# Midterm Exam I

**☞ Due Sunday, October 4th at 12:00PM noon Pacific. ☜**

## Instructions

You have 48 hours to complete this exam. Take as much of that time as you need. We've designed the exam with the expectation that it will take you around three hours to finish.

Please type your answers the way you type up your problem sets. There's a LaTeX template available on Canvas if you'd like to use it, though it's not required. Once you're finished, submit your answers on Gradescope. Please leave appropriate buffer time to ensure your submission comes in by the deadline. As with the problem sets, we'll grade the last version you submit before the deadline, so feel free to periodically submit what you have just in case something comes up.

## Honor Code Policies

You are required to abide by the Honor Code policies outlined in the Honor Code Policies handout available on Canvas. We'd like to call particular attention to the following rules.

***This midterm exam must be completed individually.*** It is a violation of the Stanford Honor Code to communicate with any other humans about this exam, to solicit solutions to this exam, or to share your solutions with others.

This exam is open-book, so you are free to make use of all course materials on Canvas. You are also permitted to search online for conceptual information (for example, by visiting Wikipedia). However, you are not permitted to communicate with other humans about the exam or to solicit help from others. For example, you ***must not*** communicate with other students in the course about the exam, and you ***must not*** ask questions on sites like Chegg or Stack Overflow.

All work done with the assistance of any material in any way (other than provided CS103 course materials) must include a detailed citation (e.g., "I visited the Wikipedia page for $X$ on Problem 1 and made use of insights $A$, $B$, and $C$"). ***Copying solutions is never acceptable***, even with citation, and is always a violation of the Honor Code. If by chance you encounter solutions to a problem, navigate away from that page before you feel tempted to copy. ***Because of the revise-and-resubmit policy, there is no reason to violate your conscience to complete a take-home exam.***

If you become aware of any Honor Code violations by any student in the class, your commitments under the Stanford Honor Code obligate you to inform course staff.

## Grading

To earn a satisfactory grade on this exam, you need to earn a raw score of **90%** or above. If your score is lower than this, you will be asked to revise your answers and resubmit by the following Sunday at 12:00PM noon Stanford time. Course staff will be available to coach you on understanding where your work needs improvement and how to proceed.

***You can do this.* Best of luck on the exam!**

## Problem One: All Squared Away

Let's begin with a new definition. A natural number $n$ is called a ***perfect square*** if there's an integer $k$ such that $n = k^2$. For example, 16 is a perfect square because $16 = 4^2$, but 15 is not.

For reference, the first positive perfect squares are

$$\begin{array}{ccccc} 1, & 4, & 9, & 16, & 25, \\ 36, & 49, & 64, & 81, & 100, \\ 121, & 144, & 169, & 196, & 225, \\ 256, & 289, & 324, & 361, & 400, \\ & & \ldots & & \end{array}$$

Your task is to prove the following theorem:

***Theorem:*** If $n$ is an odd natural number and $n \geq 3$,
then there is an $m \in \mathbb{N}$ where $m > 0$ and $m^2 + mn$ is a perfect square.

Rather than diving headfirst into the proof, let's first take a minute to explore what the theorem says.

i. Fill in the following blanks with positive natural numbers. No justification is necessary. We've filled the one of the blanks for you.

- When $n = 3$, we can write $n = 2 \cdot$ ___ $+ 1$, and one choice of $m$ that works is ____.
- When $n = 5$, we can write $n = 2 \cdot$ ___ $+ 1$, and one choice of $m$ that works is ____.
- When $n = 7$, we can write $n = 2 \cdot$ ___ $+ 1$, and one choice of $m$ that works is ____.
- When $n = 9$, we can write $n = 2 \cdot$ ___ $+ 1$, and one choice of $m$ that works is **16**.
- When $n = 11$, we can write $n = 2 \cdot$ ___ $+ 1$, and one choice of $m$ that works is ____.

*Start by filling in the rows for n = 3 and n = 5. As a hint, the numbers in those rows are all less than five, so in the absolute worst case you can just try each of the numbers in the range to see which ones work. The value of m in the row for n = 7 is less than ten.*

*Filling in the last row may require you to make an educated guess based on the pattern that emerges and to use a calculator to check your work.*

ii. Fill in the following blank. No justification is required.

When $n = 2k + 1$, pick $m =$ ________.

iii. Prove the theorem.

*This isn't the first time you've seen a result linking odd integers and perfect squares. Check out the first lecture on mathematical proofs for an example of how you might want to structure this. (The presentation for that section starts at 41:15; the final proof of the theorem is presented in Slide 100.)*

## Problem Two: Never Never Land

Regular, ordinary first-order logic has two quantifiers: $\forall$ and $\exists$. We can express the four Aristotelian forms using $\forall$ and $\exists$ like this:

| All $A$'s are $B$'s | Some $A$ is a $B$ | No $A$'s are $B$'s | Some $A$ is not a $B$ |
|---|---|---|---|
| $\forall x. (A(x) \to B(x))$ | $\exists x. (A(x) \land B(x))$ | $\forall x. (A(x) \to \neg B(x))$ | $\exists x. (A(x) \land \neg B(x))$ |

Now, let's imagine we lived in a world in which these quantifiers didn't exist, and instead we only had one quantifier, N. The quantifier N is the "never" quantifier, and the expression

$$\mathrm{N}x.\ [\textit{some formula}]$$

means "[*some formula*] is never true, regardless of what choice of $x$ we pick." For example, the expression

$$\mathrm{N}x.\ x \in \emptyset$$

says "there are no elements in the empty set."

Show how to express the four Aristotelian forms using first-order logic formulas in which

- you *are* allowed to use the N quantifier, but
- you *are not* allowed to use the $\forall$ or $\exists$ quantifiers.

You may use the predicates $A(x)$ and $B(x)$, but aside from this should not use any other predicates, functions, or constant symbols. No justification is required.

| All $A$'s are $B$'s | Some $A$ is a $B$ | No $A$'s are $B$'s | Some $A$ is not a $B$ |
|---|---|---|---|
| | | | |

## Problem Three: Tiny Sets

This question is all about a type of set called a tiny set, and some unexpected properties of tiny sets.

Let's begin with a formal definition. A set $S$ is called ***tiny*** if the following properties hold for $S$:

1. There is at least one element $X \in S$.
2. For any $X \in S$, the object $X$ is a set.
3. For any set $X \in S$ and any set $Y$ where $Y \subseteq X$, we have $Y \in S$.
4. For any sets $X \in S$ and $Y \in S$, we have $X \cup Y \in S$.

These four properties tell us quite a bit about what tiny sets must look like.

i. For each of the sets given below, tell us whether it's a tiny set. If it isn't a tiny set, identify one of the four above rules that the set doesn't obey. (If it doesn't obey multiple rules, you can pick any one of them.) No justification is required.

Is $\emptyset$ a tiny set?

☐ Yes, this is a tiny set.

☐ No, this is not a tiny set. In particular, it does not obey rule ______.

Is $\{\emptyset, 1, 2, 3\}$ a tiny set?

☐ Yes, this is a tiny set.

☐ No, this is not a tiny set. In particular, it does not obey rule ______.

Is $\{\emptyset, \{1, 2, 3\}\}$ a tiny set?

☐ Yes, this is a tiny set.

☐ No, this is not a tiny set. In particular, it does not obey rule ______.

Is $\wp(\{1, 2, 3\})$ a tiny set?

☐ Yes, this is a tiny set.

☐ No, this is not a tiny set. In particular, it does not obey rule ______.

Tiny sets have some interesting properties.

ii. Prove that if $S$ is a tiny set, then $\emptyset \in S$.

*The interplay of some number of the four rules above will let you conclude this. As a hint, start with Rule 1. In fact, if you don't use Rule 1, it's not possible to prove this result!*

*(Continued on the next page...)*

iii. Fill in the blanks in the template below to complete the proof of the following theorem: if $S$ and $T$ are tiny sets, then so is $S \cap T$.

The relative sizes of the blanks do not necessarily indicate how much you need to write. Also, feel free to expand the final blanks in each paragraph into as many sentences as you need.

***Theorem:*** If $S$ and $T$ are tiny sets, then $S \cap T$ is also a tiny set.

***Proof:*** Consider any tiny sets $S$ and $T$. We will prove that $S \cap T$ is tiny by showing that it satisfies the four rules for tiny sets.

To show that $S \cap T$ obeys Rule 1, we will prove that there is an $X$ where $X \in S \cap T$. Pick $X =$ ______. We see that $X \in S$ and that $X \in T$ because _______. Therefore, we see that _______, as required.

To show that $S \cap T$ obeys Rule 2, pick any $X \in S \cap T$. We will prove that ________.
________________________________________________________.

To show that $S \cap T$ obeys Rule 3, choose an arbitrary _______ and _______. We will prove that _______. ____________________________________________________.

Finally, we will prove that $S \cap T$ obeys Rule 4. ________________________________
____________________________________________________________________________.

We've shown that $S \cap T$ obeys Rules 1 – 4, so $S \cap T$ is tiny, as required. ■

*As you're writing up this result, remember to write up your proof by focusing on individual elements of the relevant sets, along the lines of what we did in lecture and what you saw on PS1.*

*As a hint, what do you know about A if $A \in B \cap C$? What do you need to prove to prove that $A \in B \cap C$?*