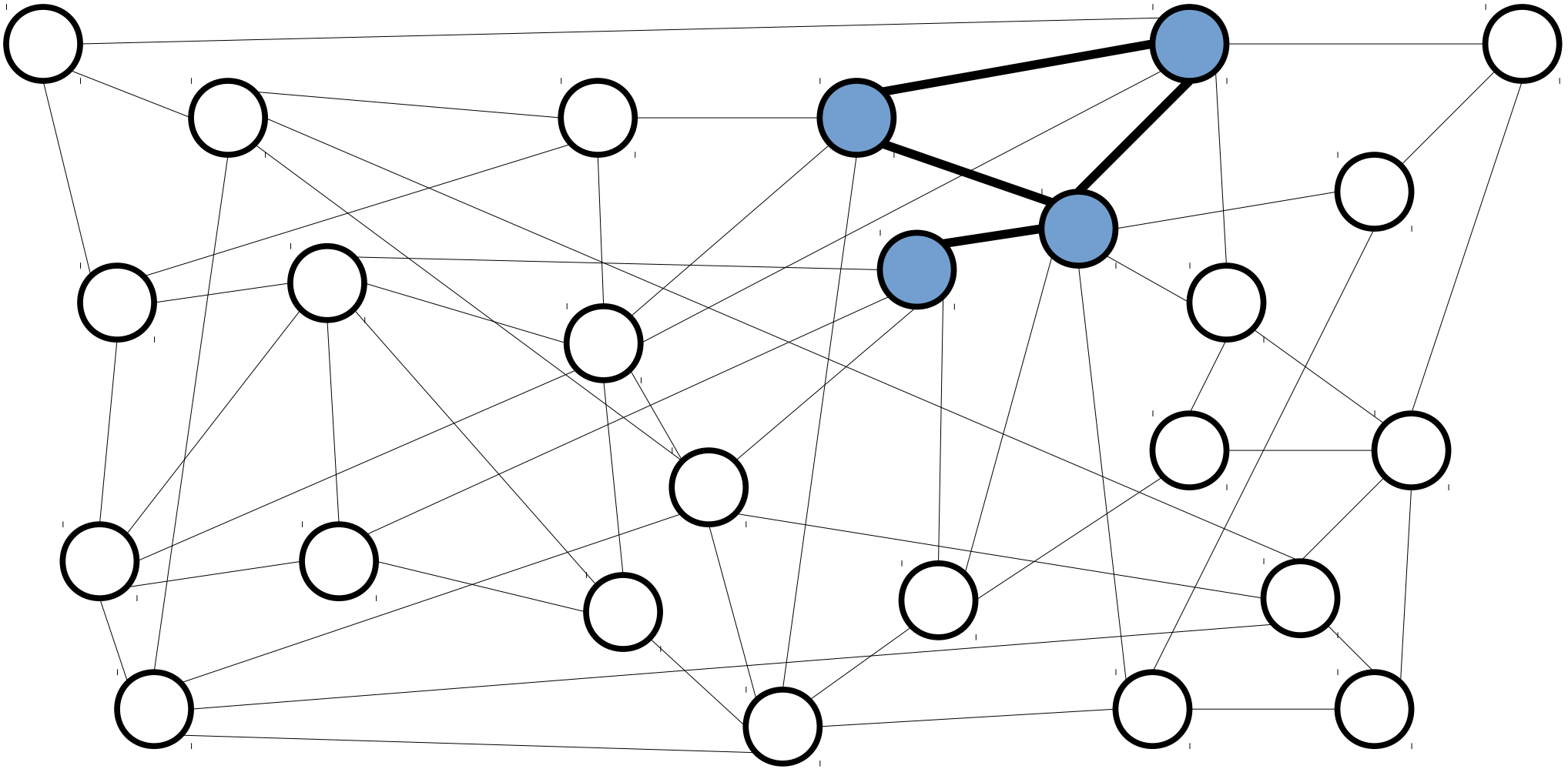
does *not* have a 4-Clique (has 3-Clique though):





QUICK REACTION: Does this graph contain a 4-clique?

Answer at **PollEv.com/cs103** or text **CS103** to **22333**
once to join, then **Y, N,** or **? (for "I don't know")**.



WITH A HINT: Does this graph contain a 4-clique?

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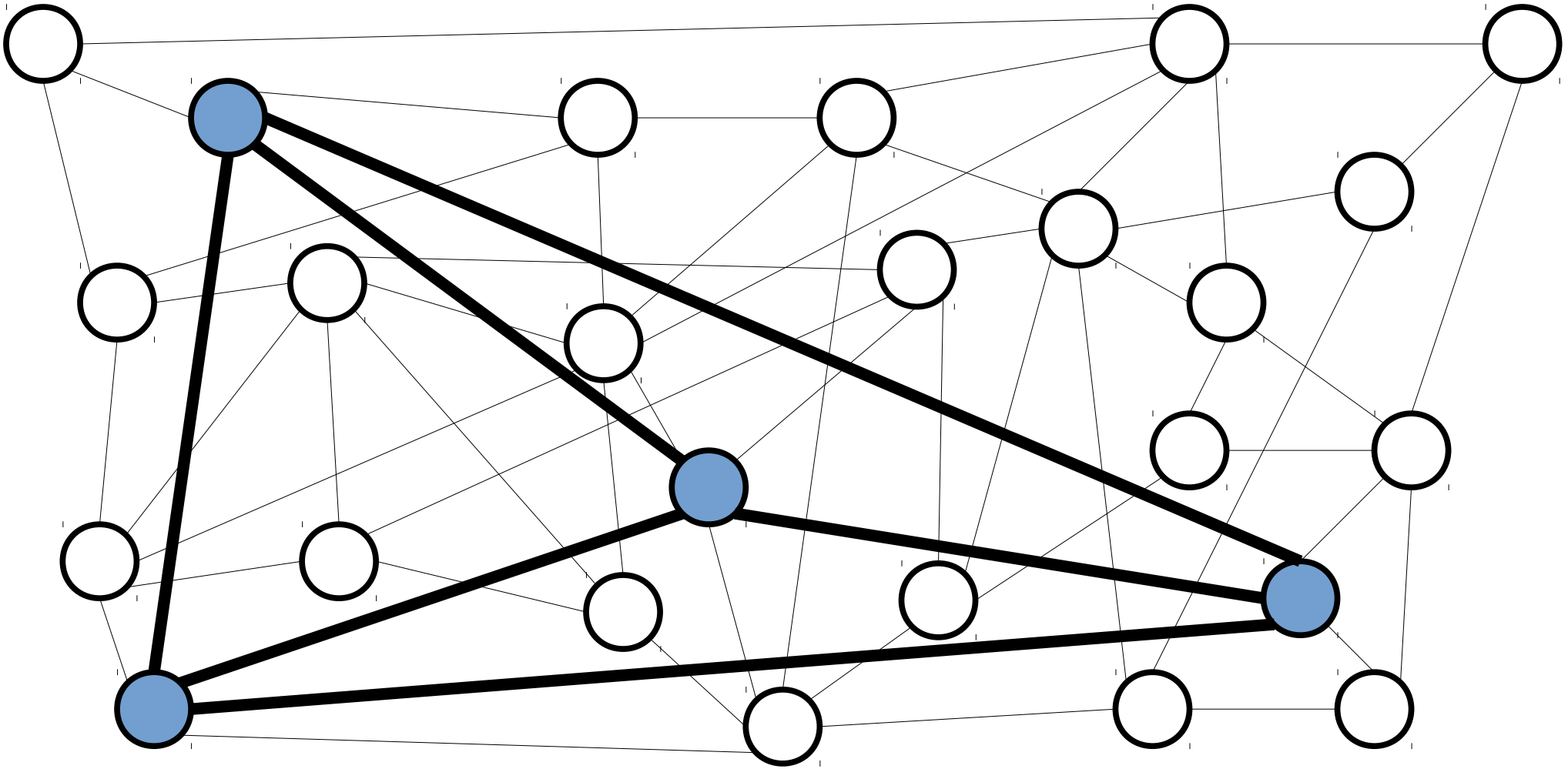

WITH A *NEW* HINT: Does this graph contain a 4-clique?

Answer at **PollEv.com/cs103** or text **CS103** to **22333**
once to join, then **Y, N,** or **? (for "I don't know")**.

Key Intuition:

A language L is in **RE** if, for any string w, if you are *convinced* that $w \in L$, there is some piece of evidence you could provide to convince someone else.

Discussion Question:

A language L is in **RE** if, for any string w, if you are *convinced* that $w \in L$, there is some piece of evidence you could provide to convince someone else.

What about for a $w \notin L$? What would a piece of evidence for that look like?

More examples of helpful hints vs unhelpful hints

Verification

		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

Does this Sudoku puzzle have a solution?

Verification

2	5	7	9	6	4	1	8	3
4	9	1	8	7	3	6	5	2
3	8	6	1	2	5	9	4	7
6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
1	6	3	2	5	7	4	9	8
5	7	8	4	9	6	2	3	1
9	2	4	3	8	1	7	6	5

Does this Sudoku puzzle have a solution?

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2	5	7	9	6	4	1	8	3
4	9	1	8	7	3	6	5	2
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6	4	5	7	3	2	8	1	9
7	1	9	5	4	8	3	2	6
8	3	2	6	1	9	5	7	4
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9	2	4	3	8	1	7	6	5

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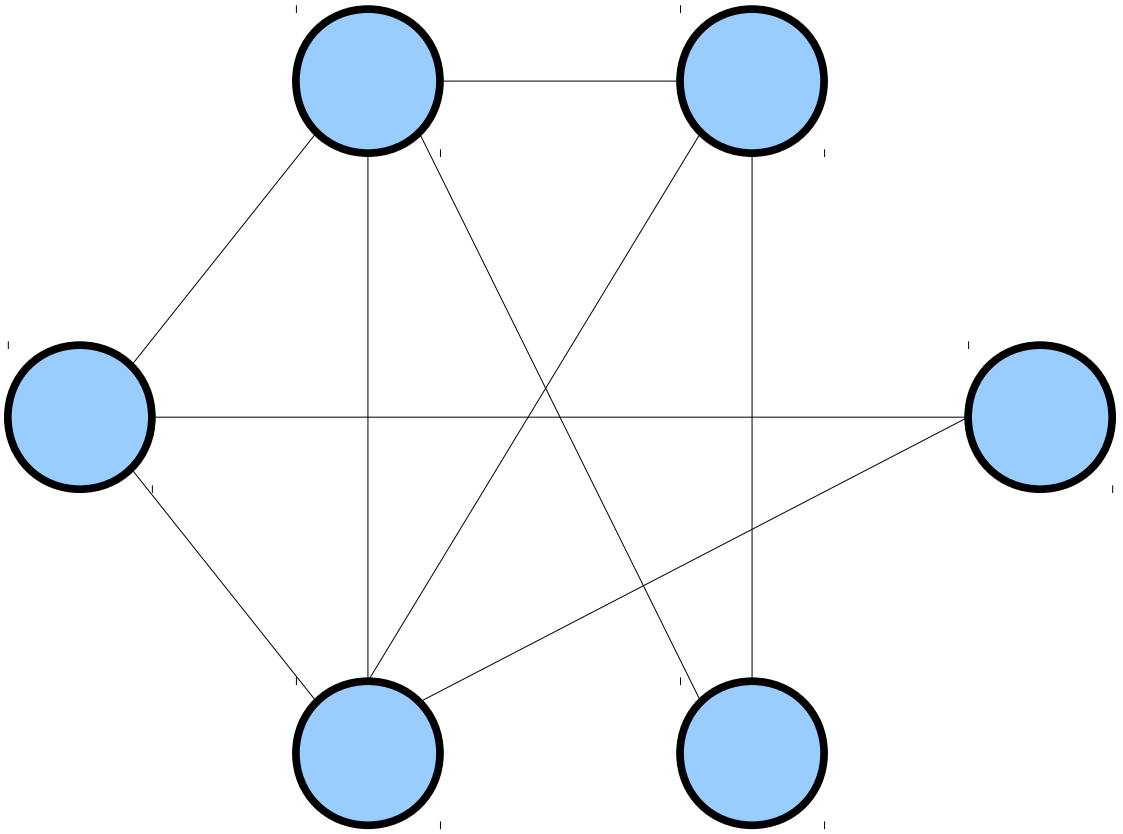
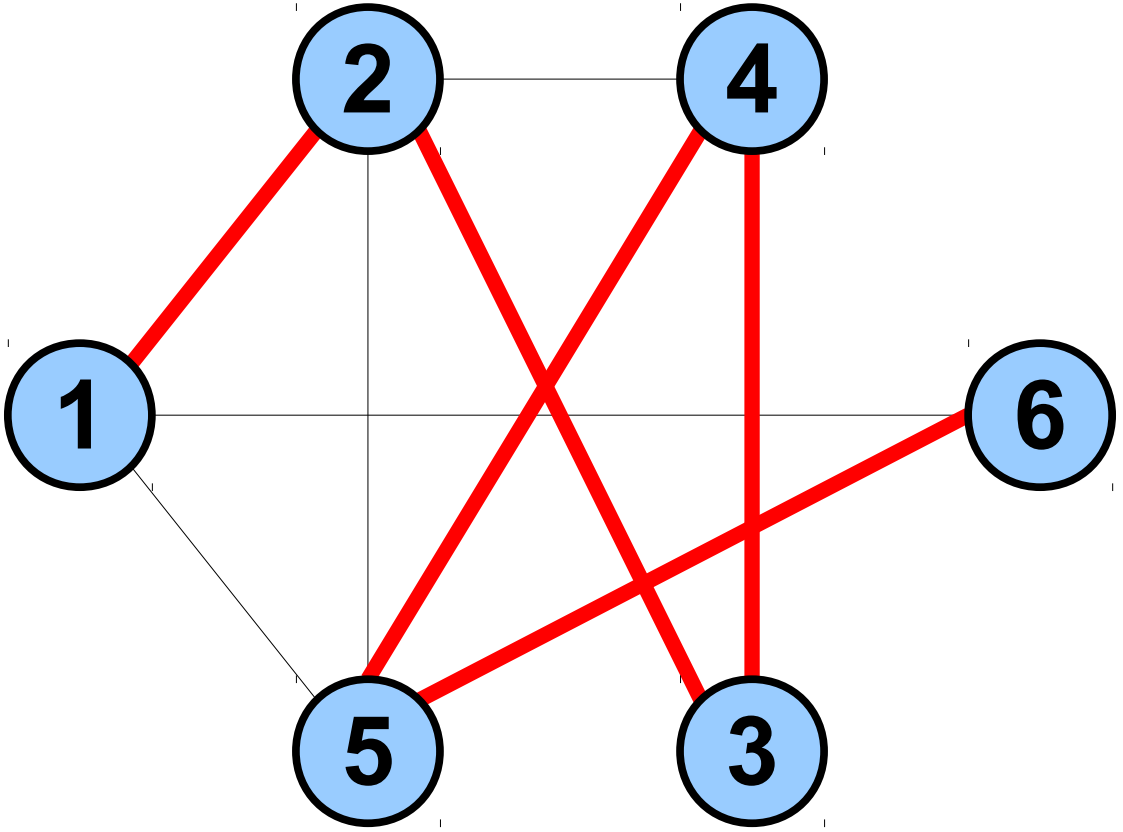


Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification



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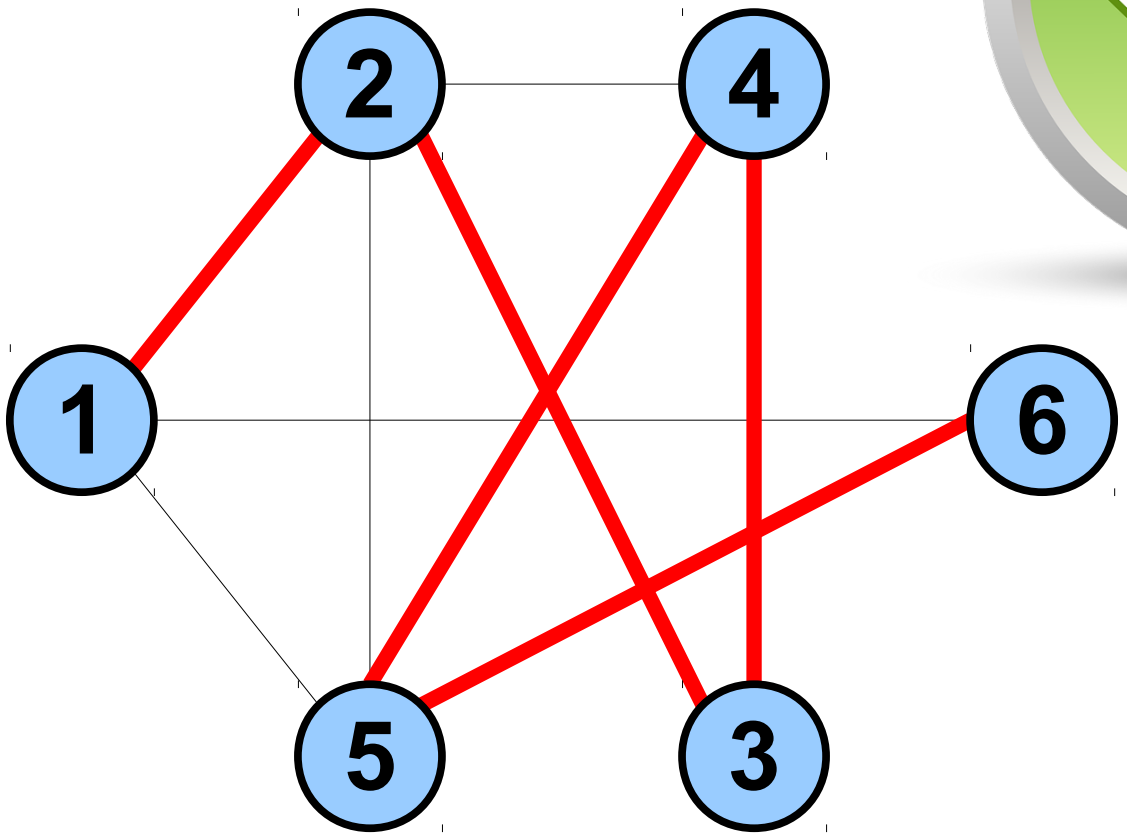

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	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

Does this Sudoku puzzle have a solution?

Verification

1	1	7	1	6	1	1	1	1
1	1	1	1	1	3	1	5	2
3	1	1	1	1	5	9	1	7
6	1	5	1	3	1	8	1	9
1	1	1	1	4	1	1	2	1
8	1	2	1	1	1	5	1	4
1	1	3	2	1	7	1	1	8
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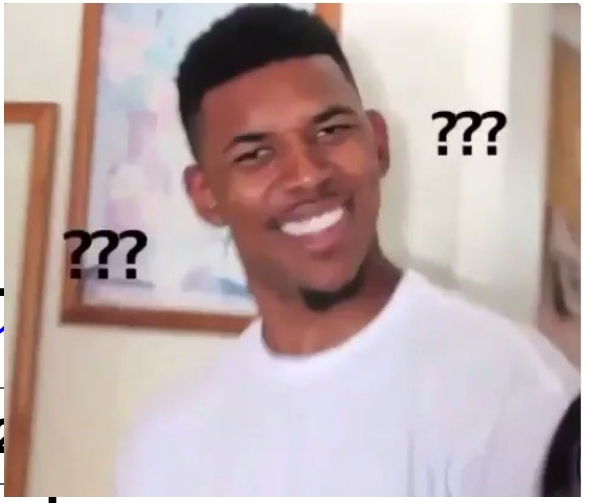
1	1	4	1	8	1	7	1	1

Does this Sudoku puzzle have a solution?

Verification



1	1	7	1	6	1	1	1	1
1	1	1	1	1	3	1	5	2
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6	1	5	1	3	1	8	1	9
1	1	1	1	4	1	1	2	1
8	1	2	1	1	1	5	1	4
1	1	3	2	1	7	1	1	8
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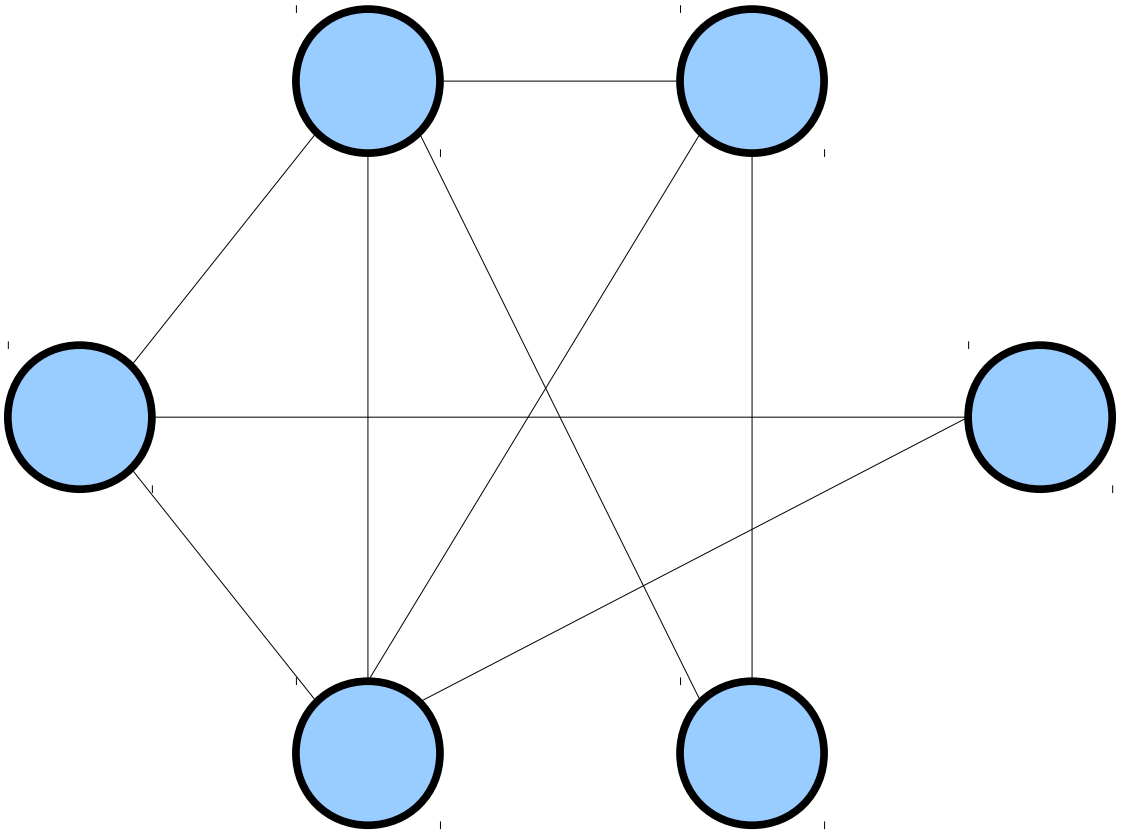
1	1	4	1	8	1	7	1	1

Does this Sudoku puzzle have a solution?

Verification



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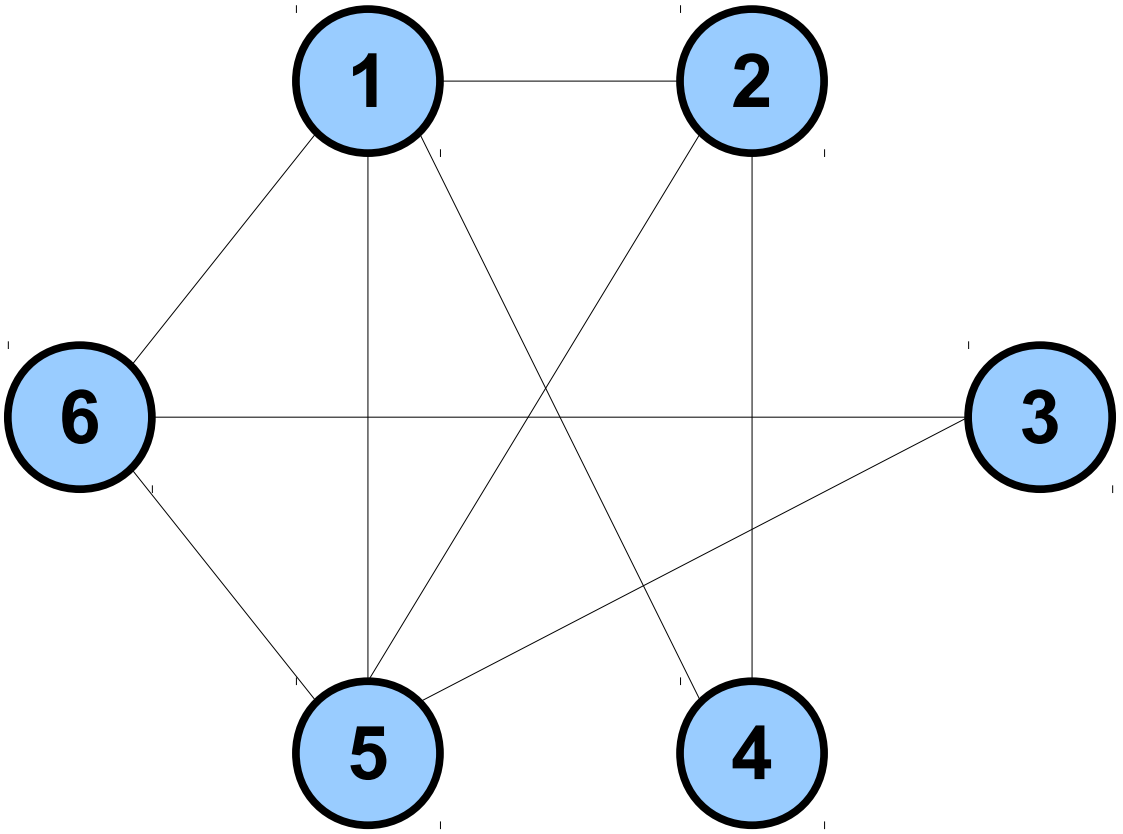

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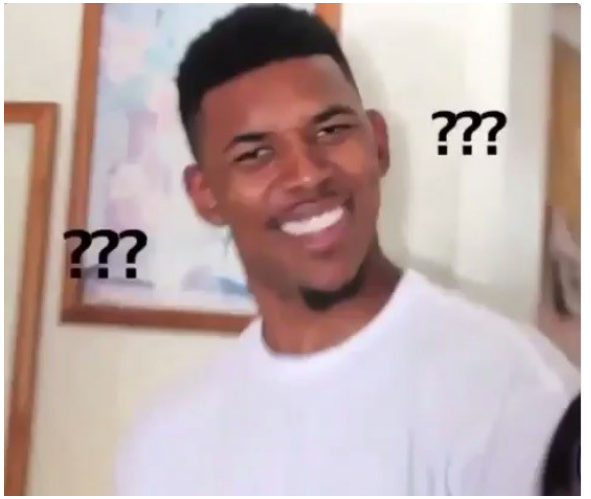
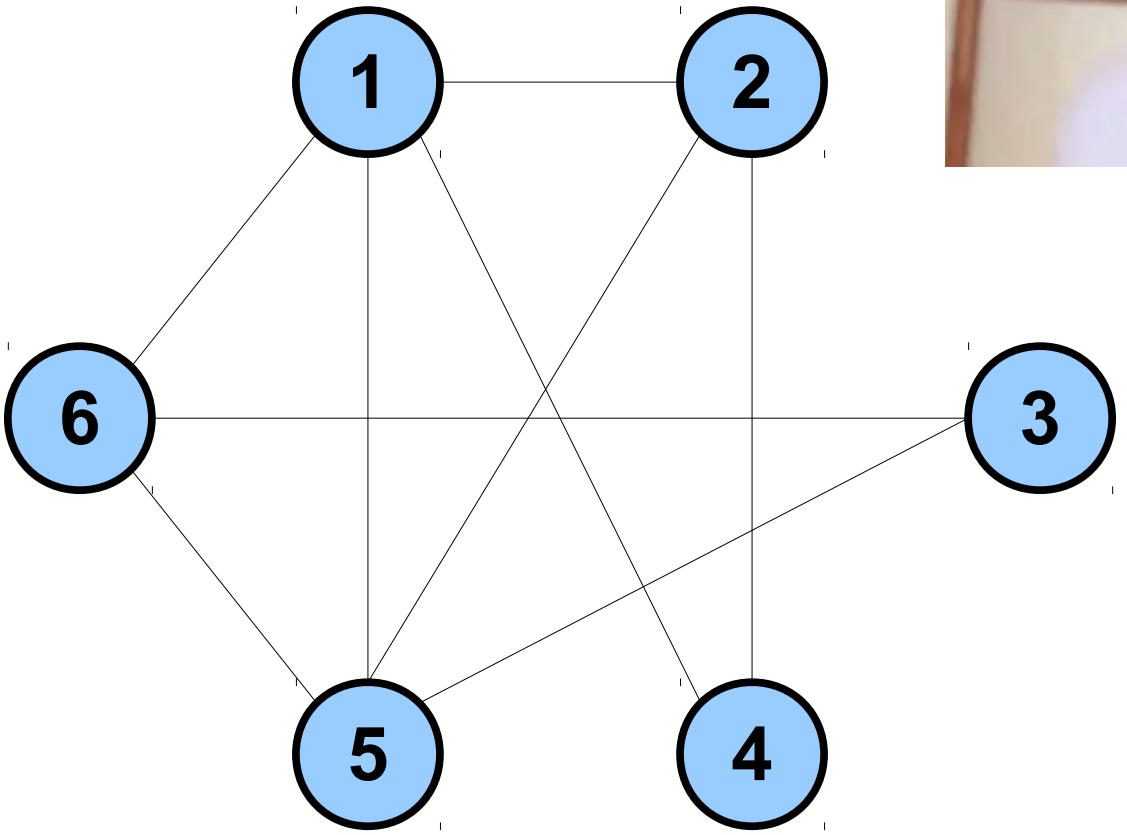
Verification





Does this graph have a ***Hamiltonian path*** (a simple path that passes through every node exactly once?)

Verification

- In each of the preceding cases, we were given some problem and some evidence supporting the claim that the answer is “yes.”
- Given the correct evidence, we can be certain that the answer is indeed “yes.”
- Given incorrect evidence, we aren't sure whether the answer is “yes.”
  - Maybe there's *no* evidence saying that the answer is “yes,” because the answer is no!
  - Or maybe there is some evidence, but just not the evidence we were given.
- Let's formalize this idea.

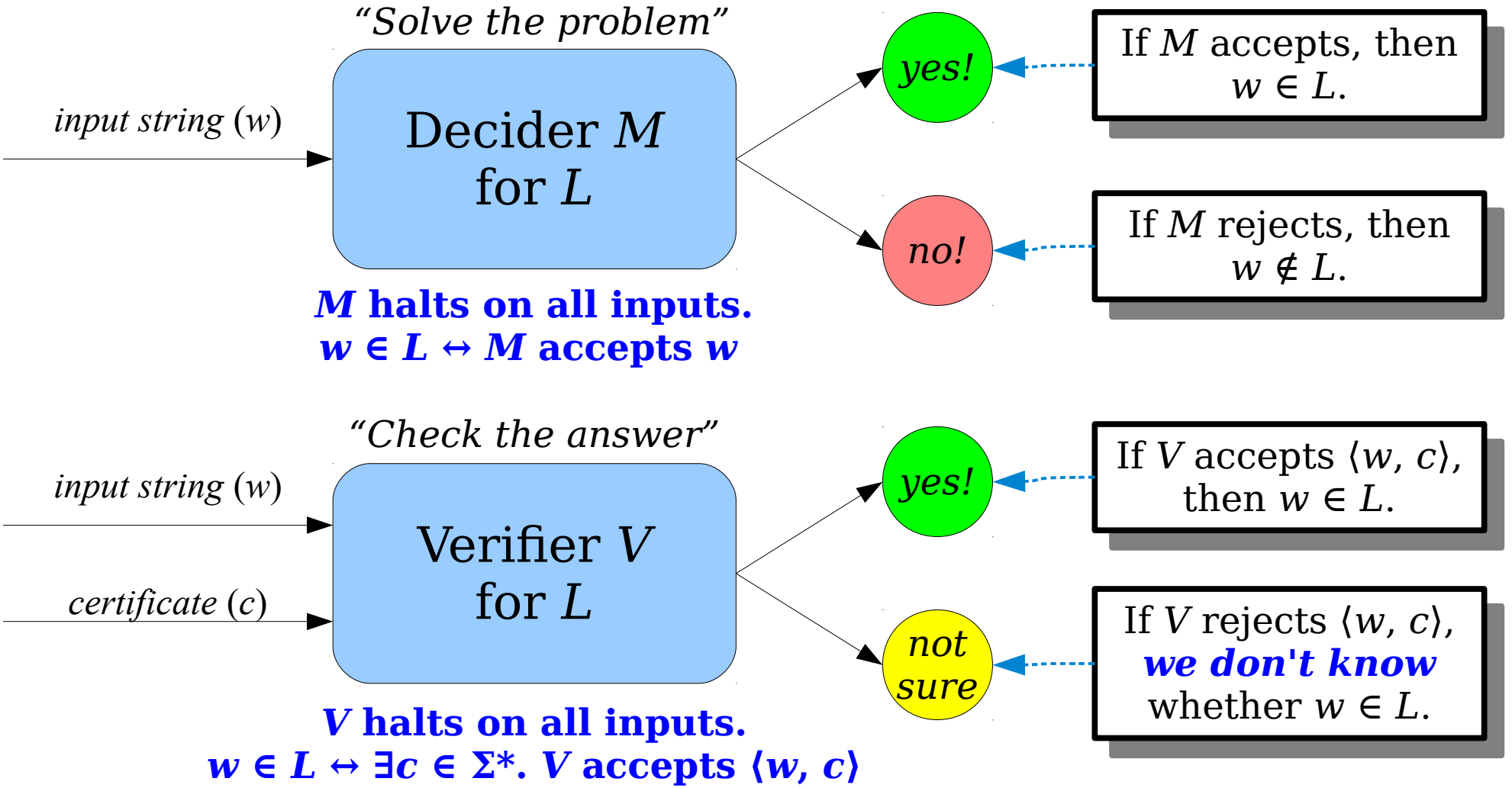
Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

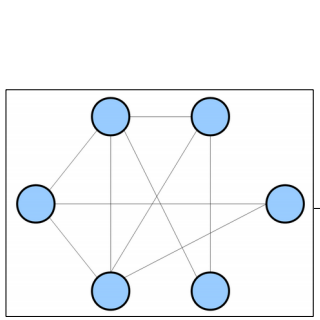
$$w \in L \iff \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle$$

- A string c where V accepts $\langle w, c \rangle$ is called a ***certificate*** for w.
- Intuitively, what does this mean?

Deciders and Verifiers



Deciders and Verifiers



(w)

"Solve the problem"

Decider M for L

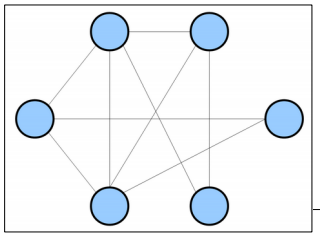
yes!

no!

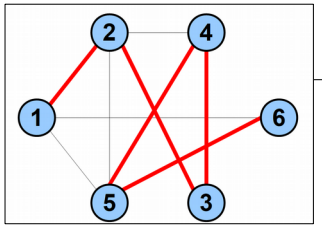
If M accepts, then $w \in L$.

If M rejects, then $w \notin L$.

M halts on all inputs.
$w \in L \leftrightarrow M$ accepts w



(w)



(c)

"Check the answer"

Verifier V for L

yes!

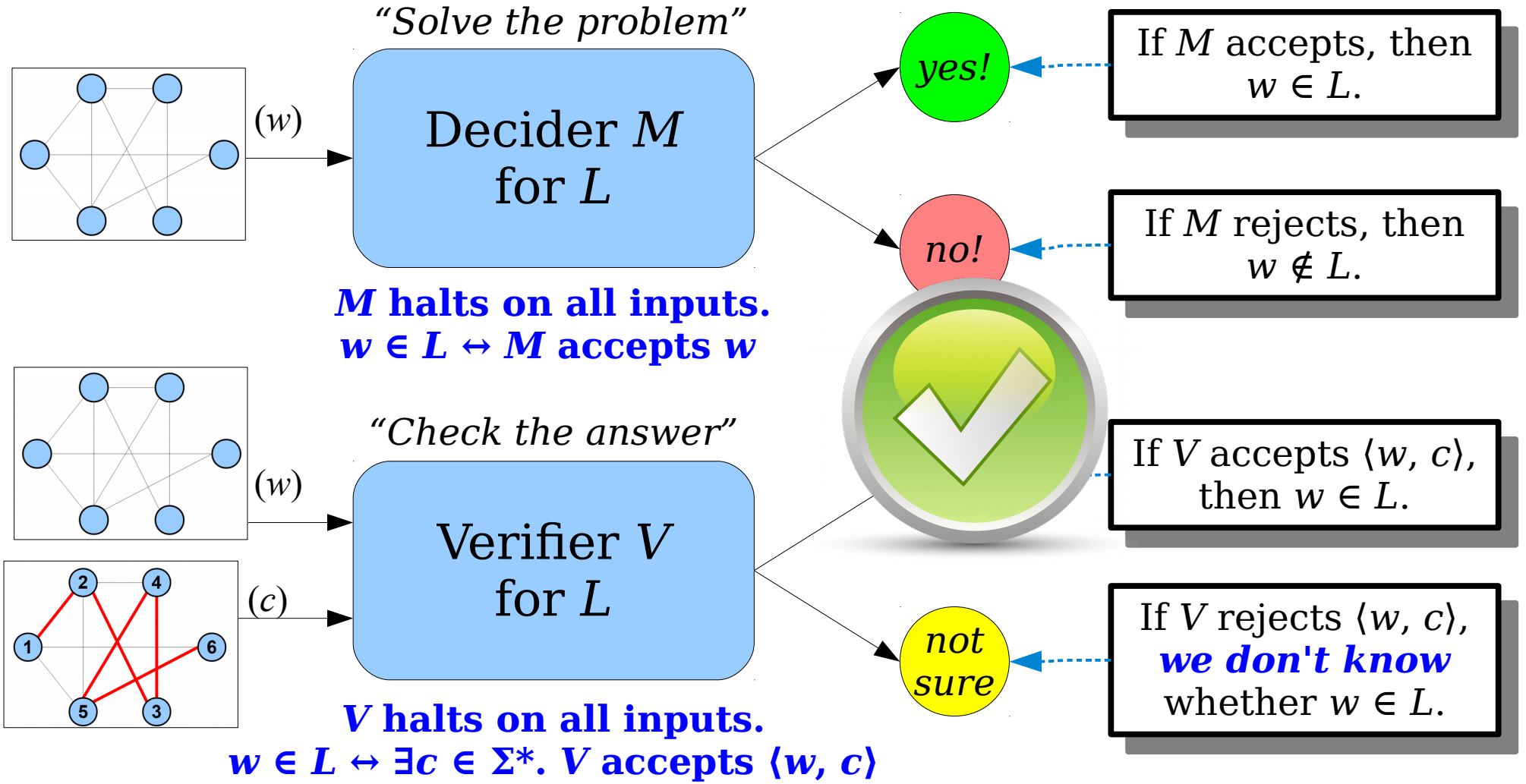
not sure

If V accepts $\langle w, c \rangle$, then $w \in L$.

If V rejects $\langle w, c \rangle$, ***we don't know*** whether $w \in L$.

V halts on all inputs.
$w \in L \leftrightarrow \exists c \in \Sigma^*$. V accepts $\langle w, c \rangle$

Deciders and Verifiers



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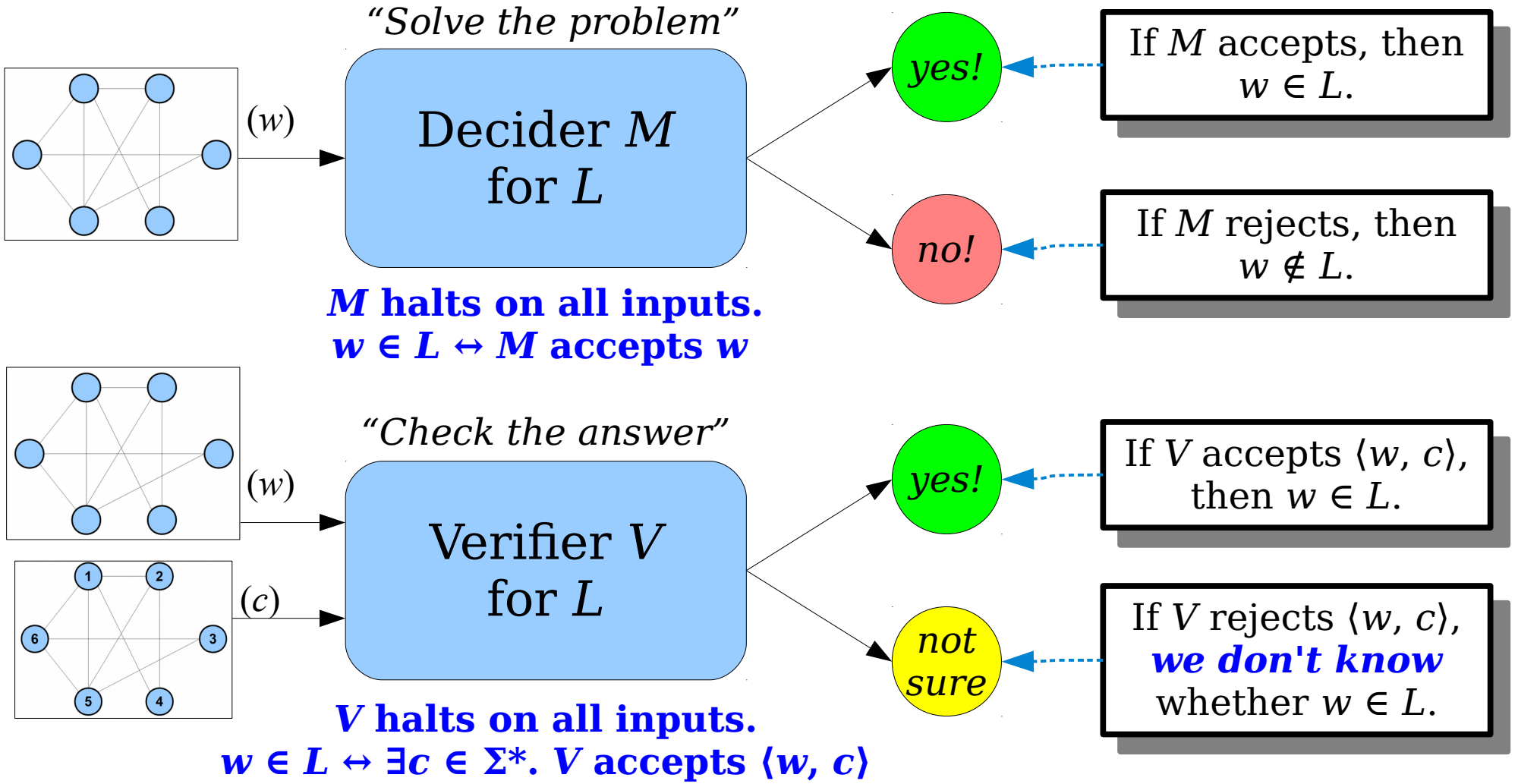
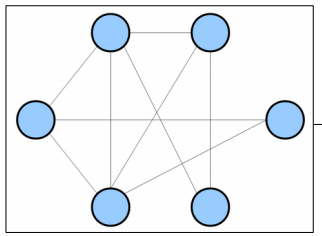


Deciders and Verifiers



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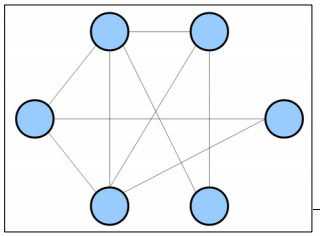

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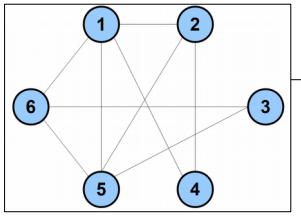

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Verifier V for L

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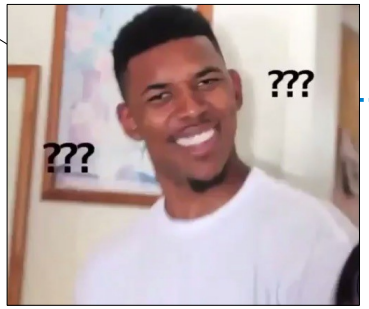

If V rejects $\langle w, c \rangle$, ***we don't know*** whether $w \in L$.

V halts on all inputs.
$w \in L \leftrightarrow \exists c \in \Sigma^*$. V accepts $\langle w,$

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
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  - For any string $w \in \Sigma^*$, the following is true:

$$\boldsymbol{w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle}$$

- Some notes about V:
  - If V accepts $\langle w, c \rangle$, then we're guaranteed $w \in L$.
  - If V does not accept $\langle w, c \rangle$, then either
    - $w \in L$, but you gave the wrong c, or
    - $w \notin L$, so no possible c will work.

Verifiers

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- More notes about V:
  - Notice that c is existentially quantified.
  - Notice V is required to halt *always* (like a decider).

Verifiers

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- More notes about V:
  - Notice that $\mathscr{L}(V) \neq L$. *(Good question to hold on to for a second: what <u>is</u> $\mathscr{L}(V)$?)*
  - The job of V is just to check certificates, not to decide membership in L.

Verifiers

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- A note about c:
  - Figuring out what would make a good certificate (should it be a number of steps to take, an equation-solving variable assignment, a set of graph nodes, an array of numbers to fill in a whole Sudoku board?) is custom work to do for each different language L.

Some Verifiers

- Let L be the following language:

$$L = \{ \langle n \rangle \mid n \in \mathbb{N} \text{ and the hailstone sequence terminates for } n \}$$

```
bool checkHailstone(int n, int c) {
    for (int i = 0; i < c; i++) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
        if (n == 1) return true;
    }
    return n == 1;
}
```

Some Verifiers

Does this always halt?

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for $n \}$

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Some Verifiers

For one given $\langle n \rangle \in L$ (say 11), how many different values of c will work to cause the verifier to accept?

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for $n \}$

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```

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How many of these statements are true of $\mathscr{L}(V)$?

- $\mathscr{L}(V) = L$
- $\mathscr{L}(V) \subseteq L$
- $L \subseteq \mathscr{L}(V)$

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for $n \}$

```
bool checkHailstone(int n, int c) {
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}
```

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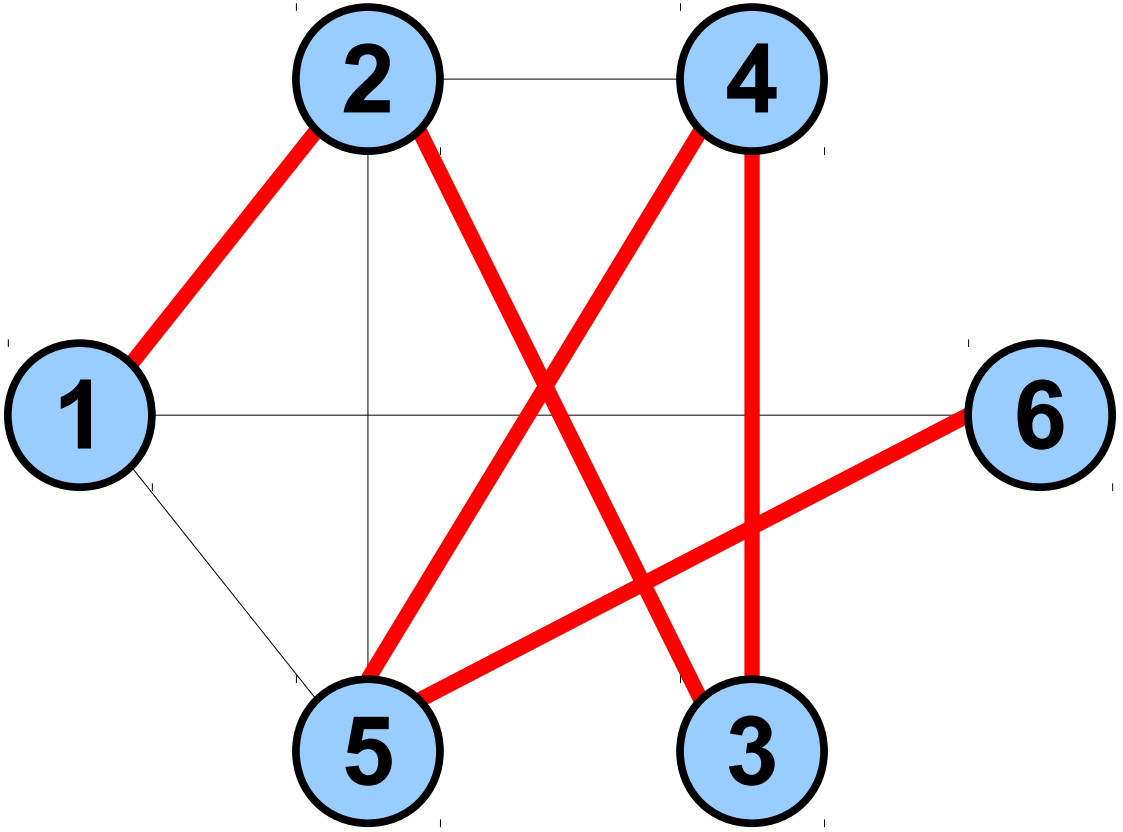
Some Verifiers

- Let L be the following language:

$$L = \{ \langle G \rangle \mid G \text{ is a graph and } G \text{ has a Hamiltonian path} \}$$

- (A Hamiltonian path is a simple path that visits every node in the graph.)
- Let's see how to build a verifier for L.

Verification



Is there a simple path that goes through every node exactly once?

Verifier Example: Hamiltonian Path

- Let L be the following language:

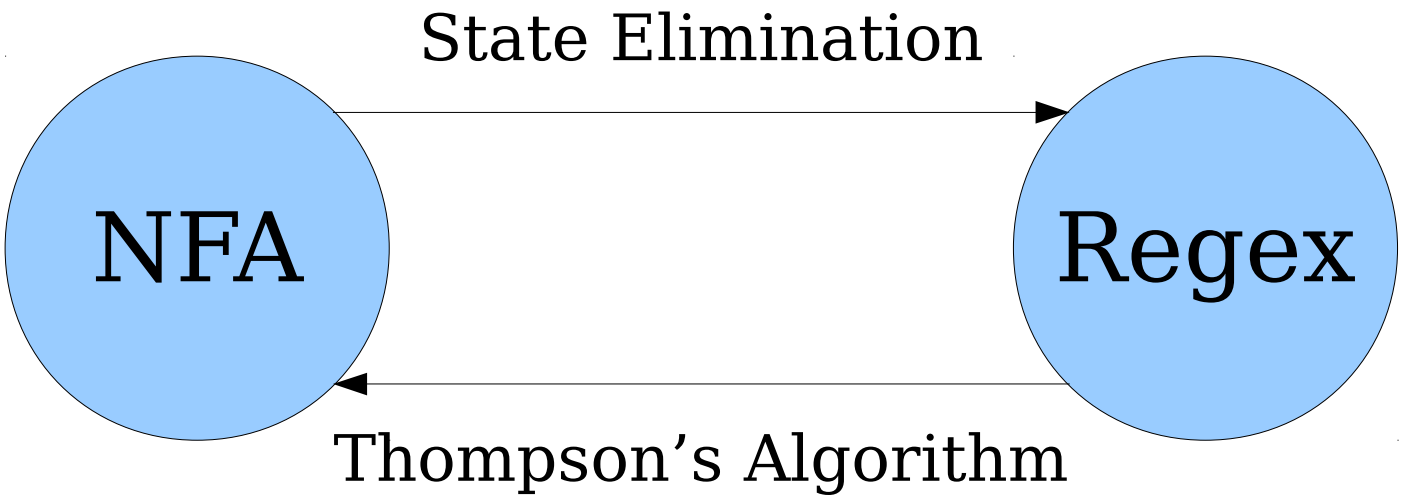
$$L = \{ \langle G \rangle \mid G \text{ is a graph with a Hamiltonian path } \}$$

```
bool checkHamiltonian(Graph G, vector<Node> c) {
   if (c.size() != G.numNodes()) return false;
   if (containsDuplicate(c)) return false;

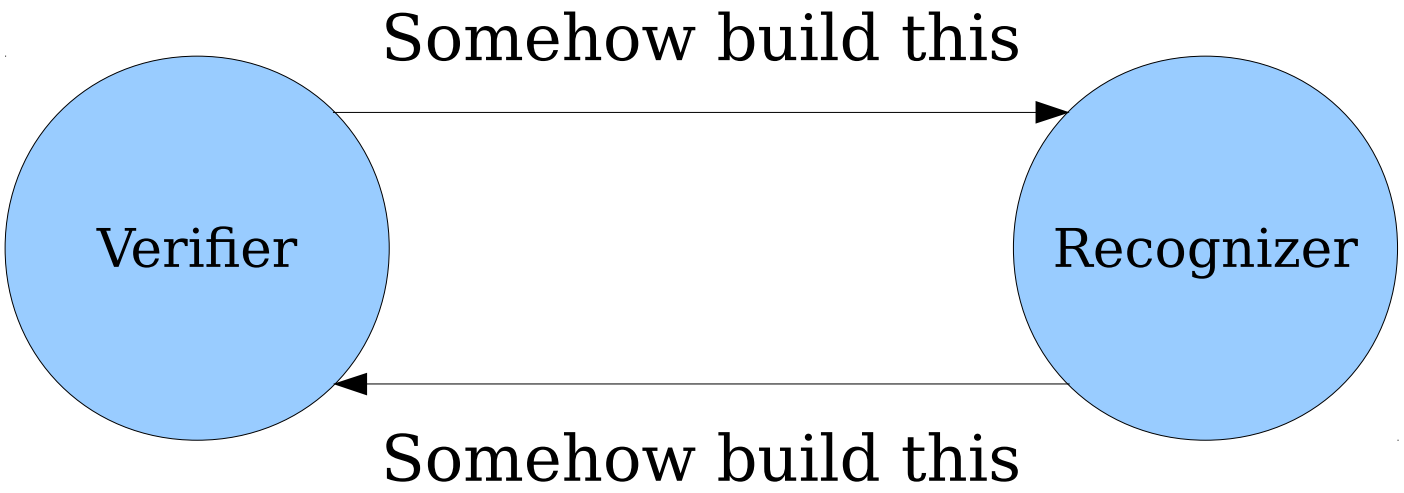
   for (size_t i = 0; i < c.size() - 1; i++) {
       if (!G.hasEdge(c[i], c[i+1])) return false;
   }
   return true;
}
```

- Do you see why $\langle G \rangle \in L$ iff there is a c where `checkHamiltonian(G, c)` returns true?
- Do you see why `checkHamiltonian` always halts?

Where We've Been



Where We're Going Today



Verifier for A_{TM}?

- Consider A_{TM}:

  $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$.

- This is a ***canonical*** example of an undecidable language. There's no way, in general, to tell whether a TM M will accept a string w.
- Although this language is undecidable, it's an **RE** language, and it's possible to build a verifier for it!

What would make a good certificate for a verifier for A_{TM}?

- Consider A_{TM}:

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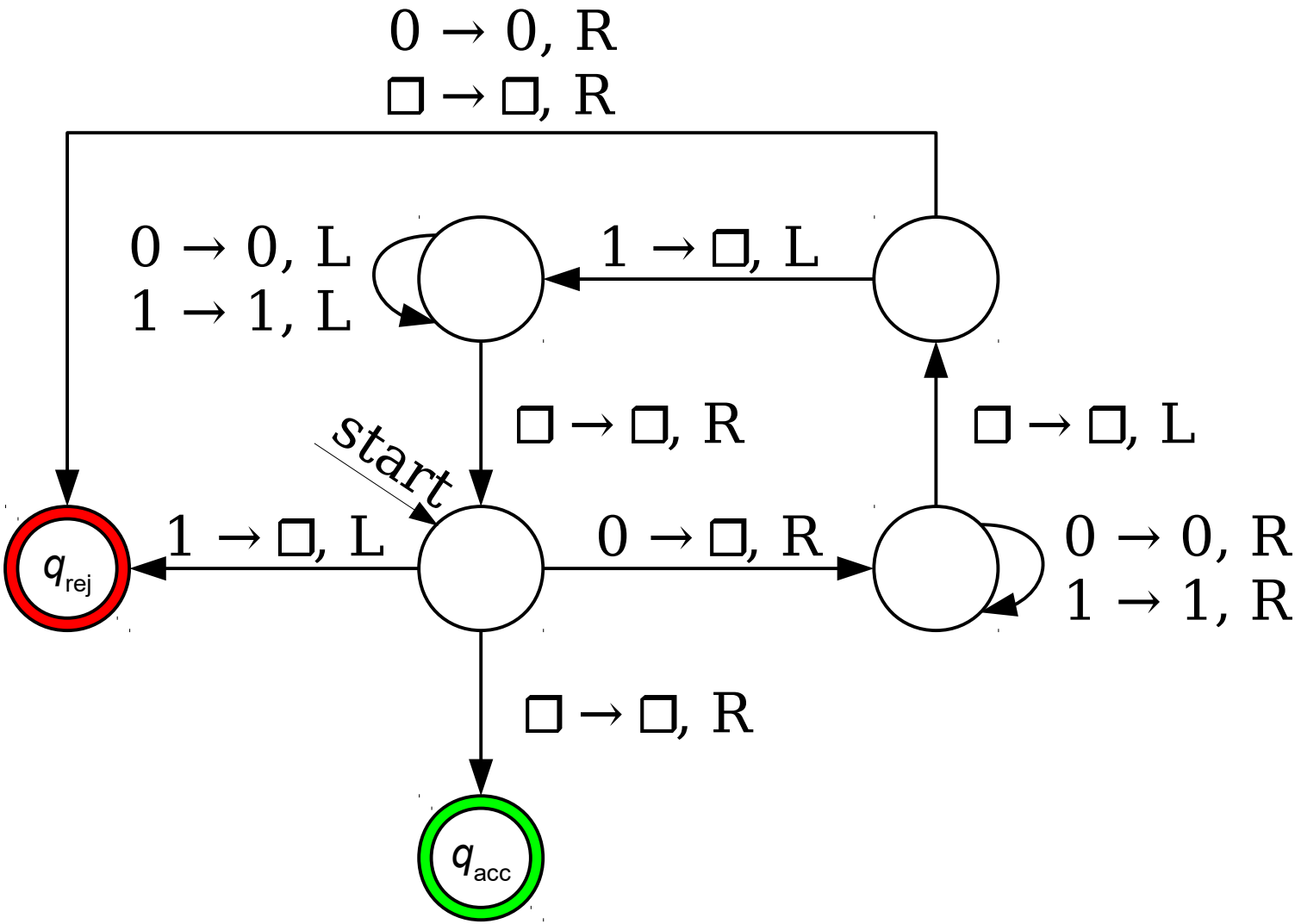
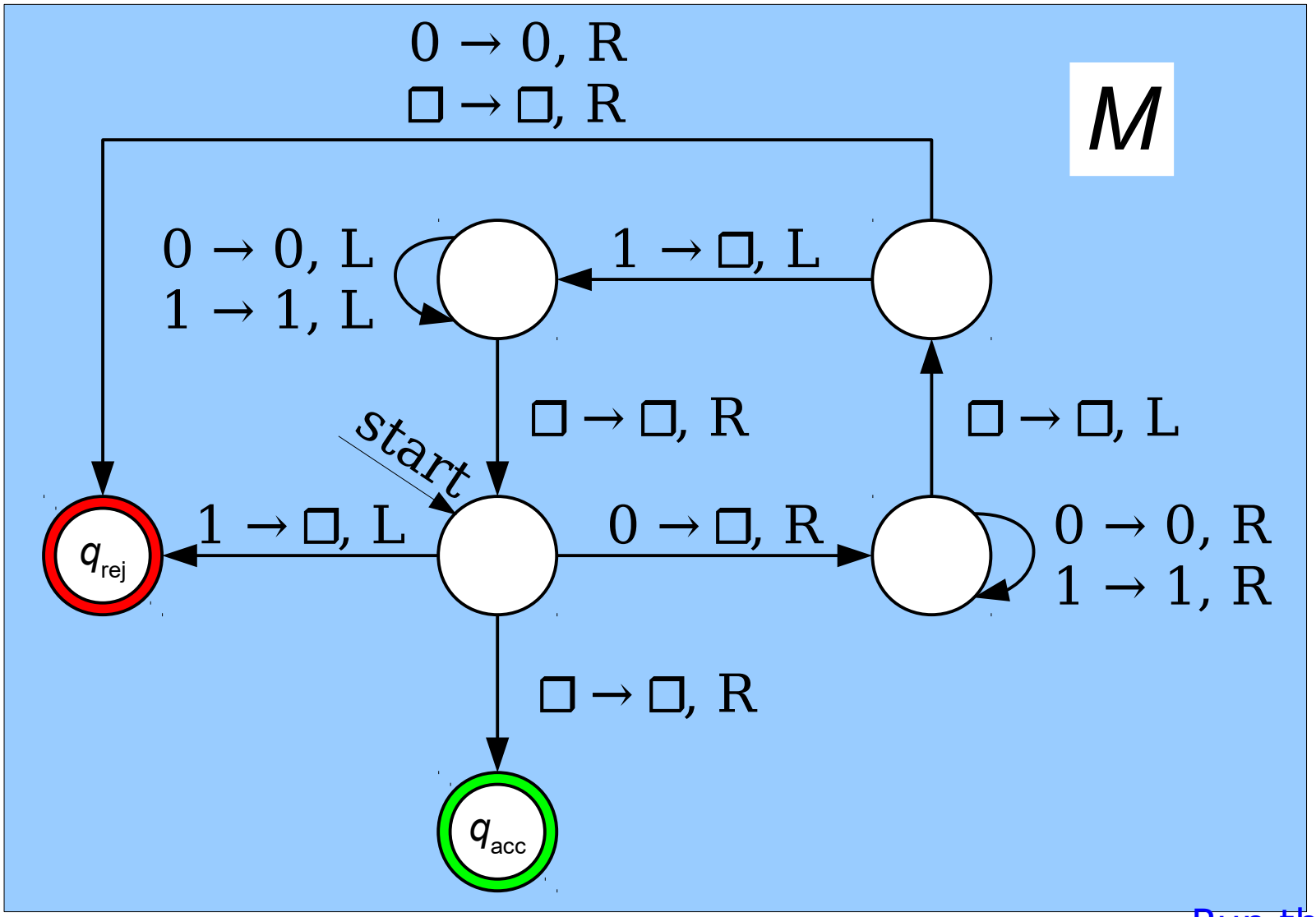
- Although this language is undecidable, it's an **RE** language, and it's possible to build a verifier for it!

Answer at **PollEv.com/cs103** or text **CS103** to **22333** once to join, then **an idea**



Run this TM for fifteen steps.

…			0	0	1	1			…



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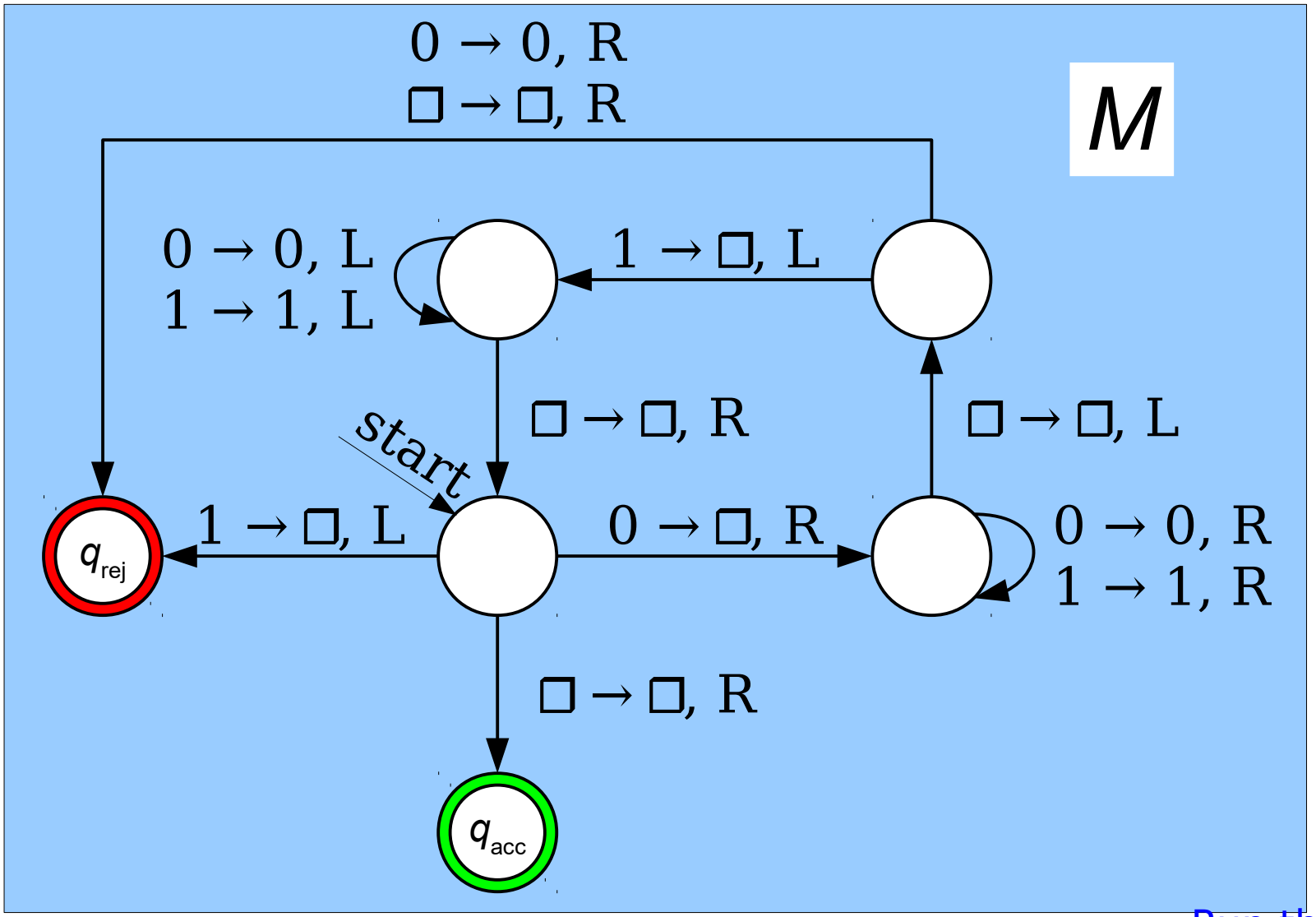
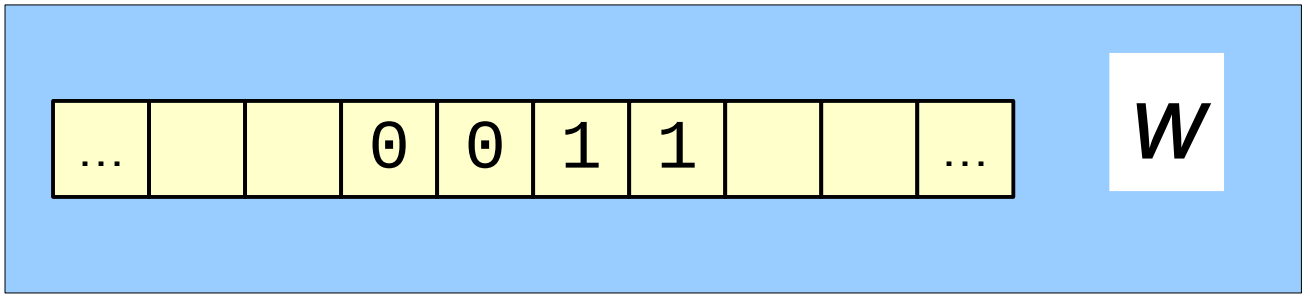
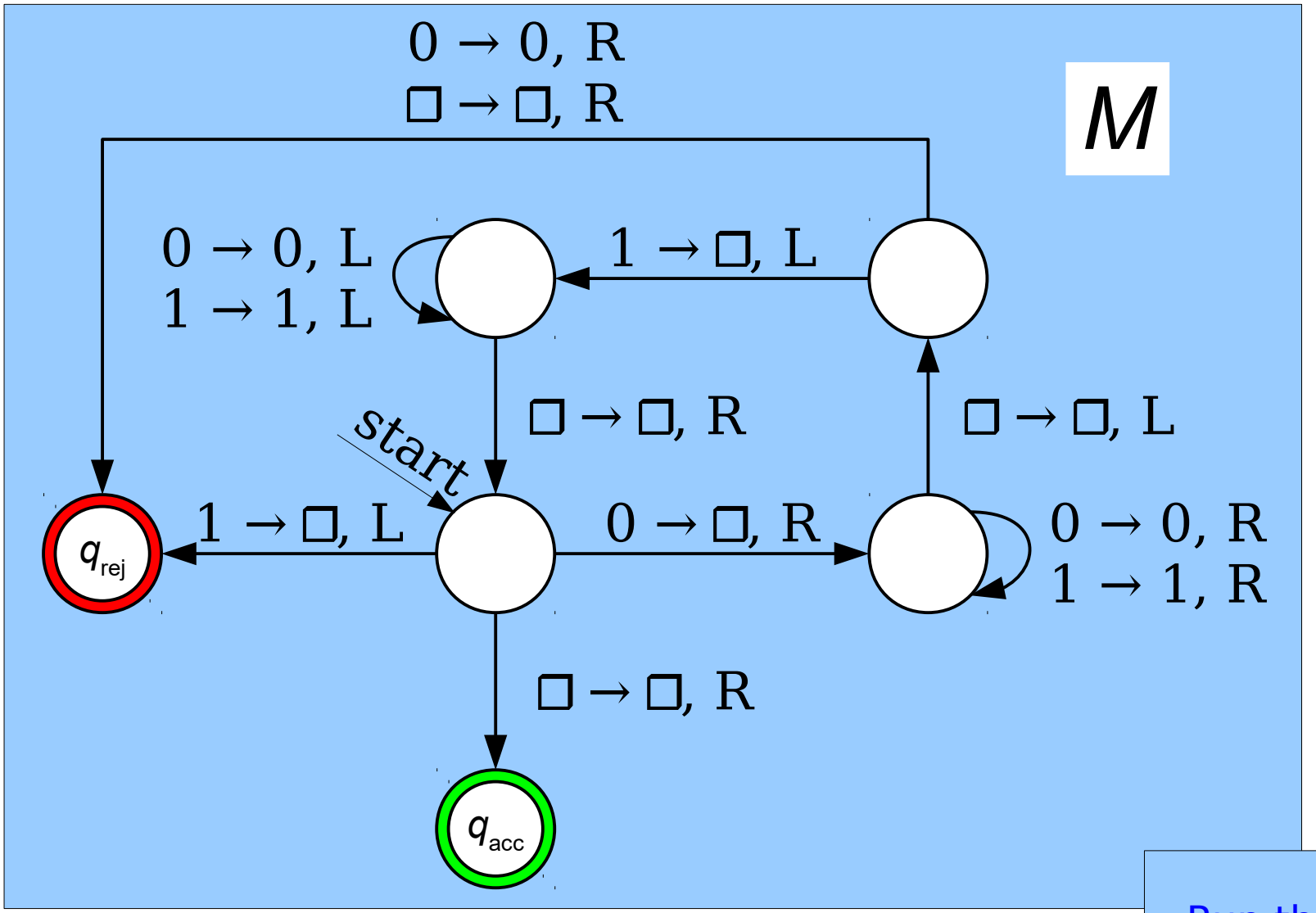
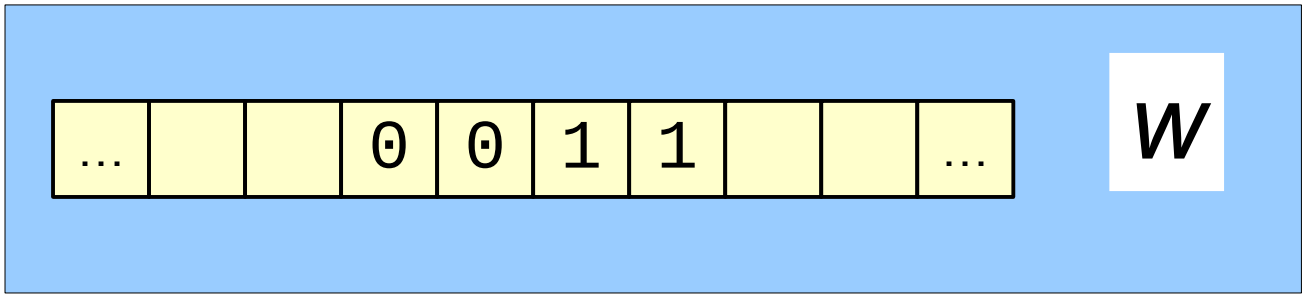
…			0	0	1	1			…





Run this TM for fifteen steps.





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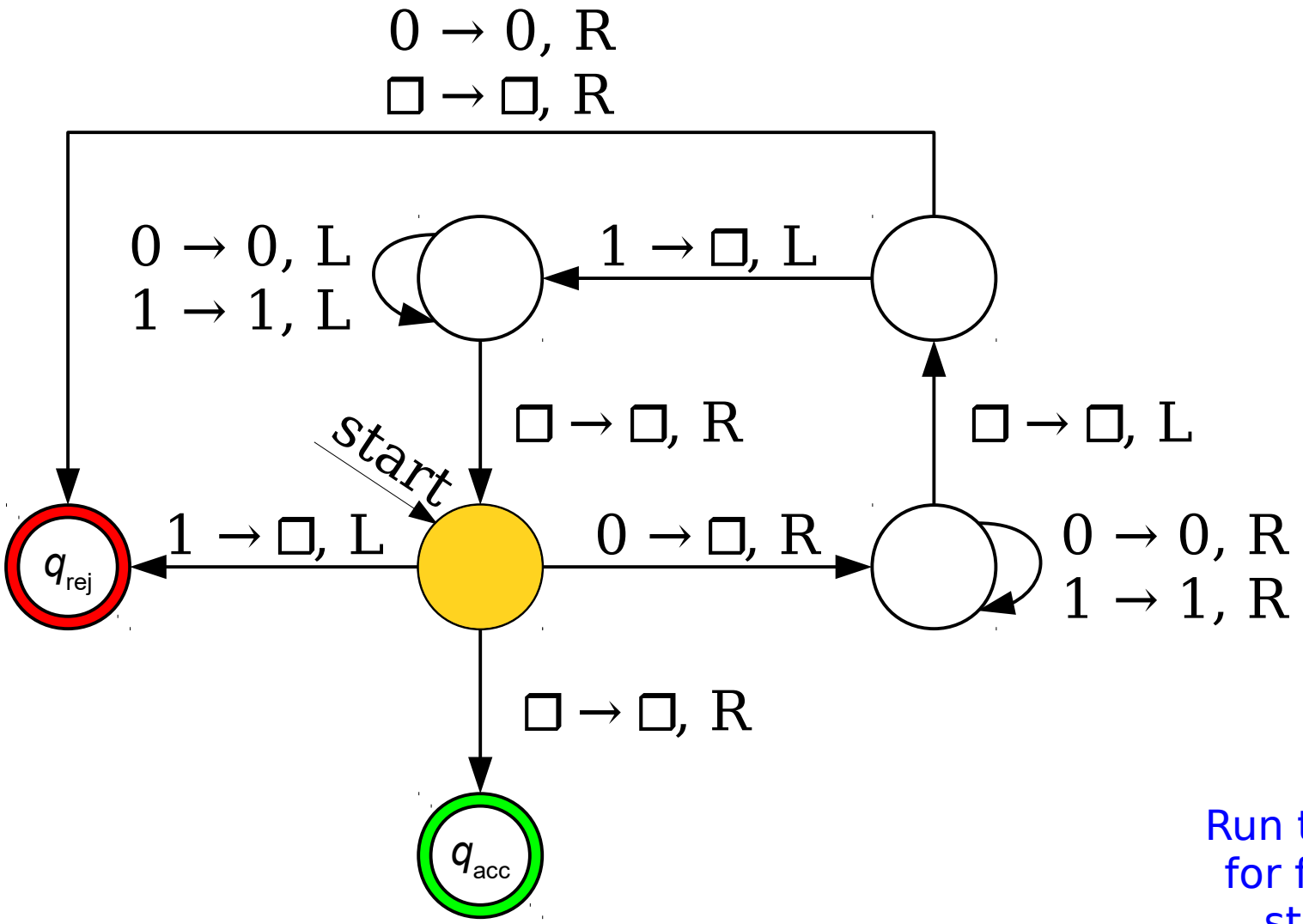
c



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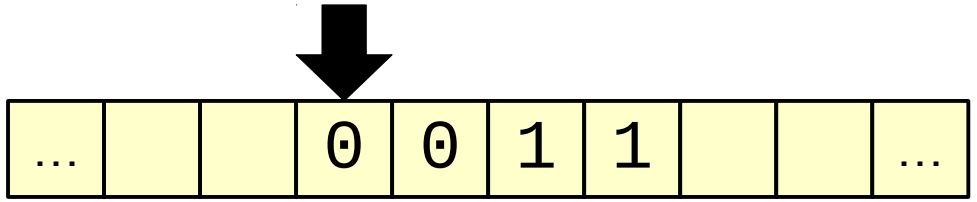

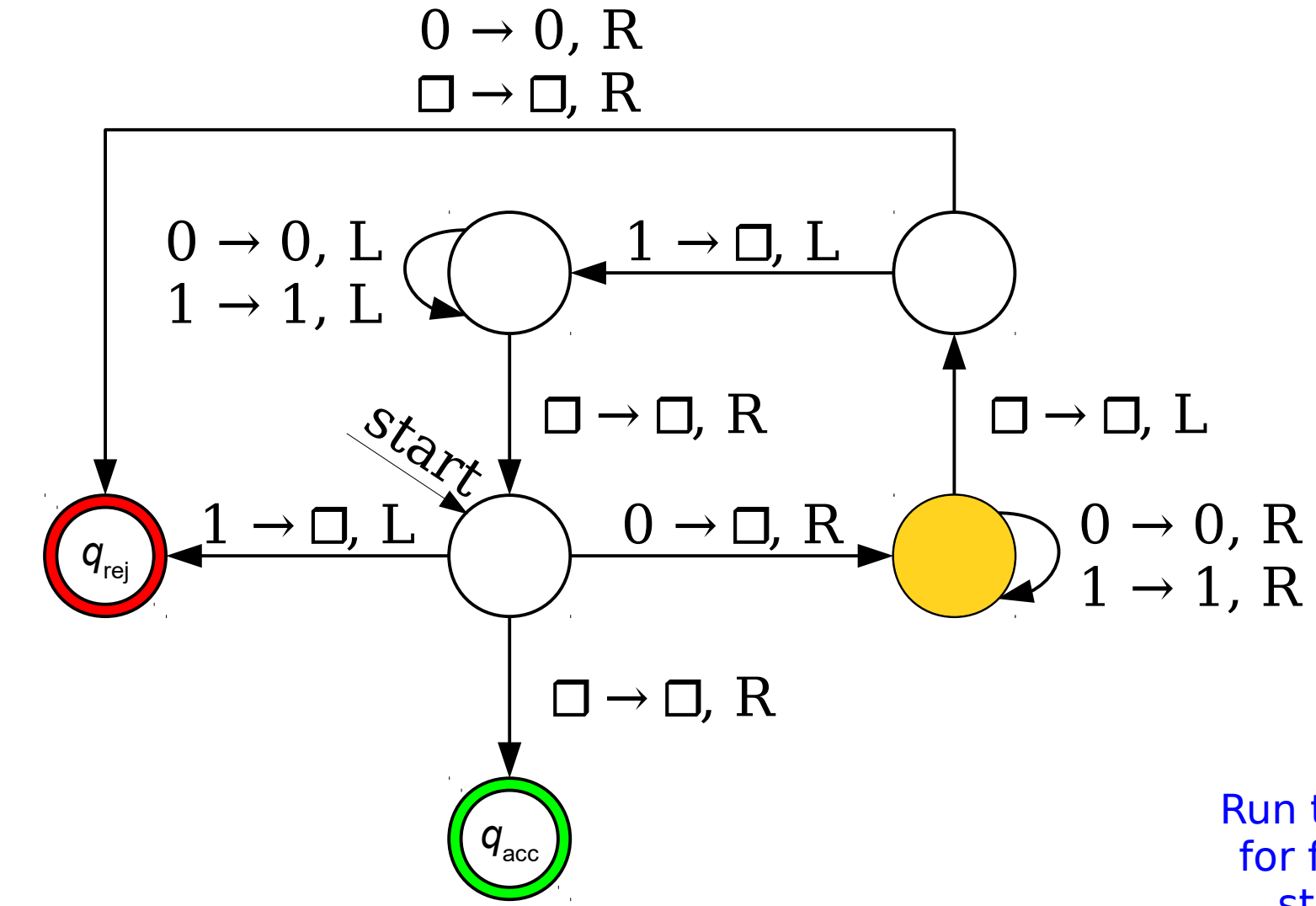

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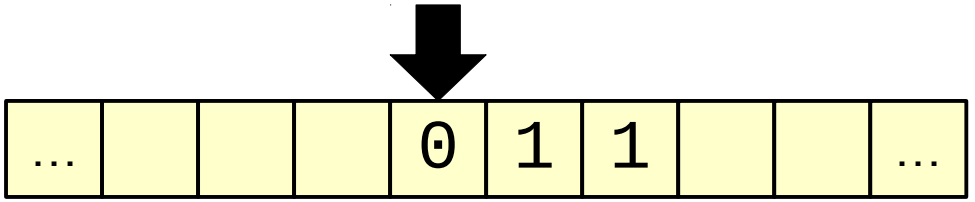

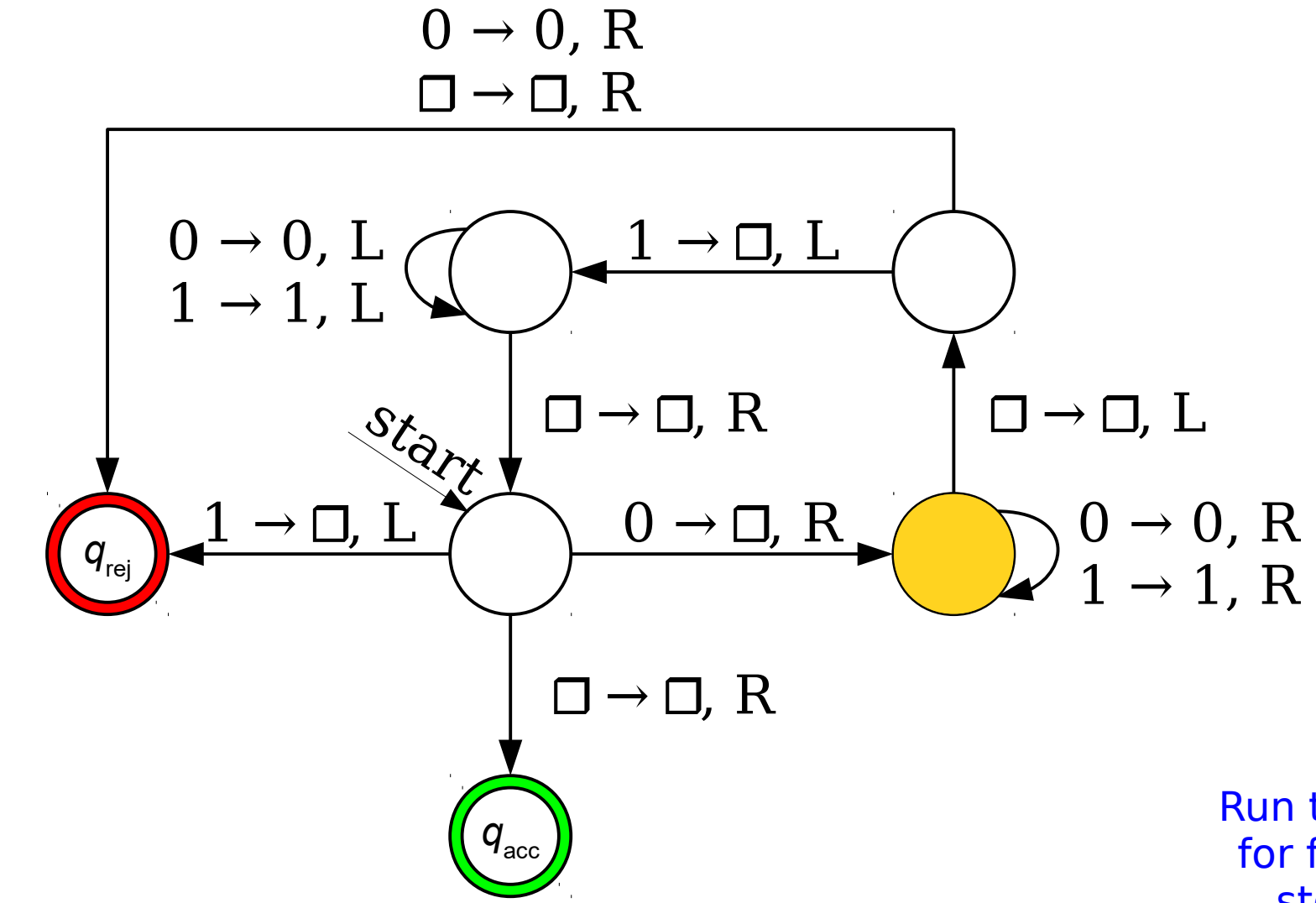

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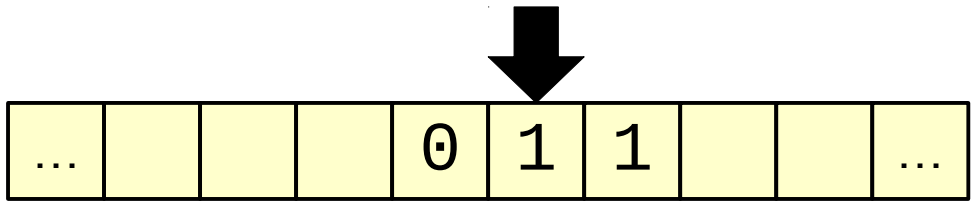

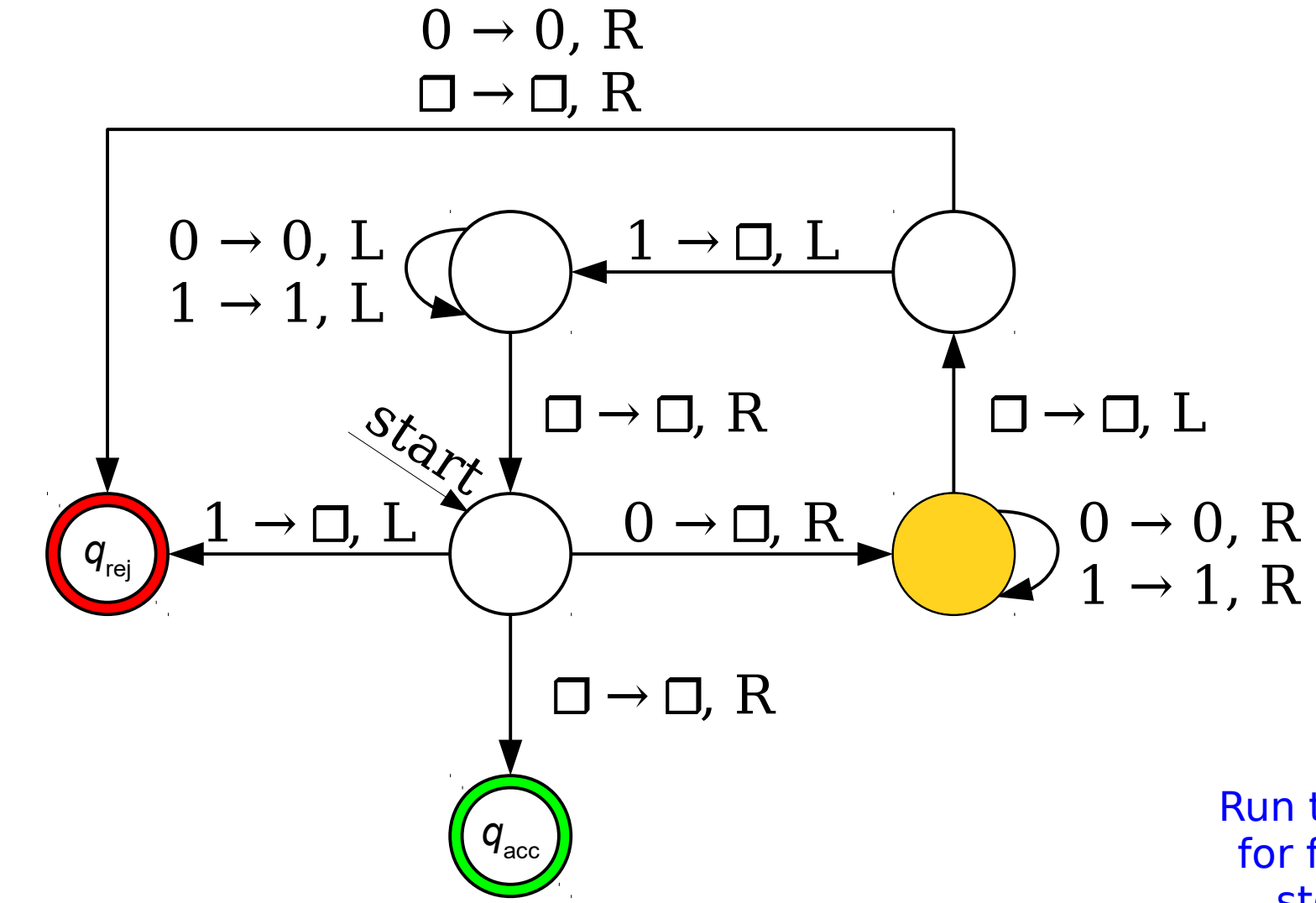

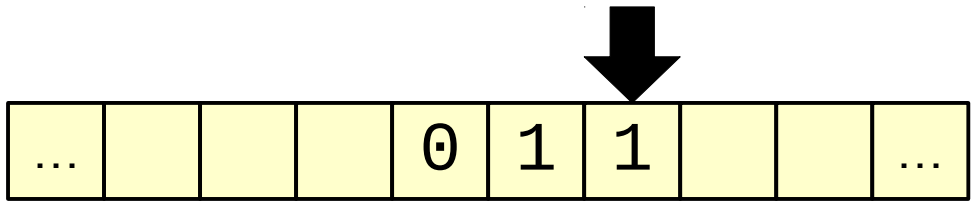

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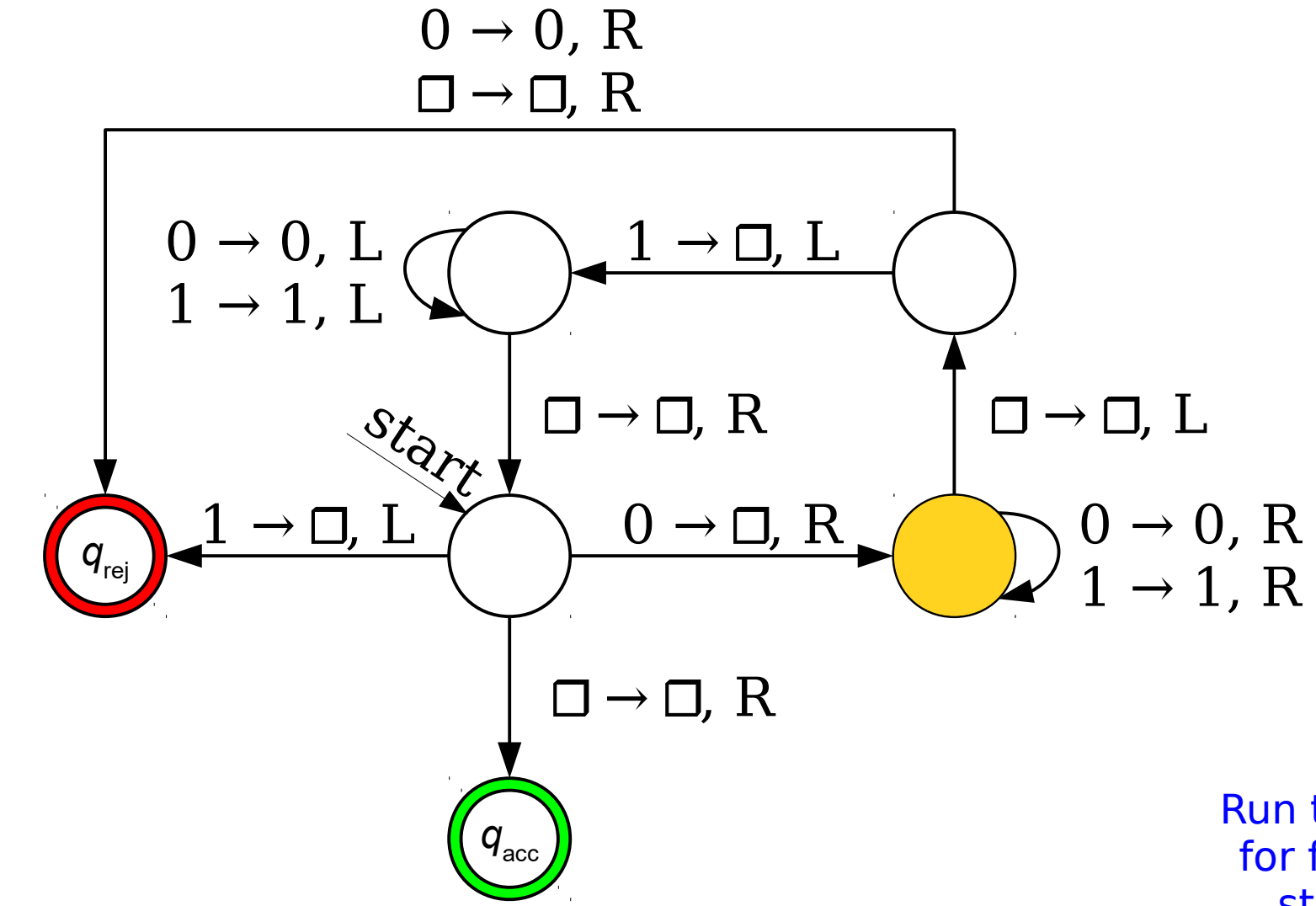

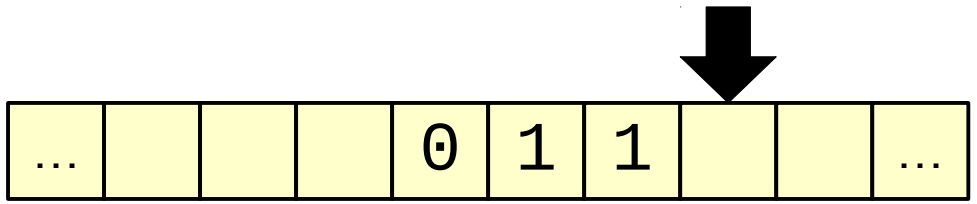

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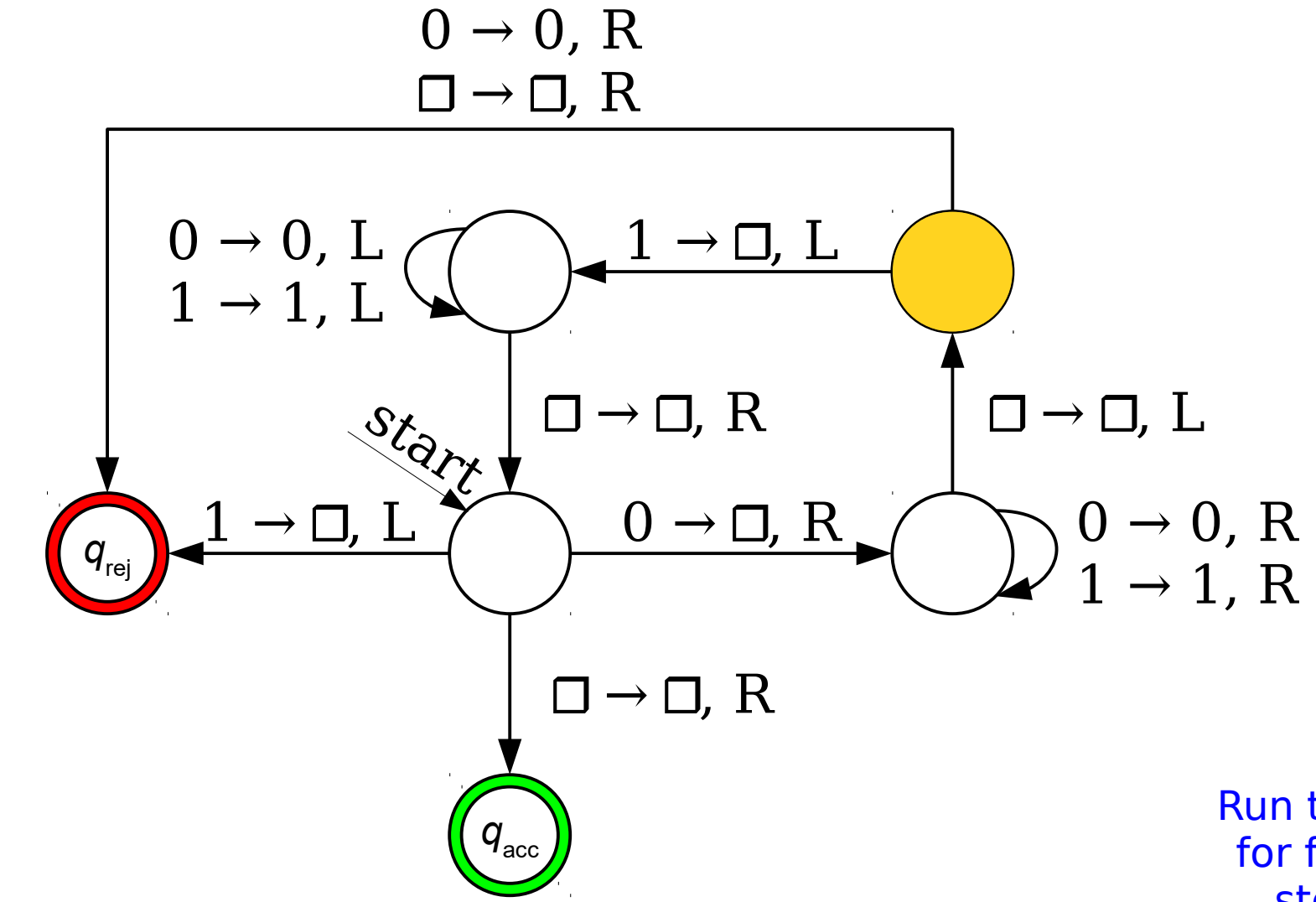

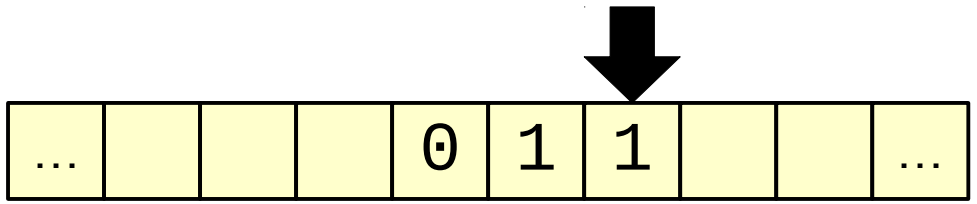

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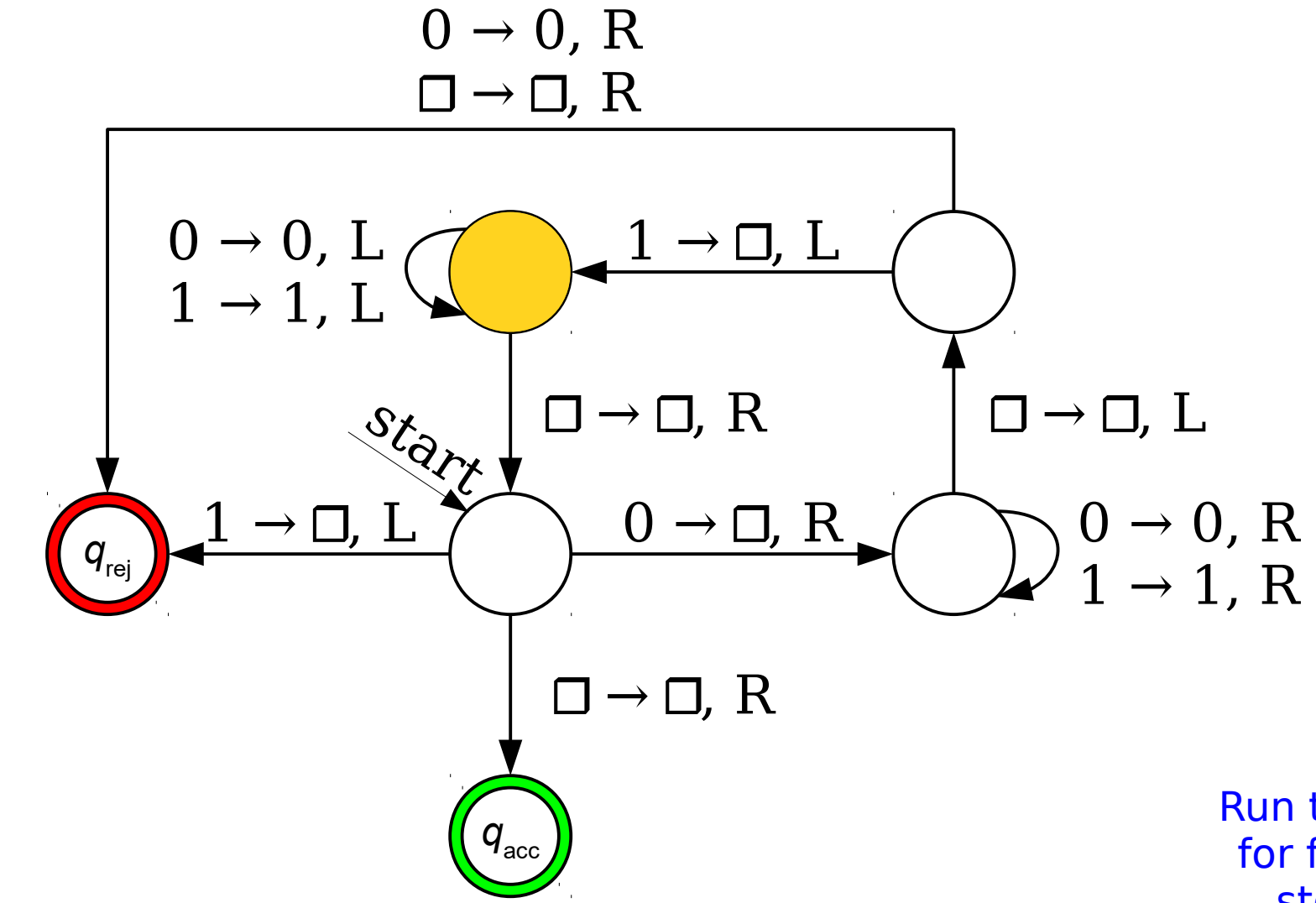

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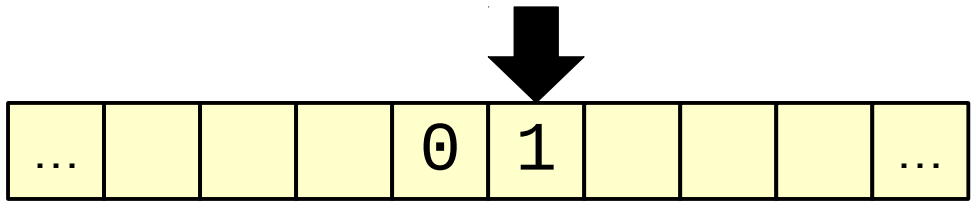

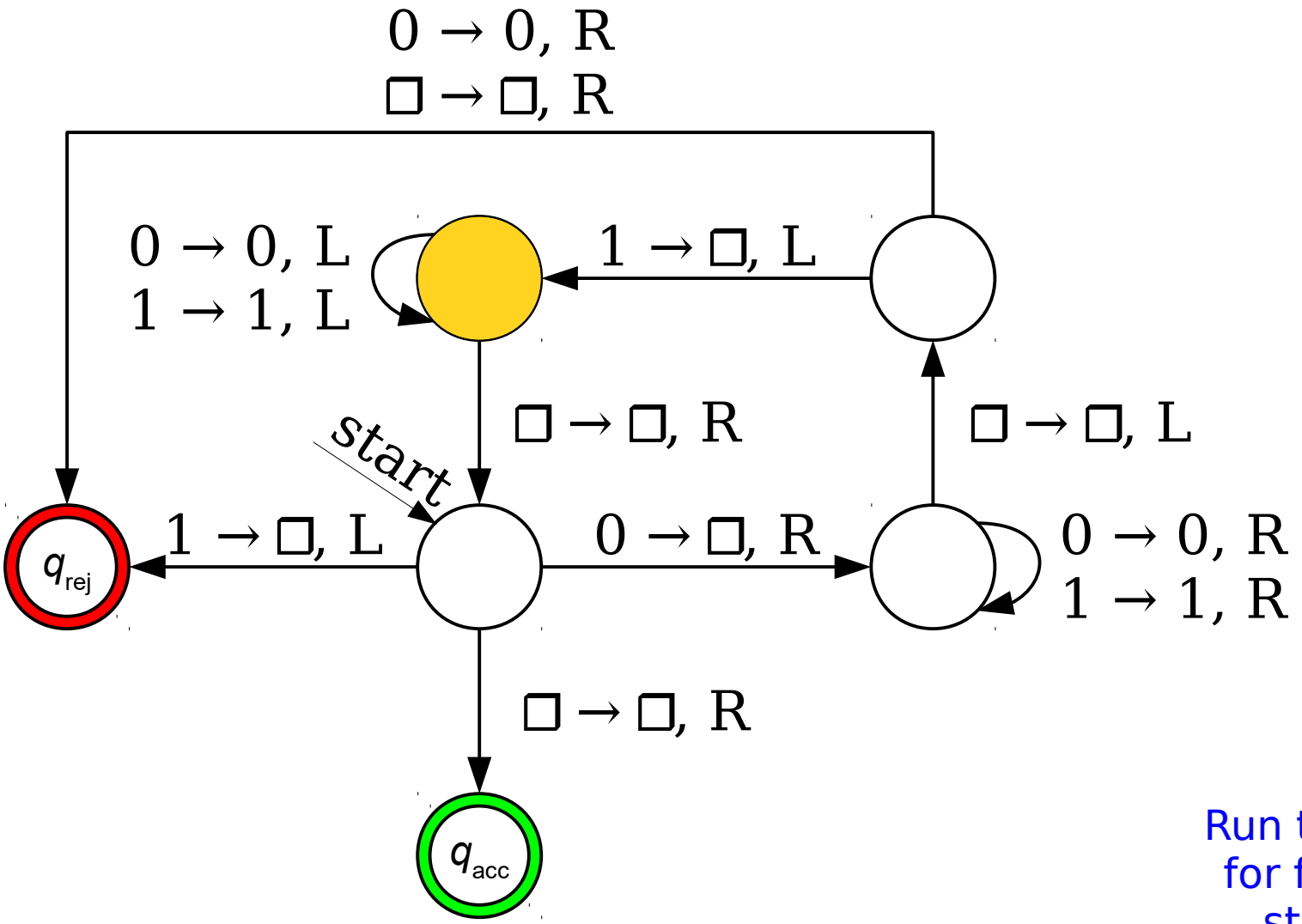

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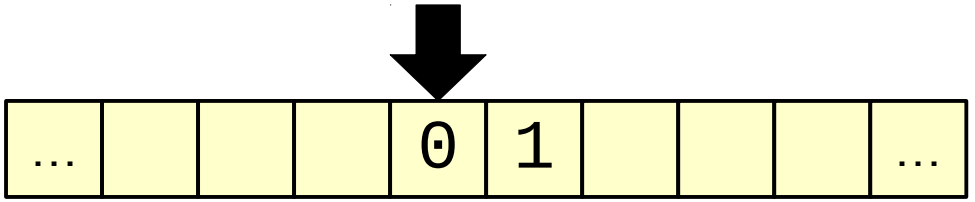

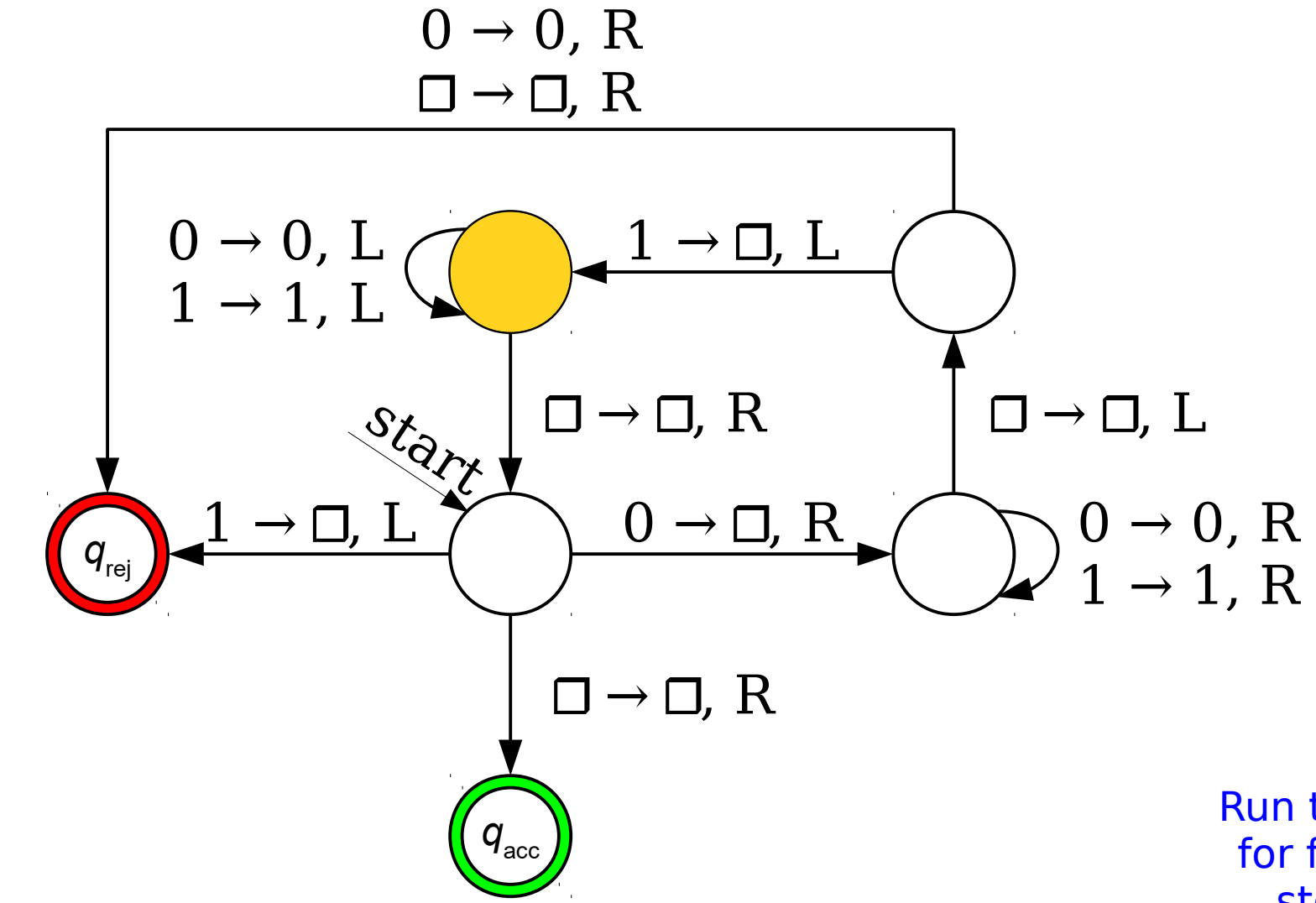

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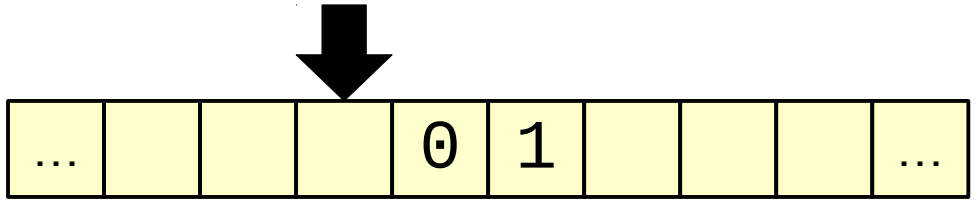

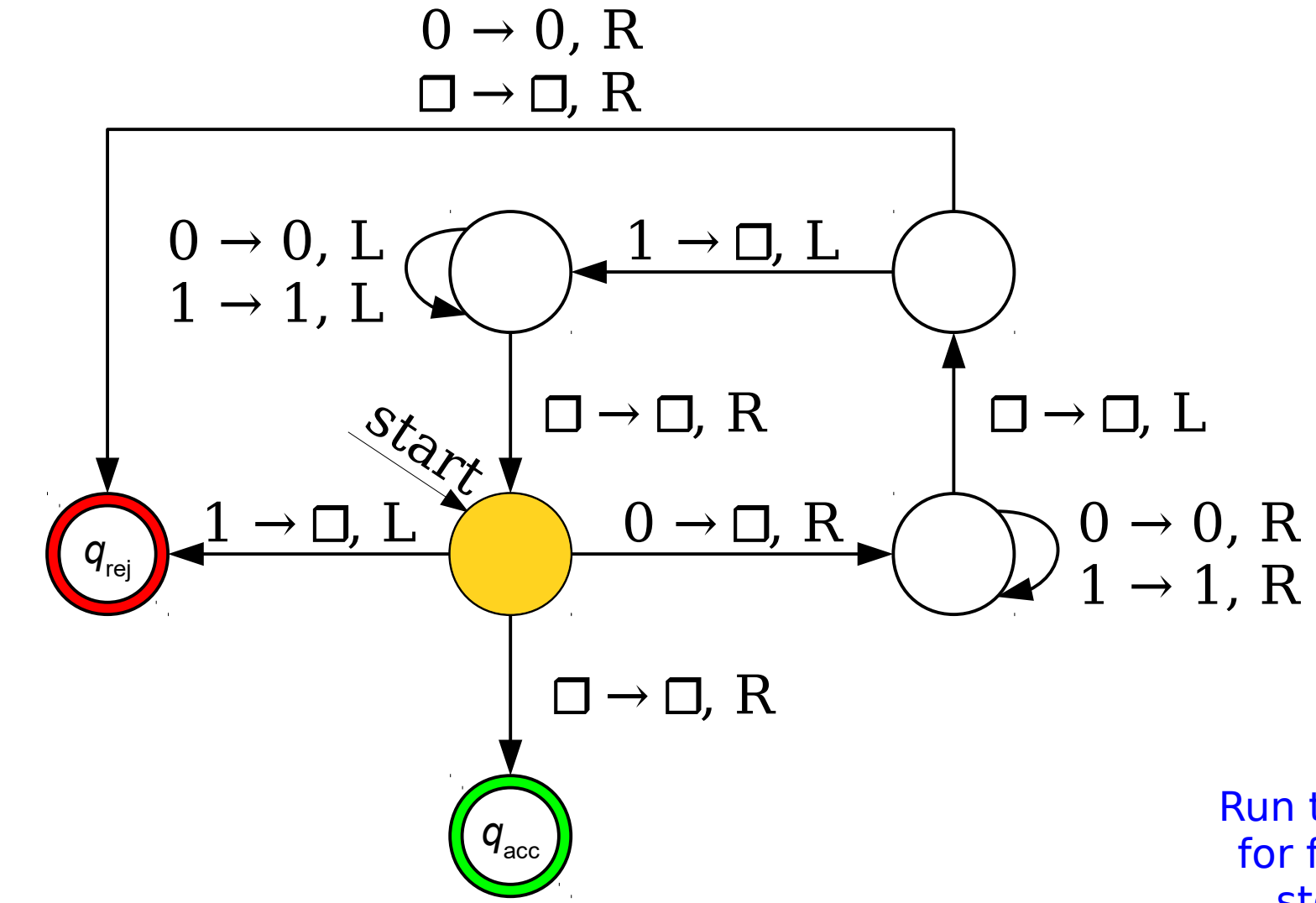

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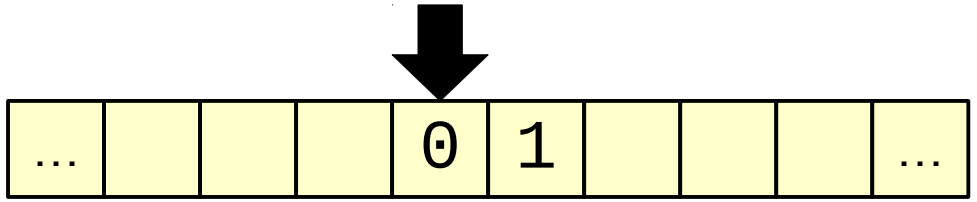

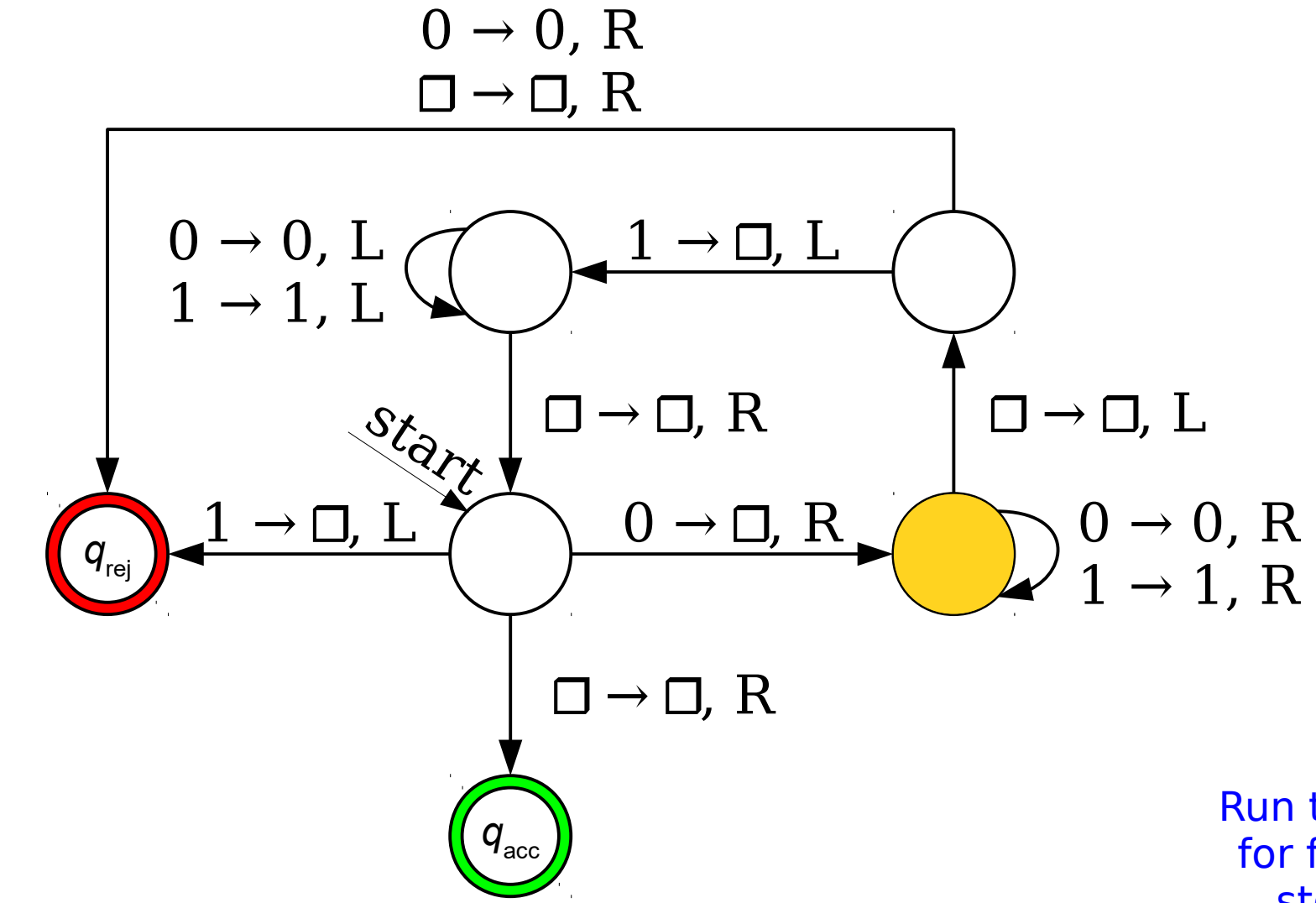

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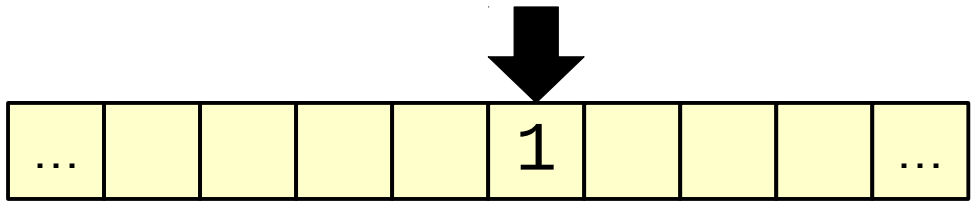

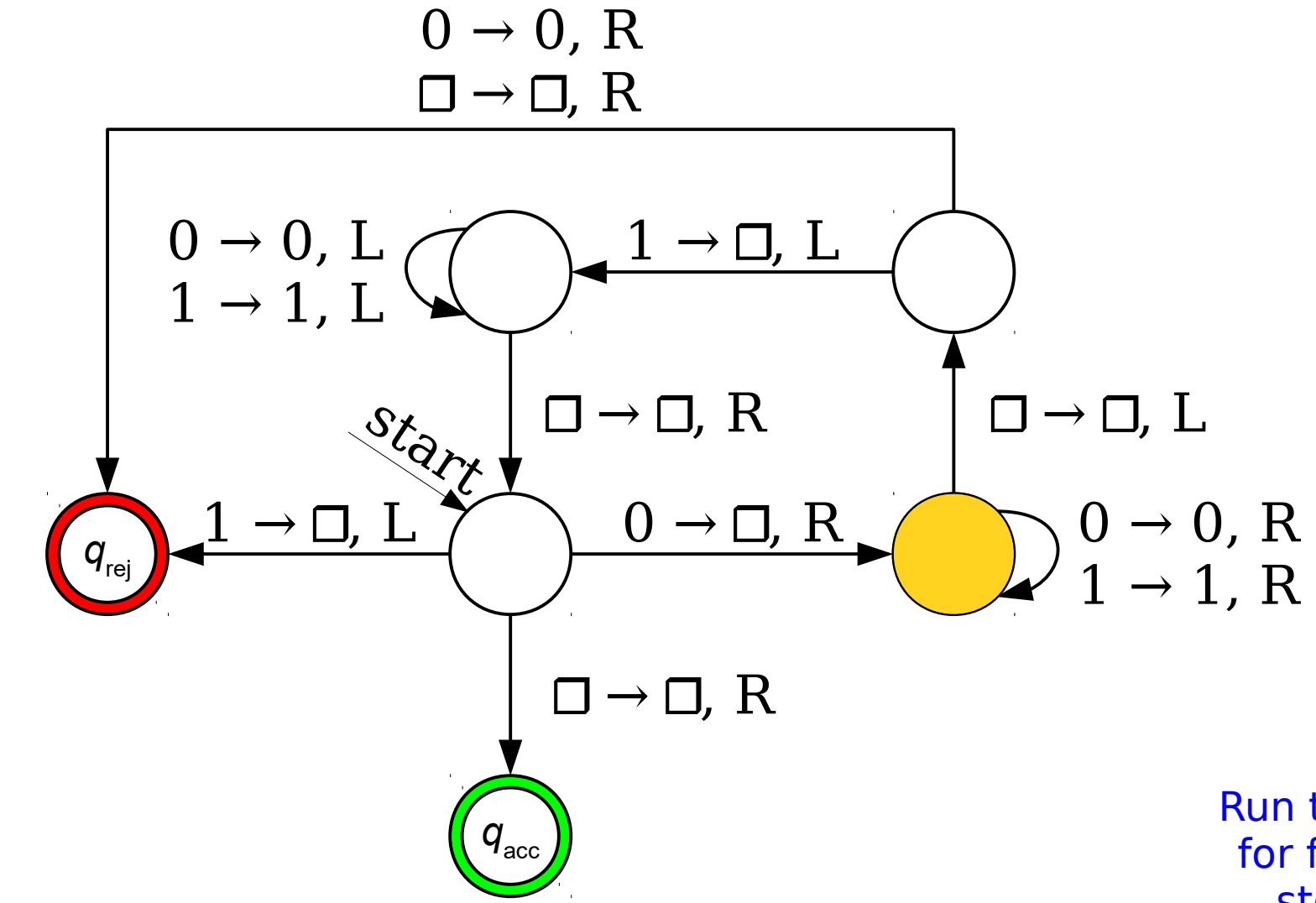

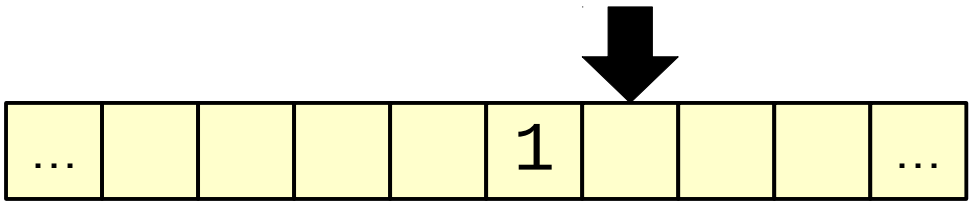

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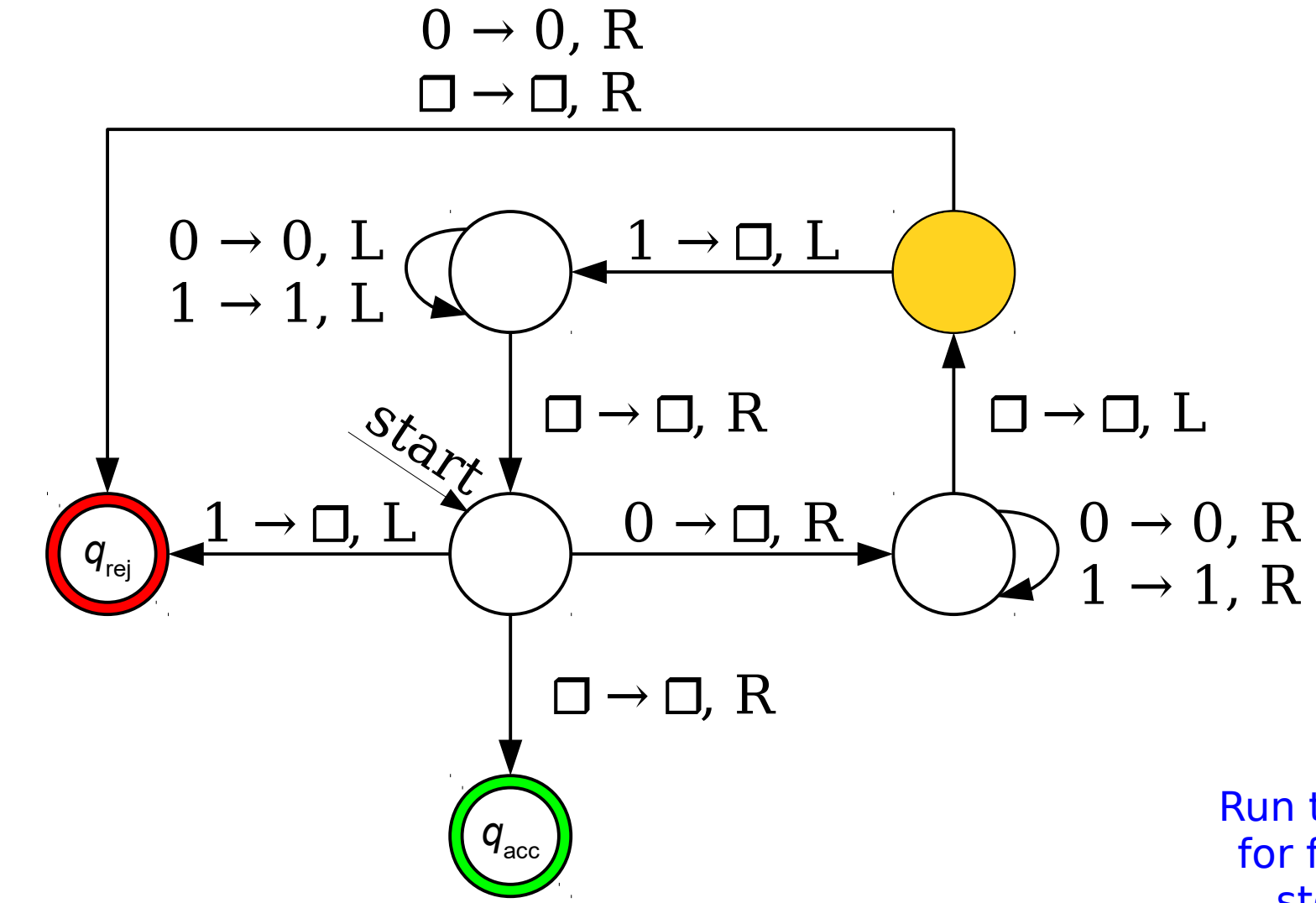

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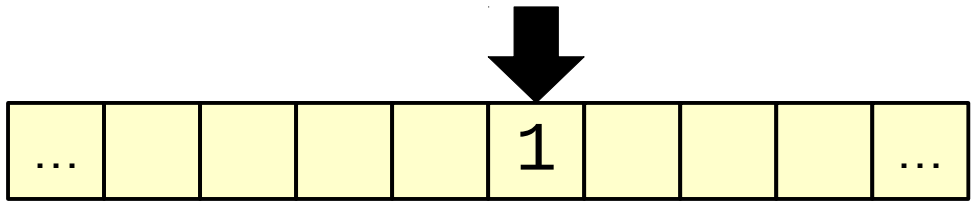

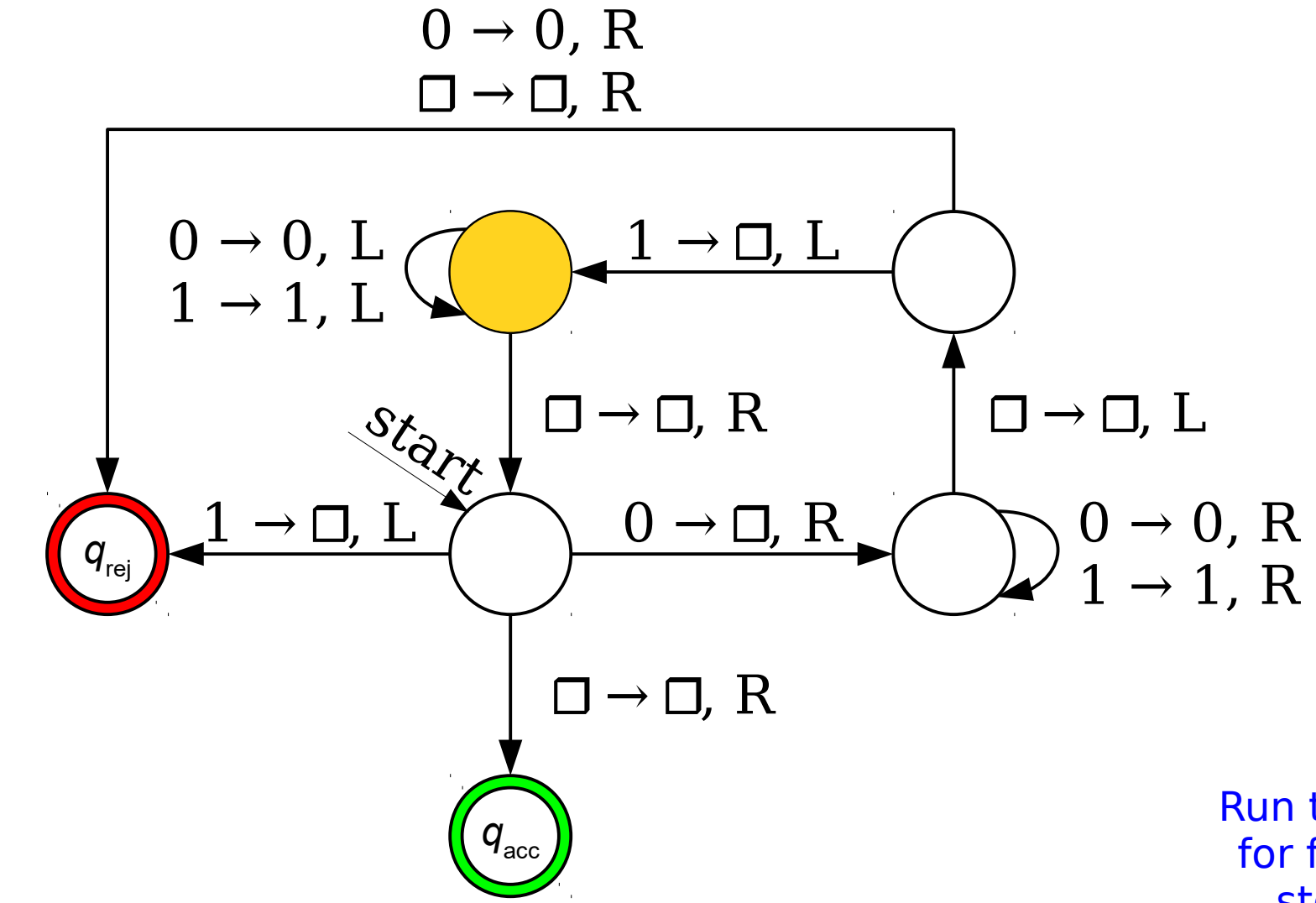

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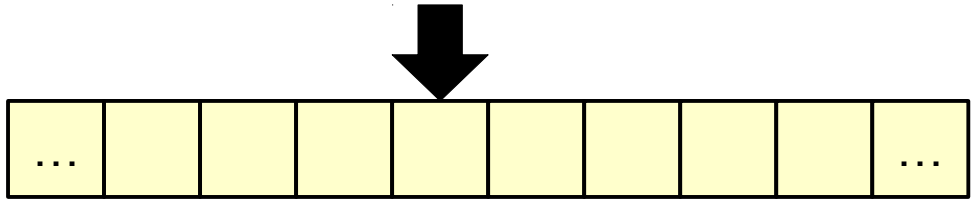

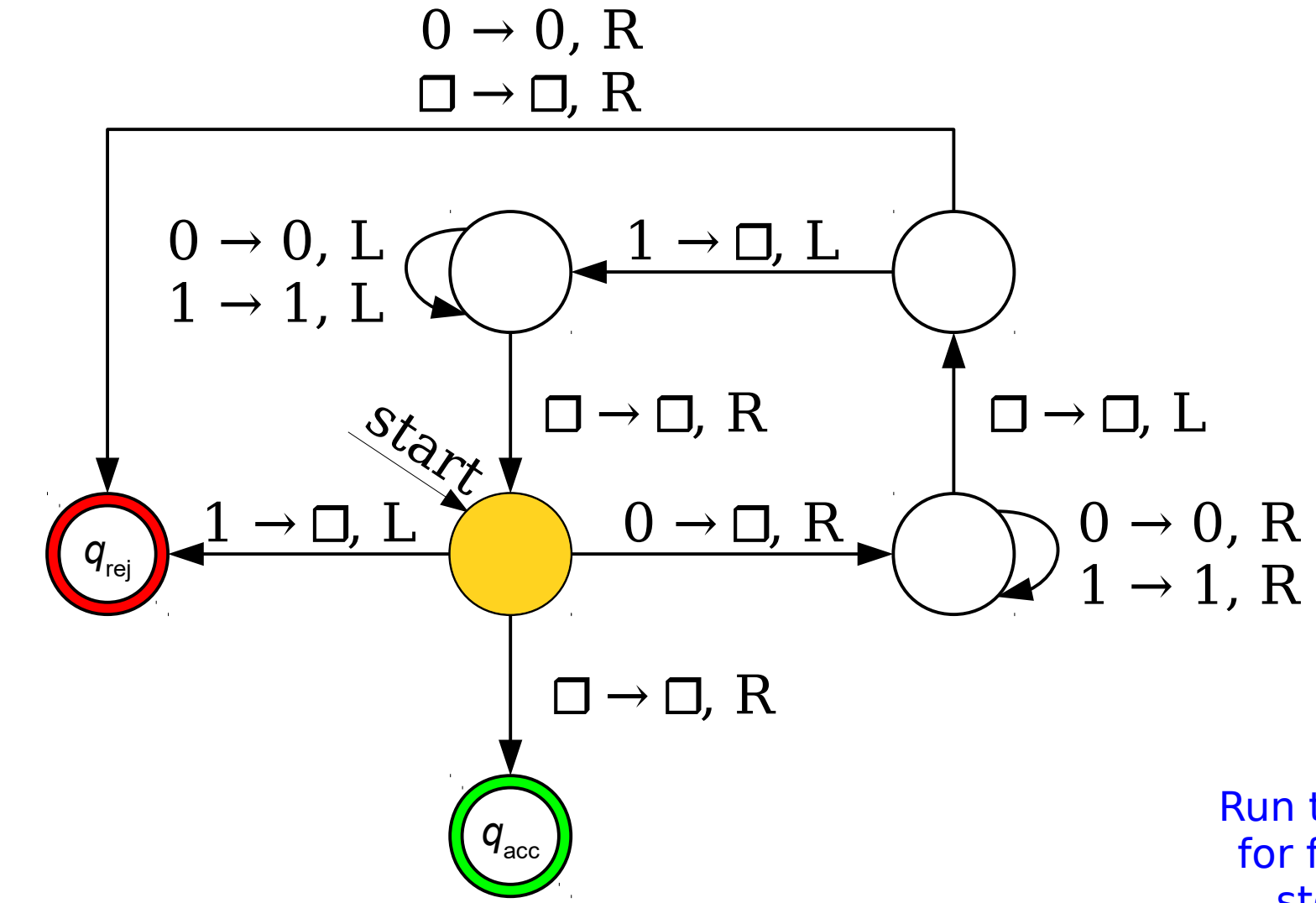

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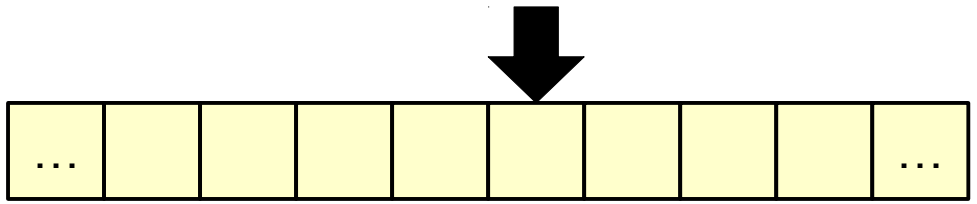

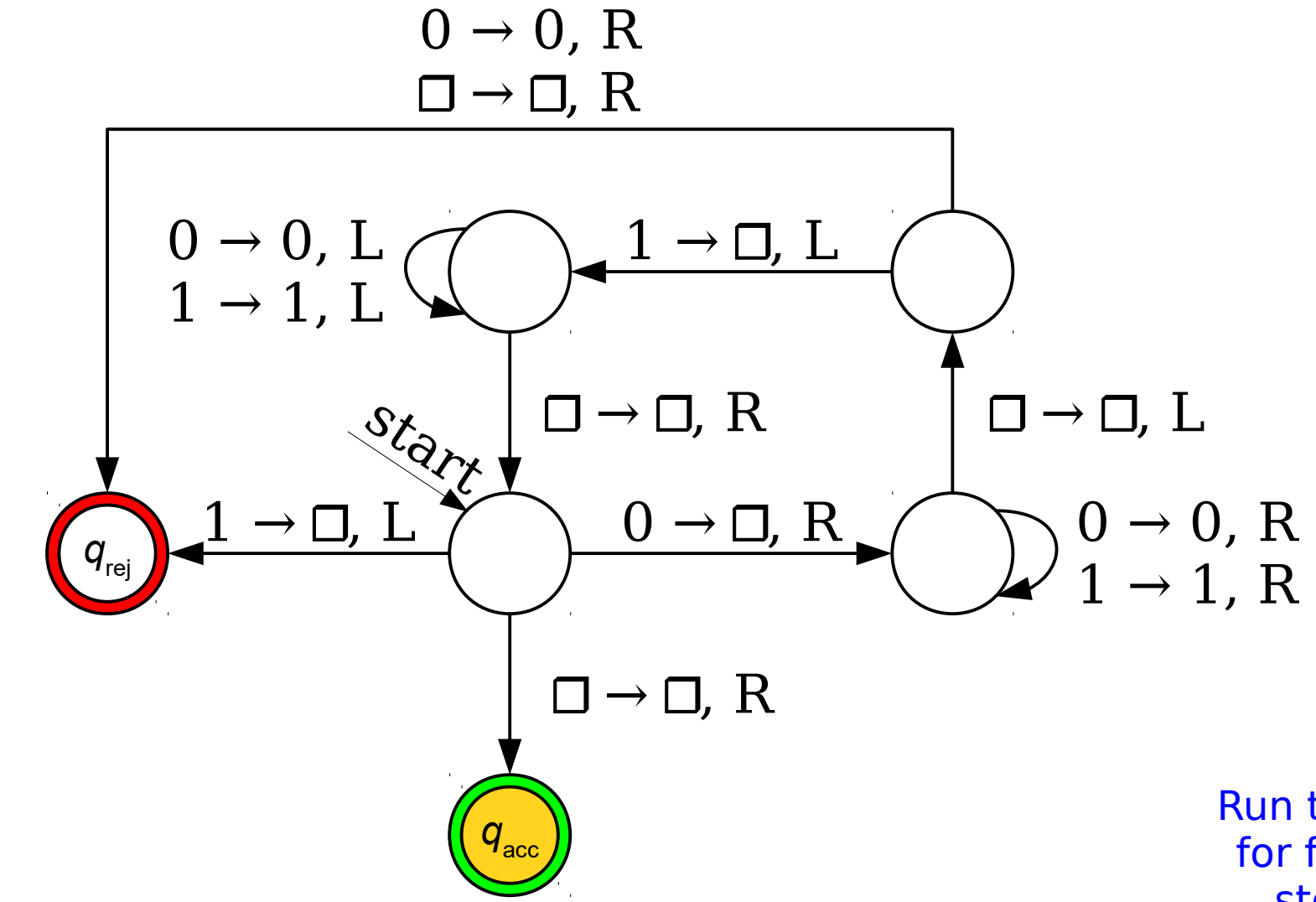

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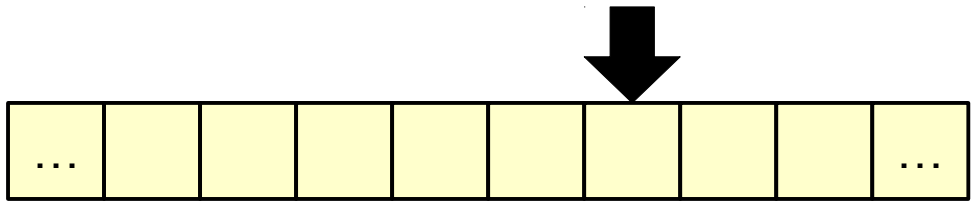


A Verifier for A_{TM}

- Recall $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$

```
bool checkWillAccept(TM M, string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
       simulate the next step of M running on w;
    }
    return whether M is in an accepting state;
}
```

- Do you see why M accepts w iff there is some c such that `checkWillAccept(M, w, c)` returns true?
- Do you see why `checkWillAccept` always halts?

Equivalence of Verifiers and Recognizers

What languages are verifiable?

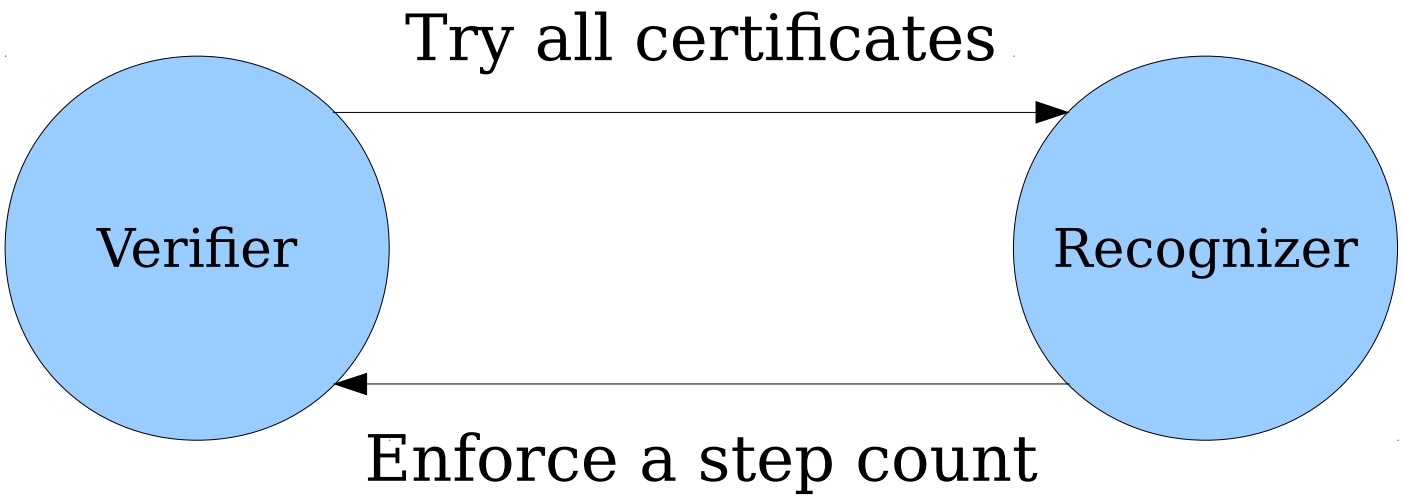
Let V be a verifier for a language L. Consider the following function given in pseudocode:

```
bool mysteryFunction(string w) {
    int i = 0;
    while (true) {
        for (each string c of length i) {
            if (V accepts ⟨w, c⟩) return true;
        }
        i++;
    }
}
```

What set of strings does `mysteryFunction` return **`true`** on?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **your answer**.

Equivalence of Verifiers and Recognizers



Theorem: If L is a language, then there is a verifier for L if and only if $L \in \mathbf{RE}$.

Verifiers and **RE**

- ***Theorem:*** If there is a verifier V for a language L, then $L \in \mathbf{RE}$.
- ***Proof goal:*** Given a verifier V for a language L, find a way to construct a recognizer M for L.

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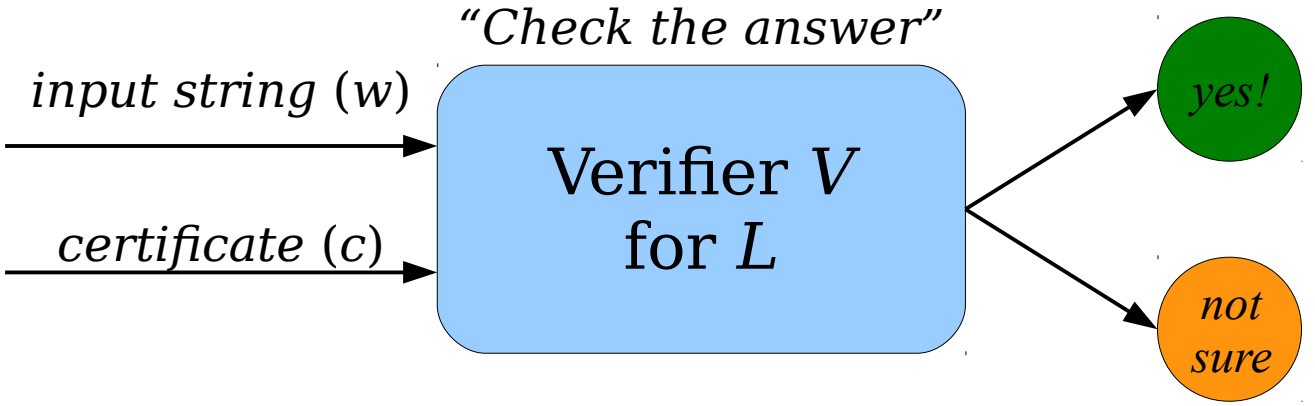

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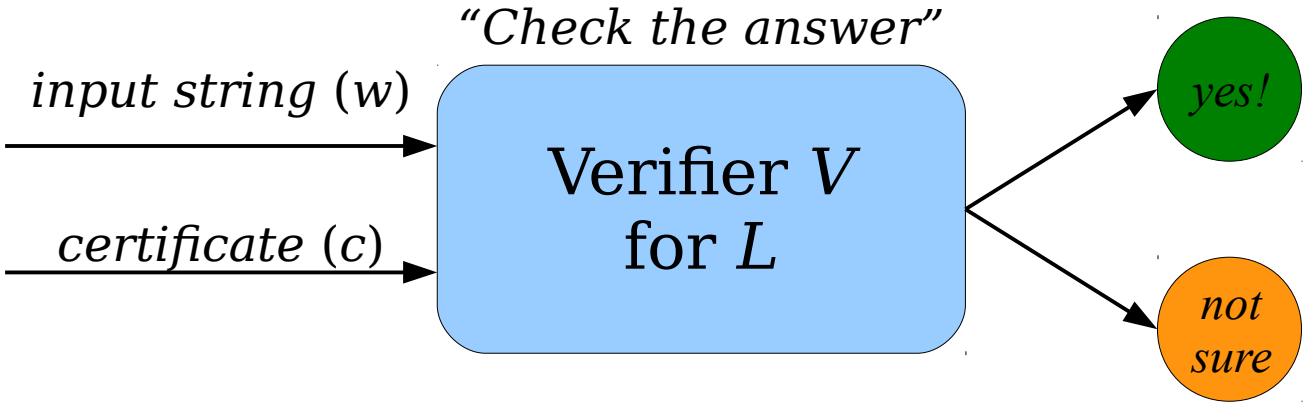
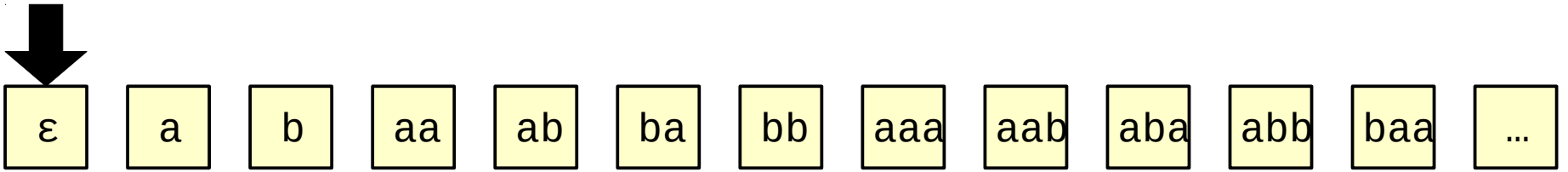


We will try all possible certificates (values of c)



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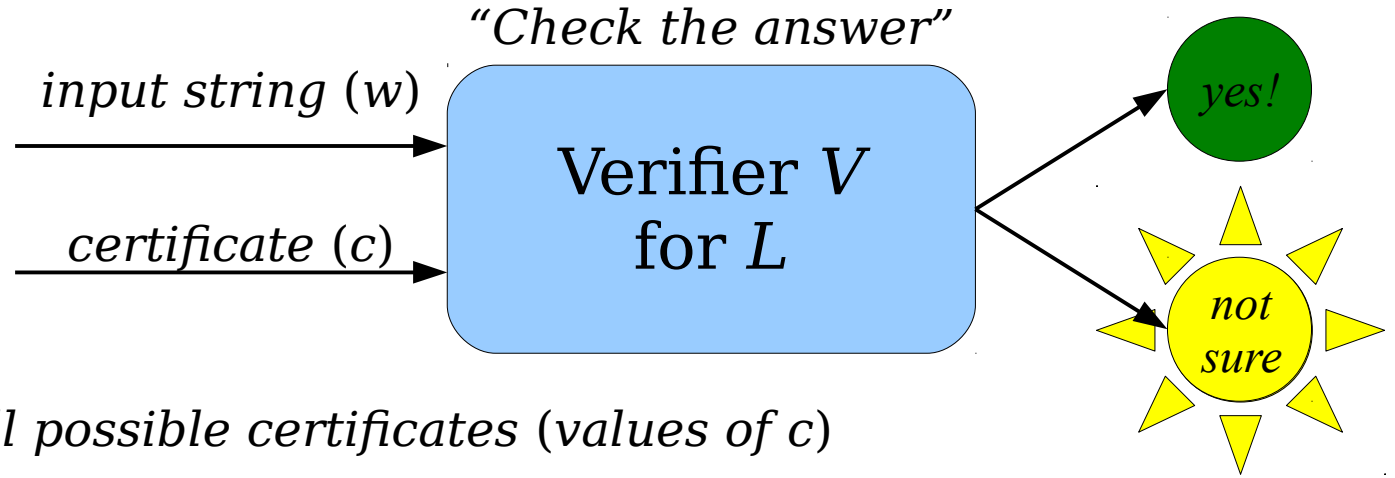

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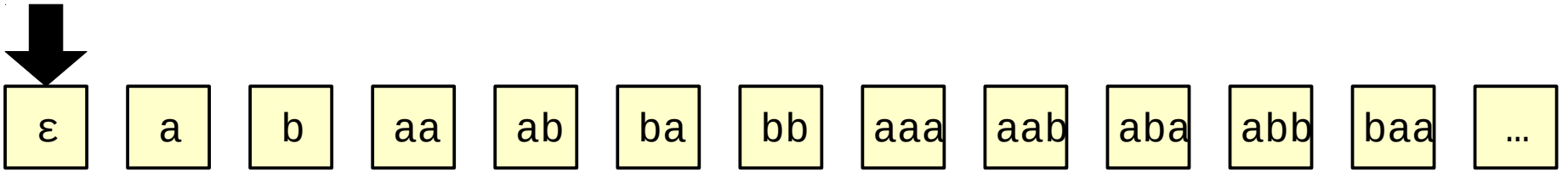

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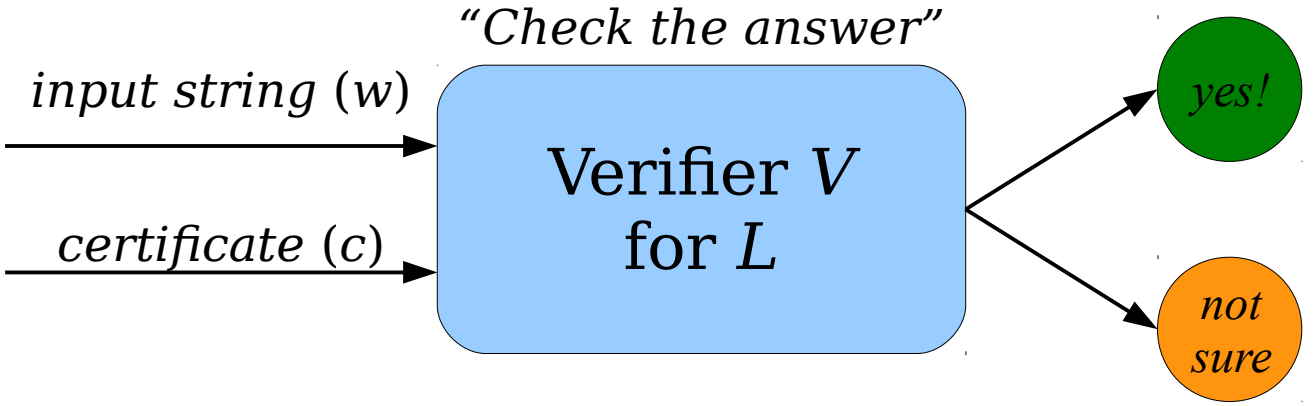

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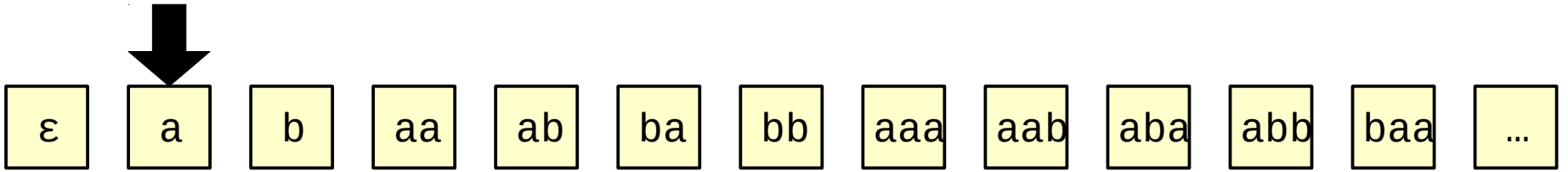

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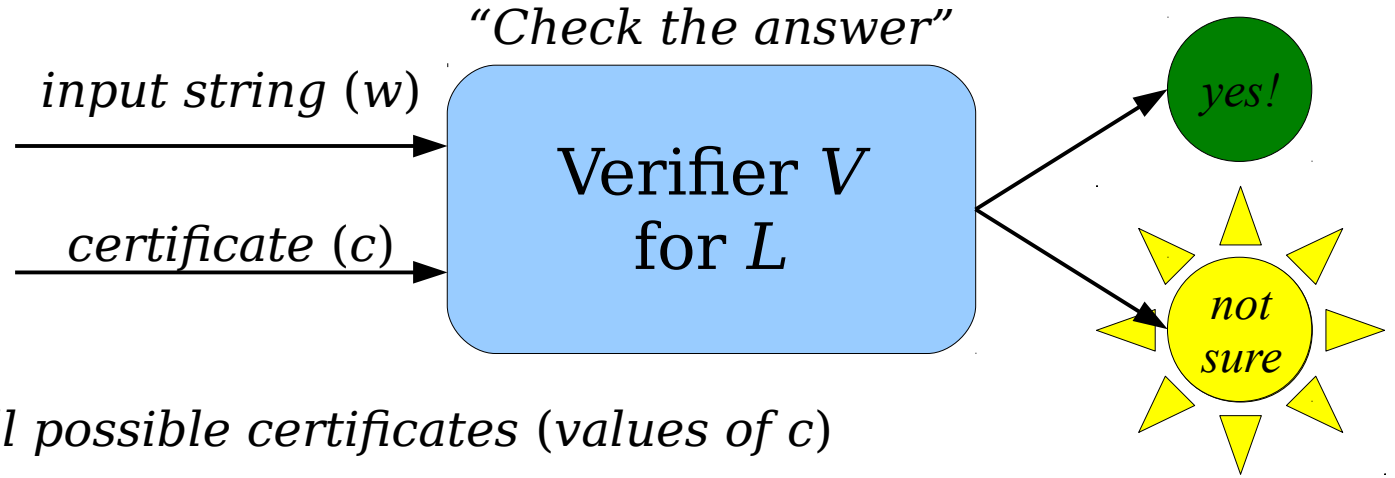

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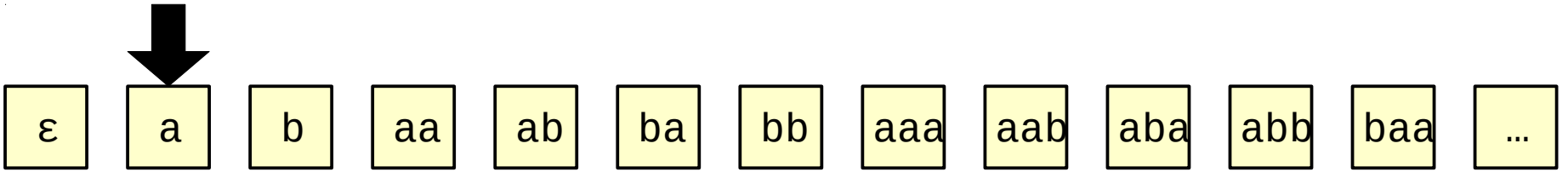

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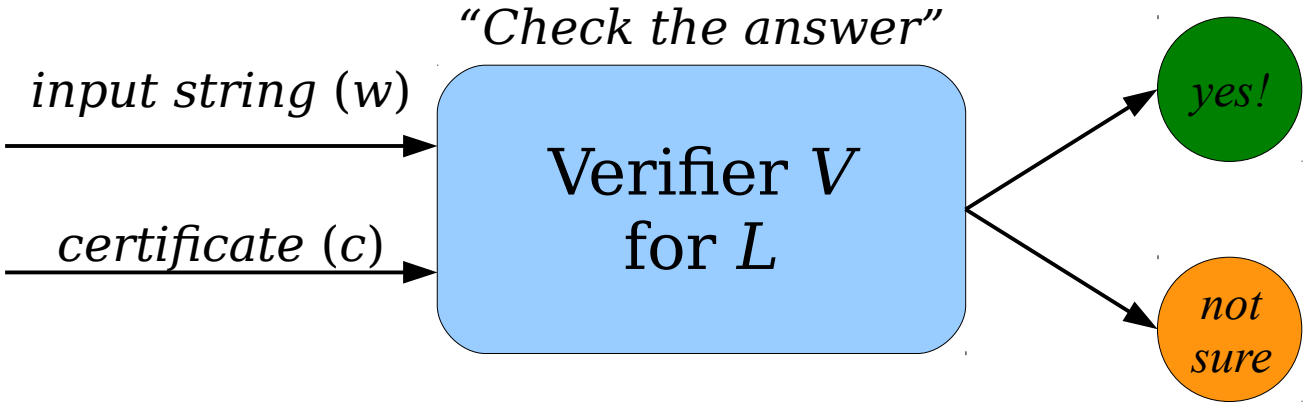

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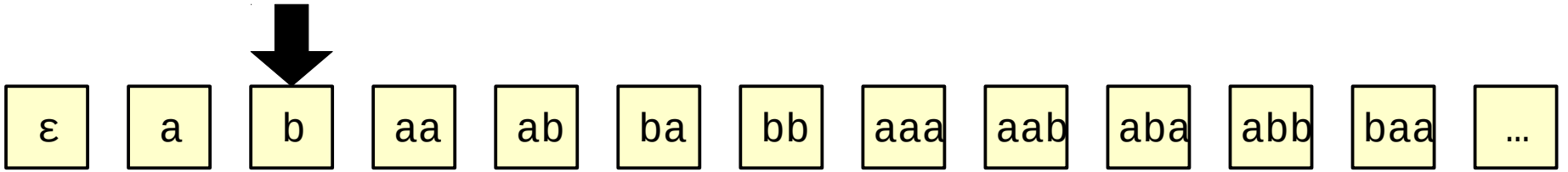

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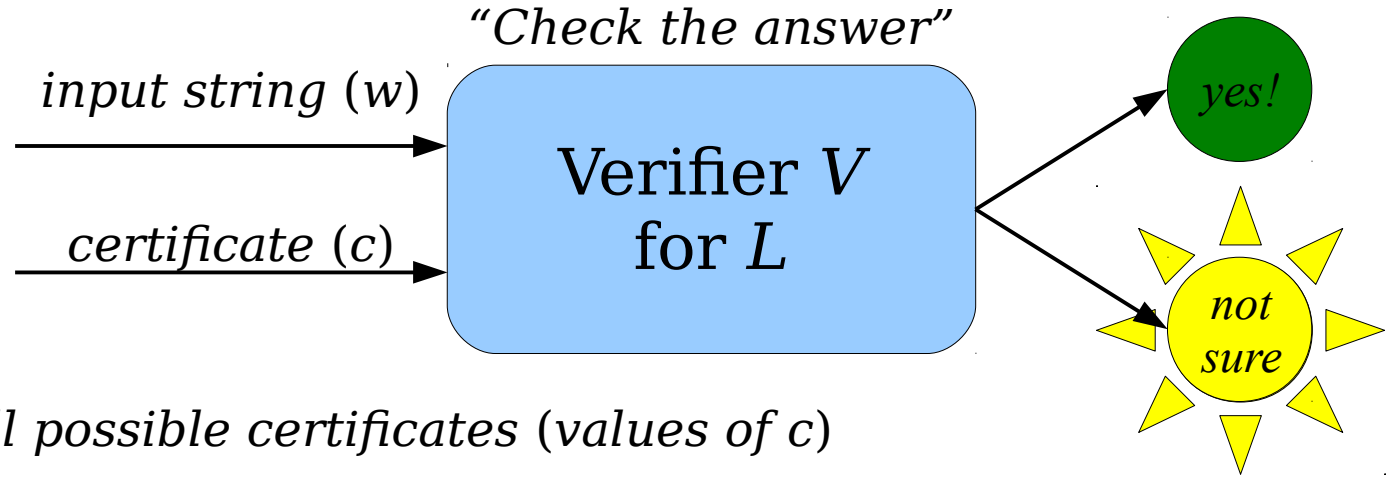

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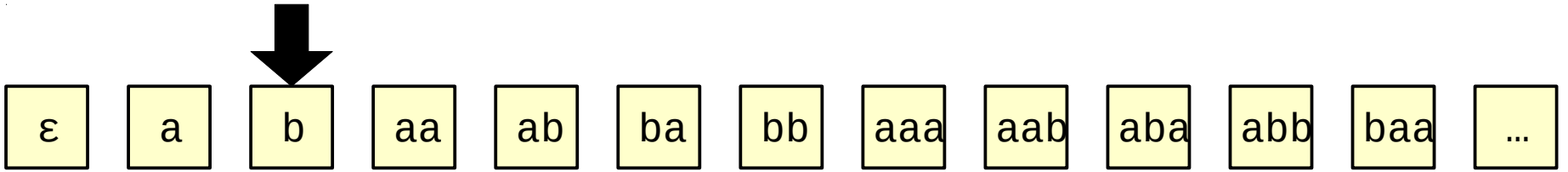

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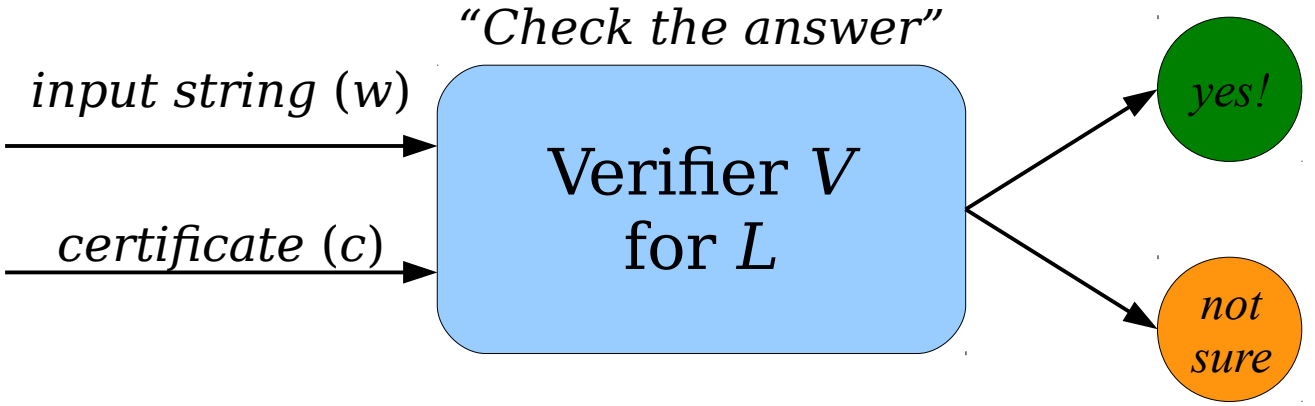

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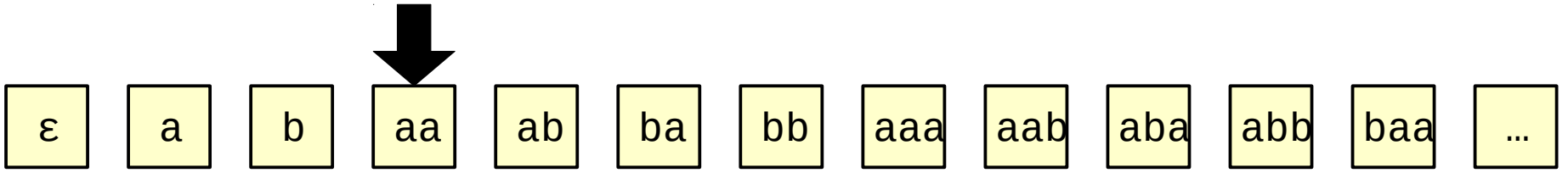

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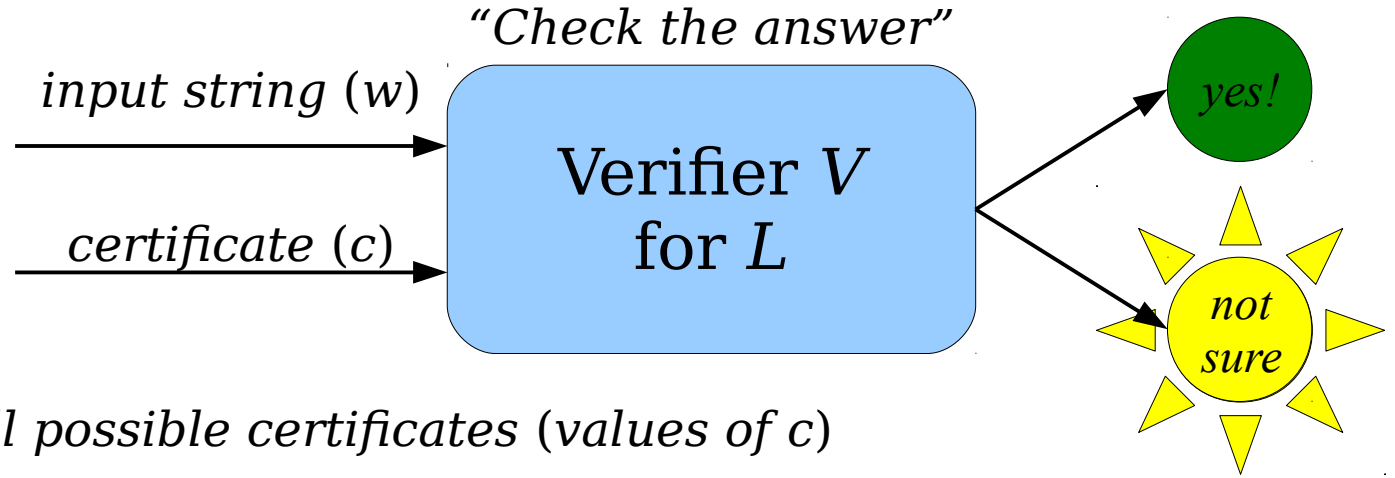

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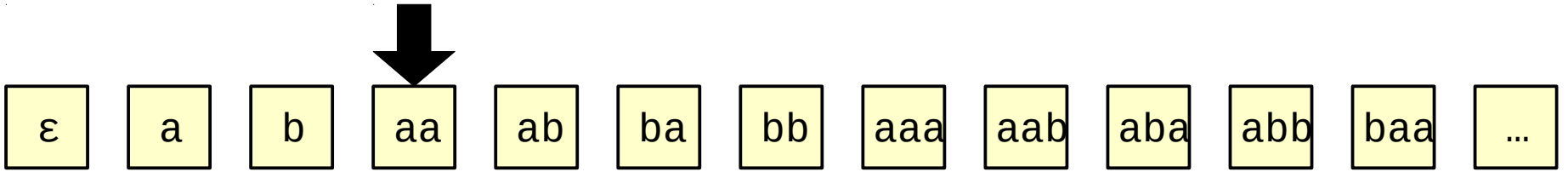

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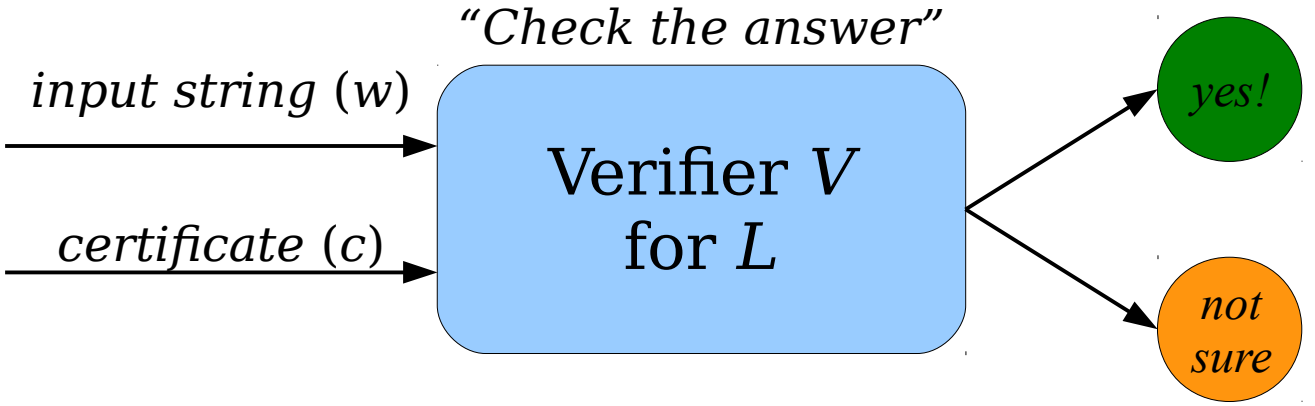

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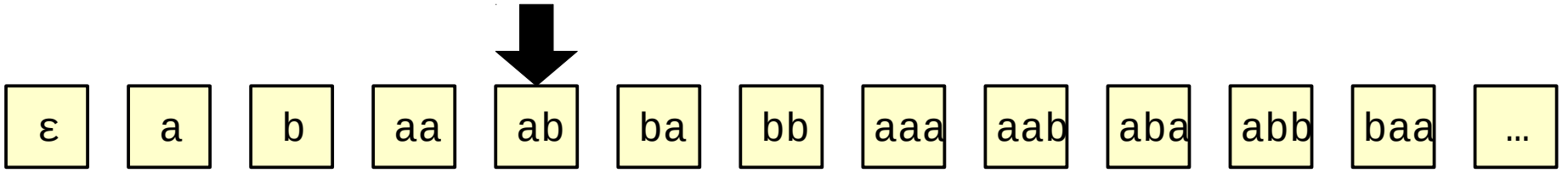

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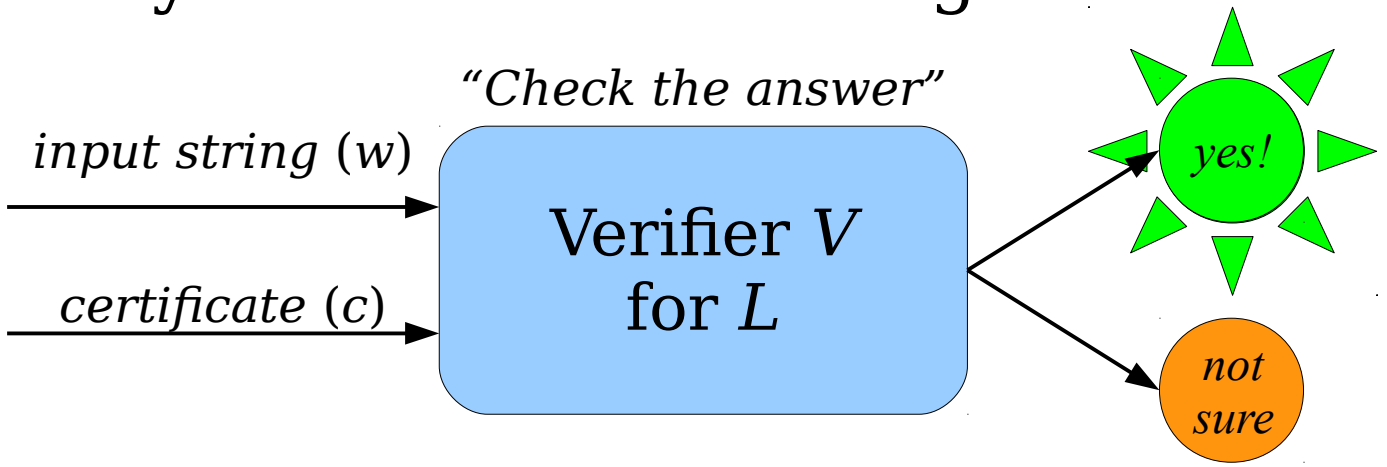

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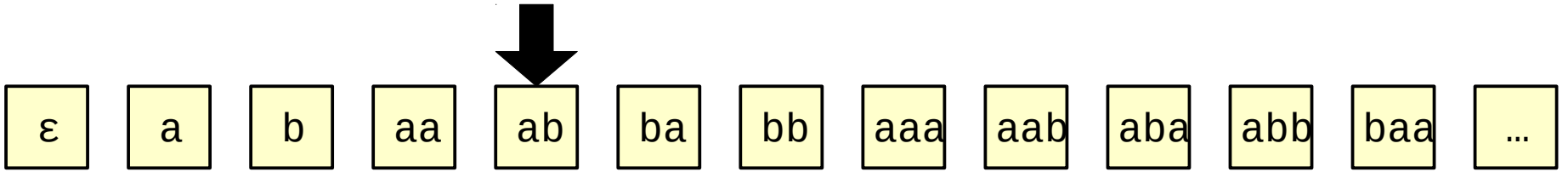

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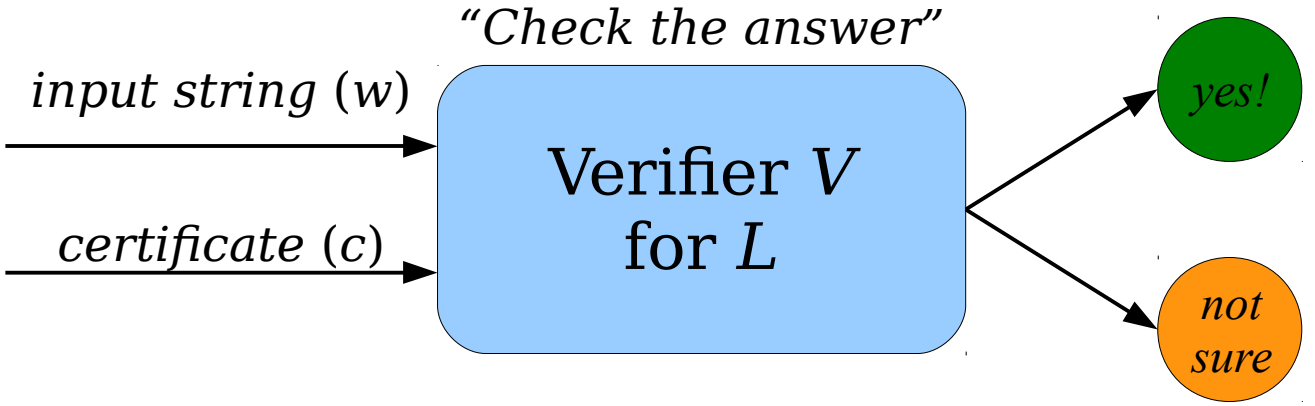

ε	a	b	aa	ab	ba	bb	aaa	aab	aba	abb	baa	…

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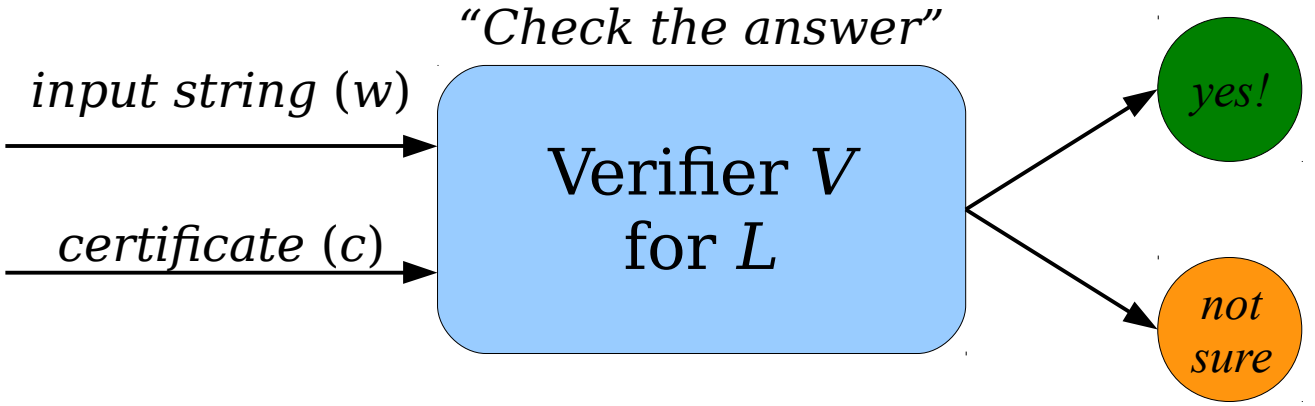

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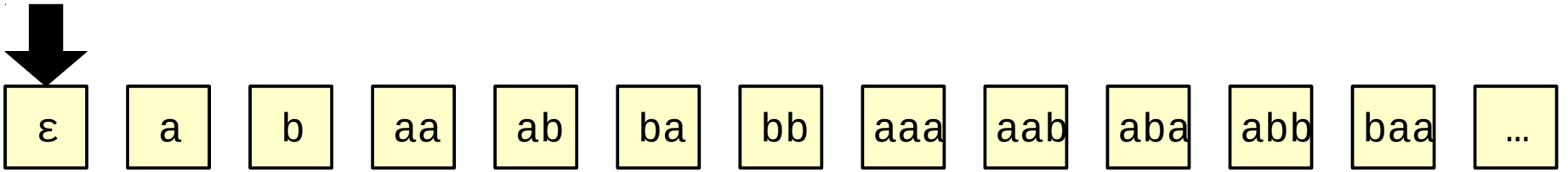

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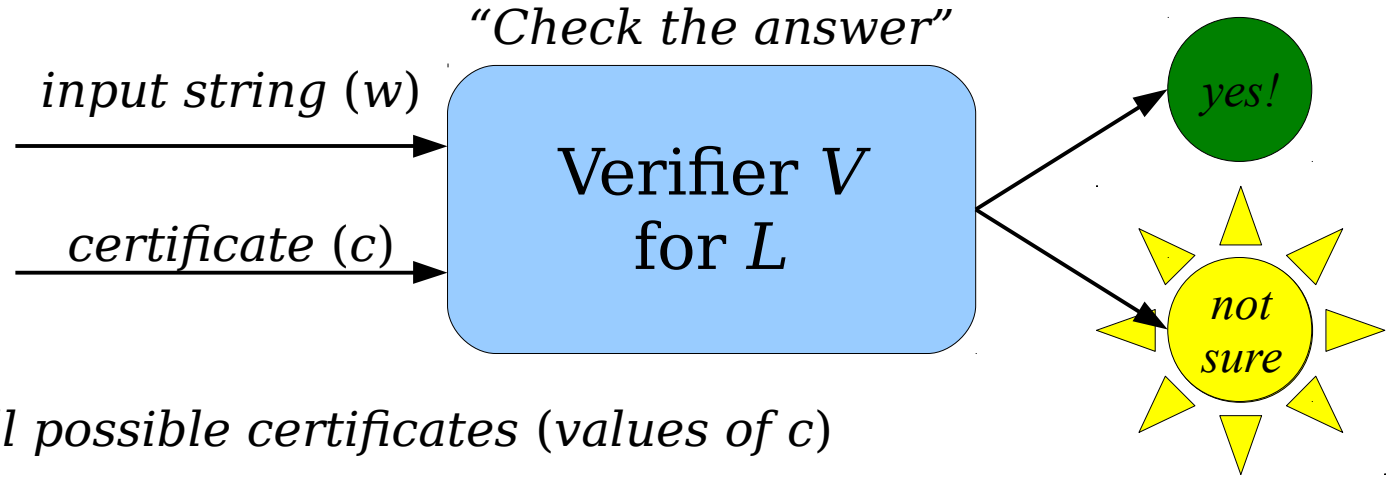

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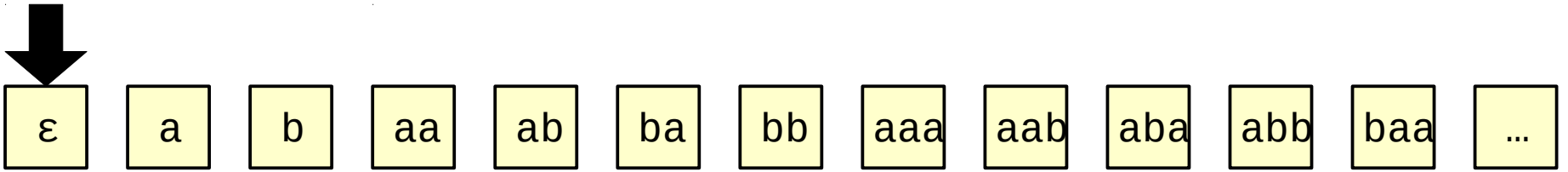

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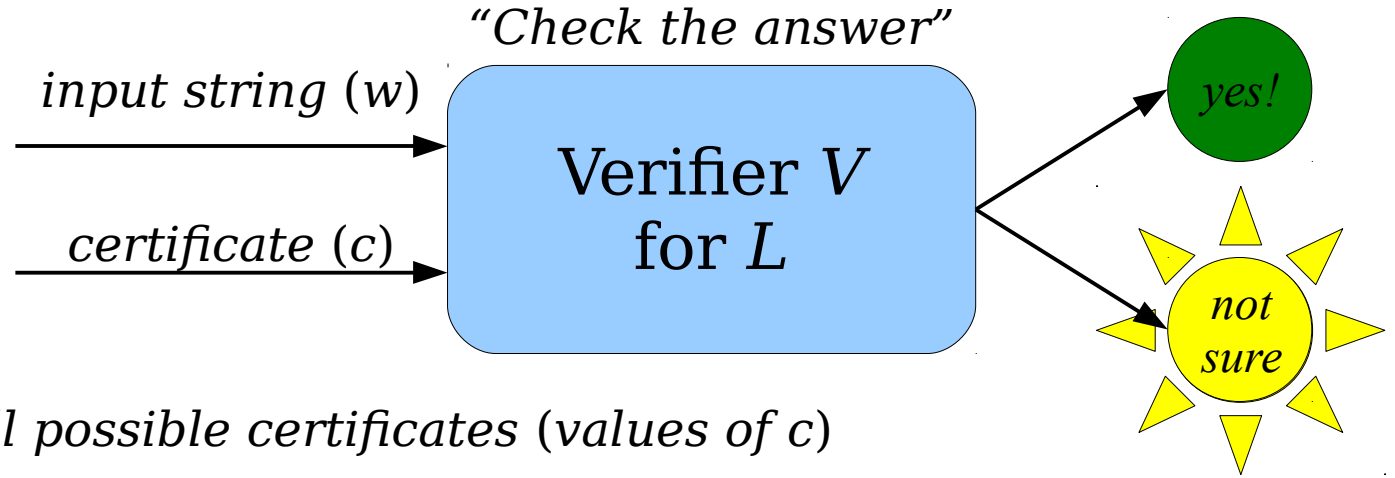

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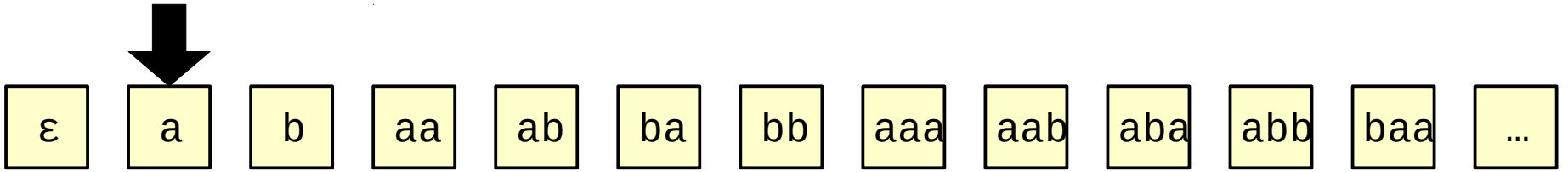

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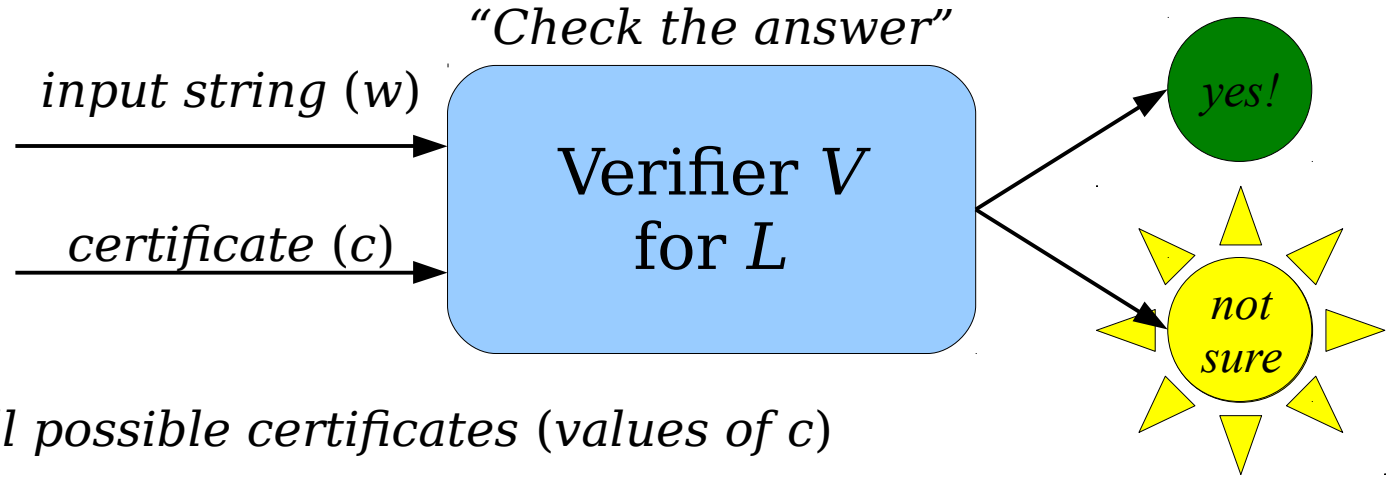

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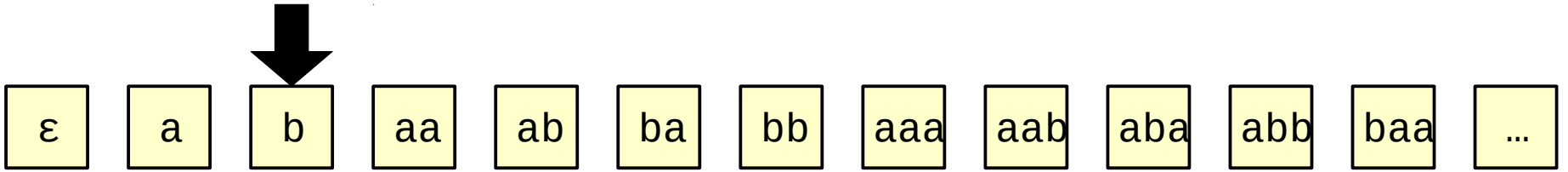

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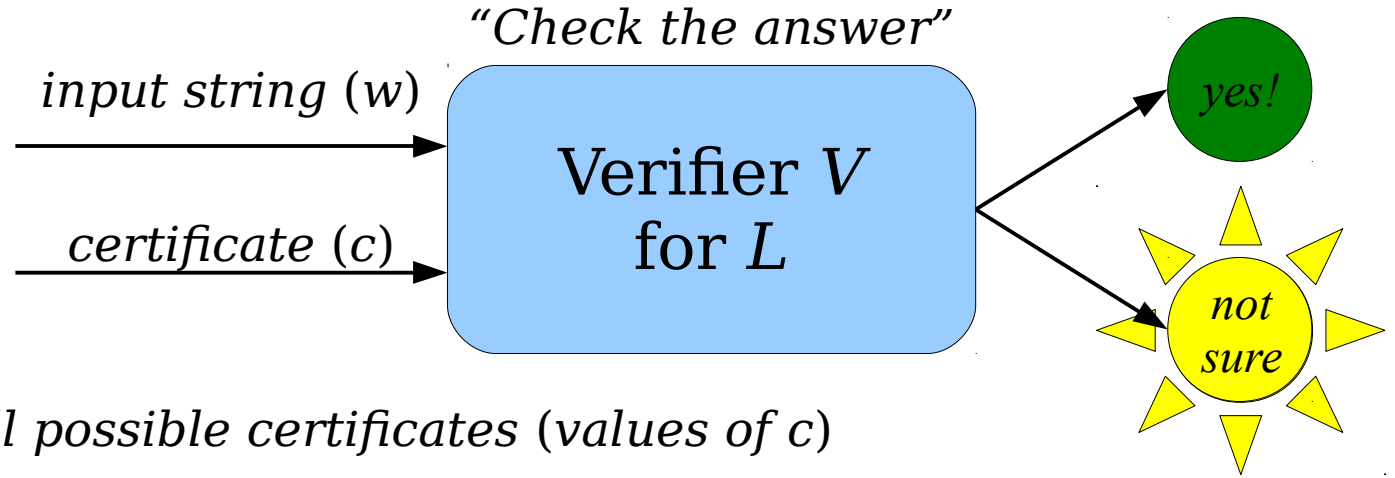

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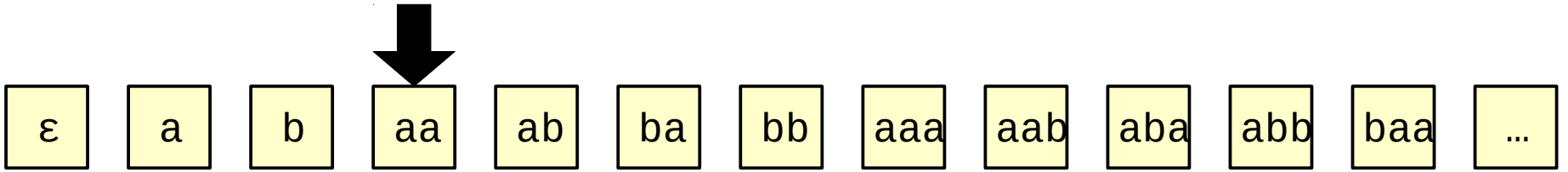

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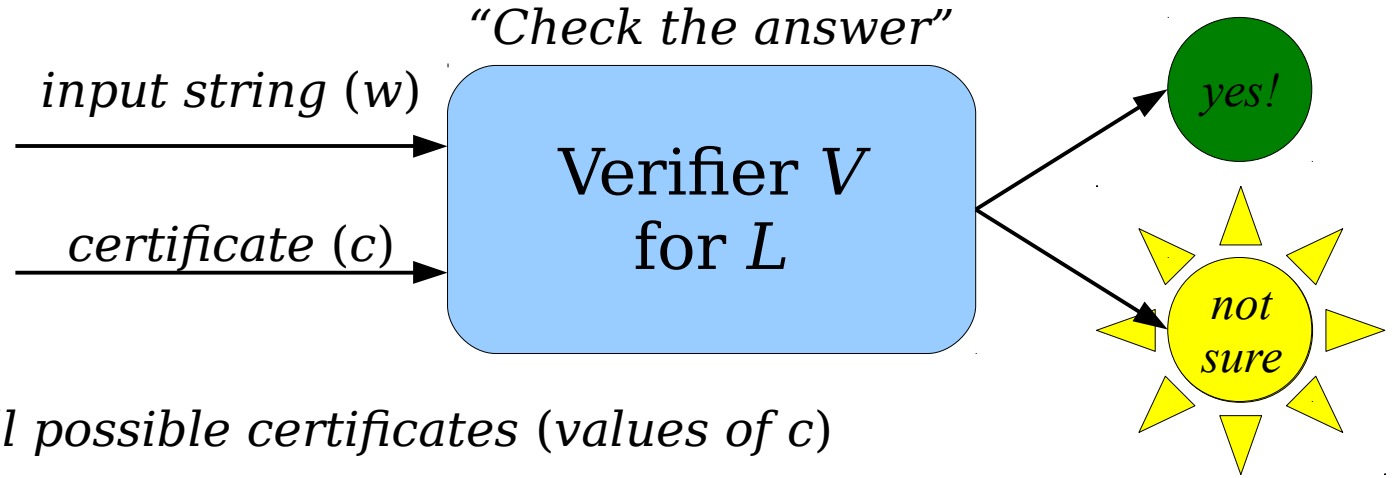

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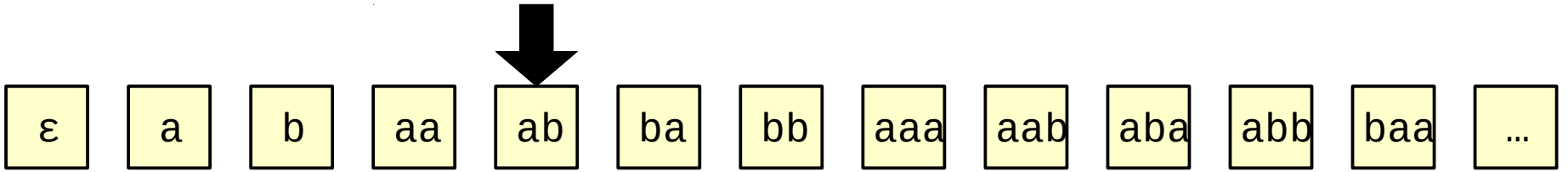

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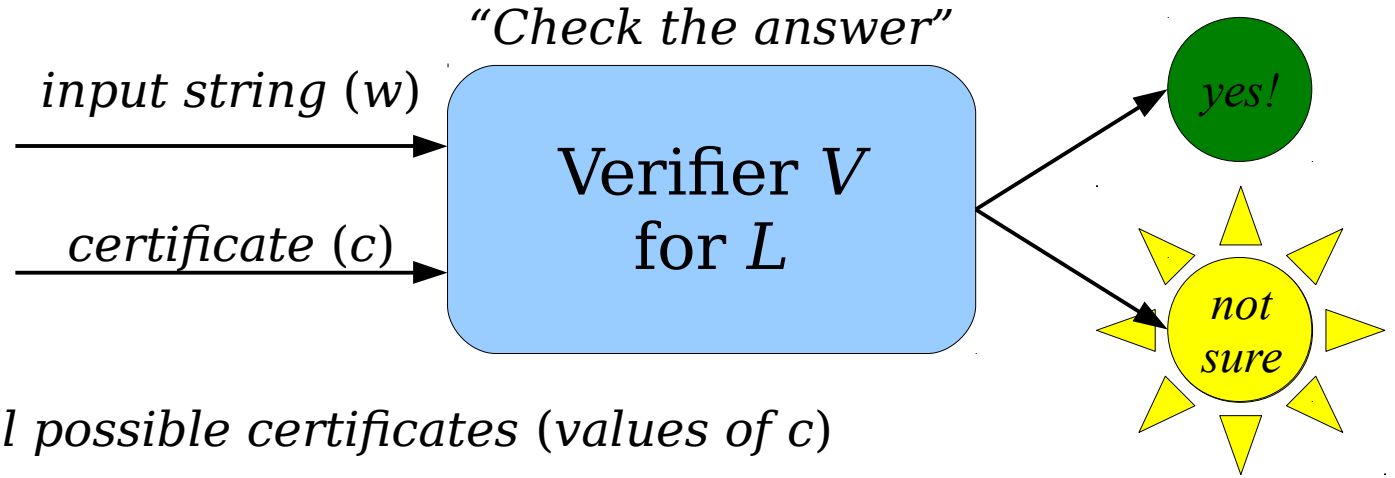

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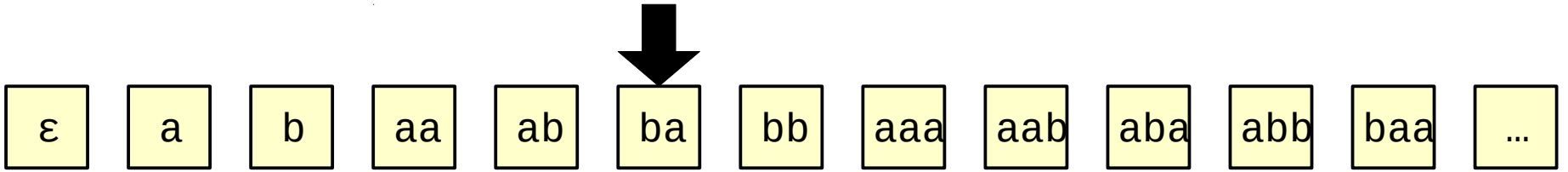

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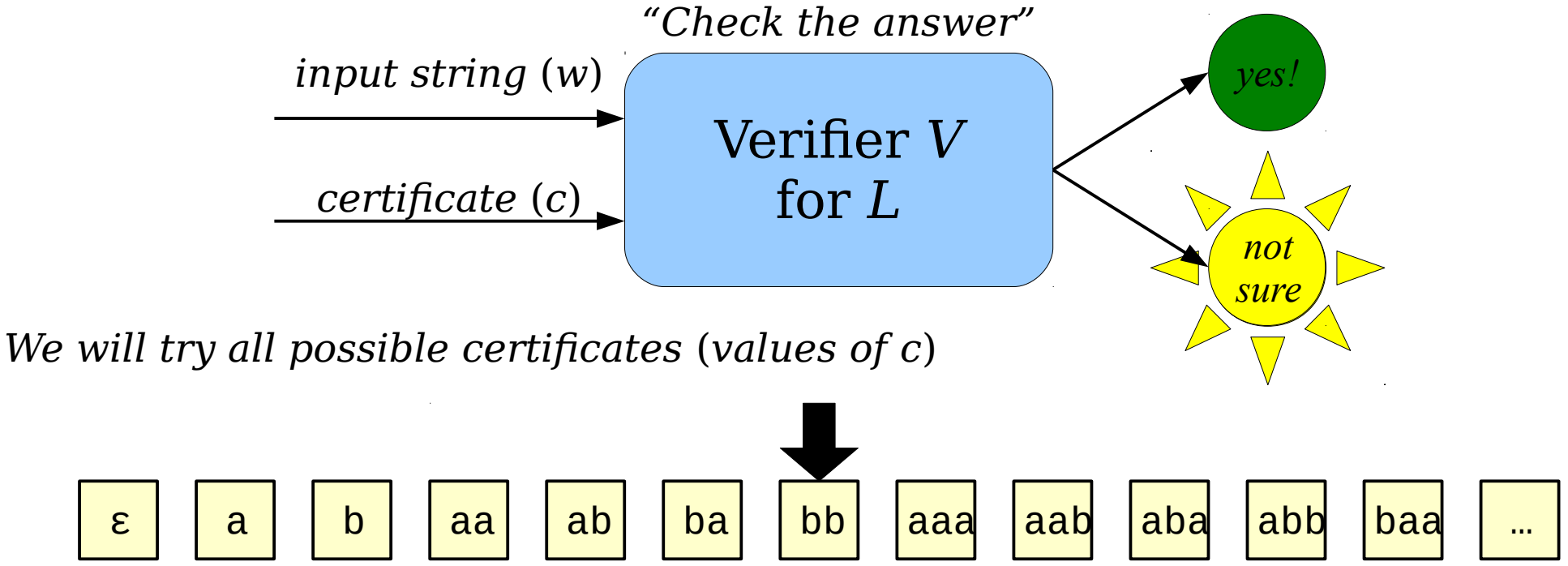

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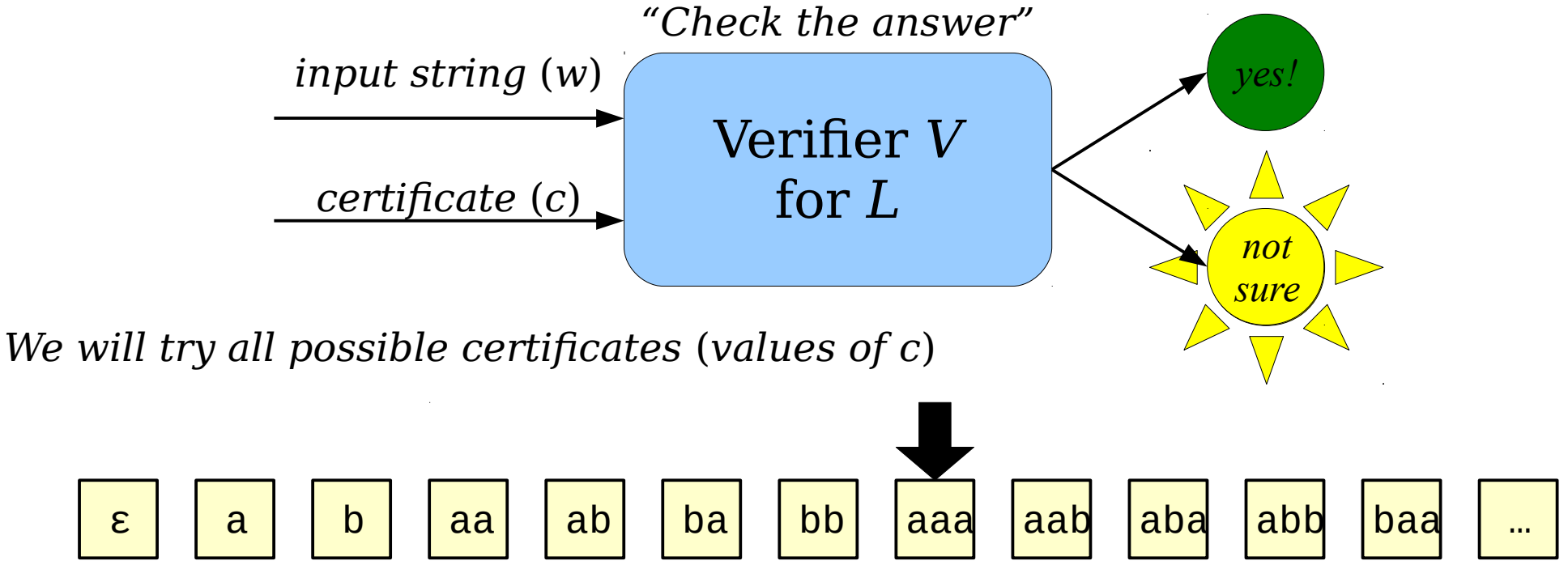

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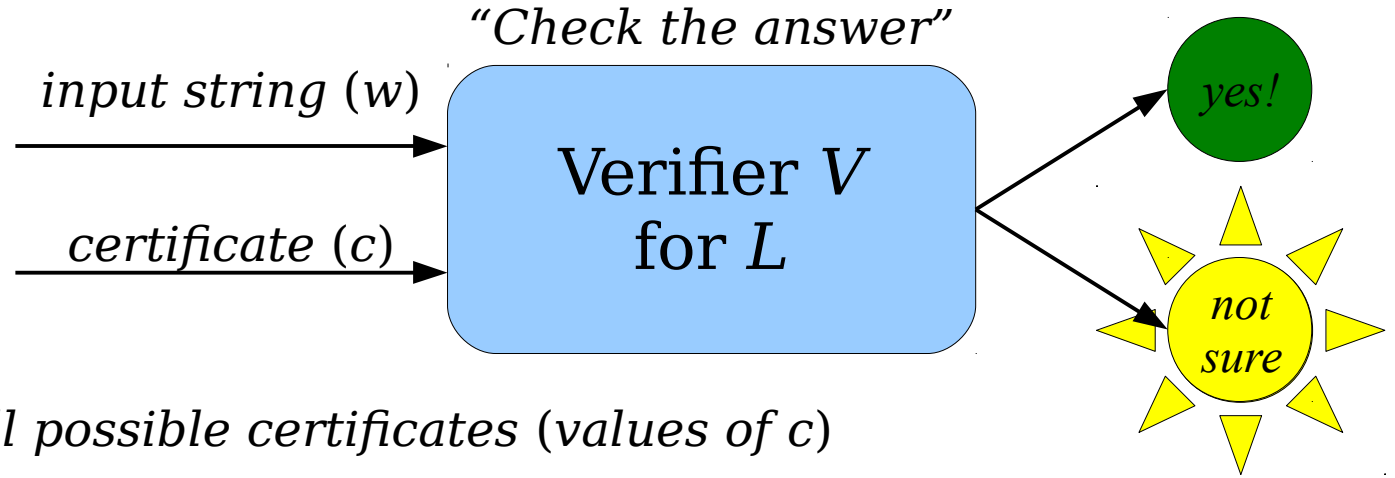

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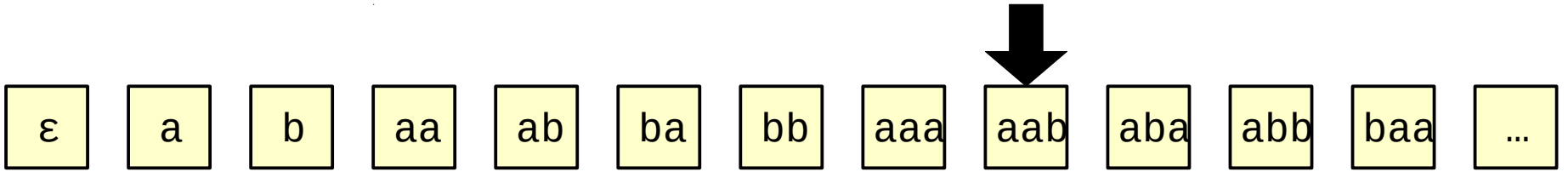

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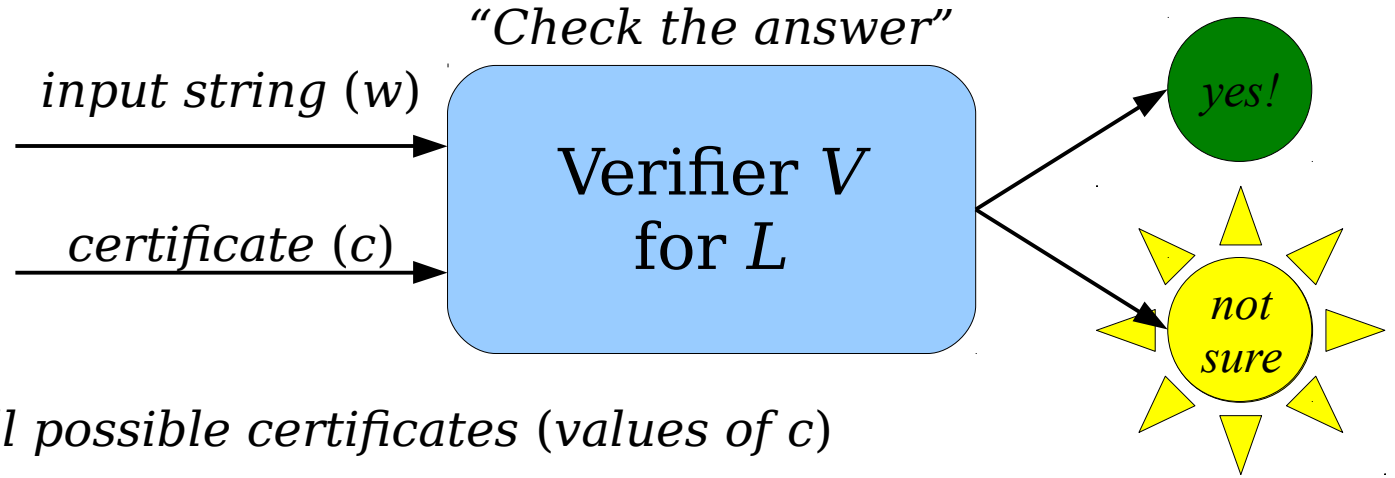

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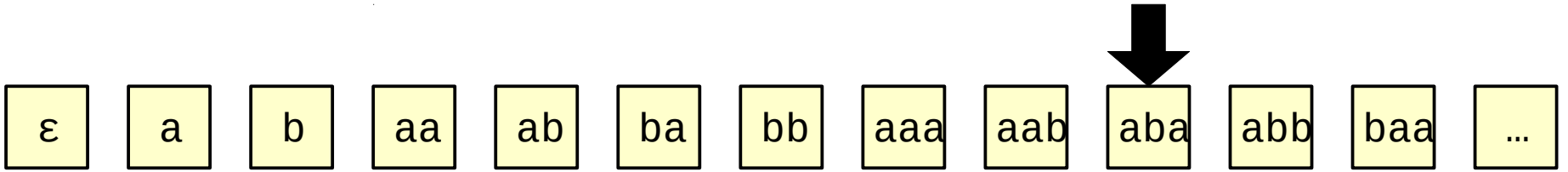

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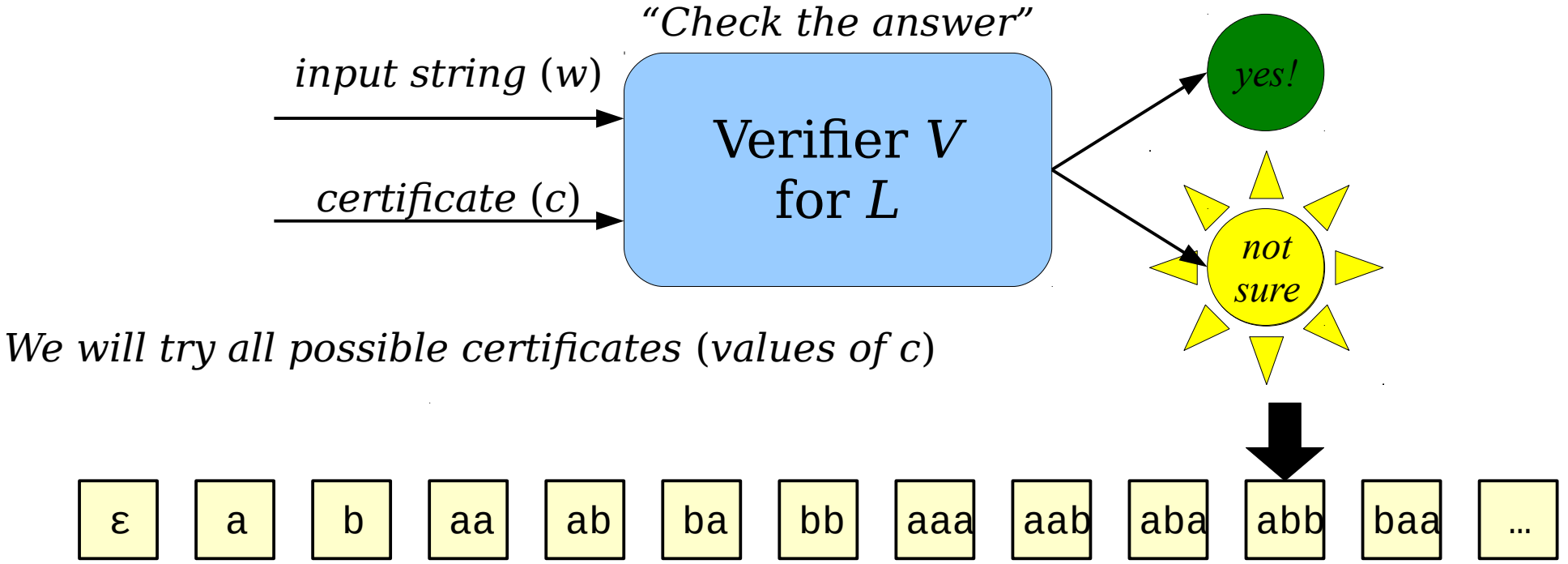

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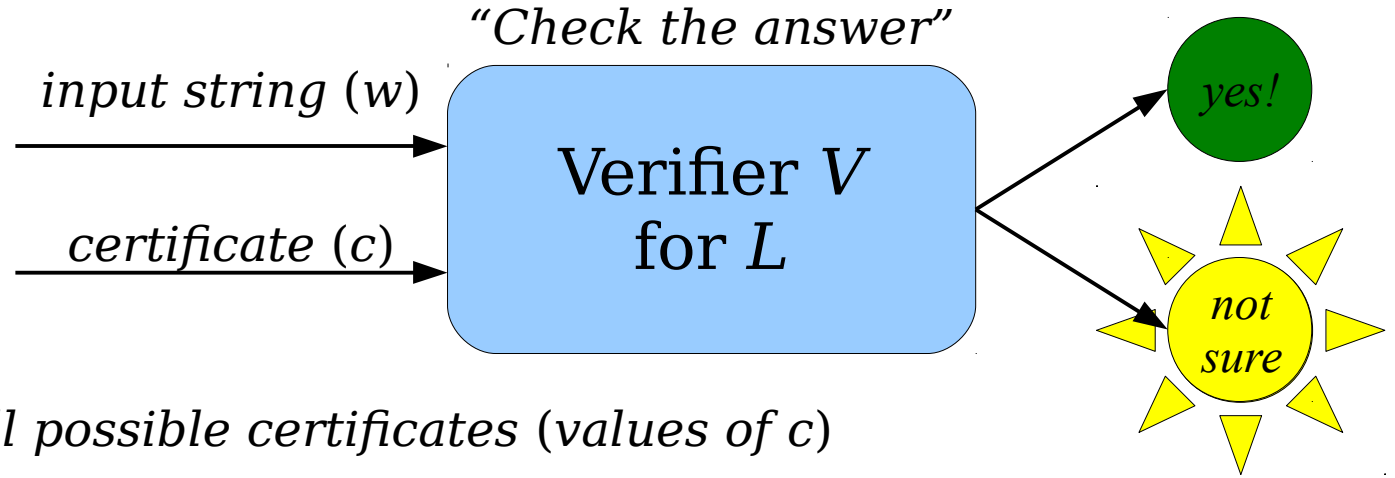

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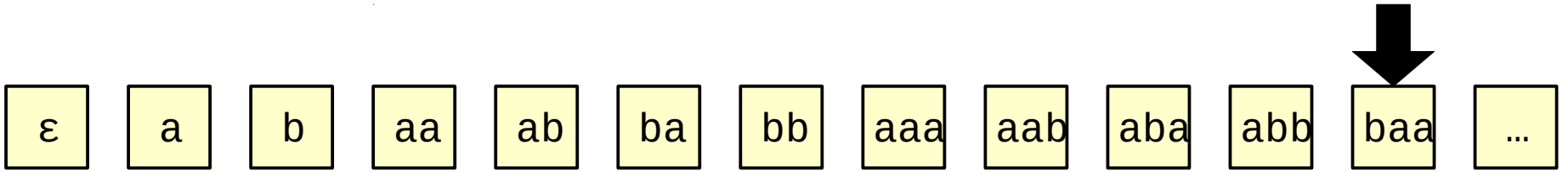

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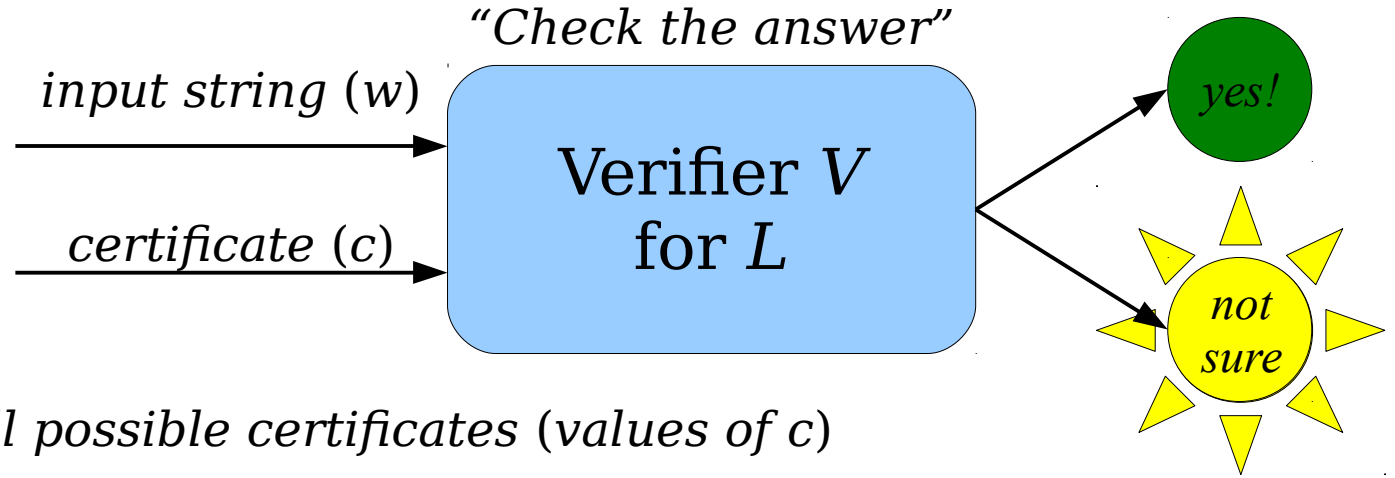

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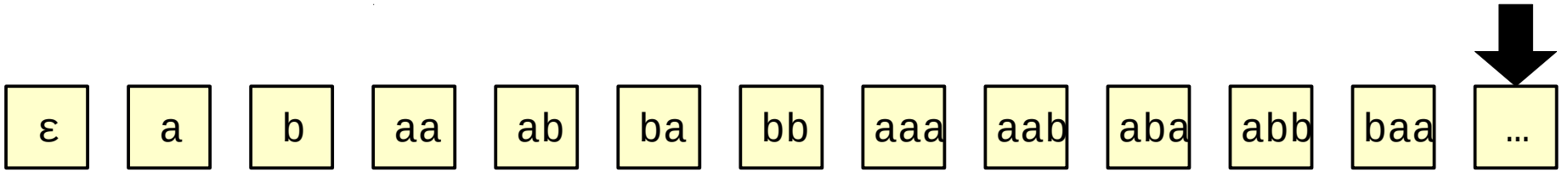

Verifiers and **RE**

- ***Theorem:*** If V is a verifier for L, then $L \in \mathbf{RE}$.
- ***Proof sketch:*** Consider the following program:

```
bool isInL(string w) {
    int i = 0;
    while (true) {
        for (each string c of length i) {
            if (V accepts ⟨w, c⟩) return true;
        }
        i++;
    }
}
```

If $w \in L$, there is some $c \in \Sigma^*$ where V accepts $\langle w, c \rangle$. The function `isInL` tries all possible strings as certificate, so it will eventually find c (or some other certificate), see V accept $\langle w, c \rangle$, then return true. Conversely, if `isInL(w)` returns true, then there was some string c such that V accepted $\langle w, c \rangle$, so $w \in L$. ■

Verifiers and **RE**

- ***Theorem:*** If $L \in$ **RE**, then there is a verifier for L.
- ***Proof goal:*** Beginning with a recognizer M for the language L, show how to construct a verifier V for L.
- The challenges:
  - A recognizer M is not required to halt on all inputs. A verifier V must always halt.
  - A recognizer M takes in one single input. A verifier V takes in two inputs.
- We'll need to find a way of reconciling these requirements.

Recall: If M is a recognizer for a language L, then M accepts w iff $w \in L$.

Key insight: If M accepts a string w, it always does so in a finite number of steps.

Idea: Adapt the verifier for A_{TM} into a more general construction that turns any recognizer into a verifier by running it for a fixed number of steps.

Verifiers and **RE**

- ***Theorem:*** If $L \in$ **RE**, then there is a verifier for L.
- ***Proof sketch:*** Consider the following program:

```
bool checkIsInL(string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on W;
    }
    return whether M is in an accepting state;
}
```

Notice that `checkIsInL` always halts, since each step takes only finite time to complete. Next, notice that if there is a c where `checkIsInL(w, c)` returns true, then M accepted w after running for c steps, so $w \in L$. Conversely, if $w \in L$, then M accepts w after some number of steps (call that number c). Then `checkIsInL(w, c)` will run M on w for c steps, watch M accept w, then return true. ■

RE and Proofs

- Verifiers and recognizers give two different perspectives on the "proof" intuition for **RE**.
- Verifiers are explicitly built to check proofs that strings are in the language.
  - If you know that some string w belongs to the language and you have the proof of it, you can convince someone else that $w \in L$.
- You can think of a recognizer as a device that "searches" for a proof that $w \in L$.
  - If it finds it, great!
  - If not, it might loop forever.

RE and Proofs

- If the **RE** languages represent languages where membership can be proven, what does a non-**RE** language look like?
- Intuitively, a language is *not* in **RE** if there is no general way to prove that a given string $w \in L$ actually belongs to L.
- In other words, even if you knew that a string was in the language, you may never be able to convince anyone of it!