Unsolvable Problems

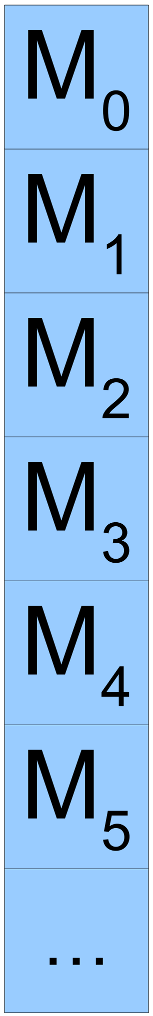
Finding Non-**RE** Languages

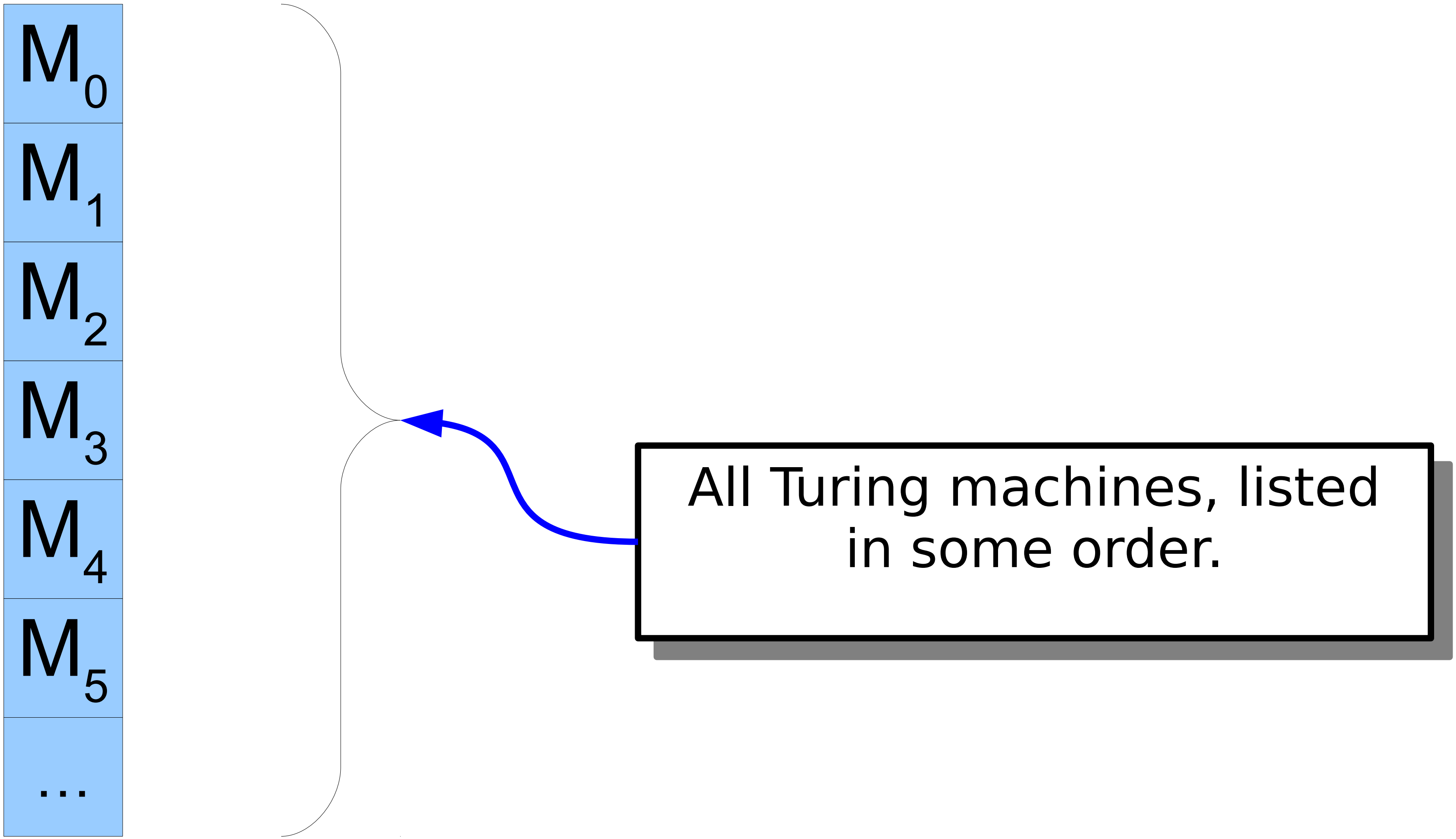
Finding Non-**RE** Languages

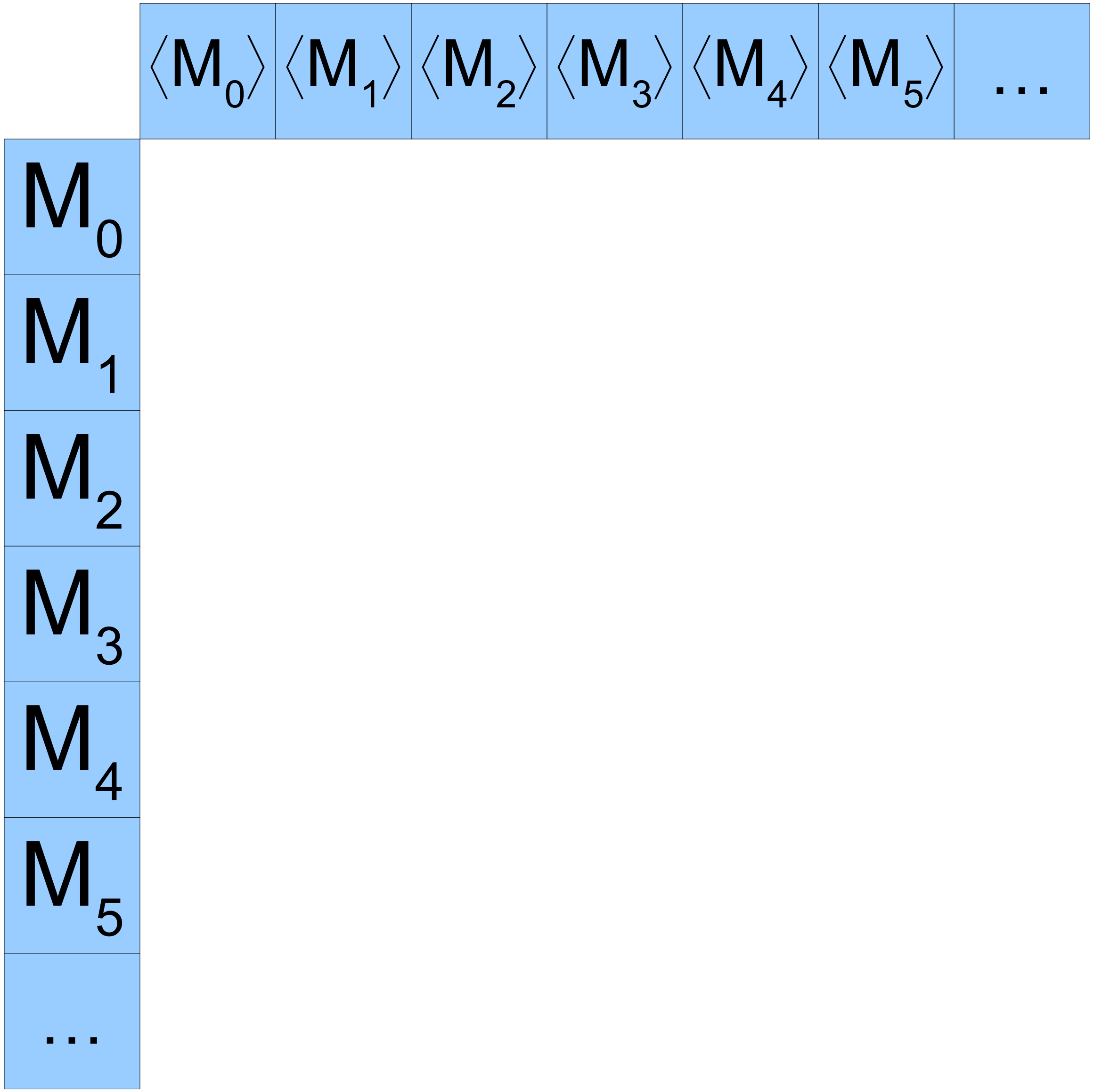
- Remember **RE** but non-**R** (undecidable) languages are those where we can reliably identify strings in the language, but cannot readily identify strings that are *not* in the language.
- Non-**RE** languages will be those where we cannot even readily identify strings that *are* in the language!
- How might we find an example of a non-**RE** language?

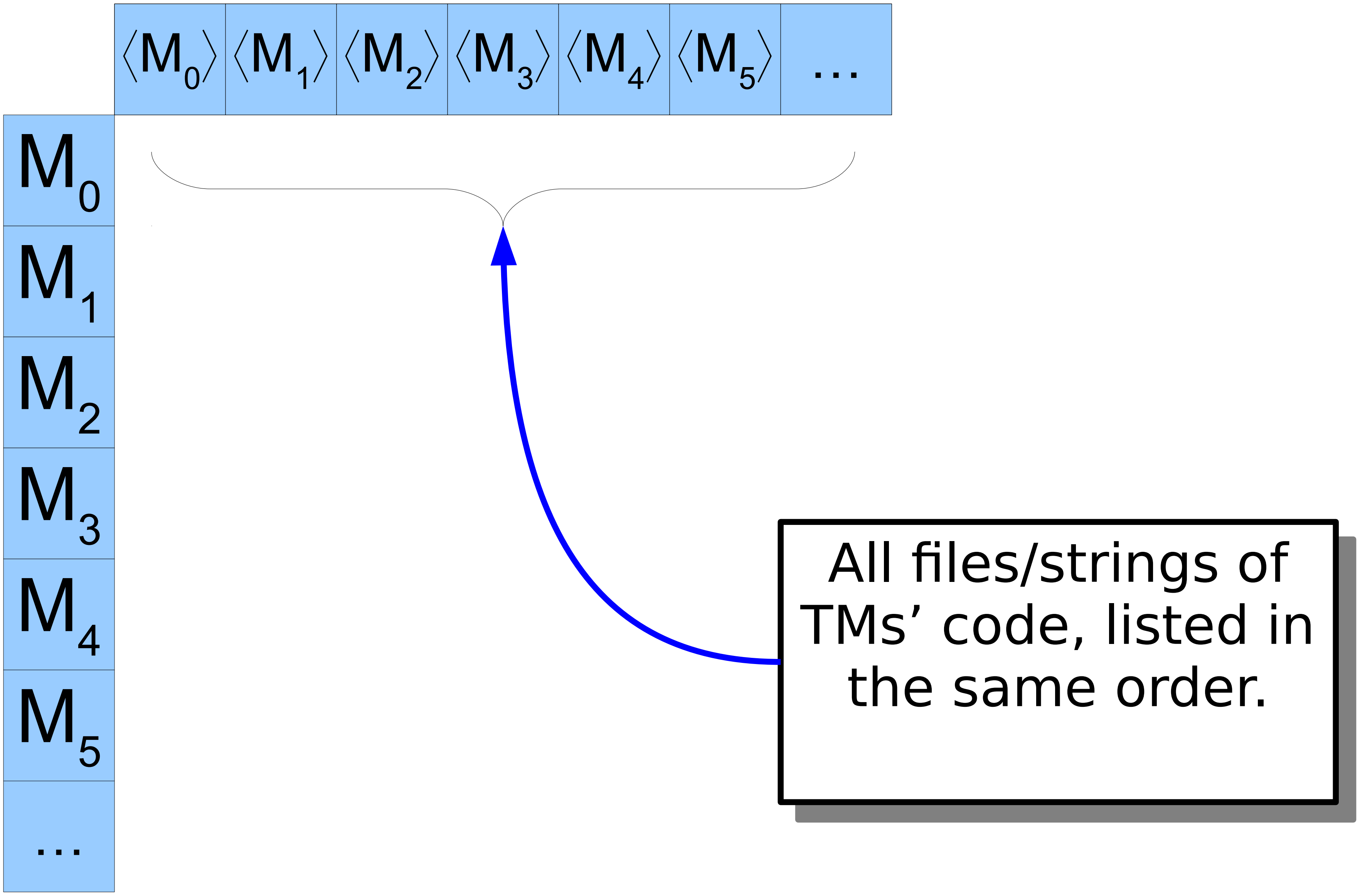
Languages, TMs, and TM Encodings

- What happens if we list off all Turing machines, looking at how those TMs behave when given other TM codes (as strings, so various $\langle M_x \rangle$ strings) as input?


M_0
M_1
M_2
M_3
M_4
M_5
…


M_0
M_1
M_2
M_3
M_4
M_5
…
All Turing machines, listed in some order.


⟨M0⟩
⟨M1⟩
⟨M2⟩
⟨M3⟩
⟨M4⟩
⟨M5⟩
…
M0
M1
M2
M3
M4
M5
…



⟨M_0⟩ ⟨M_1⟩ ⟨M_2⟩ ⟨M_3⟩ ⟨M_4⟩ ⟨M_5⟩ …
M_0
M_1
M_2
M_3
M_4
M_5
…
All files/strings of TMs’ code, listed in the same order.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1							
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2							
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3							
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc		
M_3							
M_4							
M_5							
…							

What is $\mathscr{L}(M_0)$?

A. Σ^*

B. $\{\langle M_0 \rangle, \langle M_3 \rangle, \langle M_4 \rangle, \ldots\}$

C. $\{\langle M_0 \rangle\}$

D. $\{\langle M_0 \rangle, \langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle, \ldots\}$

E. Something else.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A - E**.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3							
M_4							
M_5							
…							

$\mathscr{L}(M_0) = \{\langle M_0 \rangle, \langle M_3 \rangle, \langle M_4 \rangle, \dots\}$. And we can't see the rest of the table for M_2, but it accepts everything so far, so it's at least possible that its language is $\mathscr{L}(M_2) = \Sigma^*$.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc			
M_2	Acc	Acc	Acc	Acc			
M_3							
M_4							
M_5							
…							

Aside: we aren't really worrying about the existence of other strings that aren't TM code right now, but you could also think of it including those strings, so $\mathscr{L}(M_0) = \{\langle M_0 \rangle, \langle M_3 \rangle, \langle M_4 \rangle, \ldots\} \cup \{w \mid w$ is a string that isn't a TM's code, and M_0 accepts $w\}$

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4							
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5							
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…							

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…							

Quick check:
How many of the TMs on this chart so far do NOT accept their own code as a string?
(Enter a number.)

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **a number**.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

What are we going to do next?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **your answer**.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

Acc	Acc	Acc	No	Acc	No	…

	⟨M₀⟩	⟨M₁⟩	⟨M₂⟩	⟨M₃⟩	⟨M₄⟩	⟨M₅⟩	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

Flip all “accept” to “no” and vice-versa

	⟨M_0⟩	⟨M_1⟩	⟨M_2⟩	⟨M_3⟩	⟨M_4⟩	⟨M_5⟩	…
M_0	**Acc**	No	No	Acc	Acc	No	…
M_1	Acc	**Acc**	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	**Acc**	Acc	Acc	Acc	…
M_3	No	Acc	Acc	**No**	Acc	Acc	…
M_4	Acc	No	Acc	No	**Acc**	No	…
M_5	No	No	Acc	Acc	No	**No**	…
…	…	…	…	…	…	…	**…**

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	**Acc**	No	No	Acc	Acc	No	…
M_1	Acc	**Acc**	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	**Acc**	Acc	Acc	Acc	…
M_3	No	Acc	Acc	**No**	Acc	Acc	…
M_4	Acc	No	Acc	No	**Acc**	No	…
M_5	No	No	Acc	Acc	No	**No**	…
…	…	…	…	…	…	…	**…**

No	No	No	Acc	No	Acc	…

What TM has this behavior?

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…
	No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	⟨M_0⟩	⟨M_1⟩	⟨M_2⟩	⟨M_3⟩	⟨M_4⟩	⟨M_5⟩	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	⟨M_0⟩	⟨M_1⟩	⟨M_2⟩	⟨M_3⟩	⟨M_4⟩	⟨M_5⟩	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

No TM has this behavior!

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

"The language of all TMs that do not accept their own description."

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

{ $\langle M \rangle$ | M is a TM that does not accept $\langle M \rangle$ }

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	…
M_0	Acc	No	No	Acc	Acc	No	…
M_1	Acc	Acc	Acc	Acc	Acc	Acc	…
M_2	Acc	Acc	Acc	Acc	Acc	Acc	…
M_3	No	Acc	Acc	No	Acc	Acc	…
M_4	Acc	No	Acc	No	Acc	No	…
M_5	No	No	Acc	Acc	No	No	…
…	…	…	…	…	…	…	…

No	No	No	Acc	No	Acc	…

$$\{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$$

Diagonalization Revisited

- The ***diagonalization language***, which we denote $\boldsymbol{L_D}$, is defined as

$$\boldsymbol{L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}}$$

- That is, L_D is the set of descriptions of Turing machines that do not accept themselves.

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

$L_{\mathrm{D}} = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_{\mathrm{D}} \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_{\mathrm{D}} \in \mathbf{RE}$.

$\mathbf{L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \quad (1)$$

$$L_D = \{ \langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathscr{L}(M) \}$$

Theorem: $L_D \notin \mathbf{RE}$.

Proof: By contradiction; assume that $L_D \in \mathbf{RE}$. Then there must be some recognizer R such that $\mathscr{L}(R) = L_D$.

Let M be an arbitrary TM. Since $\mathscr{L}(R) = L_D$, we know that

$$\langle M \rangle \in L_D \text{ iff } \langle M \rangle \in \mathscr{L}(R). \qquad (1)$$

Because $\mathscr{L}(R) = L_D$, we know that a string belongs to one set if and only if it belongs to the other.

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We've replaced the left-hand side of this biconditional with an equivalent statement.

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A nice consequence of a universally-quantified statement is that it should work in all cases.

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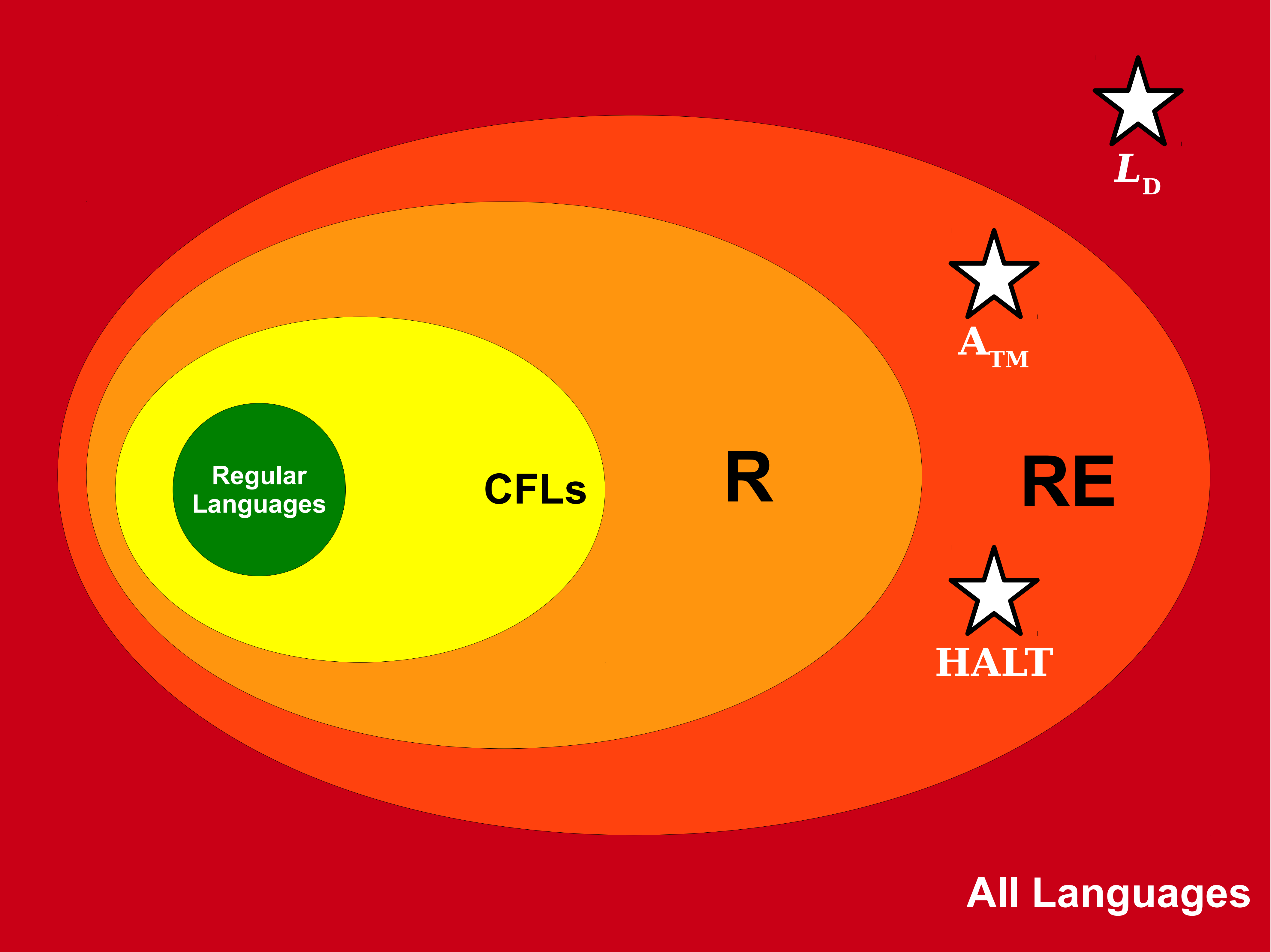


L_D
A_{TM}
Regular Languages
CFLs
R
RE
HALT
All Languages

What This Means

- On a deeper philosophical level, the fact that non-**RE** languages exist supports the following claim:

There are statements that are true but not provable.

- Intuitively, given any non-**RE** language, there will be some string in the language that *cannot* be proven to be in the language.
- This result can be formalized as a result called ***Gödel's incompleteness theorem***, one of the most important mathematical results of all time.
- Want to learn more? Take Phil 152 or CS154!

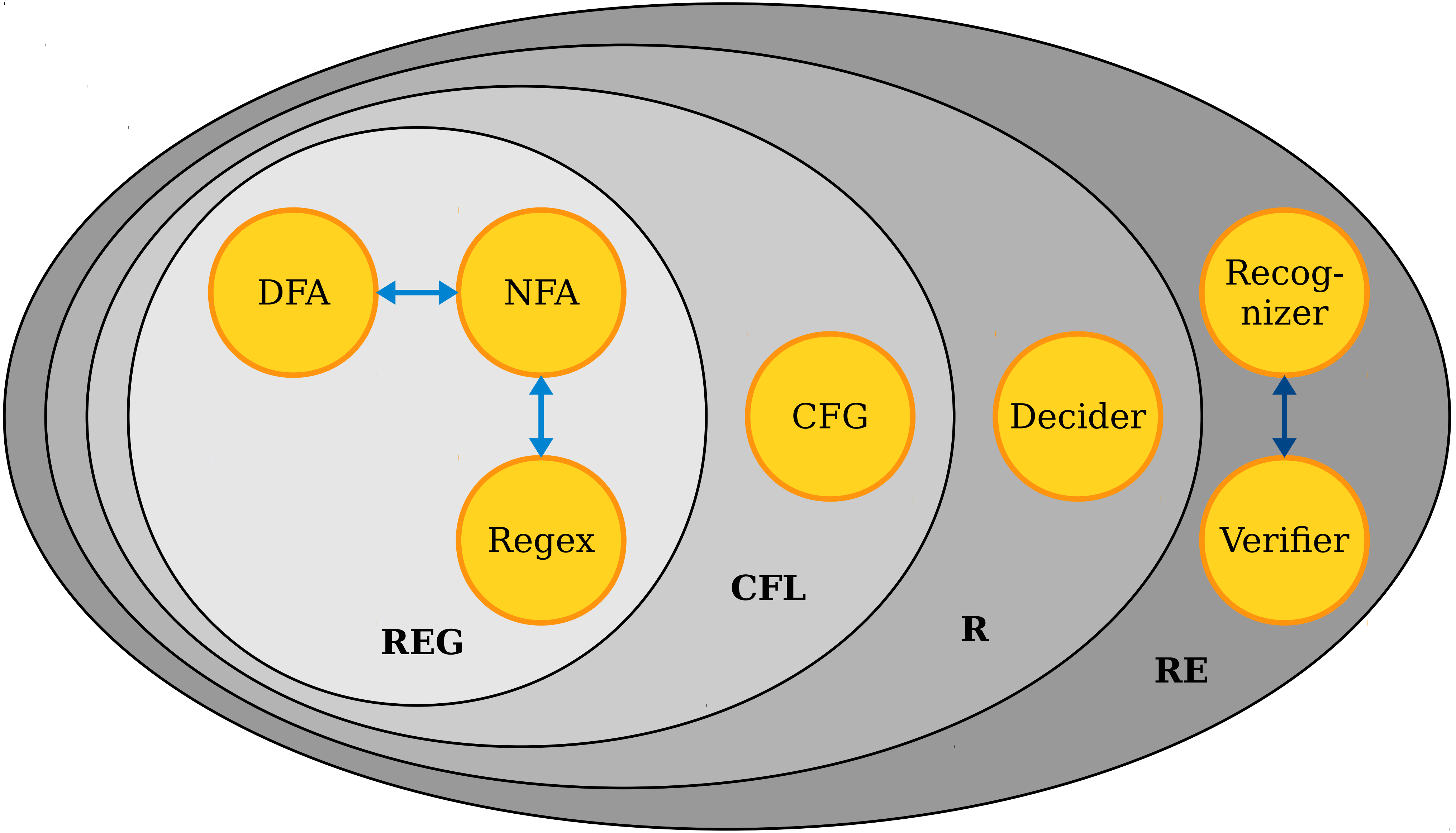
What This Means

- On a more philosophical note, you could interpret the previous result in the following way:

  There are inherent limits about what mathematics can teach us.

- There's no automatic way to do math. There are true statements that we can't prove.
- That doesn't mean that mathematics is worthless. It just means that we need to temper our expectations about it.

The Big Picture



Up to this point:

“*Can* we solve this problem?”

(**Computability Theory**)

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Starting today:

“Ok, even if we *can*, we need to consider whether the time/resources required actually make practical/feasible sense.”

(**Complexity Theory**)

Where We've Been

- The class **R** represents problems that can be solved by a computer.
- The class **RE** represents problems where “yes” answers can be verified by a computer.

Where We're Going

- The class **P** represents problems that can be solved *efficiently* by a computer.
- The class **NP** represents problems where “yes” answers can be verified *efficiently* by a computer.