Week 3 Tutorial

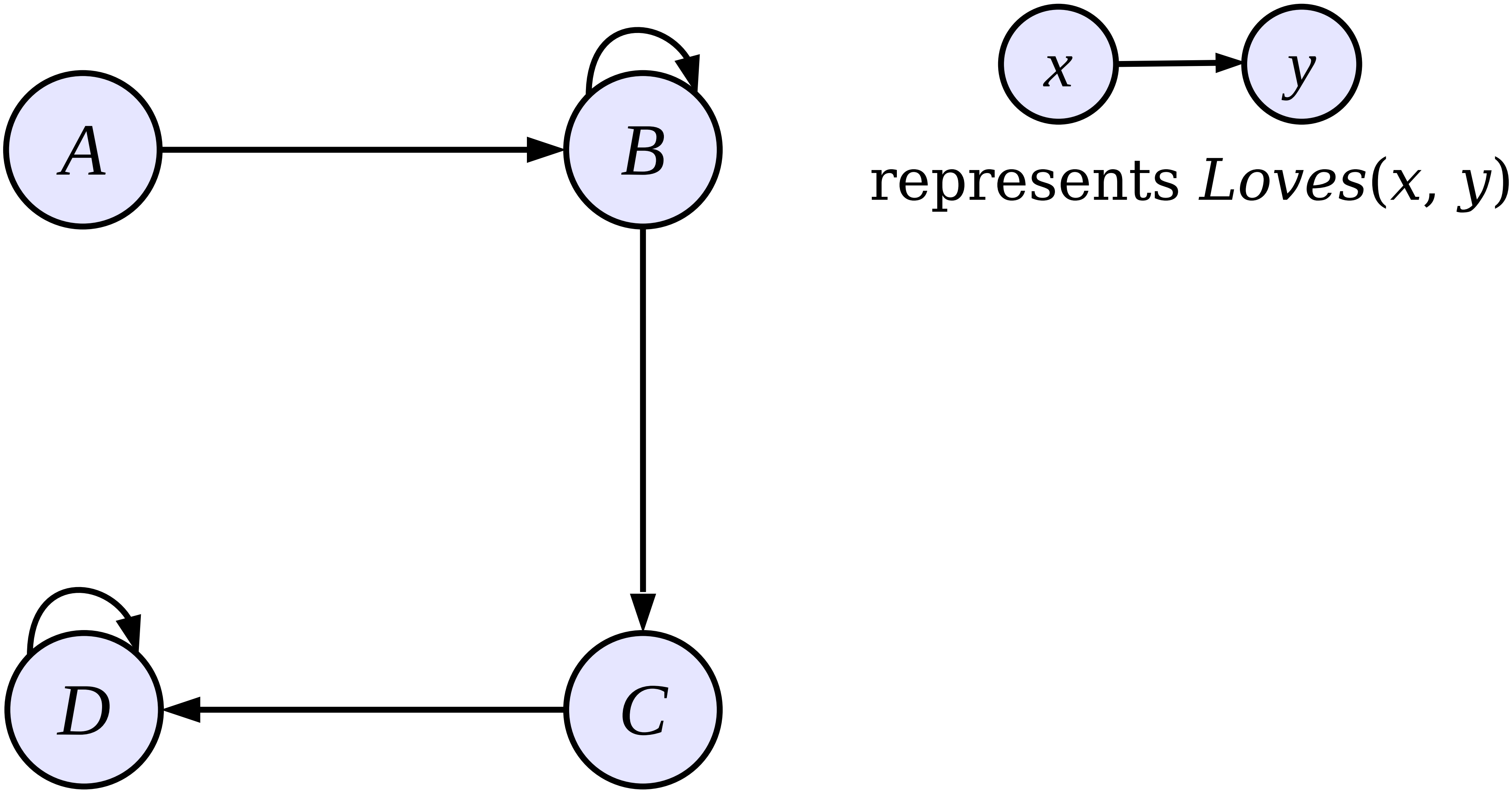
Mathematical Logic

General Announcements

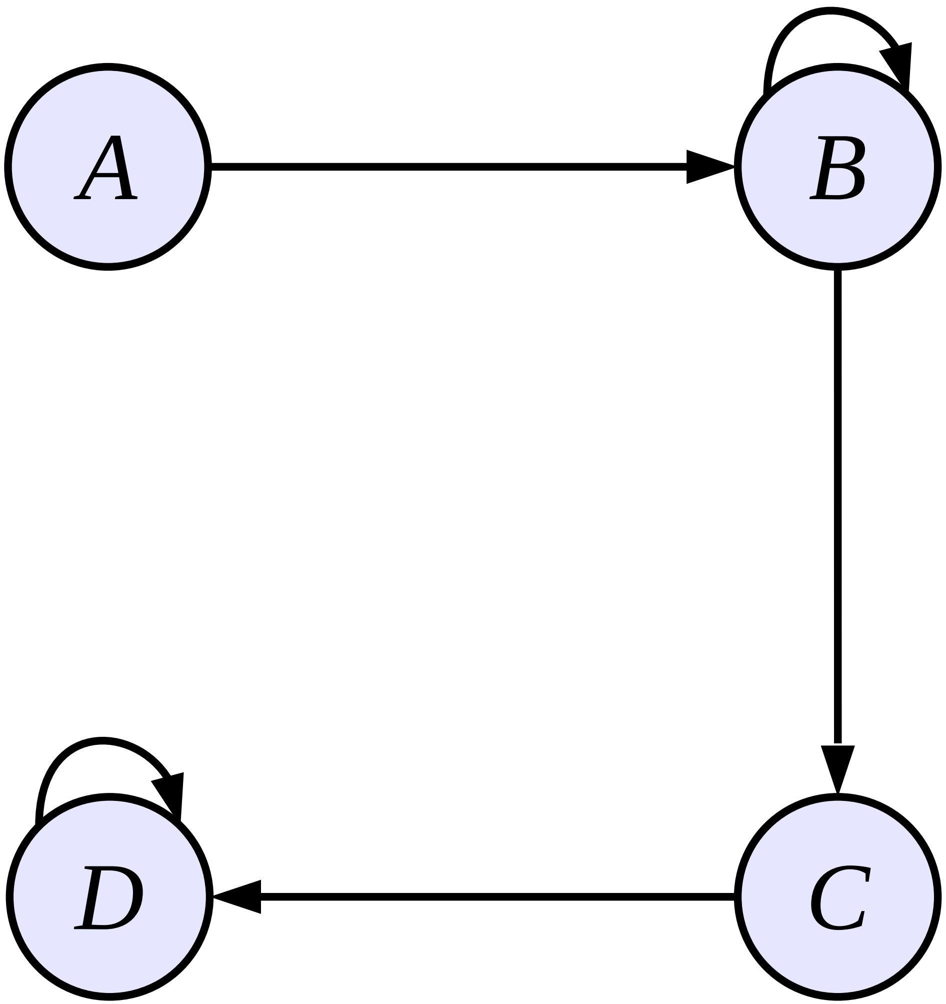
- Notes on PS1:
  - Please tag the pages for each question on Gradescope! It saves the TAs a lot of time and is a nice courtesy.
  - Please be sure to type solutions. Handwritten solutions can be hard to read. (Keith's handwriting looks markedly worse than this scrawl.)
- Please take a minute to fill out this Week 3 Check-In Form:

https://forms.gle/vvzBM9ZHEbz5bmsN9

Part 1: ***Logic Recap and Warmup***



A
B
C
D
x
y
represents Loves(x, y)



1. For each of the following statements, determine the minimum number of arrows that must be added to make the statement true.

You can't remove existing arrows. If a formula is already true, the answer will be "0". Tell us what the arrows are and briefly explain why it's the smallest number that need to be added.

a) $Loves(A, B) \rightarrow Loves(B, C)$

b) $Loves(A, C) \rightarrow Loves(B, D)$

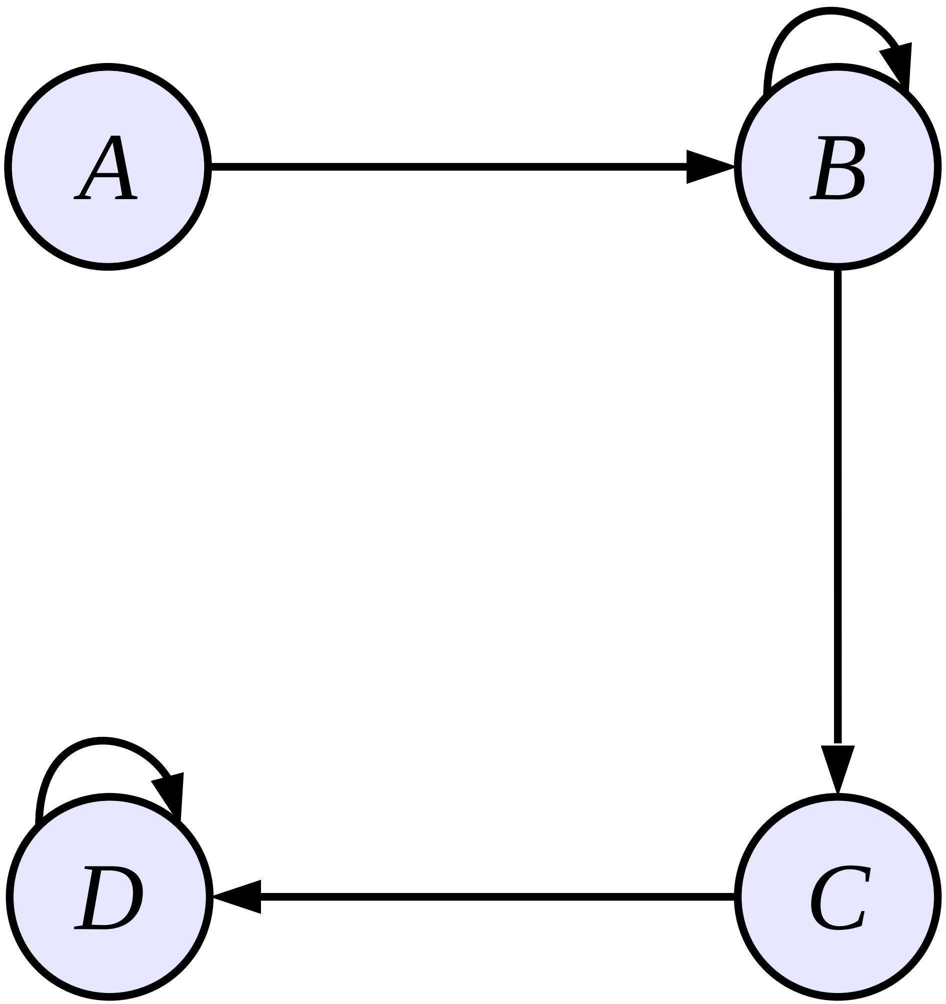
c) $Loves(A, D) \rightarrow Loves(B, C)$

d) $Loves(A, B) \rightarrow Loves(D, C)$

e) $Loves(A, C) \leftrightarrow Loves(C, A)$

f) $Loves(B, B) \leftrightarrow Loves(C, C)$

Fill in answer on Gradescope!



2. Repeat the previous exercise with the following statements:

a) $\forall x.\ \exists y.\ Loves(x, y)$

b) $\forall x.\ \exists y.\ (x \neq y \land Loves(x, y))$

c) $\exists x.\ \forall y.\ Loves(x, y)$

d) $\exists x.\ \forall y.\ (x \neq y \to Loves(x, y))$

Fill in answer on Gradescope!

Part 2: ***Negating Statements***

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Useful Resource:

"Guide to Negations," available on Canvas.

$$
\begin{array}{l}
\neg\forall p.\ (Person(p) \to \\
\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
\qquad Loves(p, q) \\
\quad) \\
)
\end{array}
$$

$$
\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}
$$

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}
$$

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}
$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\exists p.\ \neg(Person(p) \rightarrow$$
$$\quad \exists q.\ ((Person(q) \land p \neq q) \land$$
$$\qquad Loves(p, q)$$
$$\quad)$$
$$)$$

3. Finish the negation of this first-order logic formula. Your final formula should not have any negations in it except for direct negations of predicates.

Here is the formula so far in LaTeX:

```
$$\exists p. \neg (Person(p) \rightarrow \exists q.
((Person(q) \land p \neq q) \land Loves(p, q))$$
```

Fill in answer on Gradescope!

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \rightarrow \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg(A \to B)}{A \land \neg B}}
$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \rightarrow \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg(A \rightarrow B)}{A \land \neg B}}$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \rightarrow \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg(A \rightarrow B)}{A \land \neg B}}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \neg\exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg(A \to B)}{A \land \neg B}}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \neg\exists q.\ ((Person(q)\ \land\ p \neq q)\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \neg\exists q.\ ((Person(q)\ \land\ p \neq q)\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \neg\exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg\exists x.\ A}{\forall x.\ \neg A}$$

$$\exists p.\ (Person(p) \land$$
$$\neg\exists q.\ ((Person(q) \land p \neq q) \land$$
$$Loves(p, q)$$
$$)$$
$$)$$

$$\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ \neg((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg \exists x.\ A}{\forall x.\ \neg A}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ \neg((Person(q)\ \land\ p \neq q)\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land\\
&\quad \forall q.\ \neg((Person(q)\ \land\ p \neq q)\ \land\\
&\qquad Loves(p, q)\\
&\quad)\\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg(A \land B)}{A \to \neg B}}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ \neg((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg({\color{red}A} \land {\color{blue}B})}{{\color{red}A} \to \neg {\color{blue}B}}}$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ \neg((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg(A \land B)}{A \to \neg B}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ ((Person(q)\ \land\ p \neq q) \to \\
&\qquad \neg Loves(p,\ q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg(A \land B)}{A \to \neg B}}
$$

$$
\begin{array}{l}
\exists p.\ (Person(p)\ \land \\
\quad \forall q.\ ((Person(q)\ \land\ p \neq q) \to \\
\qquad \neg Loves(p,\ q) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ ((Person(q) \land p \neq q) \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ ((Person(q) \land p \neq q) \to \\
&\qquad \neg Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Part 3: ***First-Order Logic Translations***

Consider this statement:

"If someone is happy, then everyone is happy."

What is the *contrapositive* of this statement?

4. Find the contrapositive of the above statement. You may find it helpful to first write out the original statement in first order logic.

Fill in answer on Gradescope!

“If someone is happy, then everyone is happy.”

“If someone is happy, then everyone is happy.”

If someone is happy, then everyone is happy

“If someone is happy, then everyone is happy.”

someone is happy → *everyone is happy*

“If someone is happy, then everyone is happy.”

someone is happy $\rightarrow (\forall x.\ Happy(x))$

“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

"If someone is happy, then everyone is happy."

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$\neg(\forall x.\ Happy(x)) \to \neg(\exists x.\ Happy(x))$$

“If someone is happy, then everyone is happy.”

$$\neg(\forall x.\ Happy(x)) \to \neg(\exists x.\ Happy(x))$$

“If someone is happy, then everyone is happy.”

$$\neg(\forall x.\ Happy(x)) \rightarrow \neg(\exists x.\ Happy(x))$$

"If someone is happy, then everyone is happy."

$$(\exists x.\ \neg Happy(x)) \to \neg(\exists x.\ Happy(x))$$

“If someone is happy, then everyone is happy.”

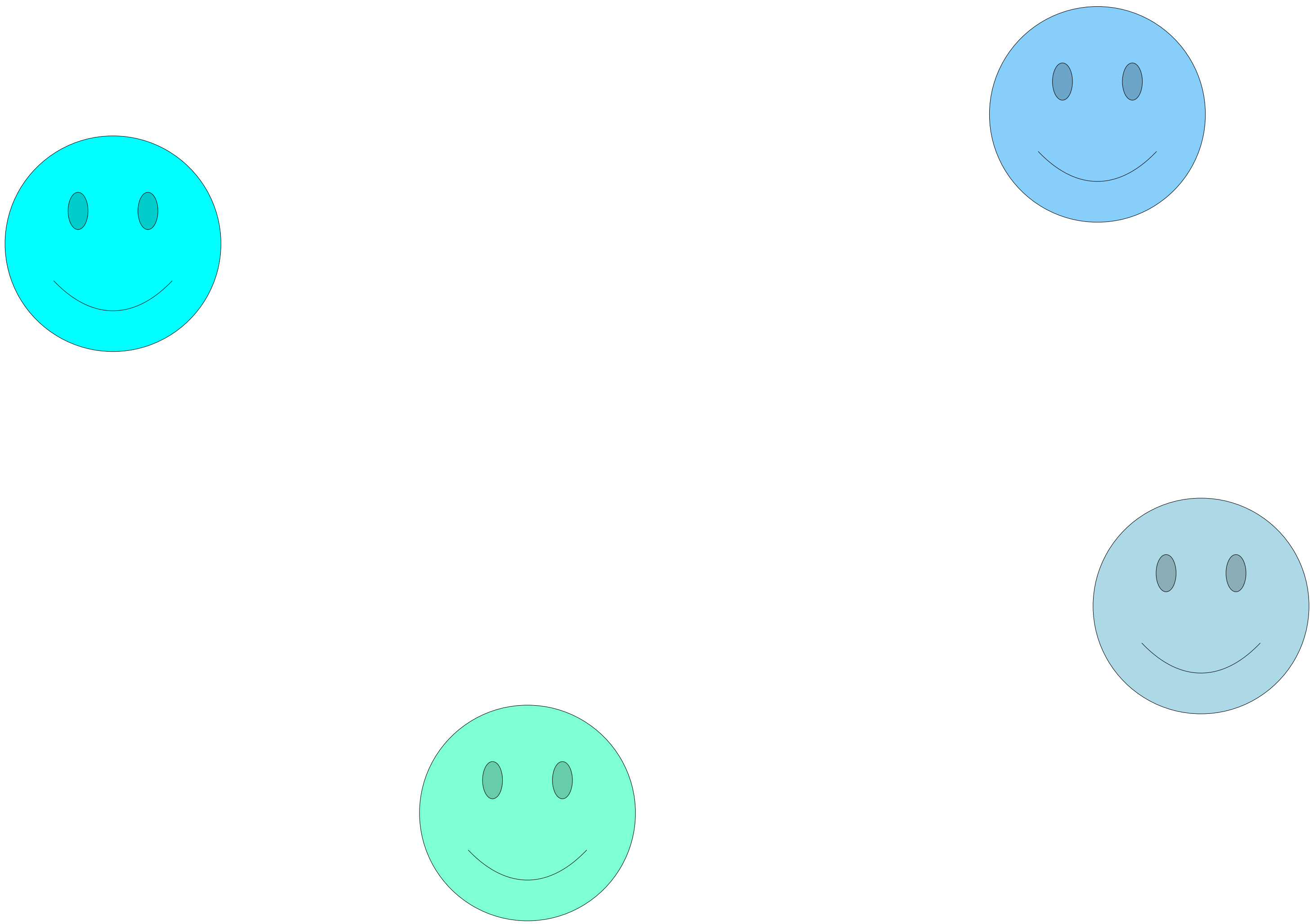
$$(\exists x.\ \neg Happy(x)) \rightarrow (\forall x.\ \neg Happy(x))$$

“If someone is happy, then everyone is happy.”

$$(\exists x.\ \neg Happy(x)) \rightarrow (\forall x.\ \neg Happy(x))$$

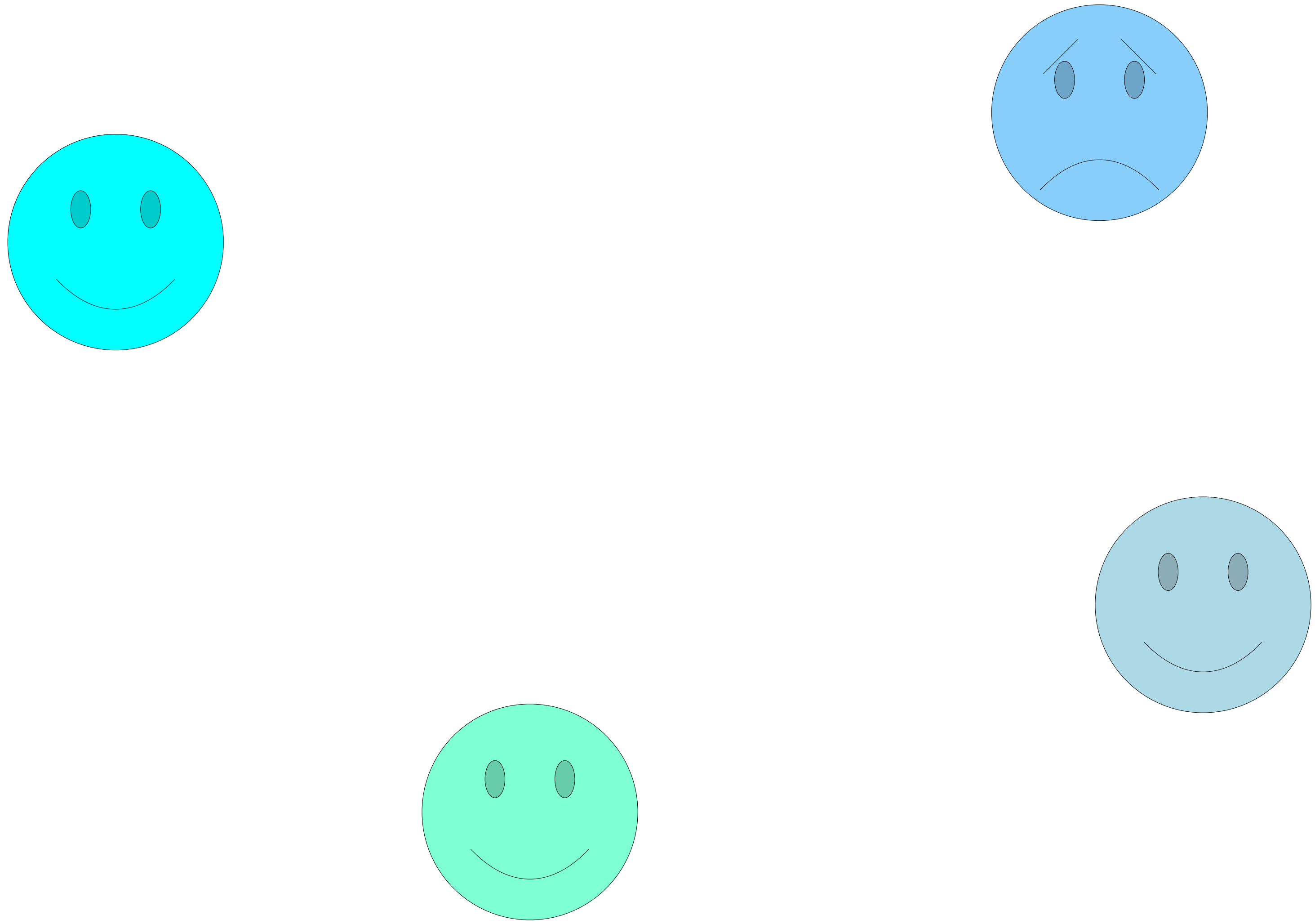
“If someone is not happy, then everyone is not happy.”

“If someone is happy, then everyone is happy.”



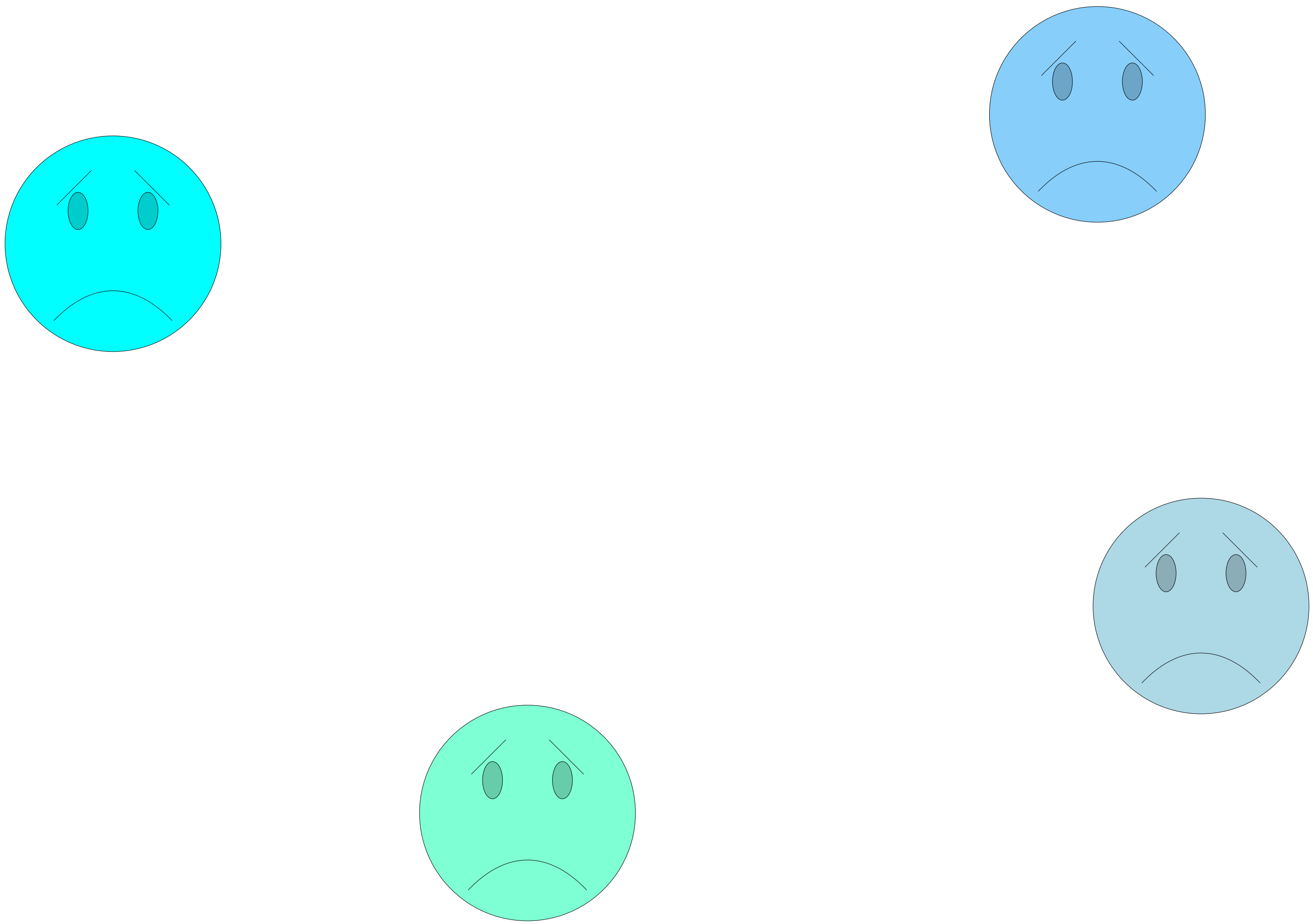
“If someone is not happy, then everyone is not happy.”

“If someone is happy, then everyone is happy.”



“If someone is not happy, then everyone is not happy.”

“If someone is happy, then everyone is happy.”



“If someone is not happy, then everyone is not happy.”

Consider this statement:

"If someone is happy, then everyone is happy."

What is the *negation* of this statement?

5. Find the negation of the above statement. You may find it helpful to first write out the original statement in first order logic.

Fill in answer on Gradescope!

“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$\neg((\exists x.\ Happy(x)) \to (\forall x.\ Happy(x)))$$

“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land \neg(\forall x.\ Happy(x))$$

“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land (\exists x.\ \neg Happy(x))$$

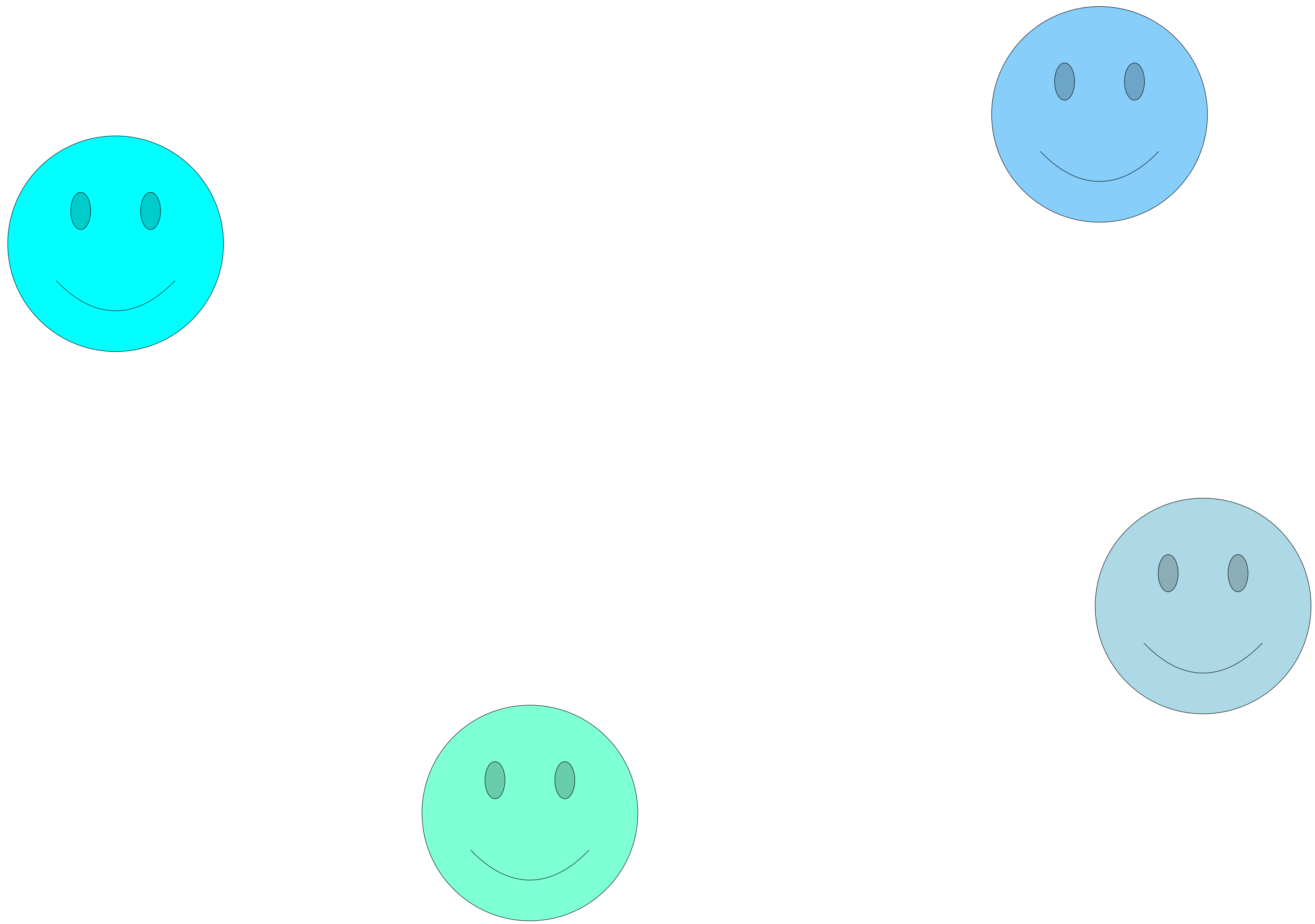
“If someone is happy, then everyone is happy.”

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land (\exists x.\ \neg Happy(x))$$

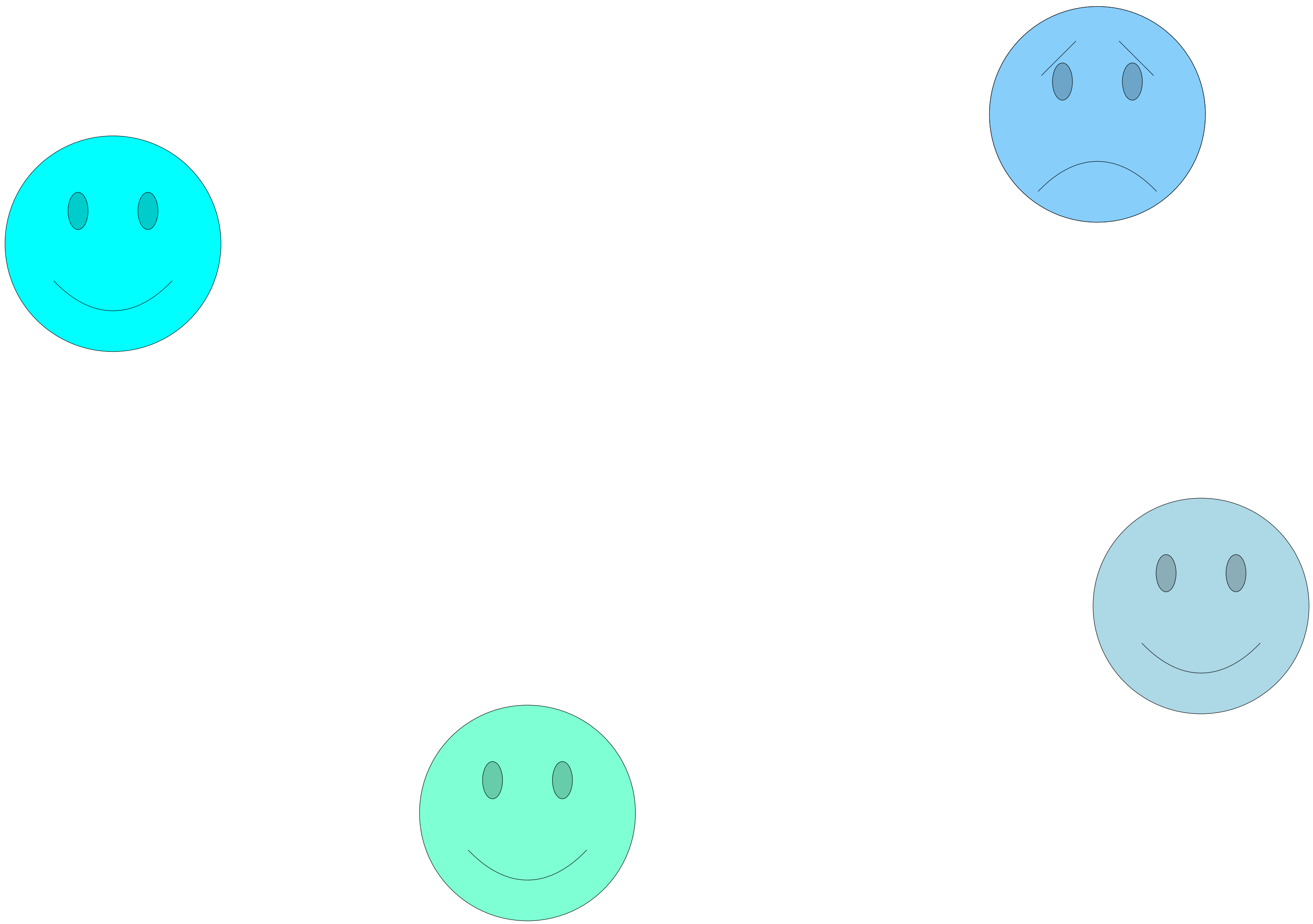
“Someone is happy and someone is not happy.”

“If someone is happy, then everyone is happy.”



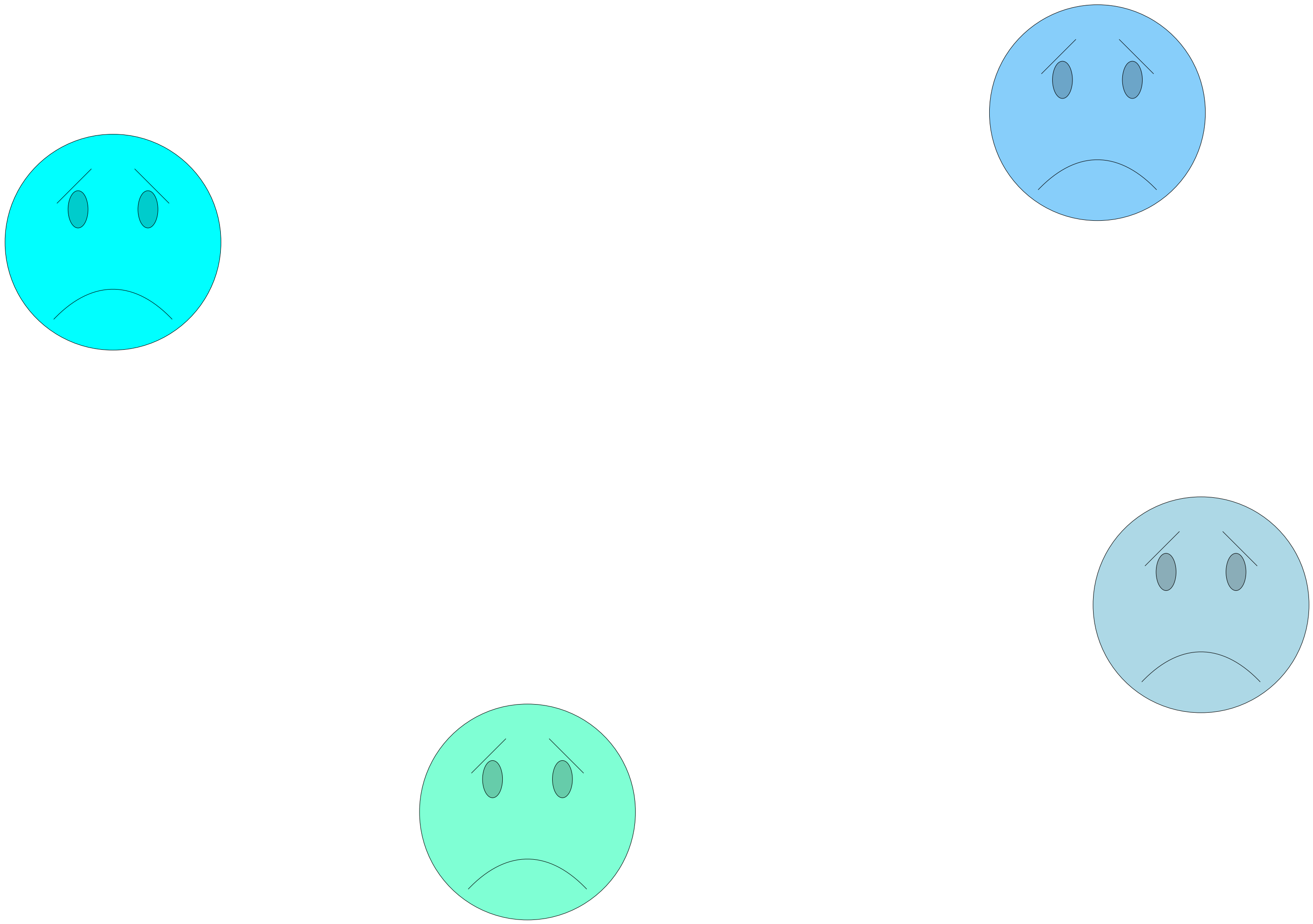
“Someone is happy and someone is not happy.”

“If someone is happy, then everyone is happy.”



“Someone is happy and someone is not happy.”

“If someone is happy, then everyone is happy.”



“Someone is happy and someone is not happy.”

Thanks for Calling In!

Stay safe, stay healthy,
and have a good week!

See you next time.