Complexity Theory

Part Two

Recap from Last Time

Up to this point:

"*Can* we solve this problem?"

(**Computability Theory**)

Up to this point:

"*Can* we solve this problem?"

(**Computability Theory**)

Starting today:

"Ok, even if we *can*, we need to consider whether the time/resources required actually make practical/feasible sense."

(**Complexity Theory**)

The Complexity Class **P**

- The ***complexity class* P** (for ***p***olynomial time) contains all problems that can be solved in polynomial time.
- Formally:

$$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$$

The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

$$\textbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$

- This means a verifier V's runtime is a polynomial in $|w|$ (that is, V's runtime is $O(|w|^k)$ for some integer k).

And now…

The

Most Important Question

in

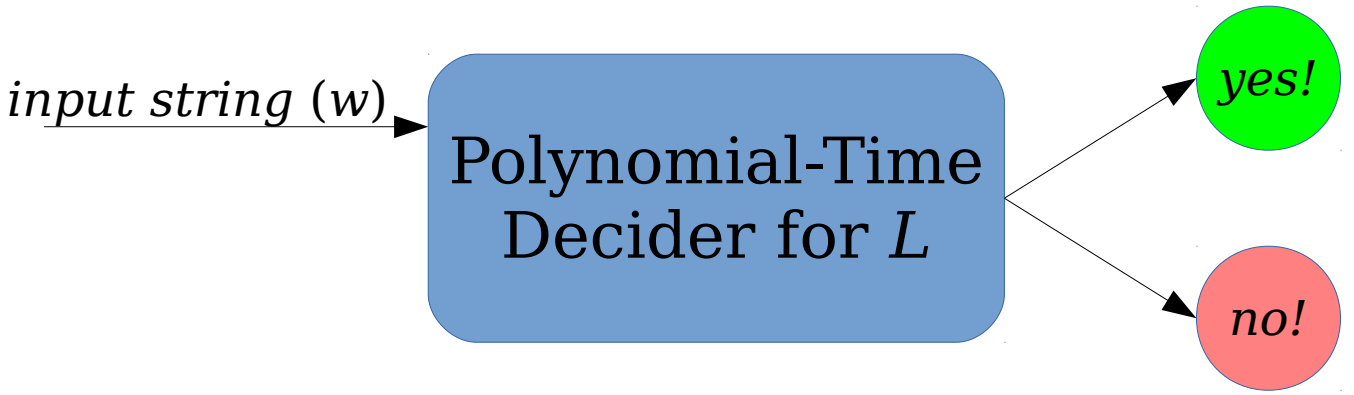
Theoretical Computer Science

What is the connection between **P** and **NP**?

Does **P** = **NP**?

P = { L | There is a polynomial-time decider for L }

NP = { L | There is a polynomial-time verifier for L }

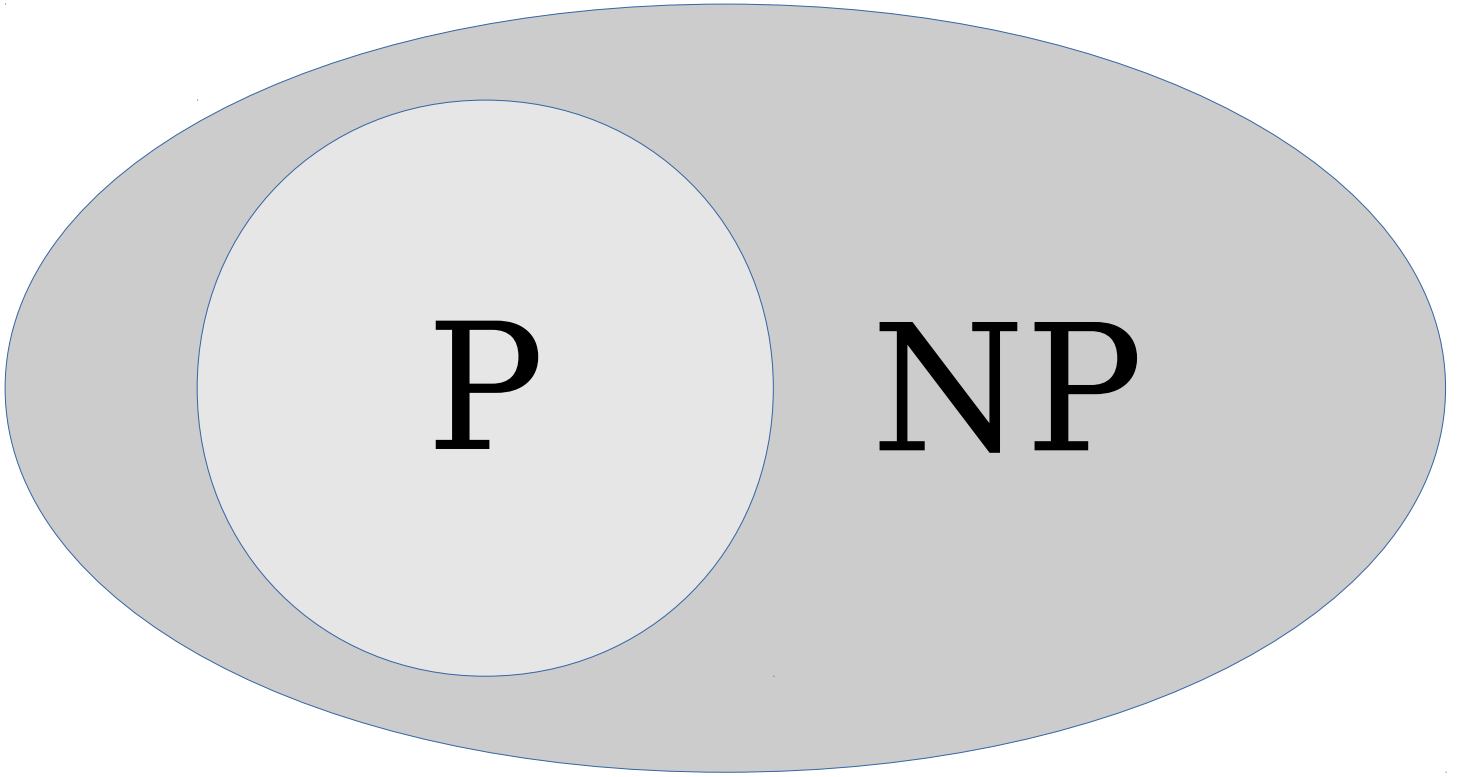


P = { L | There is a polynomial-time decider for L }

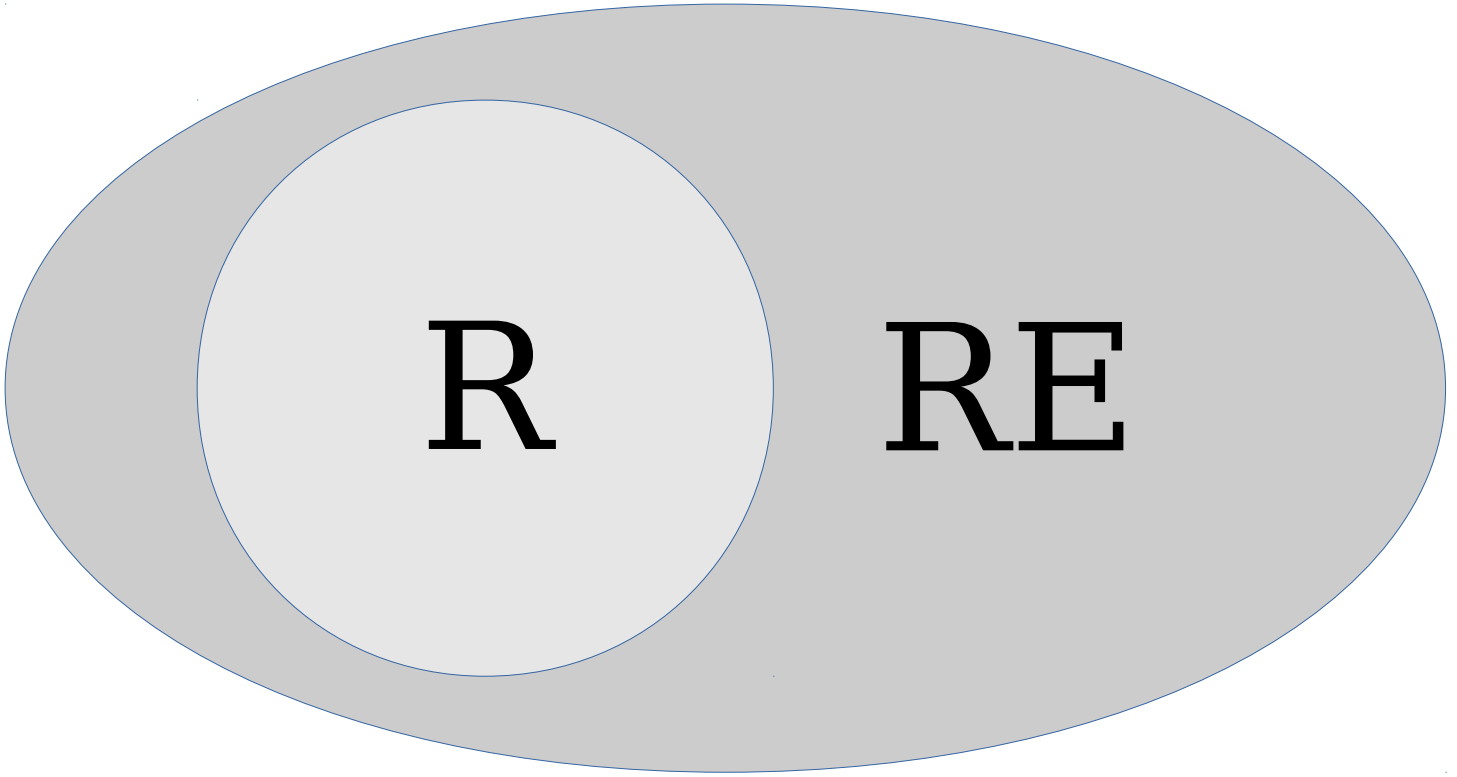
NP = { L | There is a polynomial-time verifier for L }

P ⊆ NP

Which Picture is Correct?



(remember: if you take out the timing aspect, this is the picture of languages with a decider and languages with a verifier)



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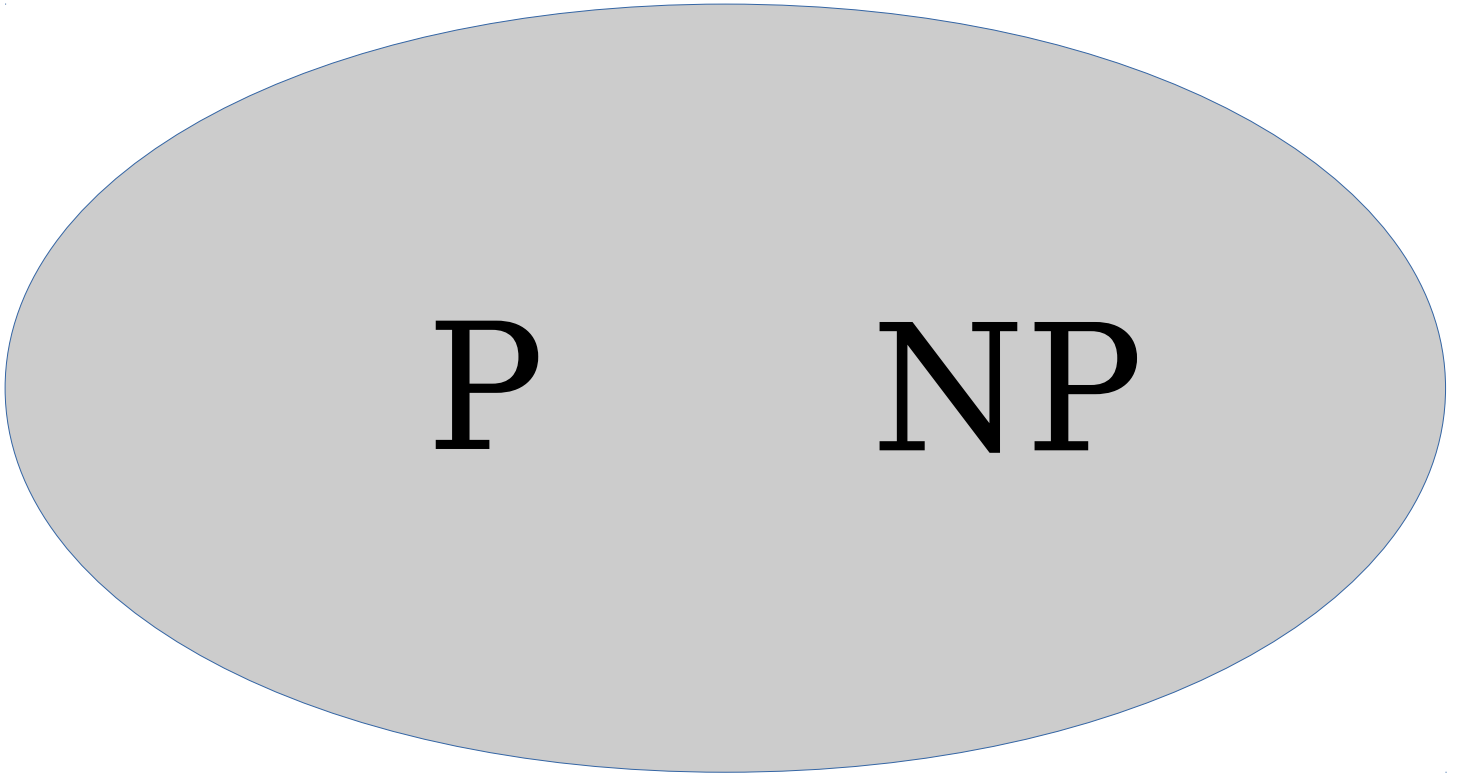


$\mathbf{P} \stackrel{?}{=} \mathbf{NP}$

- The $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ question is the most important question in theoretical computer science.
- With the verifier definition of **NP**, one way of phrasing this question is

  *If a solution to a problem can be **checked** efficiently, can that problem be **solved** efficiently?*

- An answer either way will give fundamental insights into the nature of computation.

Why This Matters

- The following problems are known to be efficiently verifiable, but have no known efficient solutions:
  - Determining whether an electrical grid can be built to link up some number of houses for some price (Steiner tree problem).
  - Determining whether a simple DNA strand exists that multiple gene sequences could be a part of (shortest common supersequence).
  - Determining the best way to assign hardware resources in a compiler (optimal register allocation).
  - Determining the best way to distribute tasks to multiple workers to minimize completion time (job scheduling).
  - ***And many more***.
- If **P = NP**, ***all*** of these problems have efficient solutions.
- If **P ≠ NP**, ***none*** of these problems have efficient solutions.

Why This Matters

- If **P = NP**:
  - A huge number of seemingly difficult problems could be solved efficiently.
  - Our capacity to solve many problems will scale well with the size of the problems we want to solve.
- If **P ≠ NP**:
  - Enormous computational power would be required to solve many seemingly easy tasks.
  - Our capacity to solve problems will fail to keep up with our curiosity.

What We Know

- Resolving **P** $\stackrel{?}{=}$ **NP** has proven ***extremely difficult***.
- In the past 45 years:
  - Not a single correct proof either way has been found.
  - Many types of proofs have been shown to be insufficiently powerful to determine whether **P** $\stackrel{?}{=}$ **NP**.
  - A majority of computer scientists believe **P** ≠ **NP**, but this isn't a large majority.
- Interesting read: Interviews with leading thinkers about **P** $\stackrel{?}{=}$ **NP**:
  - http://web.ing.puc.cl/~jabaier/iic2212/poll-1.pdf

The Million-Dollar Question

The Clay Mathematics Institute has offered a ***$1,000,000 prize*** to anyone who proves or disproves **P = NP**.

Do you think **P = NP**?

What do we know about $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$?

Adapting our Techniques

A Problem

- The **R** and **RE** languages correspond to problems that can be decided and verified, *period,* without any time bounds.
- To reason about what's in **R** and what's in **RE**, we used two key techniques:
  - ***Universality***: TMs can run other TMs as subroutines.
  - ***Self-Reference***: TMs can get their own source code.
- Why can't we just do that for **P** and **NP**?

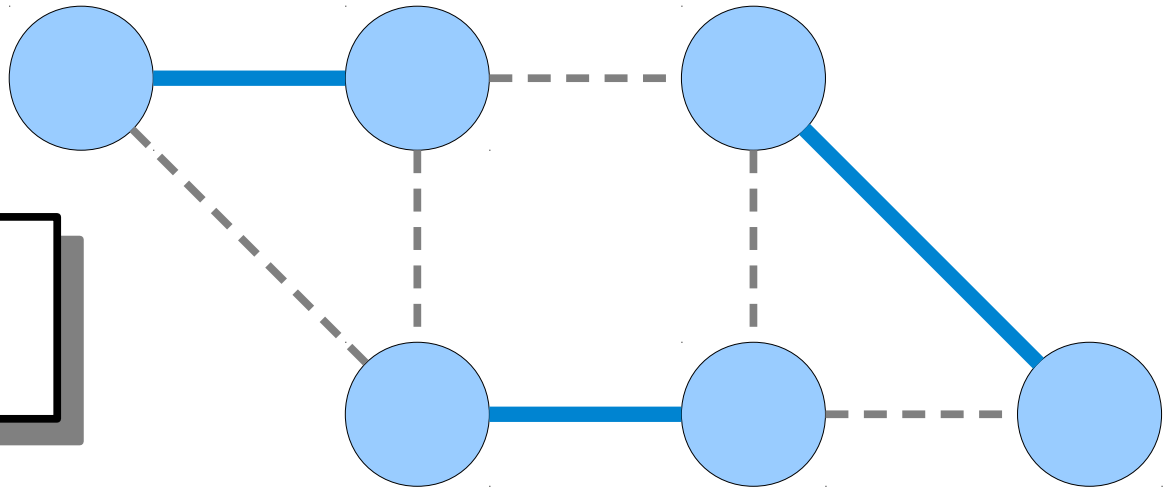
Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to reason about **P** and **NP**?

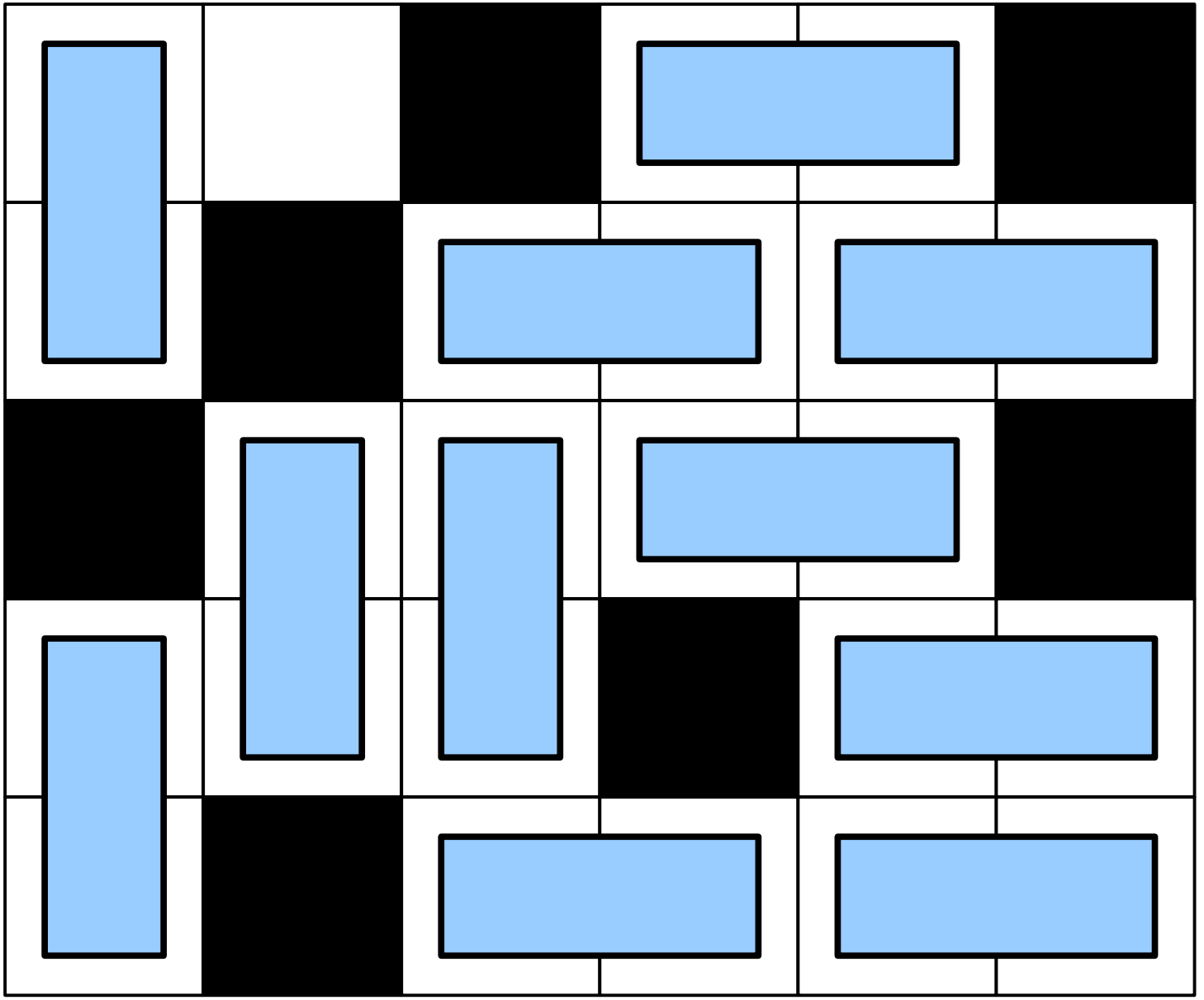
Maximum Matching

- Given an undirected graph G, a ***matching*** in G is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.

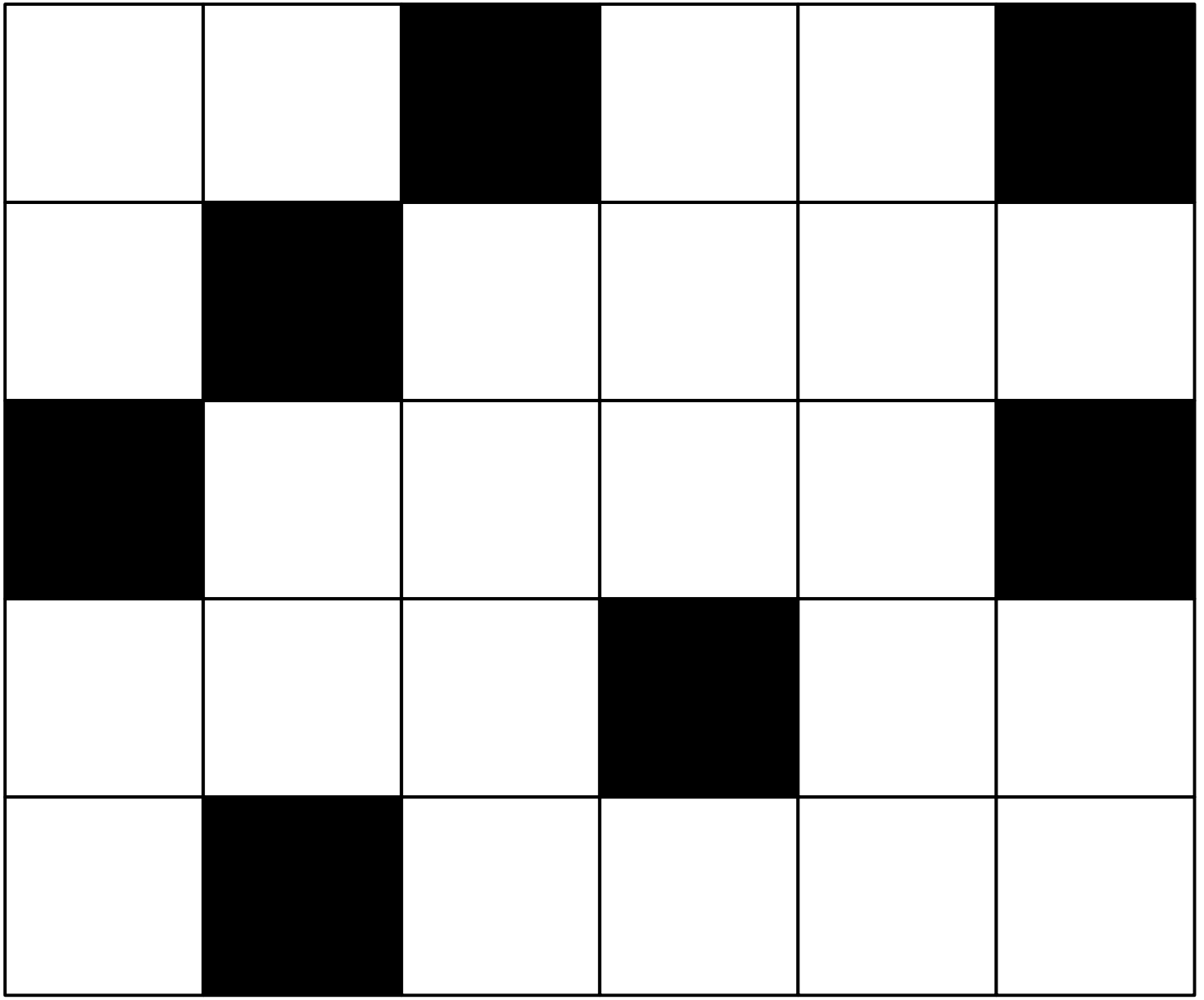


A maximum matching.

Domino Tiling



Solving Domino Tiling



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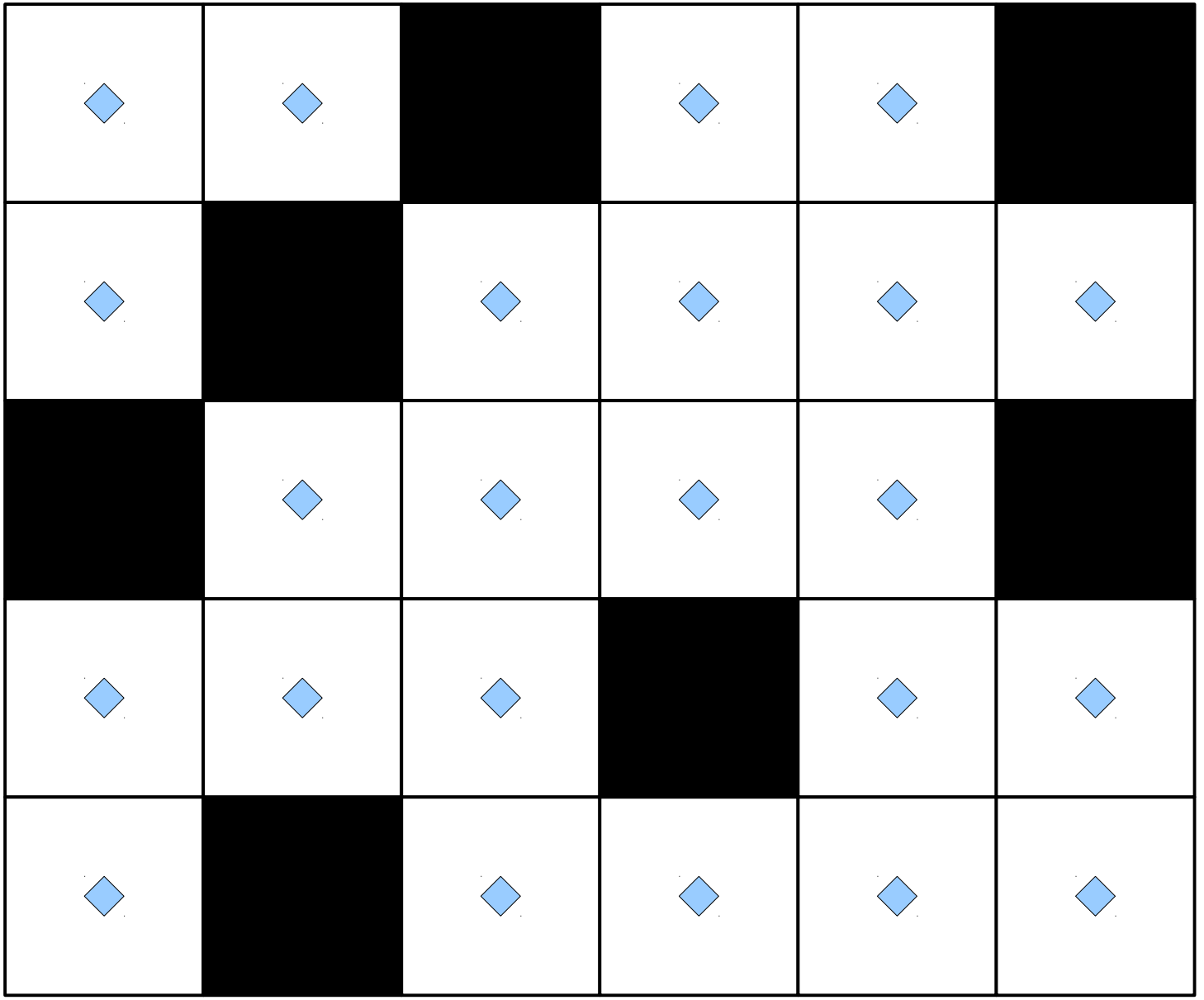

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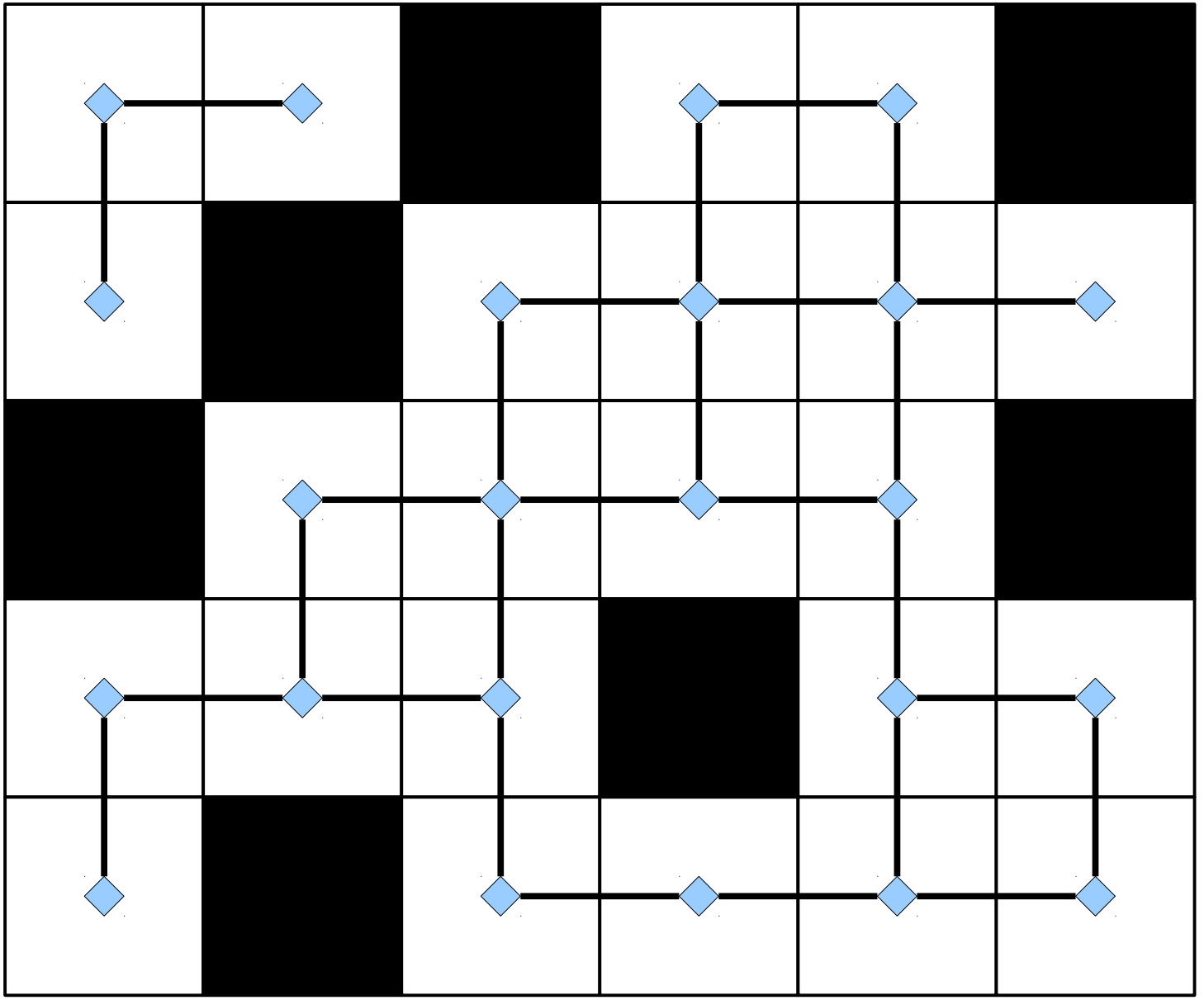

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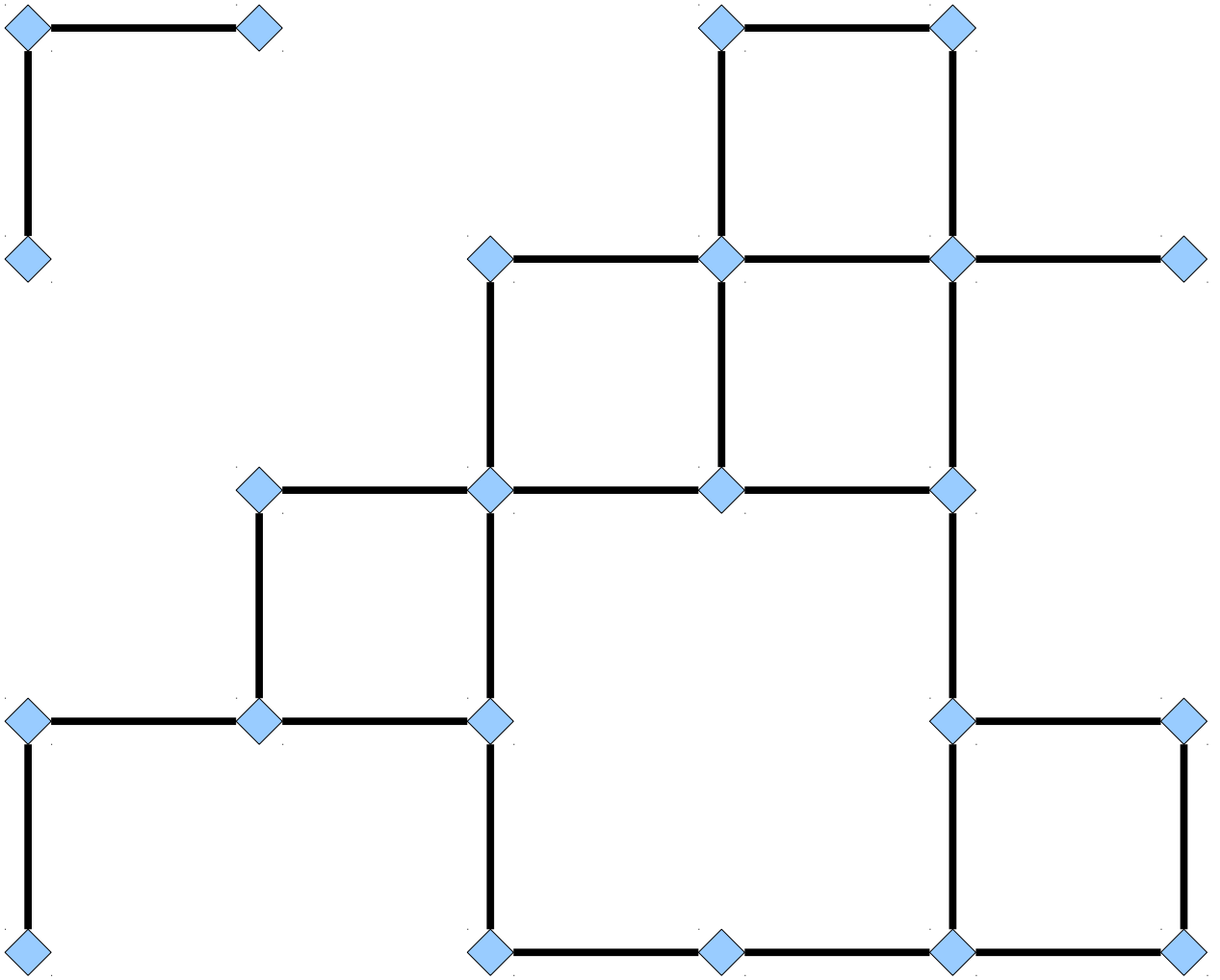

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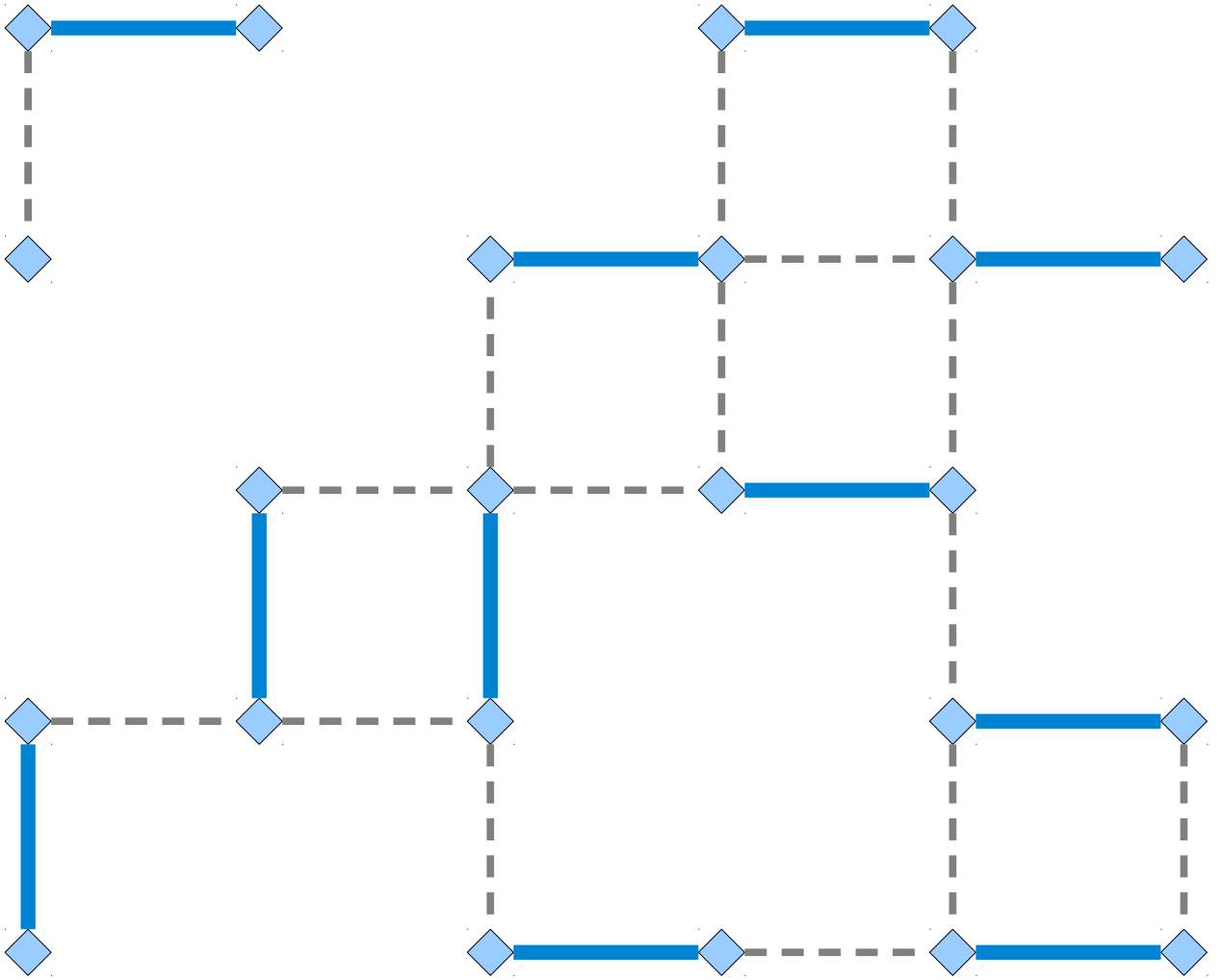

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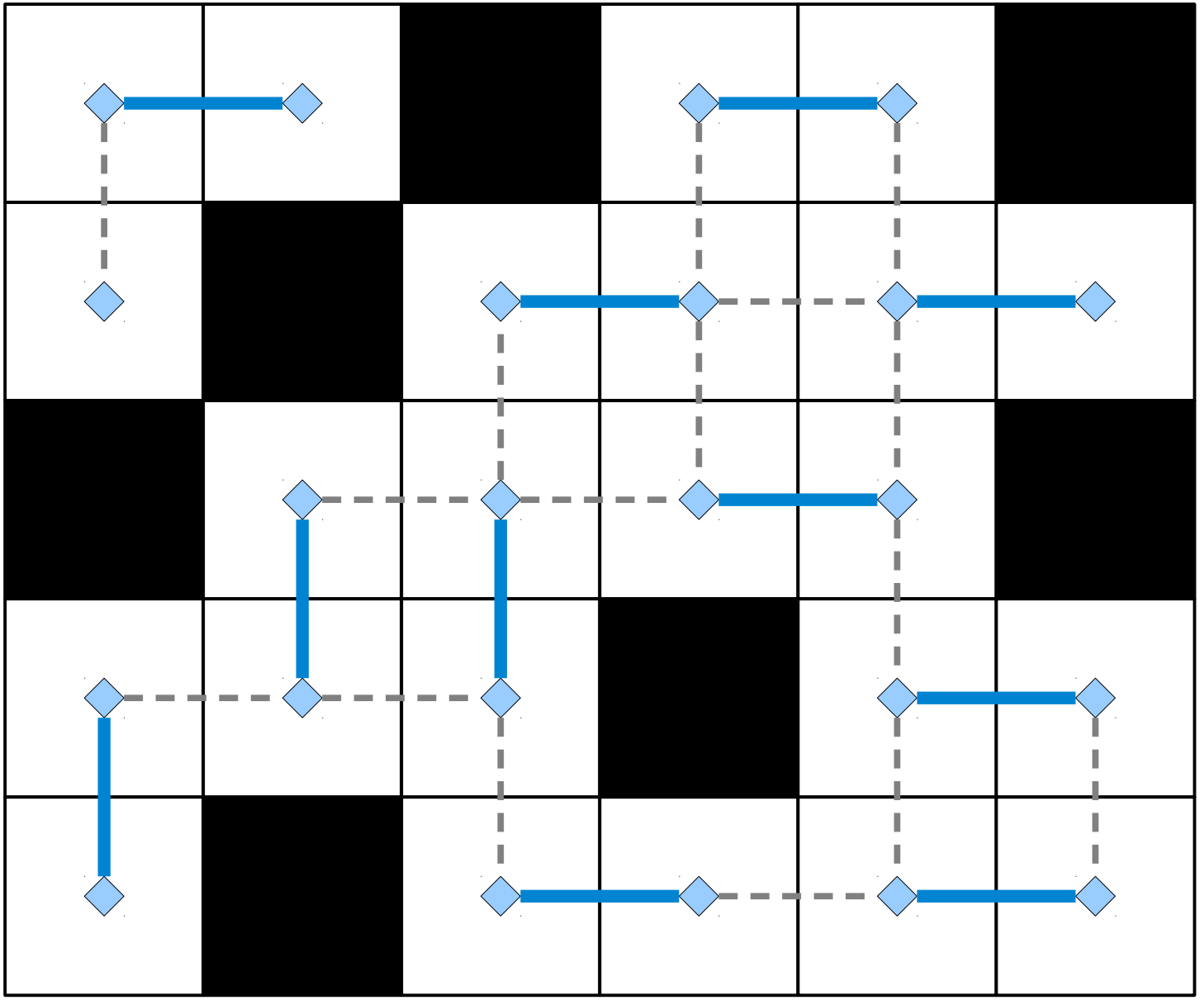

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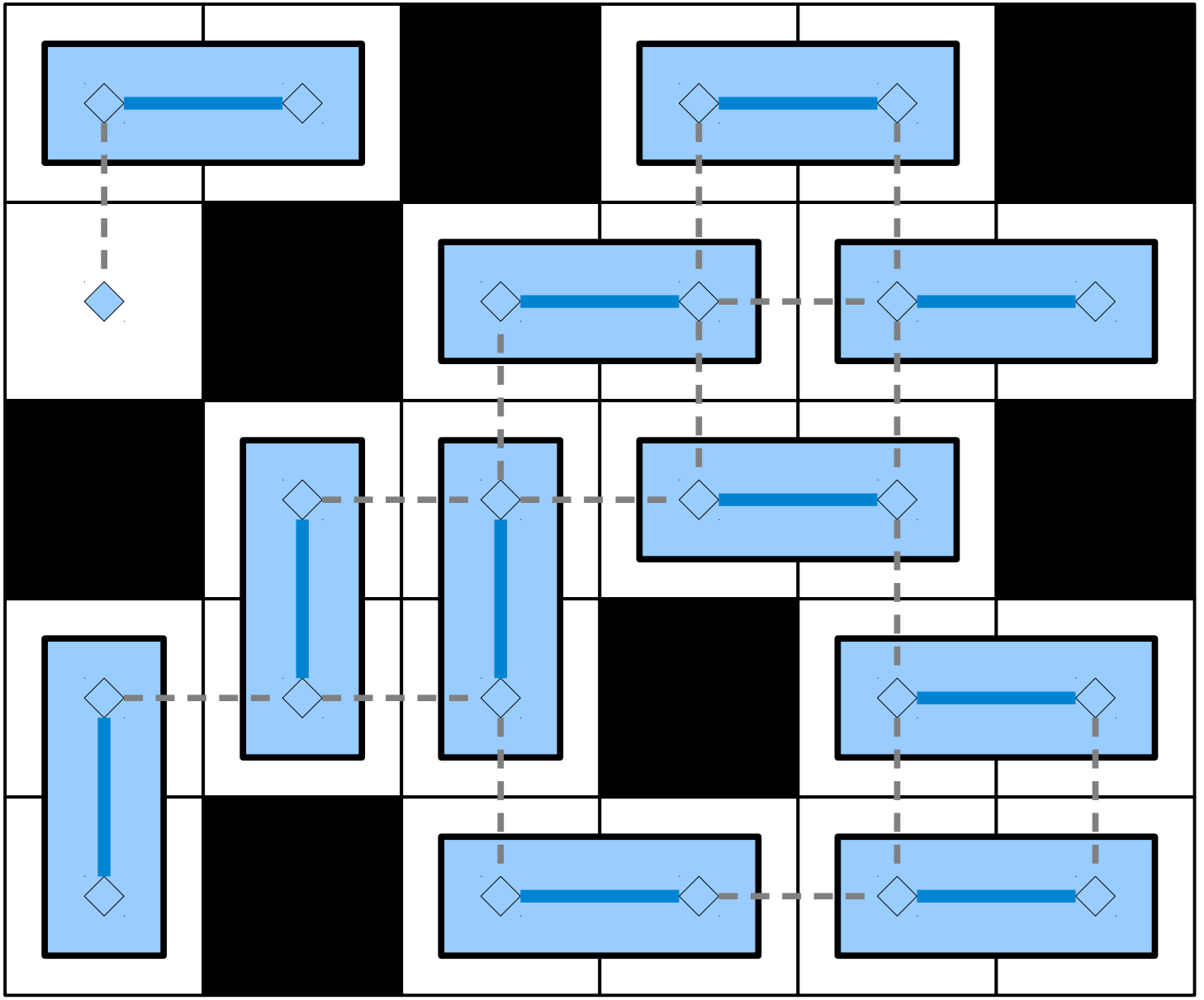

In Pseudocode

```
boolean canPlaceDominos(Grid G, int k) {
   return hasMatching(gridToGraph(G), k);
}
```

```
boolean canPlaceDominos(Grid G, int k) {
  return hasMatching(gridToGraph(G), k);
}
```

Based on this connection between maximum matching and domino tiling, which of the following statements would be more proper to conclude?

A. Finding a maximum matching isn't any more difficult (in BigO/P-NP terms) than tiling a grid with dominoes.

B. Tiling a grid with dominoes isn't any more difficult (in BigO/P-NP terms) than finding a maximum matching.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then ***A*** or ***B***.

Intuition:

Tiling a grid with dominoes can't be “harder” than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

A New Example

Reachability

- Consider the following problem:

  Given an directed graph *G* and nodes *s* and *t* in *G*, is there a path from *s* to *t*?

- It's known that this problem can be solved in polynomial time (use DFS or BFS).
- Given that we can solve the reachability problem in polynomial time, what other problems can we solve in polynomial time?

Converter Conundrums

- Suppose that you want to plug your laptop into a projector.
- Your laptop only has a VGA output, but the projector needs HDMI input.
- You have a box of connectors that convert various types of input into various types of output (for example, VGA to DVI, DVI to DisplayPort, etc.)
- ***Question:*** Can you plug your laptop into the projector?

Converter Conundrums

Connectors

RGB to USB
VGA to DisplayPort
DB13W3 to CATV
DisplayPort to RGB
DB13W3 to HDMI
DVI to DB13W3
S-Video to DVI
FireWire to SDI
VGA to RGB
DVI to DisplayPort
USB to S-Video
SDI to HDMI

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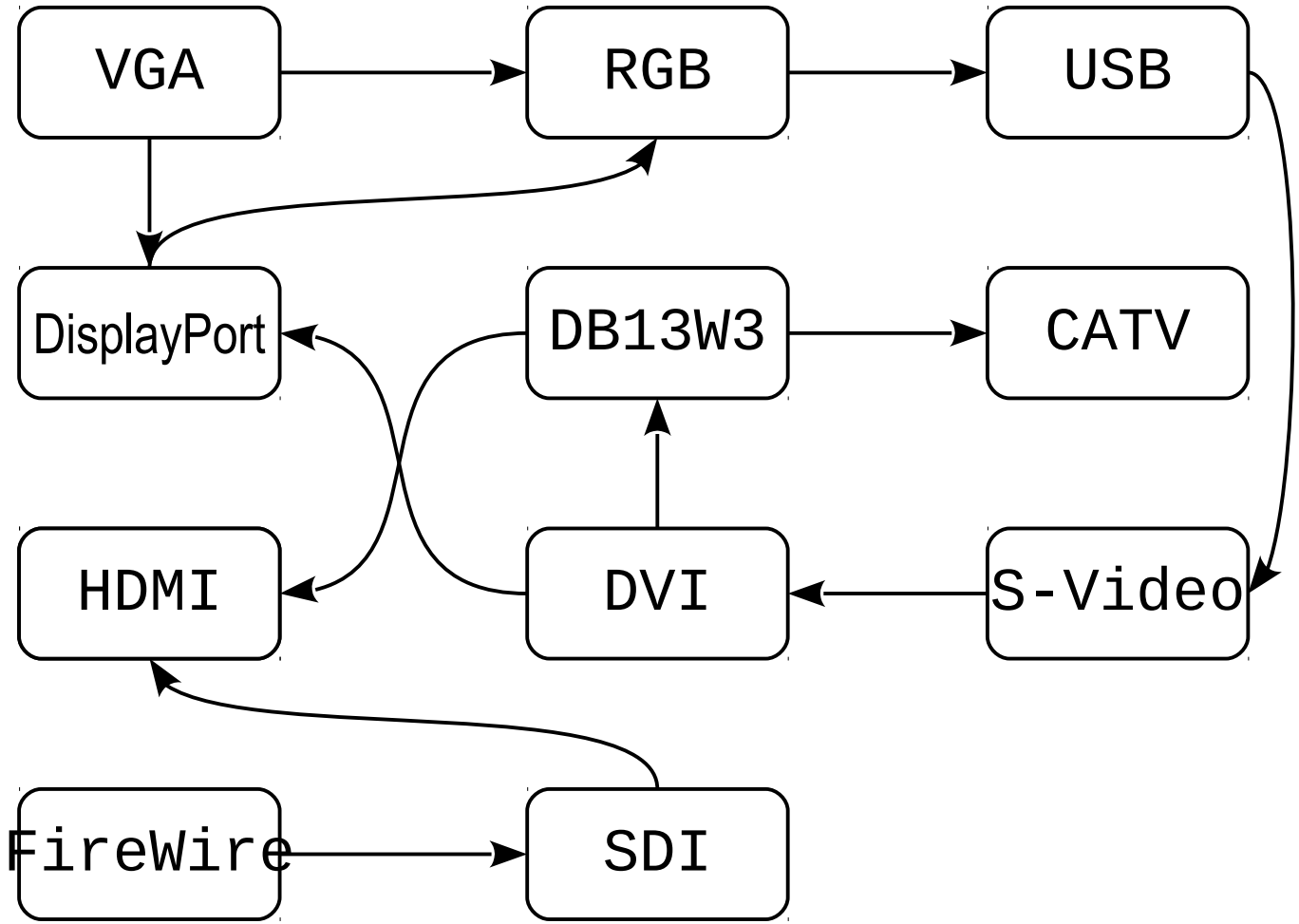

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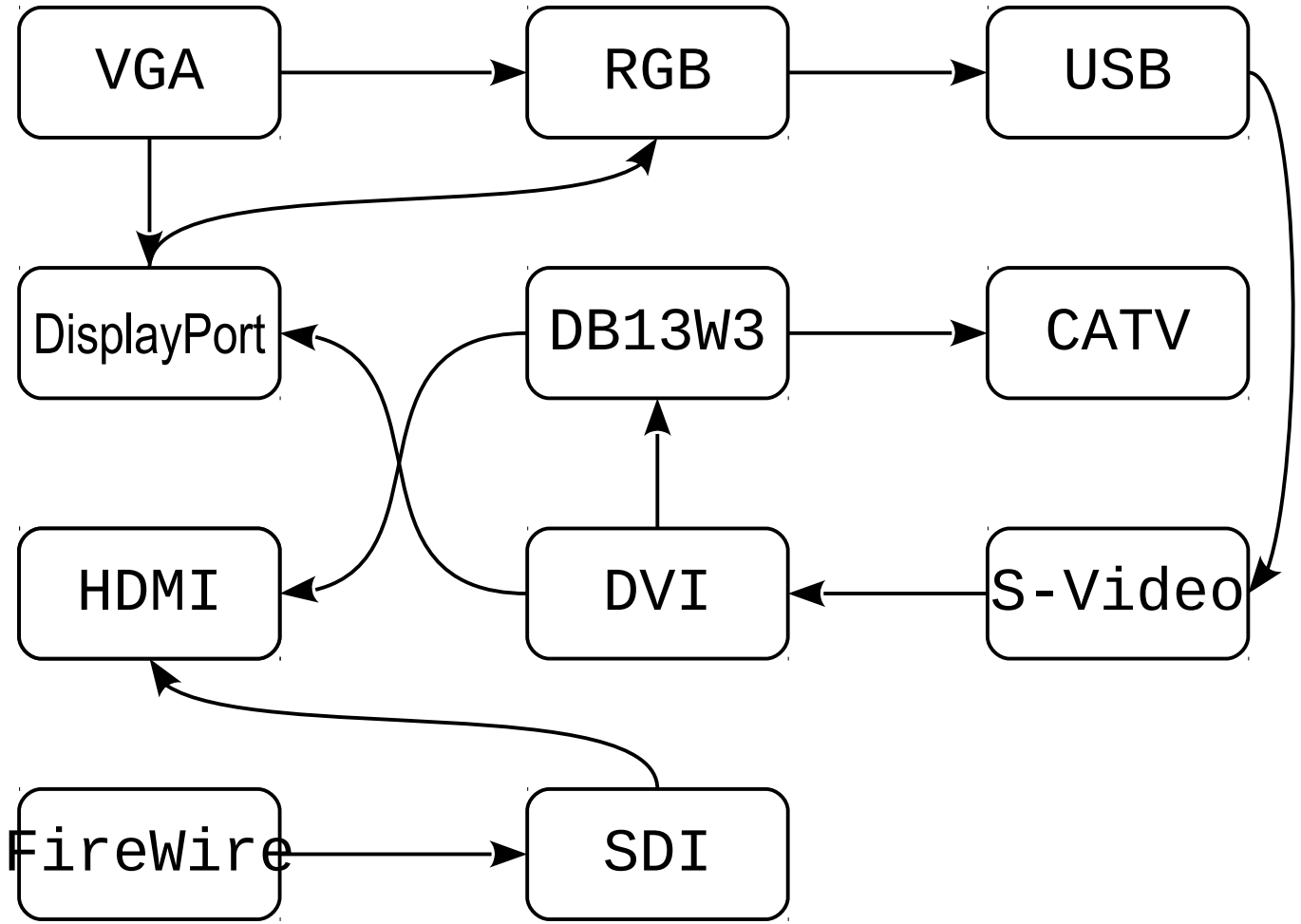

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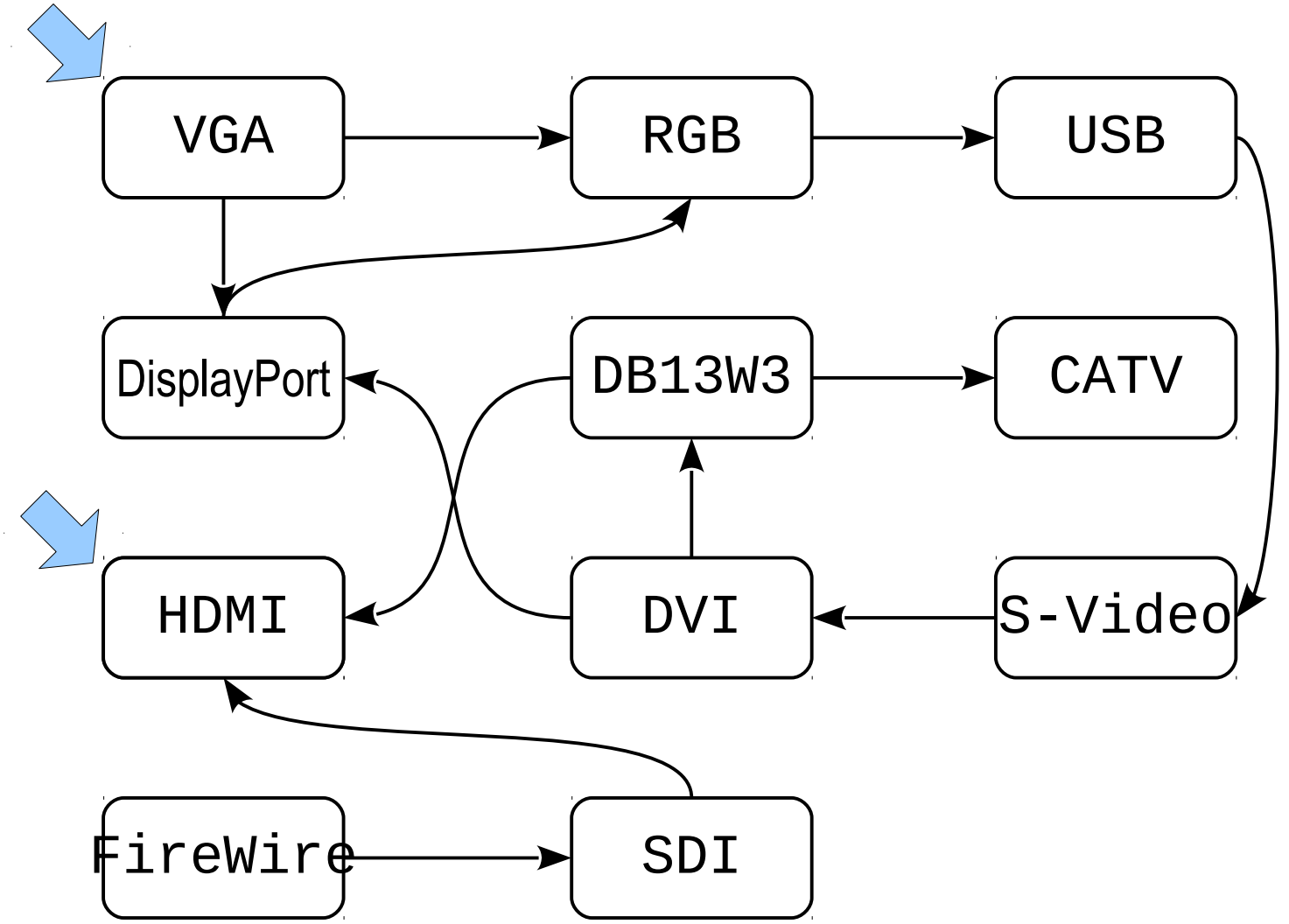

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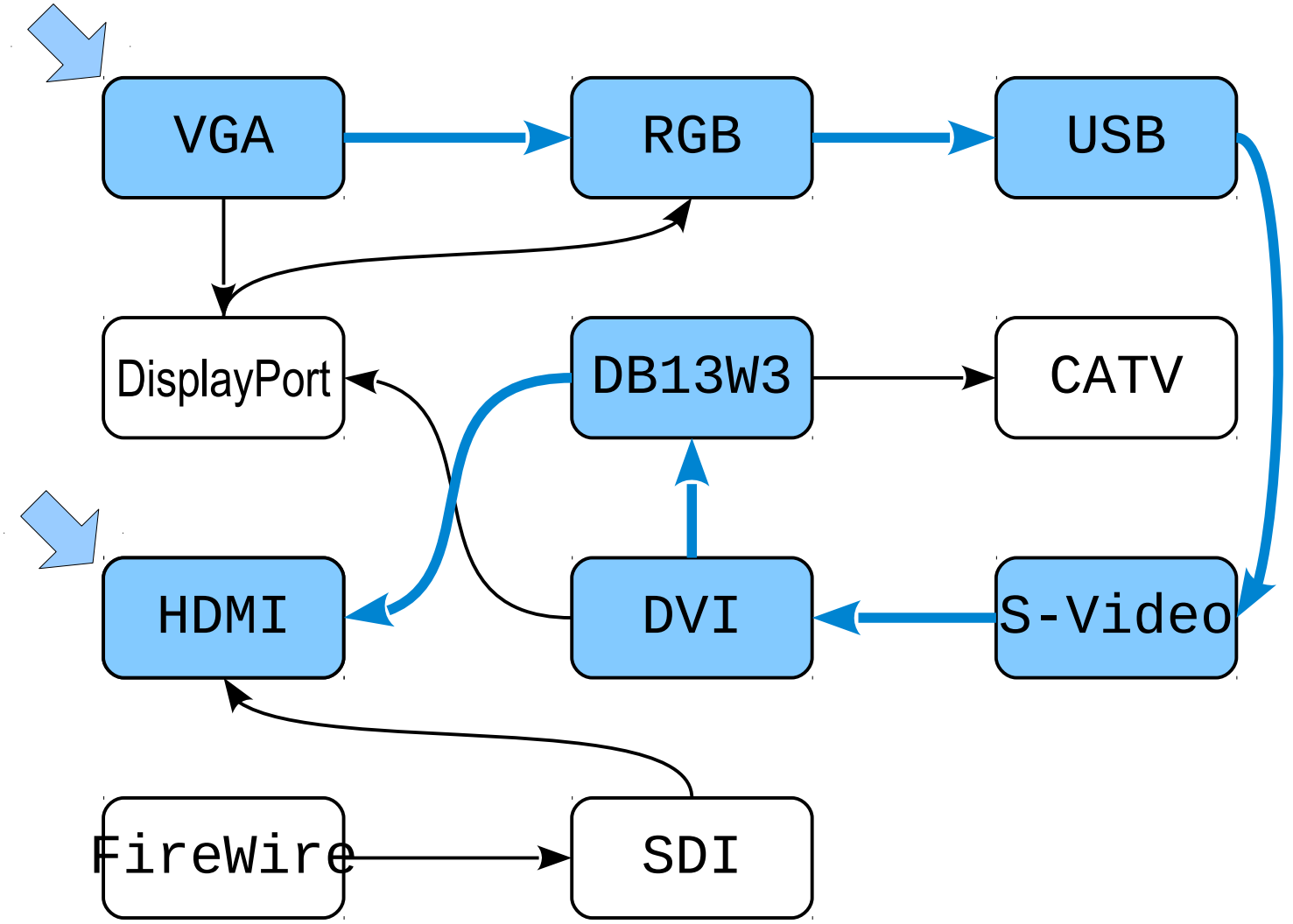

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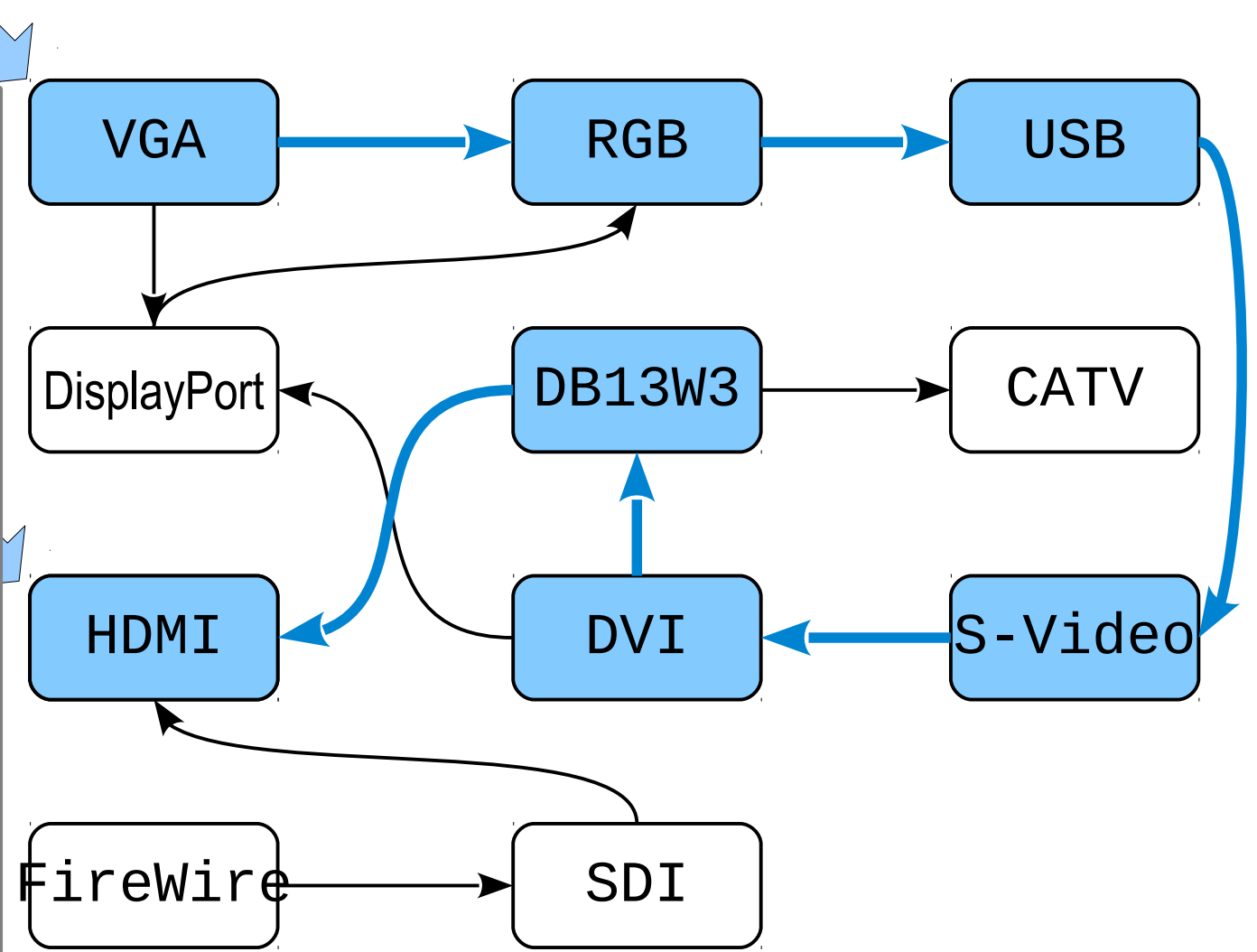

In Pseudocode

```
boolean canPlugIn(List<Plug> plugs) {
   return isReachable(plugsToGraph(plugs),
                      VGA, HDMI);
}
```

Based on this connection between plugging a laptop into a projector and determining reachability, which of the following statements would be more proper to conclude?

A. Plugging a laptop into a projector isn't any more difficult that computing reachability in a directed graph.

B. Computing reachability in a directed graph isn't any more difficult than plugging a laptop into a projector.

Answer at **PollEv.com/cs103** or
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Intuition:

Finding a way to plug a computer into a projector can't be “harder” than determining reachability in a graph, since if we can determine reachability in a graph, we can find a way to plug a computer into a projector.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

Intuition:

Problem *A* can't be "harder" than problem *B*, because solving problem *B* lets us solve problem *A*.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

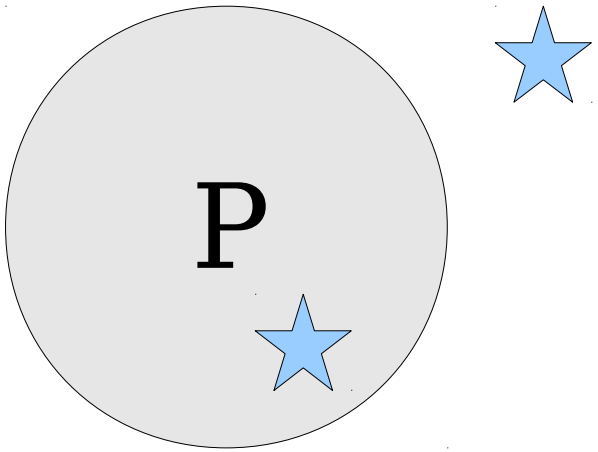
$$A \leq_{\mathrm{p}} B.$$

- We say that ***A is polynomial-time reducible to B***.

* Assuming that `transform` runs in polynomial time.

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.



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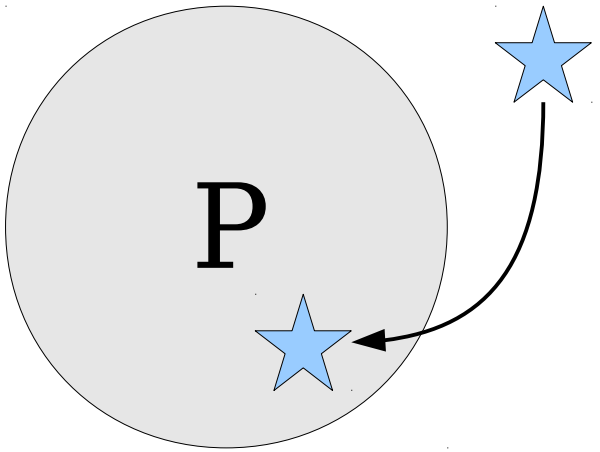

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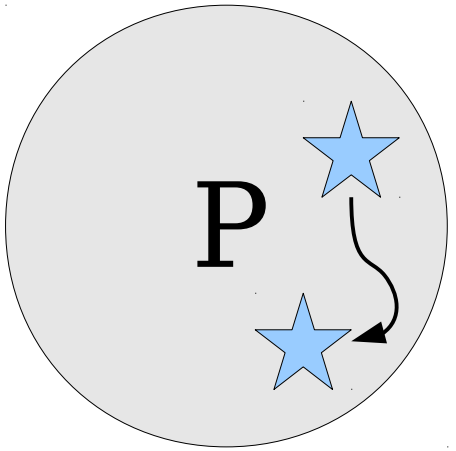

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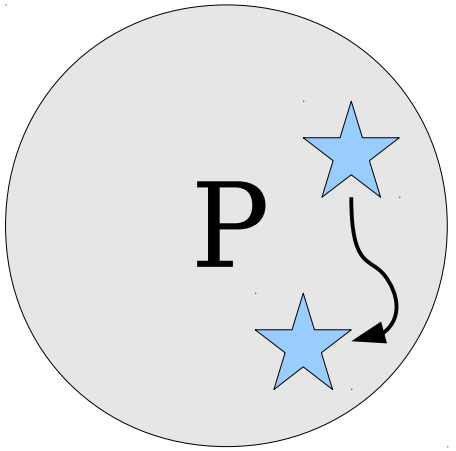

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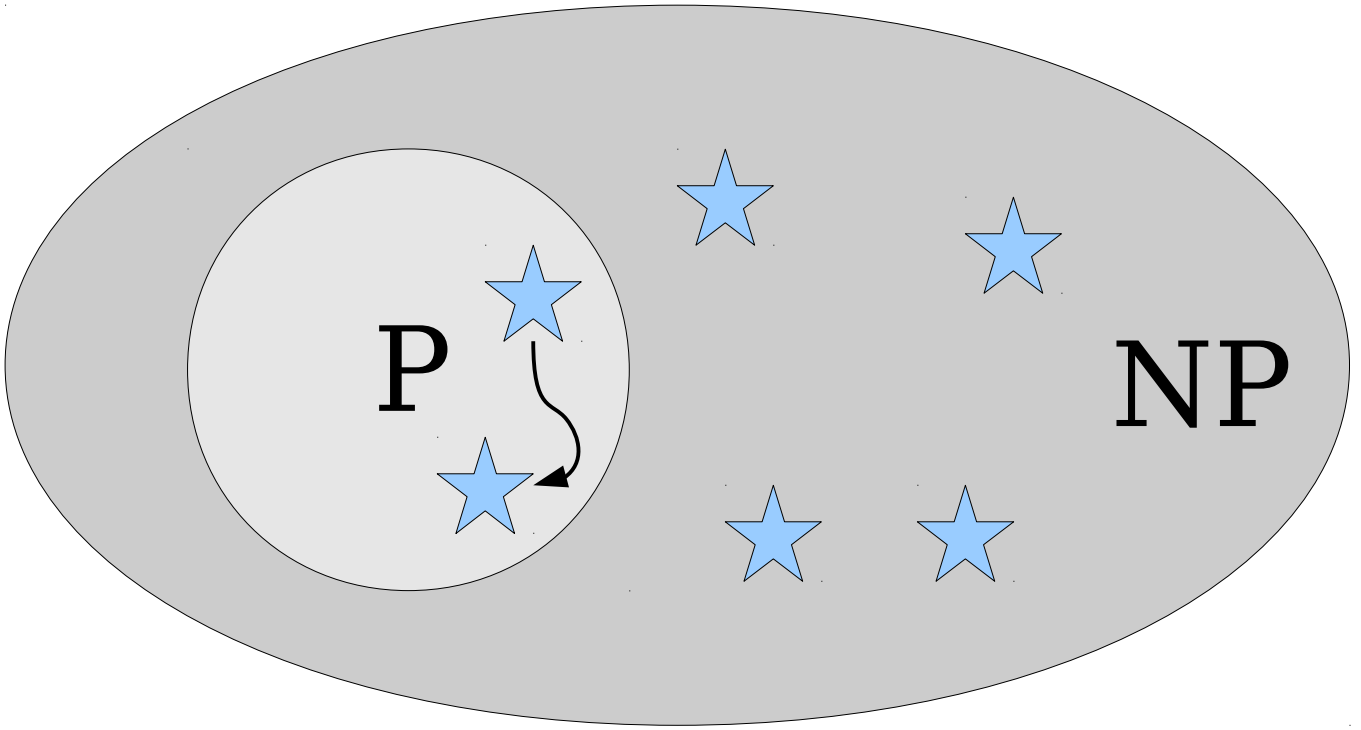

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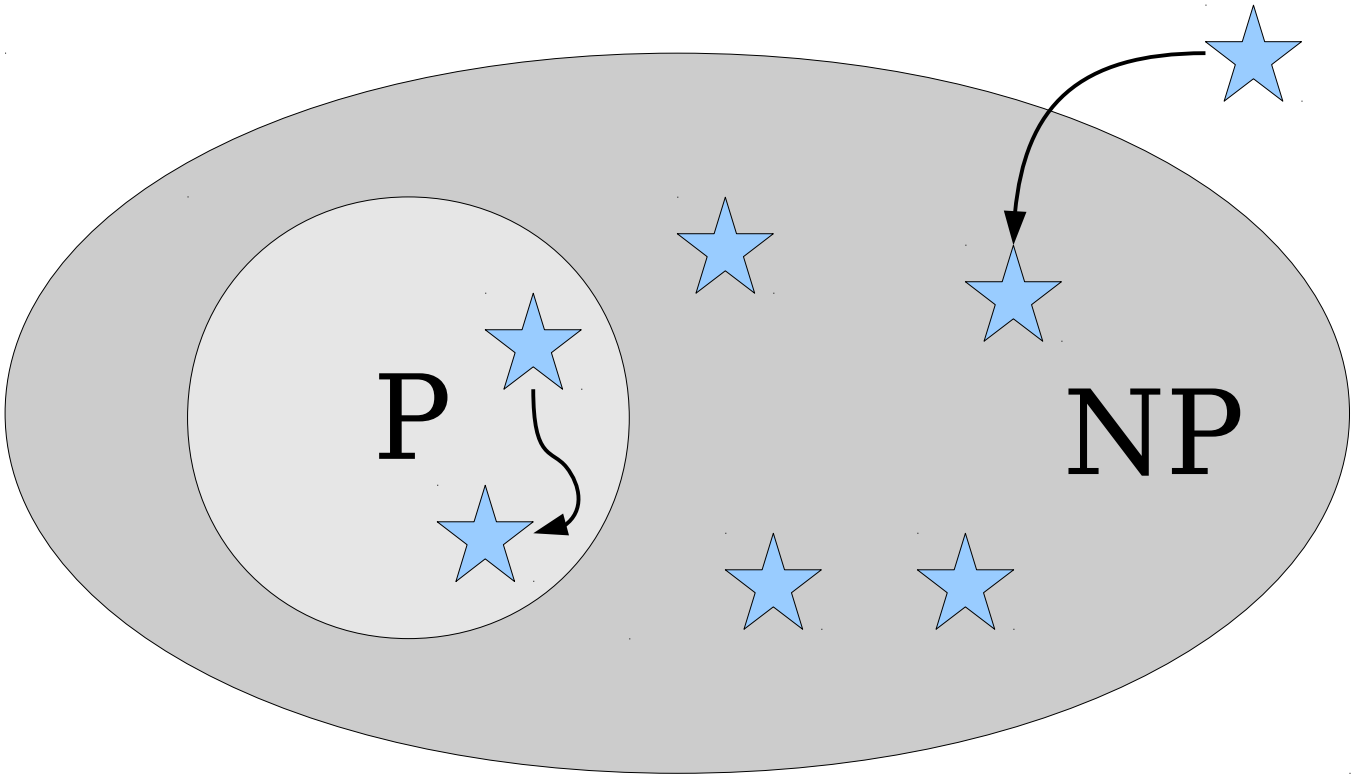

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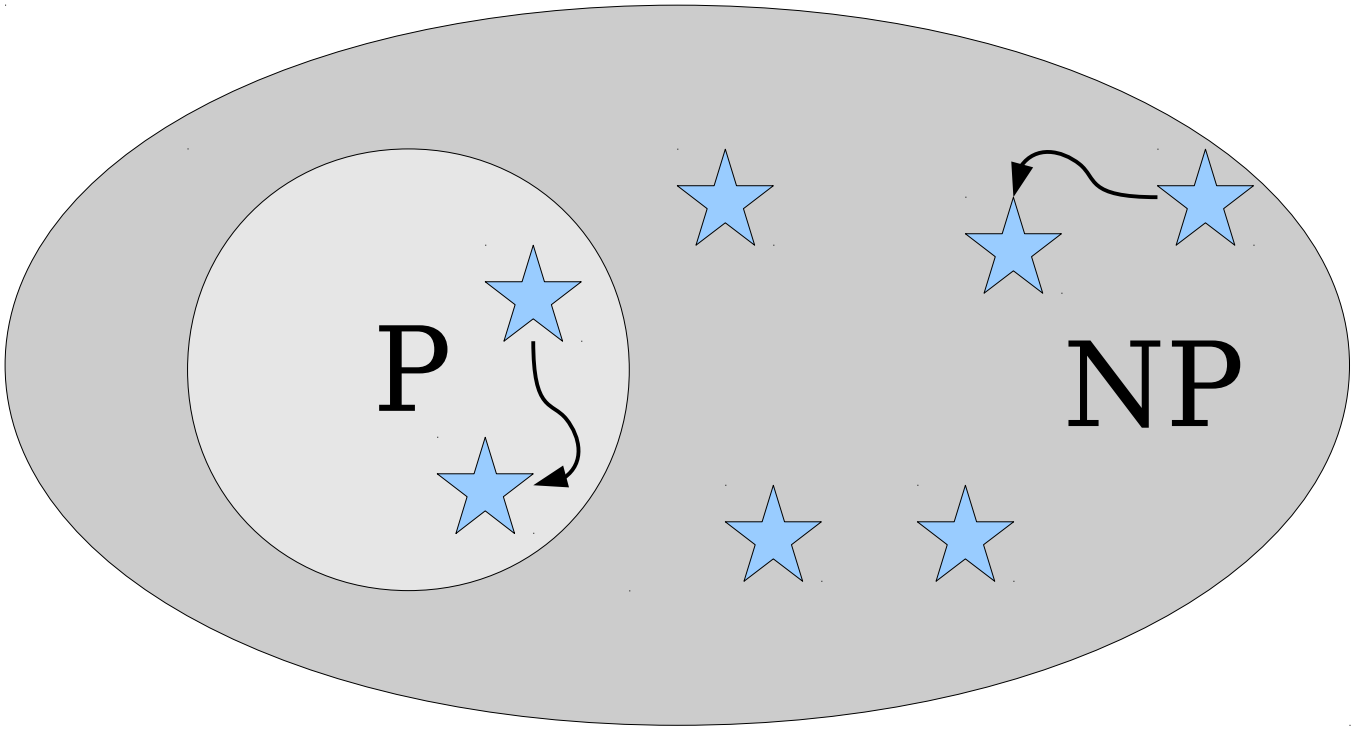


This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

NP-Hardness and **NP**-Completeness

Question: What makes a problem hard to solve?

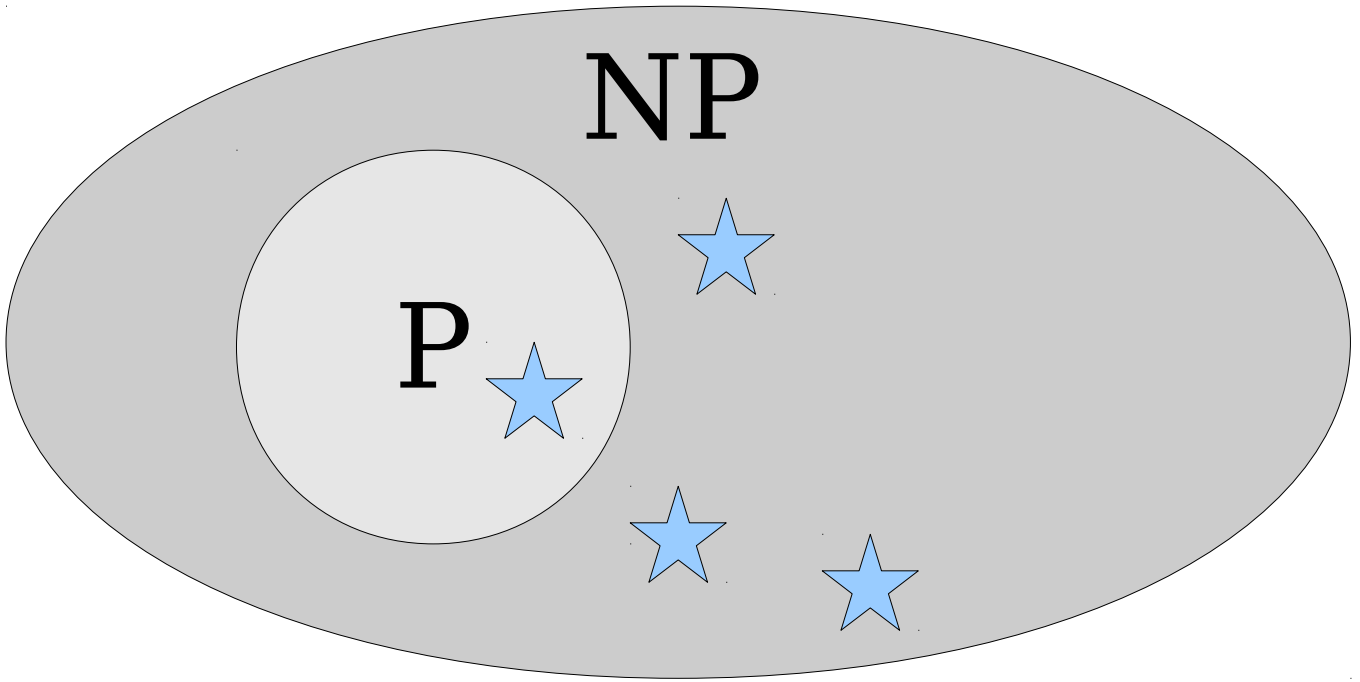
Intuition: If $A \leq_p B$, then problem B is at least as hard* as problem A.

* for some definition of “at least as hard as.”

Intuition: To show that some problem is hard, show that lots of other problems reduce to it.

NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in \mathbf{NP}$, we have $A \leq_{\mathrm{P}} L$.



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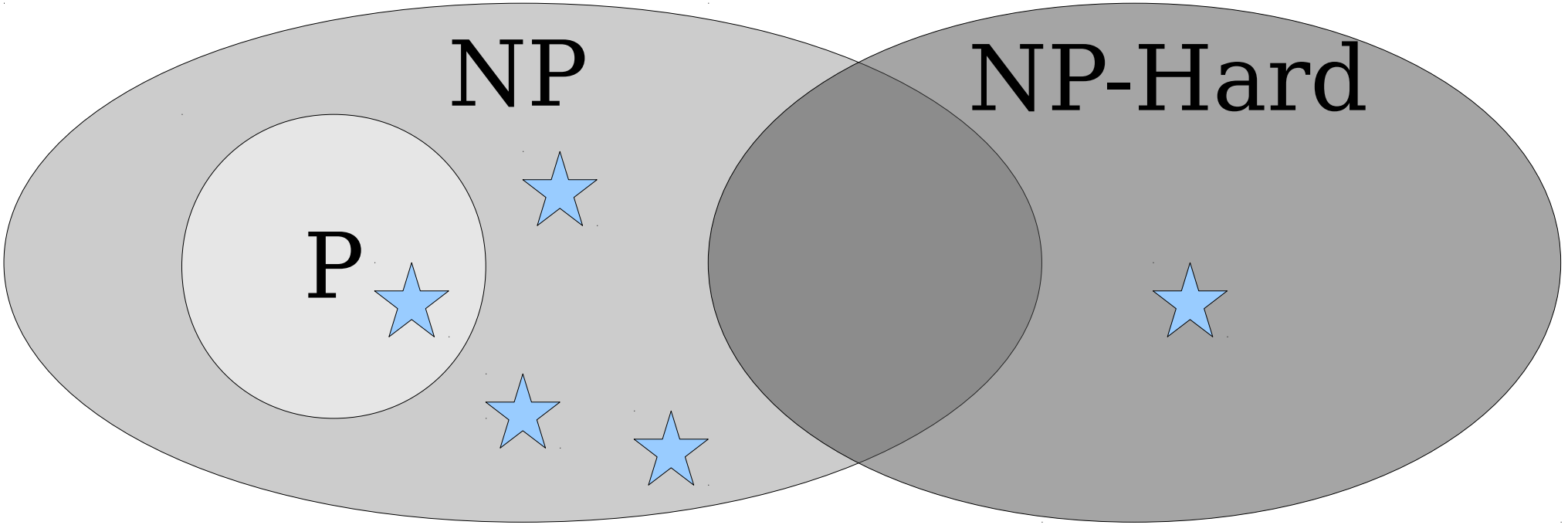

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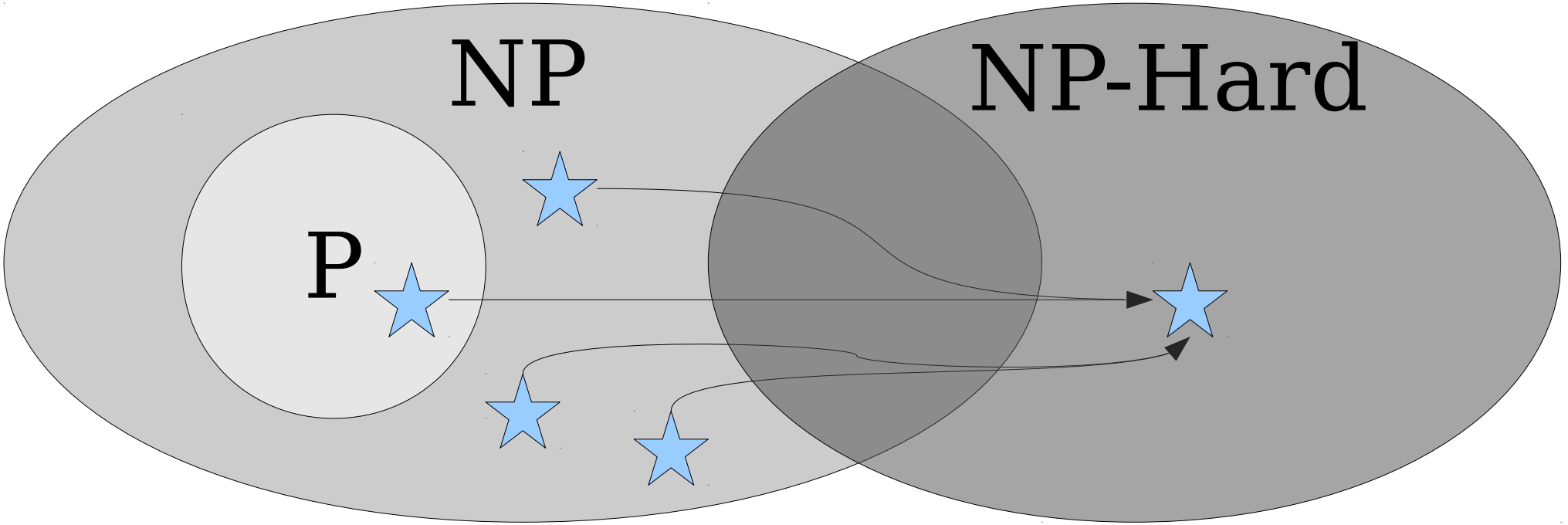
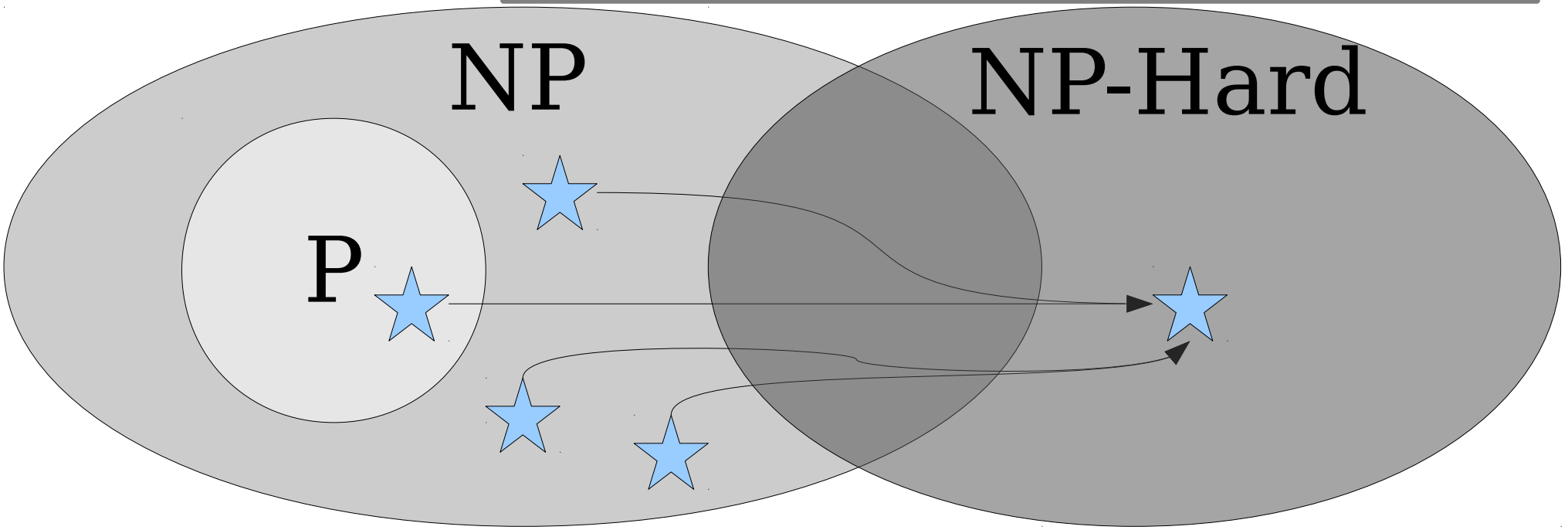


NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in \mathbf{NP}$, we have $A \leq_P L$.

Intuitively: L has to be at least as hard as every problem in **NP**, since an algorithm for L can be used to decide all problems in **NP**.



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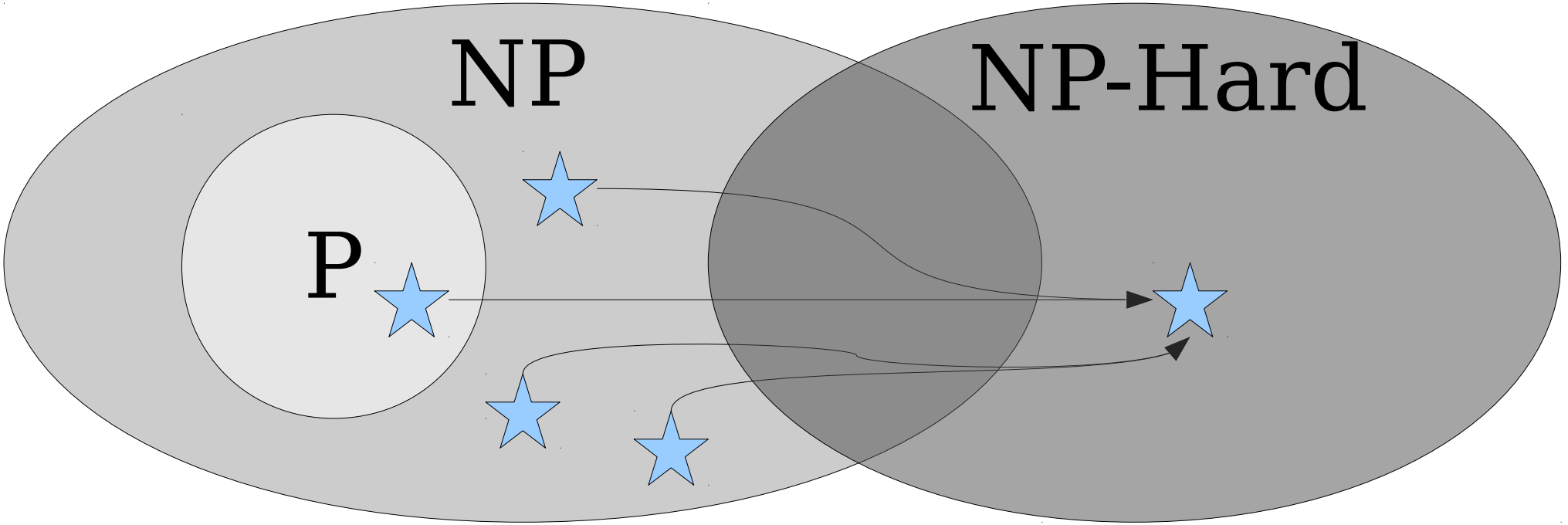

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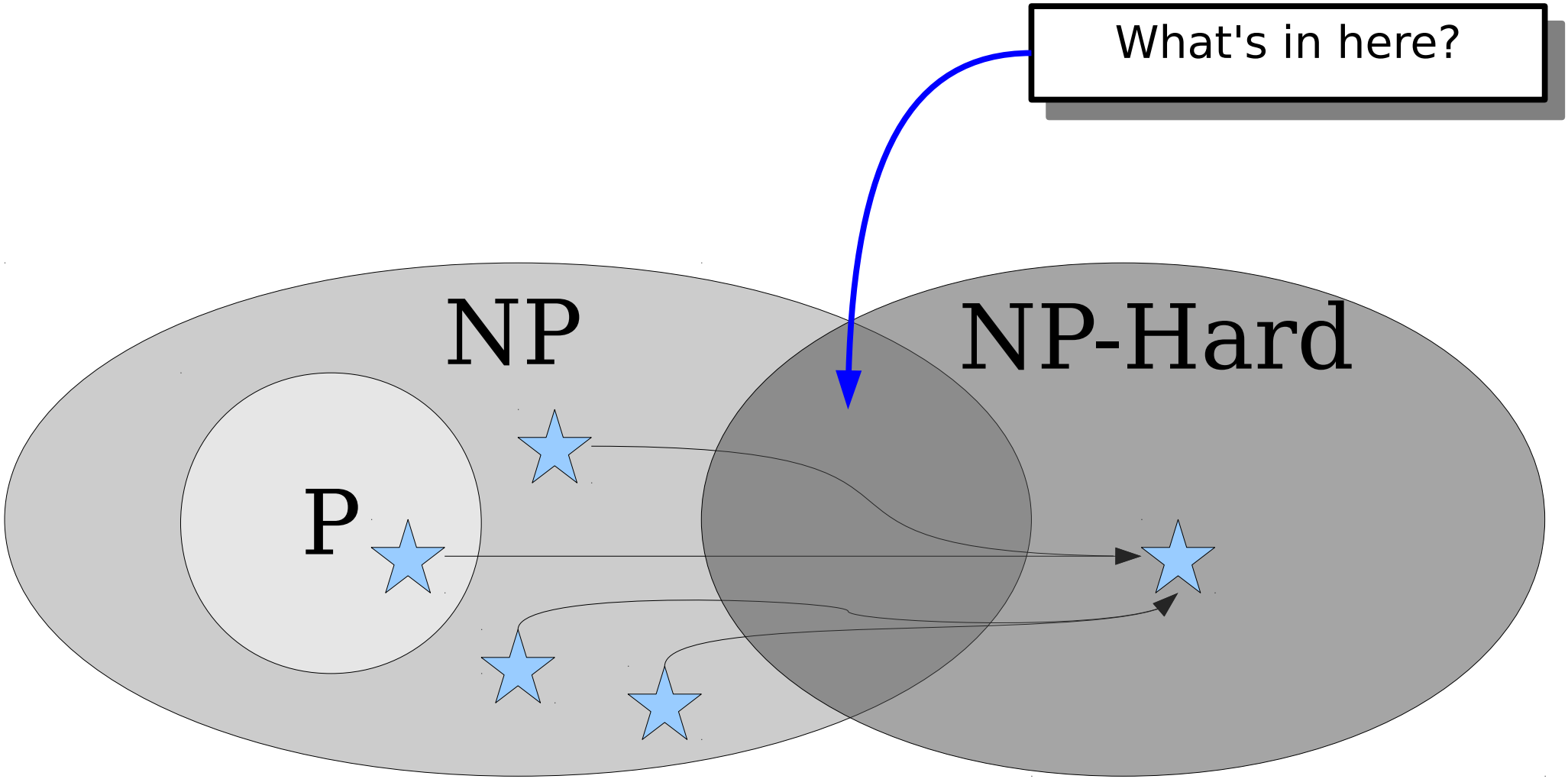
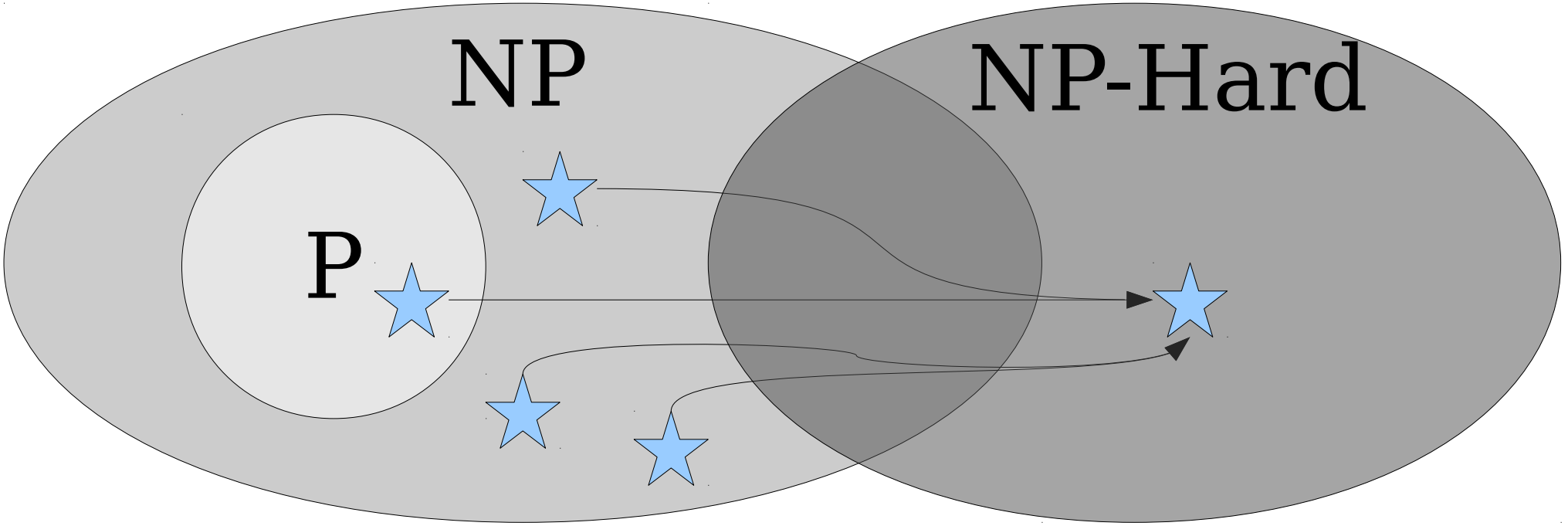


NP-Hardness

- A language L is called ***NP-hard*** if for *every* $A \in \mathbf{NP}$, we have $A \leq_{\mathrm{P}} L$.
- A language in L is called ***NP-complete*** if L is **NP**-hard and $L \in \mathbf{NP}$.
- The class ***NPC*** is the set of **NP**-complete problems.



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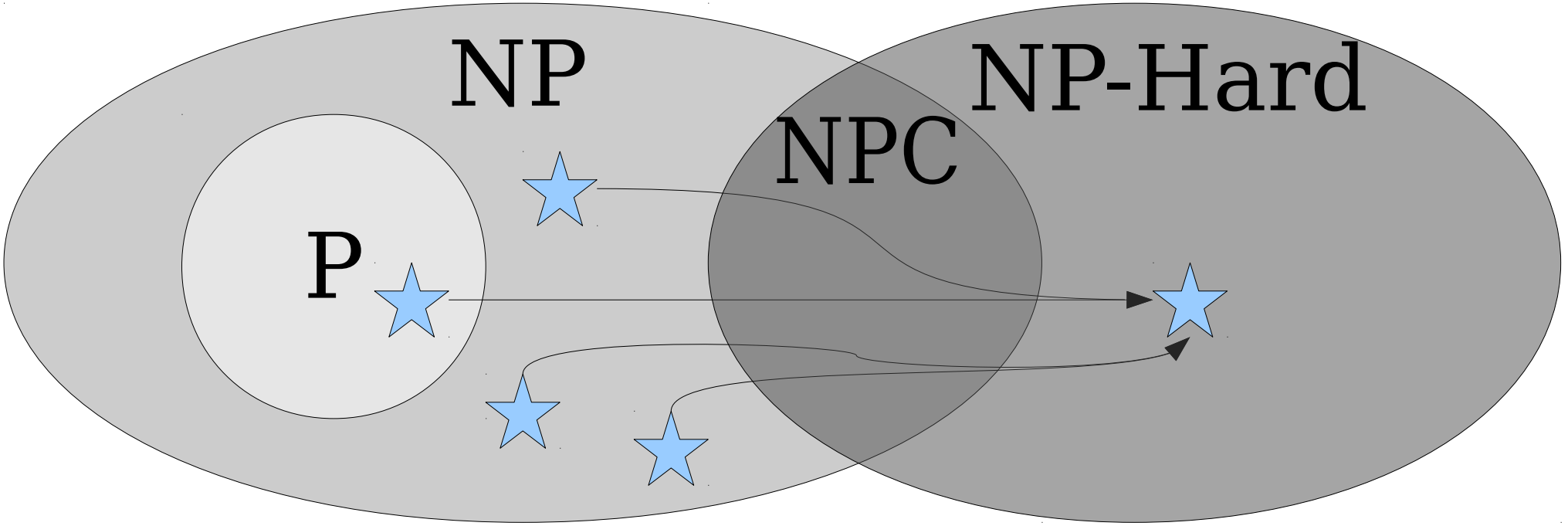

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

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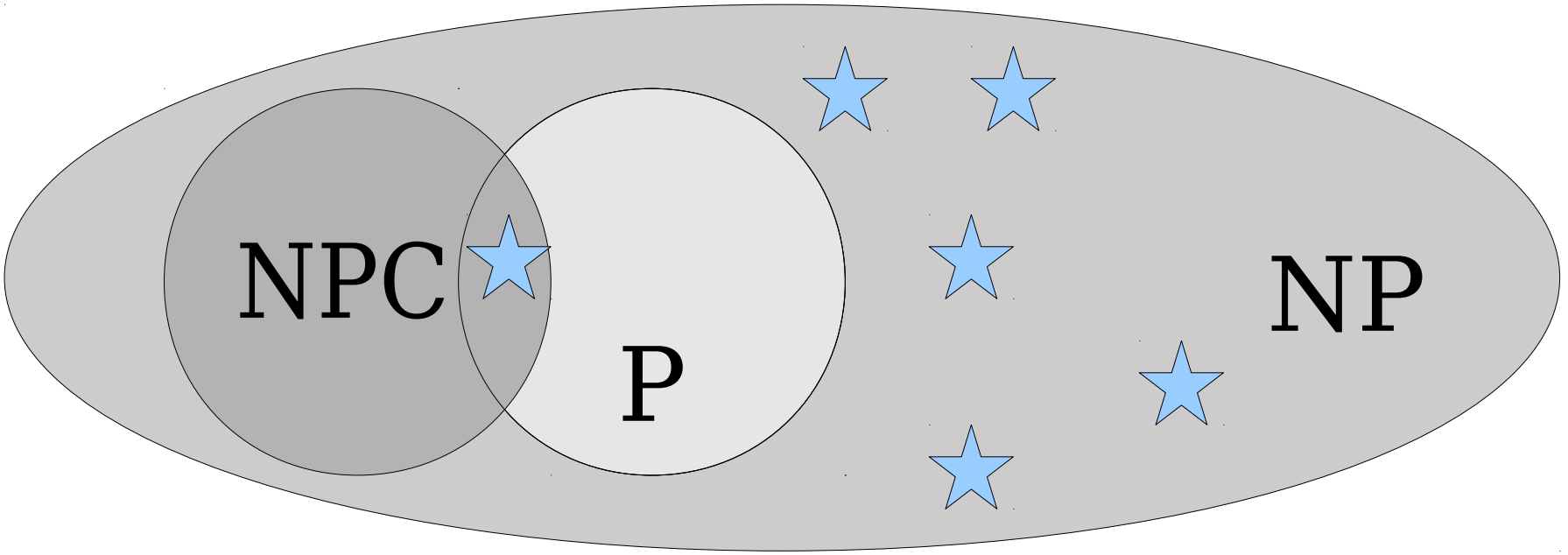
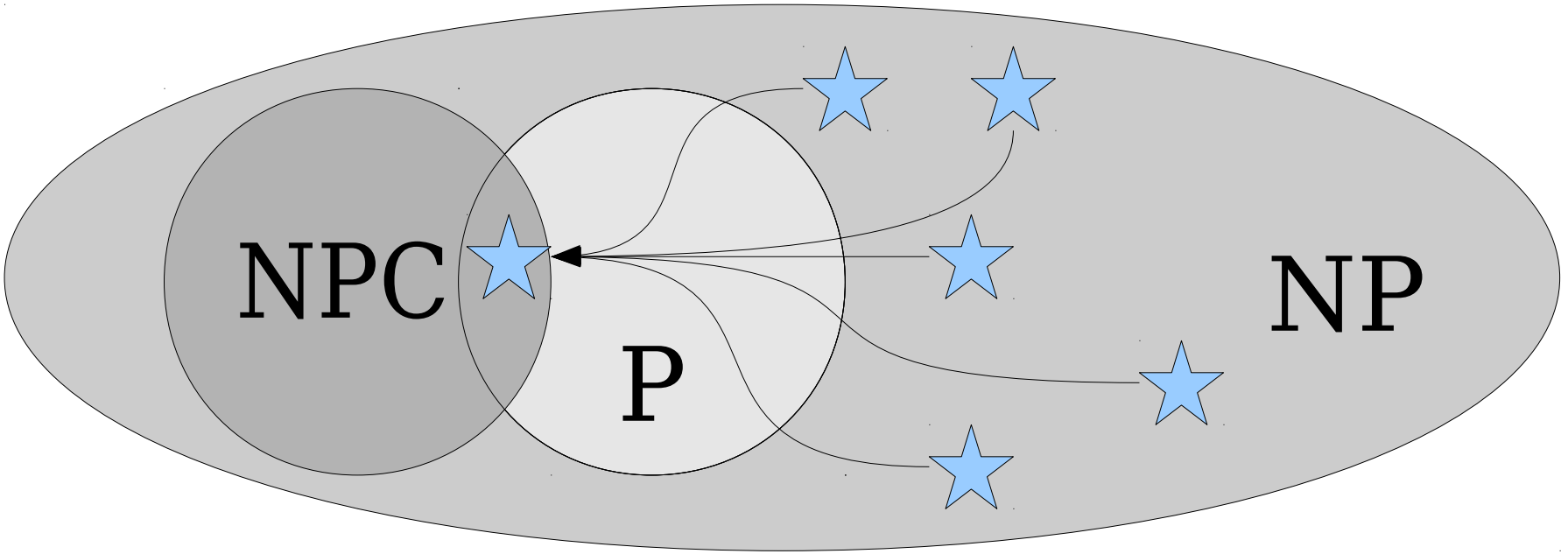


The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.



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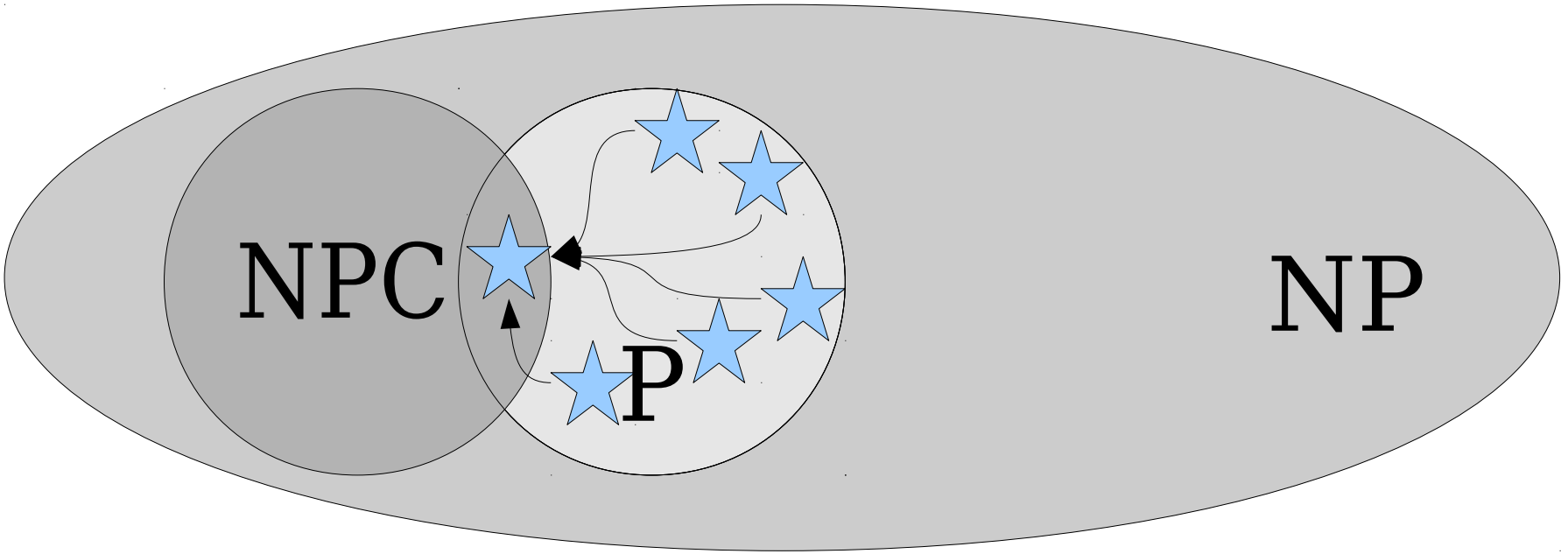

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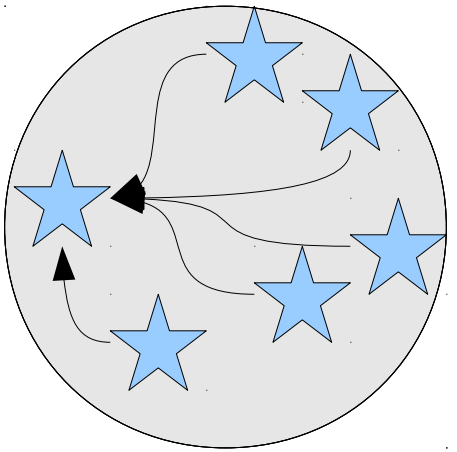
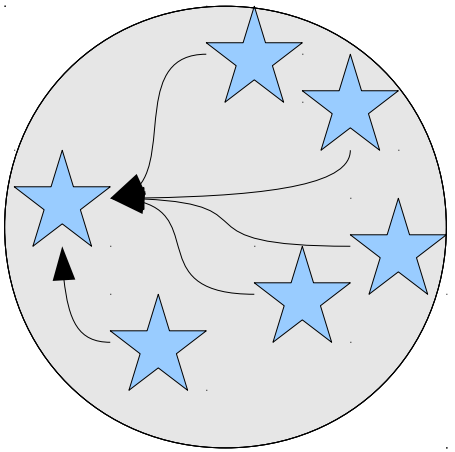


P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Proof: Suppose that L is **NP**-complete and $L \in \mathbf{P}$. Now consider any arbitrary **NP** problem A. Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in \mathbf{P}$ and $A \leq_p L$, we see that $A \in \mathbf{P}$. Since our choice of A was arbitrary, this means that $\mathbf{NP} \subseteq \mathbf{P}$, so **P** = **NP**. ■



P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in$ **NP**. Therefore, we know that $L \in$ **NP** and $L \notin$ **P**, so **P** ≠ **NP**. ■