Ten Techniques to Get Unstuck

Learning to write proofs can be a bit intimidating, especially when you're just getting started. You'll be given some problem to solve, where you might not even have the faintest idea what's being asked of you, and your goal is to produce a polished argument explaining the solution to that problem. Many first-time proofwriters imagine that the process for writing a proof looks something like this:

1. Read the problem set question.

2. Sit silently and ponder, possibly while saying "*hmmm*" and "*very interesting*."

3. Exclaim "*Eureka!*"

4. Write out a masterpiece of a proof, the kind that brings tears to the eyes of its readers.

I'll be the first to admit that the format of the lectures for CS103 can sometimes feel like this. I'm up in front talking about a bunch of theorems and slowly revealing the proofs of those results. What you're not seeing me do is go through the real proofwriting process, which looks like this:

1. Read the problem set question.

2. Panic a little when you realize you don't fully get what it's saying.

3. Start jotting things down on a sheet of paper. Make lists of things you're assuming and things you're trying to prove.

4. Review notes about all the relevant terms and definitions. Jot down related theorems that we might need and look over proofs of results kinda like the one that we're doing.

5. Throw everything at the problem. Draw pictures. Try smaller cases. Work backwards and forwards at the same time and hope you can meet in the middle.

6. Realistically, crumple up a bunch of paper and go back to step 3. Hopefully, go to step 7.

7. Write out a very rough draft of the proof, showing the argument that I came up with.

8. Realistically, in the course of doing so, discover that something doesn't quite feel right and find a flaw in the reasoning, going back to step 3. Hopefully, go to step 9.

9. Rewrite the proof and clean it up a bit.

10. Let the proof sit for a day.

11. Reread the proof and rewrite it yet again to clean it up even more.

12. Hand the third draft of the proof to a partner to read. Ignore partner's grimaces and other facial expressions as they try to decipher your reasoning.

13. Talk through the reasoning with your partner and clean up the rough parts. Realistically, go back to step 11. Hopefully, go to step 14.

14. Done!

This handout consists of a number of specific steps to take if you find that you're stuck in the course of writing a proof. If you ever find yourself staring at a blank sheet of paper, grab this handout, pick some items on it, and see if you can use those techniques to make a little bit of progress.

Technique One: Articulate a Clear Start and End Point

Imagine that a friend asks you for driving directions from Stanford to Lava Beds National Monument.[1] If you give her directions from Sacramento to Lassen Volcanic National Park,[2] while you're doing a nice thing, you're not actually doing what was asked.

Or imagine that a friend is interested in making soondubu jjigae[3] and asks you for instructions. If you tell him how to prepare the most wonderful fattoush[4] they've ever tasted, you're going to come across as kinda clueless.

And imagine, for example, that you're tasked with building a bridge across a particular treacherous river. If you go and build a bridge across the Atlantic Ocean, while everyone is going to be quite impressed, the folks trying to cross the river are going to be mighty upset that they're still stranded.

Proofwriting is in many ways analogous to the above scenarios. In a proof, you begin with a set of starting assumptions and try to craft an argument that proceeds in logical steps to arrive at a destination. If you don't have a clear sense of where you're starting and where you're going to end, chances are that you're not going to be able to write a good proof – and there's a possibility that you'll end up proving the wrong thing!

When confronted with a theorem to prove, the first step is to make sure you understand where you're starting and where you're going. The good news is that in most cases, you can look at the structure of the claim that needs to be proved and figure out what you need to show.

When writing out a proof, we recommend that you start with a blank sheet of paper and make two columns in it, like this:

What I'm assuming	*What I need to show*

Before you do anything else, start off by filling in these columns. To do so, look at the structure of the statement that you're trying to prove.

- If you're proving an *implication* of the form "if P is true, then Q is true," you *assume* that P is true, and you *need to show* that Q is true.

- If you're proving a *universally-quantified statement* of the form "for any choice of x, property P holds for x," you *assume* that x is some arbitrarily-chosen value, and you *need to show* that property P is true for x.

- If you're proving an *existentially-quantified statement* of the form "there is an x where property P holds for x," you don't *assume* anything, and you *need to show* that there is indeed some choice of x out there that works.

---

[1] You absolutely should check this place out once lockdown ends. It's amazing.
[2] A wonderful weekend road trip destination from campus. Last time I checked, it had reopened to the public.
[3] A delicious stew that's surprisingly easy to make in a dorm.
[4] Perhaps the best salad in the world.

Technique Two: Write Down Relevant Terms and Definitions

Mathematics builds on itself, and some of the most impressive results in mathematics arise from applying existing theorems in new and interesting ways.

Once you have a clear sense of where to start and where to end, write down all of the following:

- For each relevant term, how is it defined? Are there any theorems about those terms that you learned about in lecture? Have you proved anything about those terms before?

- For each relevant theorem, how was that theorem proved? What did the setup look like? How was it executed? Were there any useful pictures or intuitions that made them work?

At this point, you will likely have *lots* of things written down. If not, that's okay. Go back and reread the lecture slides to see if anything useful pops out. Or look at the CS103A materials or course reader to see if there's something you can use.

There are several benefits to writing these things down. First, writing out formal definitions can help you get a much better sense of what you need to prove. If you're trying to prove something of the form $S \subseteq T$, then expanding out this definition to realize you need to prove that for any $x \in S$, you'll find $x \in T$ might cause you recognize that you need to introduce an arbitrarily-chosen variable $x \in S$ and then somehow reason why it must belong to T.

Second, this gives you a better sense of what existing tools are available. That way, if you get stuck later on, you can run down the list of existing terms, theorems, and intuitions and see if you can make any of them work.

Third, this helps you discover whether there are any blind spots in your own understanding. In the course of writing things down, perhaps you'll find that there's some concept or term that you didn't actually understand, in which case you should stop and study up on that term before continuing.

Finally, this approach helps you get a better sense of what sorts of intuitions you might want to use in the course of solving this problem. Perhaps there's a certain picture you might want to try drawing out, or perhaps there's a non-obvious way of rewriting things that you should try out.

Technique Three: Draw Pictures!

Many of the results we'll be talking about over the course of the quarter are easiest to understand when you can visualize what it is that you're talking about. Cantor's theorem from the first day of class talks about the sizes of infinitely large sets – objects that are too huge to be seen – by visualizing them as a grid and looking down the diagonal. The proofs we did with odd and even numbers were often motivated by drawing pictures of collections of squares in different configurations. Later in the quarter, as we explore binary relations, functions, and graphs, you'll often find that drawing the right picture can turn a seemingly impossible result about abstract mathematical objects into a rather intuitive result about circles, arrows, and lines.

If you're really stuck on a problem, it sometimes helps to try to draw a schematic representation of whatever it is that you're working on. If you're unsure what that picture might look like, feel free to look over the lecture slides and course notes to see if there's any existing visualization ideas that would work well for you.

Technique Four: Try Small Cases

Many results in discrete mathematics are sweeping statements of the form "*absolutely every* object of type X will have some property Y," and when you're trying to prove results like these it can seem a bit overwhelming and abstract. It's often easier to approach problems like these by choosing concrete examples of objects of type X, seeing that they do indeed have property Y, and then trying to see if there are any patterns you can pick up. For example, take the humble claim that if n is even, then n^2 is even as well. You might pick some examples, like noticing that $6^2 = 36$, or that $4^2 = 16$, or that $10^2 = 100$. You might then try regrouping those to see that $(2 \times 3)^2 = 2 \times 18$, or that $(2 \times 2)^2 = 2 \times 8$, or than $(2 \times 5)^2 = 2 \times 50$, and then try to find some pattern linking 3 and 18, linking 2 and 8, and linking 5 and 50.

If you're trying to prove an existentially-quantified statement of the form "there is an object with properties X, Y, and Z," this approach of trying out concrete examples can be similarly helpful. Pick some random objects and see what properties they do and don't have. Make lists of them, and see where certain things work and certain other things don't work. By having some specific instances listed out, you'll be in a better position to spot some sort of pattern that previously eluded you.

And, on top of this, working out concrete examples forces you to engage with the relevant terms and definitions in ways that you previously might not have noticed. You might have an epiphany purely in the course of finding a single example simply because it required you to turn the abstract mathematical terms in the problem statement into a specific claim about a specific object.

Technique Five: Work Backwards

When you write your final, polished proof of a result, you'll start from the initial assumptions you're making and apply simple logical steps to arrive at the final destination. But that doesn't mean that when you're working on tackling the problem, you necessarily need to follow the reasoning in the same direction. In fact, in many cases, it's easier to work backwards from the end point than forward from the start point.

For example, suppose that you want to prove that if n is even, then n^2 is even. Ultimately, this means that you need to show that there is some integer m such that $n^2 = 2m$. So perhaps it would be easiest to frame the problem as searching for some natural number m that's exactly half of n^2.

You might then ask – well, where is this m going to come from? Well, you know that n itself is even, so it's got to be equal to $2k$ for some natural number k, and equating everything and simplifying gives the following:

$$2m = n^2$$

$$2m = (2k)^2$$

$$2m = 4k^2$$

$$m = 2k^2$$

And look at that – you've now got an expression for m in terms of k!

This line of exploration is useful. It tells us that there's going to be something in there about rewriting n in terms of twice something else, regrouping the terms, and somehow picking some number based on the result. But the line of reasoning shown above isn't actually valid. After all, it starts off with the assumption that $n^2 = 2m$, and if you already assume that this is the case, you know that n^2 is even! In the course of writing up the proof, you'd therefore write this up in a different order, beginning with what you know about n rather than what you know about n^2 and ultimately showing what to pick for m, rather than starting with m and solving for it.

Technique Six: Find a Related Proof

Every now and then, someone comes up with a proof that is so brilliant and so inspired that tons of other mathematicians use a similar insight to make important breakthroughs in other fields. Cantor's theorem and its core technique of diagonalization inspired a number of mathematicians and logicians in the early twentieth century, including a certain young mathematician named Alan Turing who went on to lay the foundations of computer science.

As you learn to write proofs for the first time, you might find now and then that the right way to make a breakthrough on a problem is to look at a similar proof that you've seen somewhere else (often, in lecture) and to think about how you might go about adapting it. If you find a proof idea or technique that you think might be helpful, take some time to play around with the core idea to make sure you see how it works. If you don't understand the details, that's okay! It means that there's some nuance or detail in the proof that you haven't fully internalized. So start off by taking aim at that. Ask questions to the course staff if you need clarification. Best case, you'll understand the technique well enough to apply it elsewhere. Worst case, you'll understand the technique well and find that it doesn't apply to your particular problem.

Technique Seven: Tweak the Assumptions

When you're working on writing a proof of a result, you'll start out with a set of assumptions and try to derive some ultimate conclusion. So let's suppose that you're assuming that X, Y, and Z are true and that you're trying to prove W. If you get stuck, here's a great question to ask yourself: what happens if you just assume Y and Z? Is W still true now? If so, why? If not, why not?

As you start asking these questions, you'll probably start finding out certain things change pretty noticeably. And by trying to see how things change and why, you might start discovering connections between things that weren't there before.

Technique Eight: Tweak the Conclusions

Here's a variation on the previous technique. Rather than tweaking the *assumptions* you're making about a problem, try tweaking the *conclusions* you're trying to reach.

For example, let's go back to everyone's favorite proof that if n is even, then n^2 is even as well. If you try out some concrete examples, you might notice something interesting. If you look at some sample squares of even numbers ($6^2 = 36$, $8^2 = 64$, $10^2 = 100$, $12^2 = 144$, etc.), you might notice that not only are all of the squares even, they're all multiples of four as well! Well *that*'s interesting! Why exactly is that? Maybe if you try to work out why that result is true, it might show you why the squares are even – they're even because they're all divisible by four, and four itself happens to be even.

Technique Nine: Try Another Proof Technique

In the first week of CS103, we'll cover direct proofs, proofs by contradiction, and proofs by contrapositive. Later in the quarter, we'll introduce proof by induction as another technique, along with various other ideas (the pigeonhole principle, etc.) that you can use to build up larger, more impressive results.

If you find yourself stuck and unable to make any progress on a problem, and especially if you've already tried a bunch of the other techniques, maybe it's time to switch up your proofwriting approach and try an alternative route. Haven't yet tried a proof by contradiction? Maybe it's time to do so. Didn't yet try contrapositive? Give that a shot as well. Changing from a direct proof to an indirect proof often exposes aspects of the problem that you previously wouldn't have noticed, and in some cases turns something quite tricky into something quite manageable.

Technique Ten: Sleep on It!

And finally, at some point, maybe it's time to take a break from the problem and work on something else. If you've given the problem a good effort and you haven't cracked it, it really is sometimes best to put it off until tomorrow and get some sleep. Your brain has an amazing ability to slowly make progress on problems subconsciously, and at some point in the quarter you'll almost certainly wake up with a solution to a problem in mind.

We give you a week to complete each problem set precisely because we want to give you time to think about each of the assigned problems. Read over the problem sets as soon as you get them and start taking some initial notes on them right away. That way, when you invariably get stuck on something (it happens – it's just a normal part of mathematics!), you'll have the ability to take a step back, relax, refresh yourself, and take another stab at it the next day.

Putting The Theory Into Practice: A Worked Example

Let's try out an example. For example, suppose we have the following task:

Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Prove that $A \subseteq B$.

Speaking perfectly honestly, it is not at all intuitively obvious to me why this result should be true. We have these two weirdly-defined sets, and we're supposed to somehow prove that one is a subset of the other? That's tricky.

But you know what? We also have a formal definition of what it means for one set to be a subset of another. Specifically, the definition of $S \subseteq T$ is that every element of S is an element of T. And consulting our Guide to Proofs on Sets, we can see that to prove this statement, we should set out starting off with something like this:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: Consider an arbitrary $x \in A$. We will prove that $x \in B$. [*the rest of the proof goes here…*]

This is a start, but what do we do now? Well, we haven't used all of the information available to us yet! For starters, we now have some element $x \in A$. What does that mean? The set A is specified in set-builder notation, which means that it contains all elements for which its given criteria are true. In particular, since $x \in A$, it means that those criteria must be true for x. Similarly, we know that we want to prove that $x \in B$, and since we similarly have a definition of what it means to be an element of B – it's what's specified in the set-builder notation – we can expand out our proof along the following lines:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: Consider an arbitrary $x \in A$. We will prove that $x \in B$. In other words, we will prove that x is even.

Since $x \in A$, we know that $x^2 + 4x$ is even. [*the rest of the proof goes here …*]

Let's think about where we are at this point. We started with the high-level task of proving that $A \subseteq B$. We expanded out the definitions to "unpack" this statement into one about some element x and what it must look like. And in fact, if you look at where we are right now, we're left with a very different sort of task than the one we started off with. We now have a number x where we know that $x^2 + 4x$ is even, and all we need to do is show that x is even. And we were able to get to this point without having to have a clear sense of why the result was even true in the first place.

We can take this sort of reasoning a step further. I'll admit, as I'm writing this, that I'm not 100% sure why this theorem is true. I can try out some examples, though. We've picked a choice of x where $x^2 + 4x$ is even, and we want to somehow show that x is even. Well, what might x be? For example, if $x = 3$, then $x^2 + 4x = 3^2 + 3 \cdot 4 = 9 + 12 = 21$. Oops, that would mean that $3 \notin A$, so that couldn't be our choice of x. Maybe we try $x = 0$? Then $x^2 + 4x = 0^2 + 4 \cdot 0 = 0 + 0 = 0$. So that would mean that $0 \in A$, and indeed we also see that $x \in B$ here as well. What about $x = 2$? Then $2^2 + 4 \cdot 2 = 4 + 8 = 12$, and that's even, so $2 \in A$, and since 2 is even we know that $2 \in B$ as well.

But at this point I'm just tinkering around. With a few examples it seems like the theorem is plausible, but I don't see a general pattern that explains why it would work.

We were able to get to the point where we could start playing around with these sorts of questions because we took the statement $A \subseteq B$ and, rather than playing around with its *intuitive* meaning, we engaged with its *rigorous definition*. Doing that told us that we should be looking at some x where $x^2 + 4x$ was even, then trying to think about why that would mean that x is even.

It turns out we can apply this exact same sort of reasoning a second time. We have an intuitive idea of what an even number is – it's one that ends in 0, 2, 4, 6, or 8; it's a number that can be cut cleanly in

half without leaving a remainder; etc. But we also have a formal definition of an even number, and we haven't used that formal definition yet! That's a huge opportunity, since it means that there's some more information we can integrate into this picture.

As a reminder, we've defined even numbers as follows:

n is even if there is an integer k where $n = 2k$.

Let's go back to our proof and see if we can make use of this.

Theorem: Let $A = \{\ n \in \mathbb{N} \mid n^2 + 4n \text{ is even } \}$ and let $B = \{\ n \in \mathbb{N} \mid n \text{ is even } \}$. Then $A \subseteq B$.

Proof: Consider an arbitrary $x \in A$. We will prove that $x \in B$. In other words, we will prove that x is even.

Since $x \in A$, we know that $x^2 + 4x$ is even. This means that there is an integer k such that $x^2 + 4x = 2k$. [*the rest of the proof goes here … ￼*]

Okay, that's a very useful piece of information to have, because it means that we have that loveliest of mathematical structures to work with – an equality! Once we have an equality, we can pull out all those nice tricks we learned in high school to try to manipulate it into a form that we like.

At this point, we can start playing around. We have that $x^2 + 4x = 2k$, and we want to show that x is even. Great! How exactly should we go about doing that? Well, we could try factoring the left-hand side of that equation, which would give us

$$x(x + 4) = 2k,$$

but that doesn't seem super promising. We're trying to show something about x being even, which would suggest that we probably want to isolate x by itself, but after factoring things we're stuck with a product of two terms with x in them, and it's unclear how we could proceed here. So let's make a note that we tried this, but for now we'll put it on hold and see if there's another strategy we could use.

Okay, so that didn't work. Maybe we could go back to our original equation and subtract $2k$ from both sides? That would give us

$$x^2 + 4x - 2k = 0,$$

which is a quadratic equation. We could try applying the quadratic formula here, which would give us

$$x = \frac{-4 \pm \sqrt{16 + 8k}}{2},$$

and yikes, that's not very helpful. I mean, it's not immediately clear why the square root of $16 + 8k$ is even an integer to begin with.

So maybe we should take a step back and think about what other tools we have at our disposal. The difficulties we're running into seem to stem from the fact that we've got an x^2 term that we don't know what to do with. Maybe we should be asking whether we've seen anything anywhere that would be helpful. I mean, we pulled out the quadratic formula because we know that it's a tool we can use to reason about quadratic formulas. Is there some other theorem or technique we could rely on?

Let's look back over the results we've proved over the past couple of days. Pulling out my notes on the lectures (you are taking notes, right? I mean, we did say that was a really good thing to do in our "How to Succeed in CS103" handout, so we assume you're taking our advice), we've got this list of results about odd and even numbers:

Theorem: If n is even, then n^2 is even.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Theorem: The product of any two consecutive integers is even.

Theorem: For any odd integer n, there are integers r and s where $r^2 - s^2 = n$.

Theorem: For any integer n, if n^2 is even, then n is even.

Let's go through these one at a time. Will that first one help us? This initially looks promising, since it connects n being even to n^2 being even. But the problem is that the implication flows the wrong way: it says that if you have a number n that you already know is even, then you can conclude n^2 is even. Here, we have a number n^2 and want to conclude something about n, which is the reverse of what we're look-ing for. Still, let's make a mental note about it, since we might want to return to this point later on.

What about that second one? It's probably not super useful, since we're not trying to add up odd inte-gers, and even if we were, we're ultimately stuck on an annoying bit with an x^2 term, and this doesn't seem to get rid of squares.

That third one initially might not seem very useful, since it only talks about the product of consecutive integers. But wait a minute – do we have that? We saw that we could factor $x^2 + 4 = x(x + 4)$. That is a product of two integers, but they aren't consecutive. That's too bad. It would have been nice to have been able to bust that one out. But I guess that we'll have to set it aside for now.

That fourth theorem, again, looks promising. It's got some squares in it! In fact, it's got two of them. Which might make us rethink our optimism – we only have one perfect square lying around, not two. And moreover, the structure is a bit off. That theorem says that if you start with an odd number, you can form two perfect squares from it. Here, we want to start with a perfect square and do something to it, so that's probably not a good candidate.

That leaves the very last one, and in this case, that one might be a winner. This says that if we can find an integer n where n^2 is even, we can conclude that n is even. And that looks very close to what we want to do. In our case, we have that $x^2 + 4x$ is even, and we want to conclude that x is even. If we could just get rid of that pesky $4x$ term, we'd be home free! So let's go play around with that and see what we can find.

Let's jump back to our lovely equality $x^2 + 4x = 2k$. Previously, we were trying to isolate x all by itself, and we got bogged down in doing so. But now we're saying that we don't have to do that at all, and in fact, we can just try to isolate x^2. So let's do that by subtracting $4x$ from each side to get

$$x^2 = 2k - 4x.$$

The theorem we'd like to use only works in the case where x^2 is even. Can we show that? Well, to do that, we just need to show that x^2 is twice some integer. And if you look at what we have here, it sure seems like we should be able to do just that! We can factor the right-hand side to get

$$x^2 = 2(k - 2x),$$

which means that x^2 is twice some integer – namely, $k - 2x$. It is a little bit weird that we're arguing that x^2 is even by using the fact that it's twice some term that depends on x. But then again, the definition of an even number just says it needs to be twice some integer, and really doesn't care what that integer is!

So at this point, we've laid out our line of reasoning. Looking back over all the discussion and scratch work and notes and false starts from the previous pages, we can see that the reasoning goes like this:

- Write $x^2 + 4x = 2k$, which we can do because we know that $x^2 + 4x$ is even.

- Rearrange the terms to get $x^2 = 2k - 4x = 2(k - 2x)$.

- Conclude that x^2 is even.

- Apply the theorem from lecture to conclude that x is even.

Integrating this idea into our proof looks like this:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: Consider an arbitrary $x \in A$. We will prove that $x \in B$. In other words, we will prove that x is even.

Since $x \in A$, we know that $x^2 + 4x$ is even. This means that there is an integer k such that $x^2 + 4x = 2k$. Subtracting $4x$ from both sides of that equality tells us that

$$x^2 = 2k - 4x,$$

which we can rewrite as

$$x^2 = 2(k - 2x).$$

This means that there is an integer m, namely, $k - 2x$, such that $x^2 = 2m$. Consequently x^2 is even. Therefore, using a theorem from lecture, we see that x is even, which, as mentioned earlier, is what we needed to show. ∎

And there you have it. We've got a proof of this result!

When I'm presenting proofs in lecture, I'm showing you polished, thoroughly-revised results along with one way of arriving at them. I'm not showing you all the false starts, all the dead ends, all the scratch work, or all the abandoned leads. *But real mathematics isn't like that*. Real math is about trying things out. It's about walking back from otherwise promising ideas, about making lists of strategies and crossing off ones that don't work. At the same time, that doesn't mean that there aren't some good, guiding, general principles you can use to attack problems. As you saw here, we repeatedly used the technique of **expanding out definitions** to expose structures we were then able to latch onto. We used that first to convert $A \subseteq B$ into some statement about an individual object x, and then to convert the statement "$x^2 + 4x$ is even" into an equality. Similarly, we used the technique of **writing down everything you know** to find the gem of a theorem about how if n is an integer and n^2 is even, then n is even.

… but are we really done yet? In one sense, yes we are – we've got a working proof of the result. But on the other hand, do we know that we have the *best proof* of this result? It's hard to say, because we never tried any other approaches!

Let's back way, way, up to the beginning of this proof. We're ultimately trying to show the following:

For any x, if $x \in A$, then $x \in B$.

The core of this statement is an implication, and we proved that implication here by choosing an arbitrary $x \in A$ and concluding that $x \in B$. But that's not the only way we could have structured this proof! In fact, we could have used some of the other techniques for proving implications that we saw on Friday of the first week of class. What happens if we use one of those approaches?

Let's try doing a proof by contrapositive. That would have us prove this statement instead:

For any x, if $x \notin B$, then $x \notin A$.

If we set up the proof this way, we get the following:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: We will prove this result by contrapositive by showing that, for any choice of x, if $x \notin B$, then $x \notin A$. [*the rest of the proof goes here …*]

This is going to look quite different than what we saw before, but that's okay! Let's chase this down and see what we find.

First, we can look at the format of the proof and try to think about how to start it off. Specifically, we're proving an implication – in this case, the contrapositive of the one we proved last time – and so we can start things off the way we normally do: assuming the antecedent is true. That's shown here:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: We will prove this result by contrapositive by showing that, for any choice of x, if $x \notin B$, then $x \notin A$. To do so, choose an arbitrary x where $x \notin B$. We will prove that $x \notin A$. [*the rest of the proof goes here …*]

Let's think about what to do here. We now have our choice of x, and it's specifically some value that isn't in the set B. So what does that tell us? Well, B is the set

$$\{\, n \in \mathbb{N} \mid n \text{ is even} \,\},$$

and that tells us a lot. If an object *does* belong to B, then it has to be a natural number (that's what the bit on the left of the bar means), and it has to be even (that's what the bit on the right says). So if we have this object x where x is *not* an element of B, it means that at least one of those conditions must not hold. That is, either x isn't a natural number, or x is a natural number, but it isn't even. We can't say for sure which of these is the case, since we know nothing about x other than the fact that it's not an element of B. But we've seen how to deal with things like this before – this is what a proof by cases is best at! So let's try using one here. That gives us the following:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: We will prove this result by contrapositive by showing that, for any choice of x, if $x \notin B$, then $x \notin A$. To do so, choose an arbitrary x where $x \notin B$. We will prove that $x \notin A$.

There are two possible ways that we can have $x \notin B$:

Case 1: $x \notin \mathbb{N}$. [*the rest of the proof goes here …*]

Case 2: $x \in \mathbb{N}$, but x isn't even. [*the rest of the proof goes here …*]

Let's take a minute to think through these cases. What about Case 1, where we have $x \notin \mathbb{N}$? Well, if we look at where we're going, we'll see that we want to show that $x \notin A$. How do we do that? Well, let's go back and write down everything we know. We know in this case that

$$A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\},$$

so we can ask – does this help us? Fortunately, it does! Remember that we're assuming that $x \notin \mathbb{N}$, and that means that it's not possible for x to be an element of A – the bit to the left of the vertical bar tells us that every element of A is a natural number. Let's go fill that part in.

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: We will prove this result by contrapositive by showing that, for any choice of x, if $x \notin B$, then $x \notin A$. To do so, choose an arbitrary x where $x \notin B$. We will prove that $x \notin A$.

There are two possible ways that we can have $x \notin B$:

Case 1: $x \notin \mathbb{N}$. Then, by definition of the set A, we see that $x \notin A$.

Case 2: $x \in \mathbb{N}$, but x isn't even. [*the rest of the proof goes here …*]

That wasn't so bad! So let's look at Case 2. What would it mean for us to have $x \in \mathbb{N}$ and x to not be even? Looking back at our notes, when we first defined even and odd numbers, we mentioned that every natural number is either even or odd, but not both. Great! So if x isn't even, it must be odd. In other words, we know that we're dealing with an odd number.

We still want to prove that $x \notin A$, which means that we want to prove that $x^2 + 4x$ isn't even. Again, using the fact that every natural number is either even or odd, that means that we need to prove that $x^2 + 4x$ is odd. So let's articulate that this is what we're going to prove:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: We will prove this result by contrapositive by showing that, for any choice of x, if $x \notin B$, then $x \notin A$. To do so, choose an arbitrary x where $x \notin B$. We will prove that $x \notin A$.

There are two possible ways that we can have $x \notin B$:

Case 1: $x \notin \mathbb{N}$. Then, by definition of the set A, we see that $x \notin A$.

Case 2: $x \in \mathbb{N}$, but x isn't even. This means that x is odd. We will prove that $x \notin A$ by showing that $x^2 + 4x$ is odd. *[the rest of the proof goes here …]*

And now we just have to think about how to prove this. There are many ways we could proceed here. We've seen lots of proofs like this one in lecture, and here a reasonable guess of what to do might be to just expand out the definition of an odd number and see what we find.

We could just show you the rest of the proof, but that's such a good exercise that we figure you might want to try doing it yourself. Try filling in the square braces above. Use the techniques and the method-ologies that you've seen in the preceding sections. Once you have a draft, feel free to carry on to the next page, where we've included one possible proof.

Like, seriously, don't go on to the next page until you've worked through this. It's a great exercise!

So you've done it? Really truly? If so, great! Move onward to the next page.

Here's one possible way to write this proof up:

Theorem: Let $A = \{\, n \in \mathbb{N} \mid n^2 + 4n \text{ is even} \,\}$ and let $B = \{\, n \in \mathbb{N} \mid n \text{ is even} \,\}$. Then $A \subseteq B$.

Proof: We will prove this result by contrapositive by showing that, for any choice of x, if $x \notin B$, then $x \notin A$. To do so, choose an arbitrary x where $x \notin B$. We will prove that $x \notin A$.

There are two possible ways that we can have $x \notin B$:

Case 1: $x \notin \mathbb{N}$. Then, by definition of the set A, we see that $x \notin A$.

Case 2: $x \in \mathbb{N}$, but x isn't even. This means that x is odd. We will prove that $x \notin A$ by showing that $x^2 + 4x$ is odd. Since x is odd, we know that there is an integer k such that $x = 2k + 1$. Then we see that

$$x^2 + 4x = (2k + 1)^2 + 4(2k + 1)$$
$$= 4k^2 + 4k + 1 + 8k + 4$$
$$= 4k^2 + 12k + 4 + 1$$
$$= 2(2k^2 + 6k + 2) + 1.$$

This shows that there is an integer m (namely, $2k^2 + 6k + 2$) such that $x^2 + 4x = 2m + 1$, so $x^2 + 4x$ is odd, as needed.

In either case, we see that $x \notin A$, which is what we needed to show. ∎

So there you have it – here's a totally different proof of the same result!

As you're working through problems on your own, don't despair if you get stuck. That's normal. It's a fact of life in mathematics. It doesn't reflect on your abilities or your potential. Instead, go through the techniques in this handout and see if any of them might make your life easier.

And what should you do if you've really exhausted all the possibilities? In that case, feel free to stop by our office hours to ask for help! That's what they're there for.