Week 4 Tutorial

Binary Relations and Functions

Adapted from slides created by Amy Liu.

Part 1: ***Binary Relations Warmup***

Let $\mathbb{R}^2$ denote the set of all ordered pairs of real numbers. For example $(137, 42) \in \mathbb{R}^2$, $(\pi, e) \in \mathbb{R}^2$, and $(-2.71, 103) \in \mathbb{R}^2$.

Two ordered pairs are equal if and only if each of their components are equal. That is, we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

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Now, consider the relation E over $\mathbb{R}^2$ defined as follows:

$$(x_1, y_1)\ E\ (x_2, y_2) \quad \text{if} \quad \exists k \in \mathbb{R}.\ (k \neq 0 \land (kx_1, ky_1) = (x_2, y_2)).$$

For example, $(3, 4)\ E\ (6, 8)$ because $(2 \cdot 3, 2 \cdot 4) = (6, 8)$.

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The word "if" here means "is defined as" and isn't an implication. Generally speaking, when introducing a new relation, the word "if" indicates a definition.

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For example, $(3, 4)\ E\ (6, 8)$ because $(2 \cdot 3, 2 \cdot 4) = (6, 8)$.

1. Complete the set-up for the proof that E is an equivalence relation by filling in the "assume" and "want to show" statements to prove that E is reflexive, symmetric, and transitive. For example:

Assume: (assumption for the reflexive part)
Want to show: ("want to show" for the reflexive part)

You should have 6 statements in total.

Fill in answer on Gradescope!

- [ ] We want to show each pair of real numbers relates to itself.
- [ ] Pick an arbitrary $x \in \mathbb{R}^2$. We want to show that xEx.
- [ ] Pick an arbitrary $x, y \in \mathbb{R}^2$. We want to show that $(x, y)E(x, y)$.
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Fun fact: this binary relation is related to a concept called ***homogeneous coordinates*** that's used extensively in computer graphics.

Take CS148 for more details!

Part 2: ***More Binary Relations***

Let R be a binary relation over a set A. We can define a new relation over A called the ***inverse relation of R***, denoted R^{-1}, as follows:

$$xR^{-1}y \qquad \text{if} \qquad yRx$$

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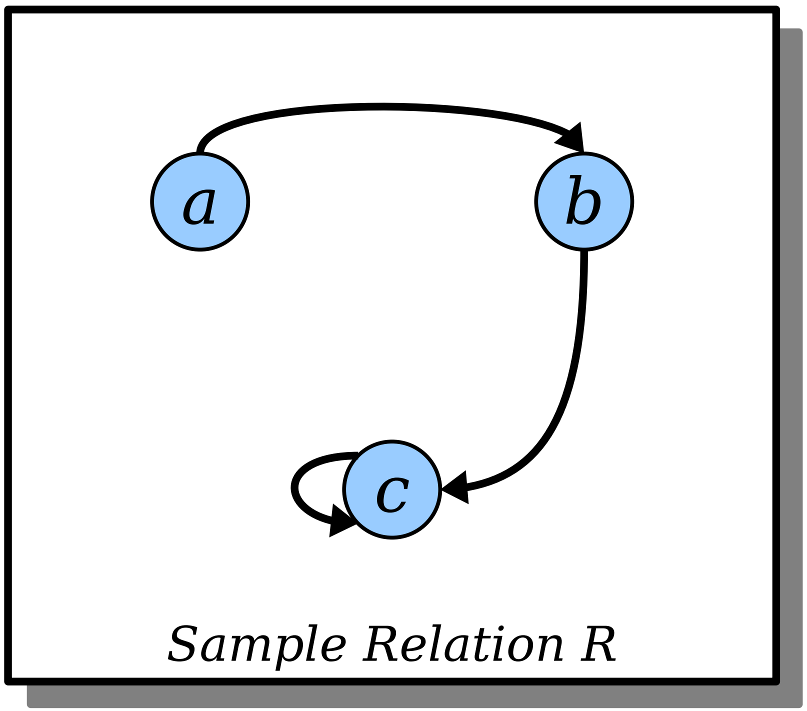

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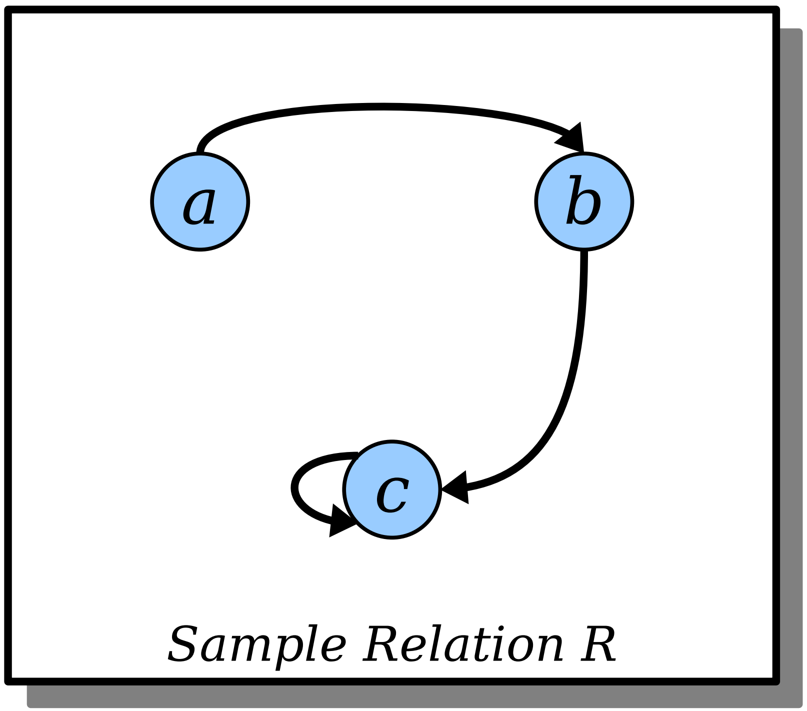

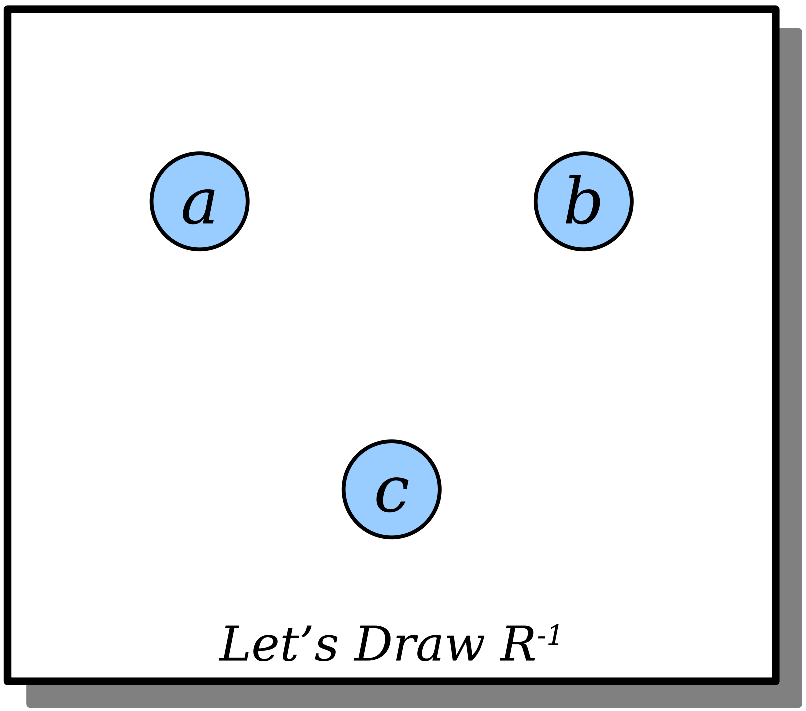

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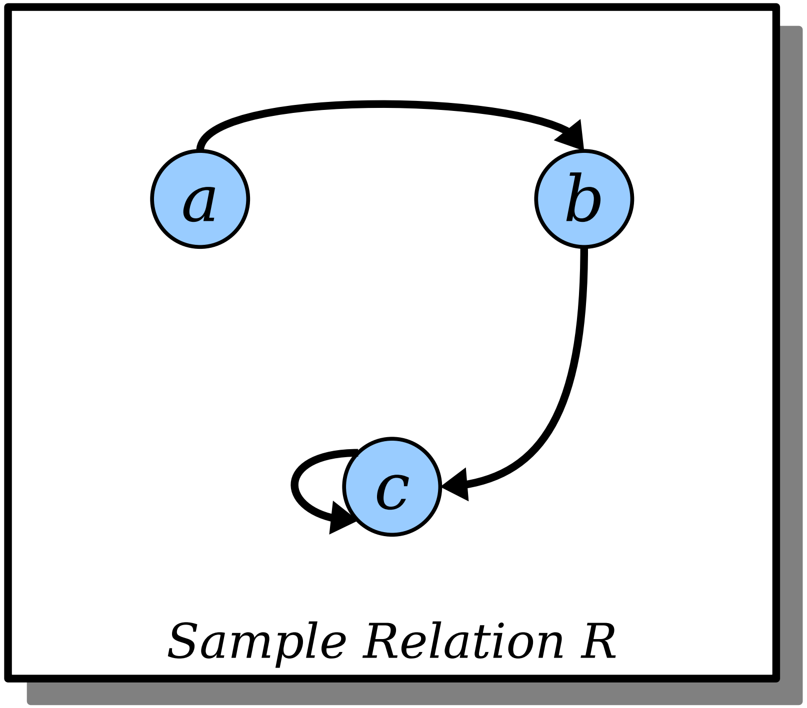

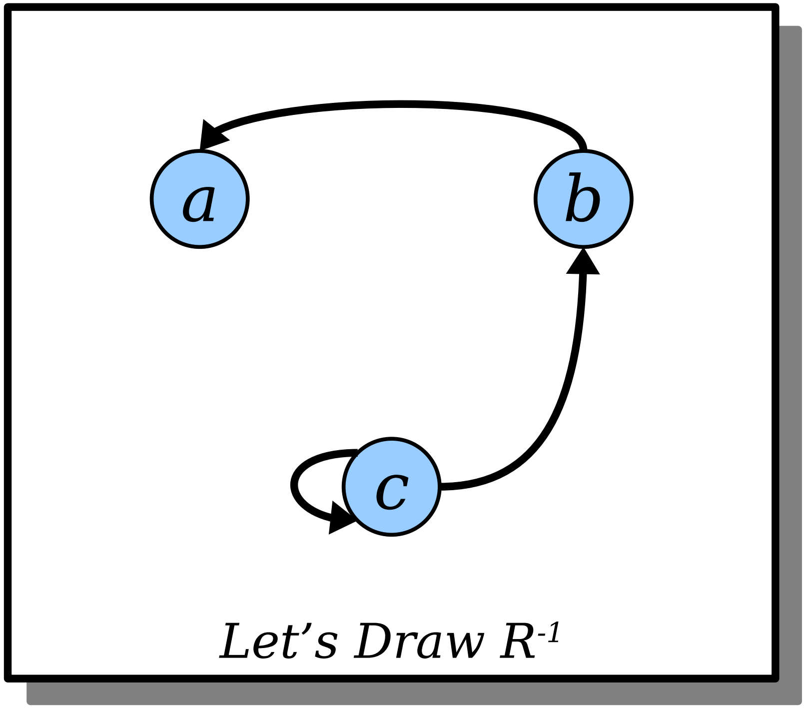

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Prove the following theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

2a) Complete the set-up for this proof by filling in the "assume" and "want to show" statements. Remember that the "assume" should include properly introducing any variables you need to state the assumption.

Assume: ______________________________

Want to show: ________________________

Fill in answer on Gradescope!

Let R be a binary relation over a set A. We can define a new relation over A called the ***inverse relation of R***, denoted R^{-1}, as follows:

$$xR^{-1}y \qquad \text{if} \qquad yRx$$

Prove the following theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume:

Pick an arbitrary relation R over a set A and assume it's an equivalence relation.

Want to Show:

We want to show that R^{-1} is also an equivalence relation over A.

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

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Assume	***Want to Show***

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume	***Want to Show***
R is an equivalence relation	

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume	***Want to Show***
R is an equivalence relation	R^{-1} is an equivalence relation

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume

R is an equivalence relation

Want to Show

R^{-1} is an equivalence relation

Relevant Definitions

$$xR^{-1}y \text{ if } yRx$$

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume

R is an equivalence relation

Want to Show

R^{-1} is an equivalence relation

2b) Expand out both the Assume and the Want to Show one step further using the definition of an equivalence relation.

Fill in answer on Gradescope!

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Want to Show

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Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume	***Want to Show***
R is an equivalence relation · R is reflexive · R is symmetric · R is transitive	R^{-1} is an equivalence relation

Relevant Definitions

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Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume

R is an equivalence relation

- R is reflexive
- R is symmetric
- R is transitive

Want to Show

R^{-1} is an equivalence relation

- R^{-1} is reflexive
- R^{-1} is symmetric
- R^{-1} is transitive

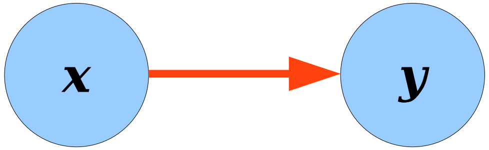
Relevant Definitions

$$xR^{-1}y \text{ if } yRx$$

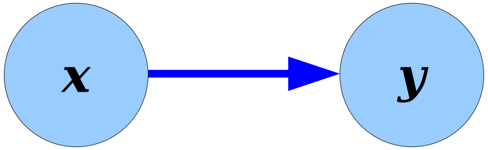
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

A great proofwriting strategy is to ***draw pictures*** – it's often easier to reason about concrete circles, lines, and arrows than abstract mathematical definitions.

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.



We'll use a ***red arrow*** to denote that $x\boldsymbol{R}y$



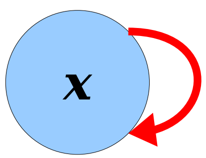
And a ***blue arrow*** to denote that $x\boldsymbol{R^{-1}}y$

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assume:

R is reflexive

$\forall x \in A.\ x\boldsymbol{R}x$



We can always draw a red self-loop

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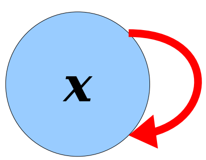

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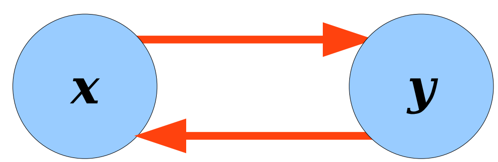
R is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x\boldsymbol{R}y \to y\boldsymbol{R}x)$



If there's a red arrow in one direction, we can draw one in the other direction

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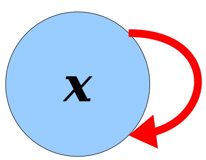

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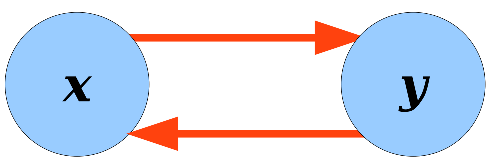
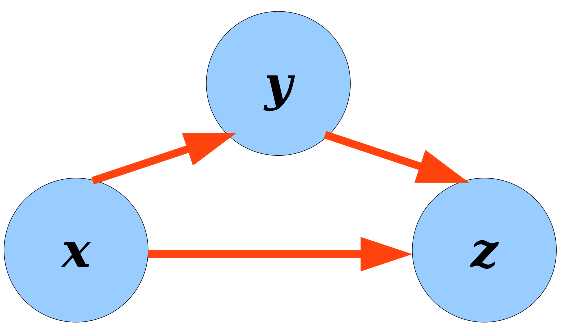


If there's a red arrow in one direction, we can draw one in the other direction

R is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(x\mathbf{R}y \land y\mathbf{R}z \to x\mathbf{R}z)$

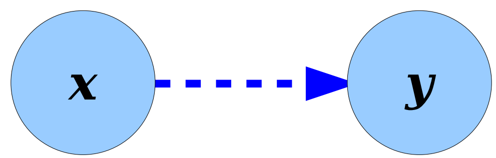


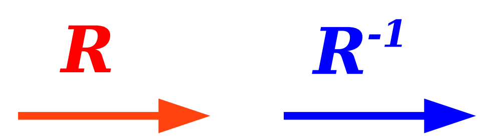
If you can get somewhere by following red arrows, you can draw a red arrow directly there

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

$x\boldsymbol{R^{-1}}y$ if $y\boldsymbol{R}x$

When can we draw a blue arrow?





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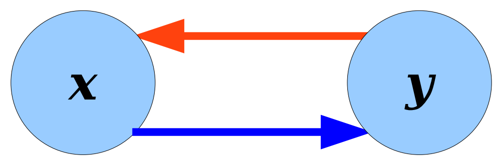
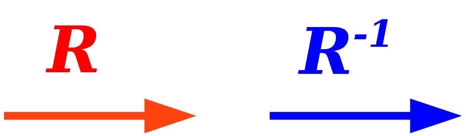
$$xR^{-1}y \text{ if } yRx$$

When can we draw a blue arrow?

If there's a red arrow going one way



Then we can draw a blue arrow going the other way



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Want to Show**:**

R^{-1} is reflexive

$\forall x \in A.\ x R^{-1} x$

$\xrightarrow{R}$ $\xrightarrow{R^{-1}}$

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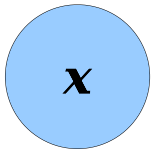

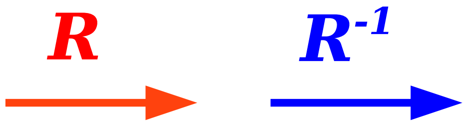

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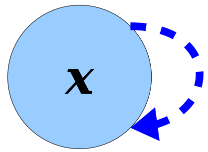
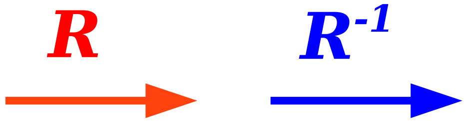


We want to always be able to draw a blue self-loop



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

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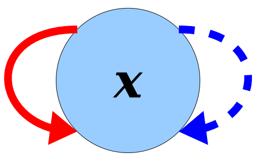
$\forall x \in A.\ xR^{-1}x$

Since we assumed R is reflexive, we can put in this red self loop



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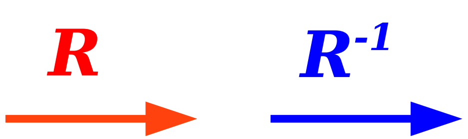

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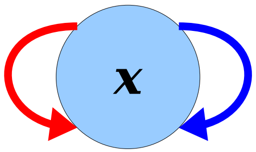
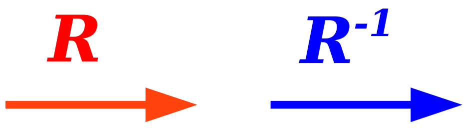


Since there's a red arrow going from x to x, we can draw a blue arrow going "the other way", from x to x



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Want to Show:

R^{-1} is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x R^{-1} y \to y R^{-1} x)$

$R \longrightarrow$ $R^{-1} \longrightarrow$

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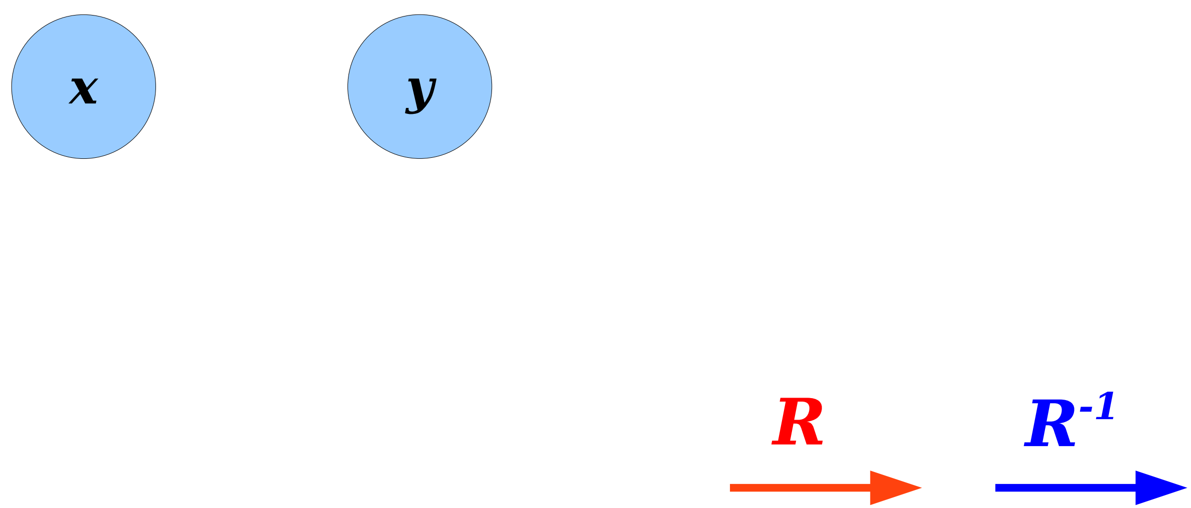

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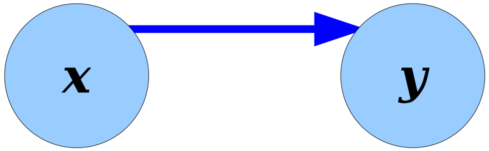

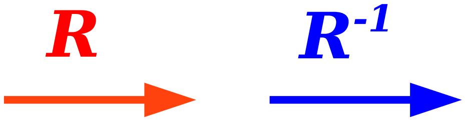

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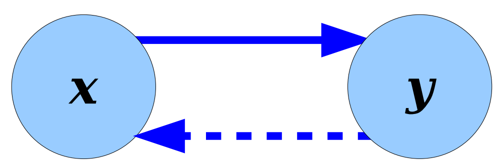
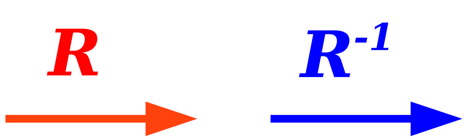


We want to say that if there's a blue arrow in one direction, we can draw one in the other direction



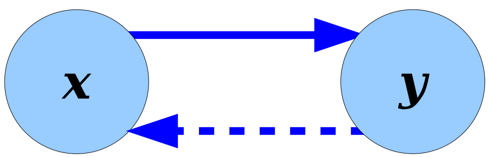
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

***Want to Show*:**

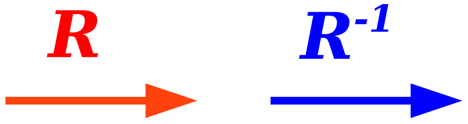
$\boldsymbol{R^{-1}}$ is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x\boldsymbol{R^{-1}}y \rightarrow y\boldsymbol{R^{-1}}x)$

So we'll assume this arrow exists



And prove that this arrow exists too



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

***Want to Show*:**

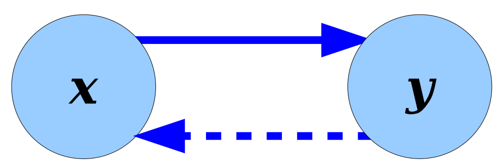
2c) Fill in the missing steps for the proof that R^{-1} is symmetric.

Fill in answer on Gradescope!

R^{-1} is symmetric

$\forall x \in A.\ \forall y \in A.\ (x R^{-1} y \rightarrow y R^{-1} x)$

So we'll assume this arrow exists

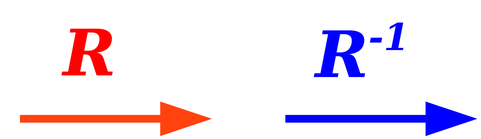


And prove that this arrow exists too

Remember that you can apply this definition

$x R^{-1} y$ if $y R x$

in the other direction too



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***Want to Show*:**

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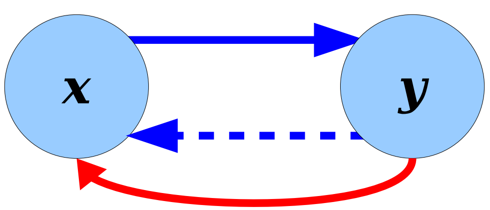
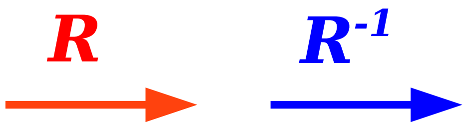
$xR^{-1}y$ if yRx



Since there's a blue arrow from x to y, we can draw a red arrow going the other way, from y to x



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

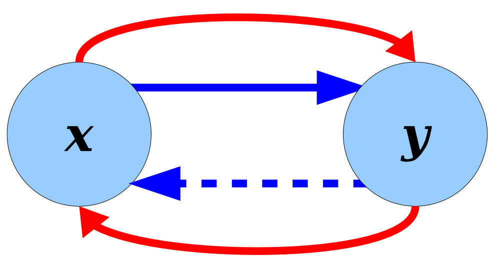
***Want to Show*:**

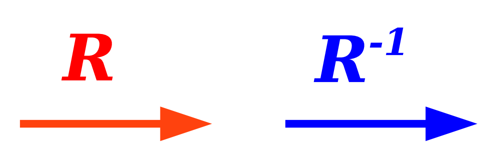
$\boldsymbol{R^{-1}}$ is symmetric

$\forall x \in A.\ \forall y \in A.$

$(x\boldsymbol{R^{-1}}y \rightarrow y\boldsymbol{R^{-1}}x)$

Since R is symmetric,
we can use this arrow
to draw a red arrow
from x to y





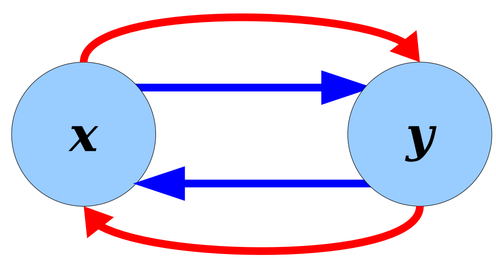
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Want to Show:

R^{-1} is symmetric

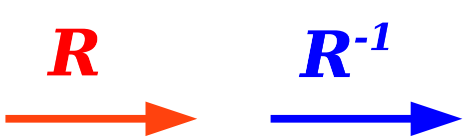
$\forall x \in A.\ \forall y \in A.$

$(xR^{-1}y \rightarrow yR^{-1}x)$



$xR^{-1}y$ if yRx

Finally, since we have a red arrow from x to y, we can apply the definition of R^{-1} again to conclude that there's a blue arrow from y to x



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

***Want to Show*:**

$\boldsymbol{R^{-1}}$ is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(x\boldsymbol{R^{-1}}y \land y\boldsymbol{R^{-1}}z \to x\boldsymbol{R^{-1}}z)$

$\boldsymbol{R}$ $\quad$ $\boldsymbol{R^{-1}}$

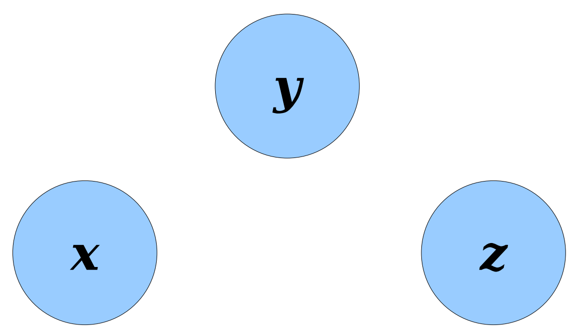
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

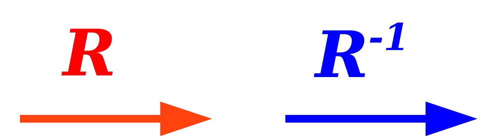
***Want to Show*:**

$\boldsymbol{R^{-1}}$ is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$

$(x\boldsymbol{R^{-1}}y \land y\boldsymbol{R^{-1}}z \to x\boldsymbol{R^{-1}}z)$



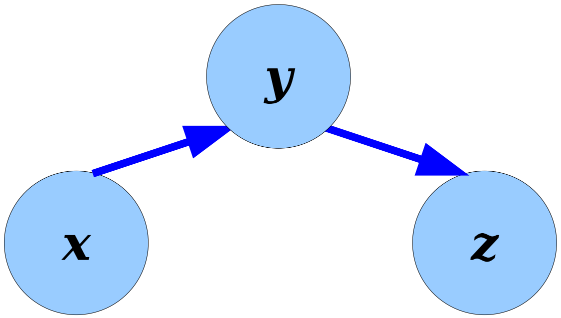


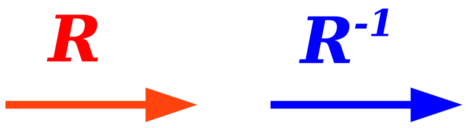
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

***Want to Show*:**

$\boldsymbol{R^{-1}}$ is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(x\boldsymbol{R^{-1}}y \land y\boldsymbol{R^{-1}}z \to x\boldsymbol{R^{-1}}z)$





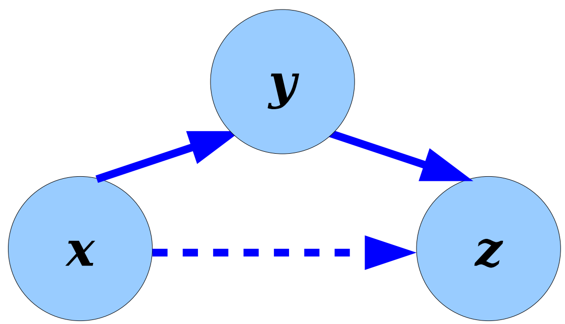
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

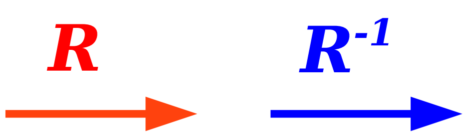
***Want to Show*:**

$\boldsymbol{R^{-1}}$ is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(x\boldsymbol{R^{-1}}y \land y\boldsymbol{R^{-1}}z \to x\boldsymbol{R^{-1}}z)$

We want to say that if we can get from x to z through an intermediary y, then we can draw an arrow straight from x to z





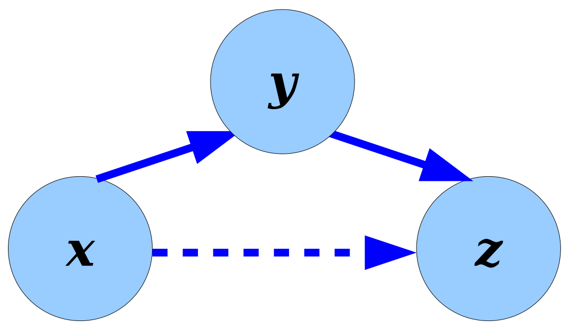
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

***Want to Show*:**

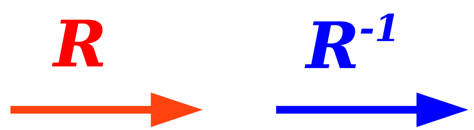
$\boldsymbol{R^{-1}}$ is transitive

$$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$$
$$(xR^{-1}y \land yR^{-1}z \rightarrow xR^{-1}z)$$

So we'll assume that these arrows exist



And prove that this arrow exists too



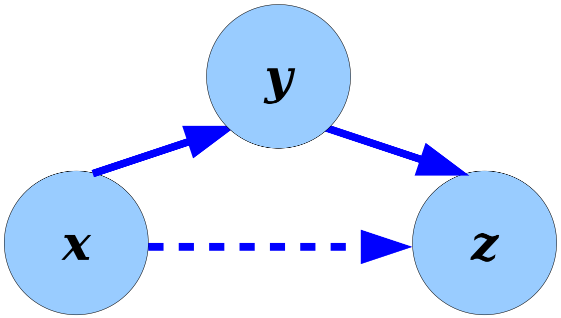
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

***Want to Show*:**

$\boldsymbol{R^{-1}}$ is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(x\boldsymbol{R^{-1}}y \land y\boldsymbol{R^{-1}}z \rightarrow x\boldsymbol{R^{-1}}z)$

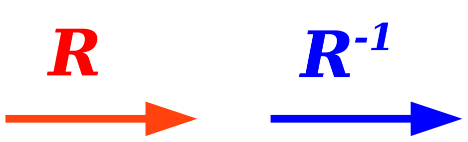
So we'll assume that these arrows exist



And prove that this arrow exists too

2d) Fill in the missing steps for the proof that R^{-1} is transitive.

Fill in answer on Gradescope!



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

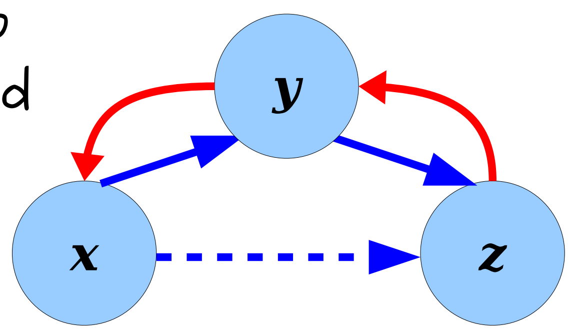
Want to Show:

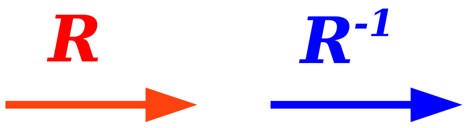
R^{-1} is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$

$(x R^{-1} y \land y R^{-1} z \rightarrow x R^{-1} z)$

We can apply the definition of R^{-1} to draw these two red arrows



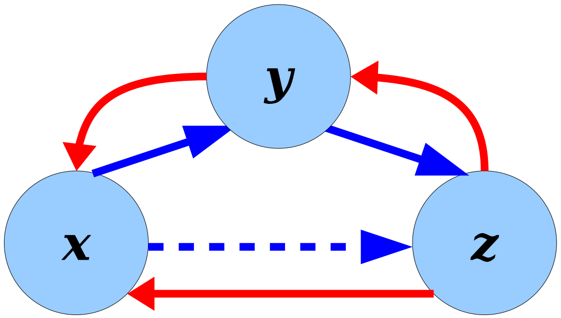


Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

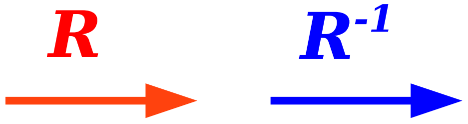
***Want to Show*:**

$\boldsymbol{R^{-1}}$ is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(x\boldsymbol{R^{-1}}y \land y\boldsymbol{R^{-1}}z \to x\boldsymbol{R^{-1}}z)$



Then since R is transitive, we can draw this arrow

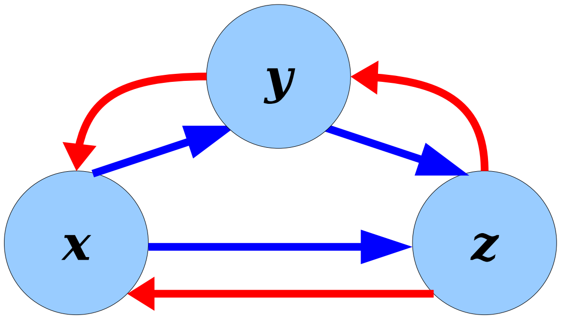


Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

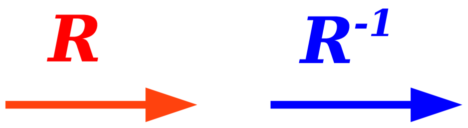
Want to Show:

$\boldsymbol{R^{-1}}$ is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(x\boldsymbol{R^{-1}}y \land y\boldsymbol{R^{-1}}z \to x\boldsymbol{R^{-1}}z)$

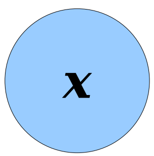


Applying the definition of R^{-1} again gives us the arrow we desire!



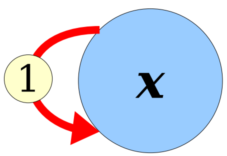
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

R^{-1} is reflexive



Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

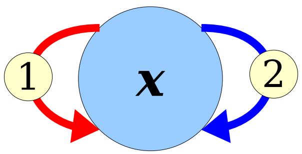
$\boldsymbol{R^{-1}}$ is reflexive



1 $x\boldsymbol{R}x$
($\boldsymbol{R}$ is reflexive)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

$\boldsymbol{R^{-1}}$ is reflexive



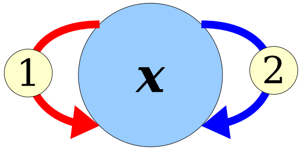
1 $x\boldsymbol{R}x$
($\boldsymbol{R}$ is reflexive)

2 $x\boldsymbol{R^{-1}}x$
(definition of $\boldsymbol{R^{-1}}$)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

$\boldsymbol{R^{-1}}$ is reflexive

$\boldsymbol{R^{-1}}$ is symmetric



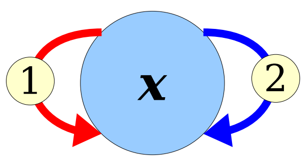
1 $x\boldsymbol{R}x$
($\boldsymbol{R}$ is reflexive)

2 $x\boldsymbol{R^{-1}}x$
(definition of $\boldsymbol{R^{-1}}$)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

$\boldsymbol{R^{-1}}$ is reflexive

$\boldsymbol{R^{-1}}$ is symmetric

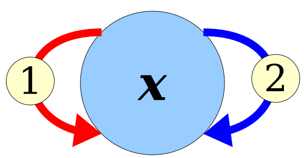


1 $x\boldsymbol{R}x$
($\boldsymbol{R}$ is reflexive)

2 $x\boldsymbol{R^{-1}}x$
(definition of $\boldsymbol{R^{-1}}$)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

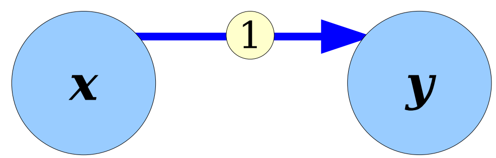
$\boldsymbol{R^{-1}}$ is reflexive



① $x\boldsymbol{R}x$
($\boldsymbol{R}$ is reflexive)

② $x\boldsymbol{R^{-1}}x$
(definition of $\boldsymbol{R^{-1}}$)

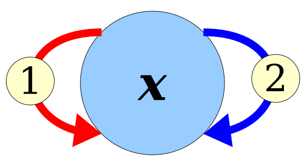
$\boldsymbol{R^{-1}}$ is symmetric



① $x\boldsymbol{R^{-1}}y$
(by assumption)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

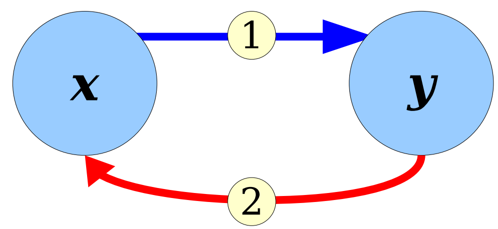
$\boldsymbol{R^{-1}}$ is reflexive



① $x\boldsymbol{R}x$
($\boldsymbol{R}$ is reflexive)

② $x\boldsymbol{R^{-1}}x$
(definition of $\boldsymbol{R^{-1}}$)

$\boldsymbol{R^{-1}}$ is symmetric

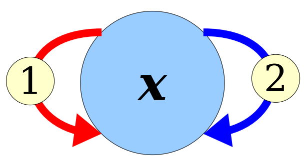


① $x\boldsymbol{R^{-1}}y$
(by assumption)

② $y\boldsymbol{R}x$
(definition of $\boldsymbol{R^{-1}}$)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

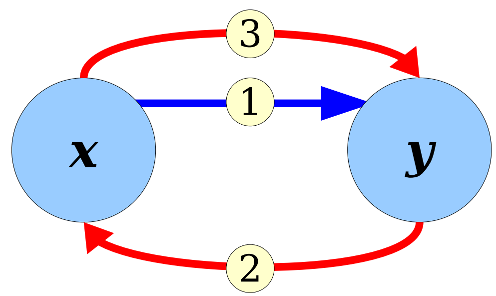
R^{-1} is reflexive



① xRx
(R is reflexive)

② $xR^{-1}x$
(definition of R^{-1})

R^{-1} is symmetric



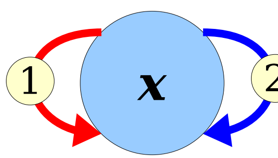
① $xR^{-1}y$
(by assumption)

② yRx
(definition of R^{-1})

③ xRy
(R is symmetric)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

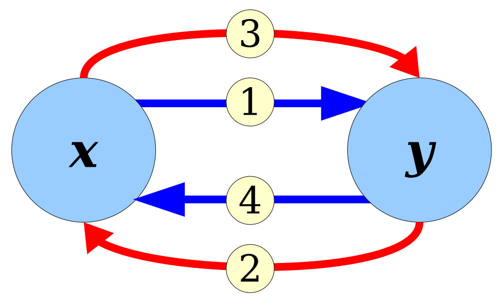
R^{-1} is reflexive



1 xRx
(R is reflexive)

2 $xR^{-1}x$
(definition of R^{-1})

R^{-1} is symmetric



1 $xR^{-1}y$
(by assumption)

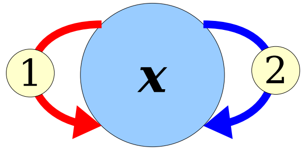
2 yRx
(definition of R^{-1})

3 xRy
(R is symmetric)

4 $yR^{-1}x$
(definition of R^{-1})

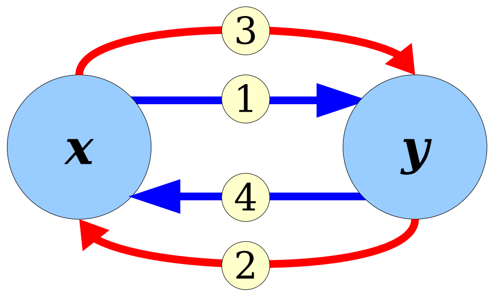
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

$\boldsymbol{R^{-1}}$ is reflexive



1. $x\boldsymbol{R}x$
   ($\boldsymbol{R}$ is reflexive)
2. $x\boldsymbol{R^{-1}}x$
   (definition of $\boldsymbol{R^{-1}}$)

$\boldsymbol{R^{-1}}$ is symmetric

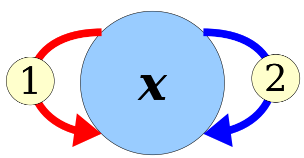


1. $x\boldsymbol{R^{-1}}y$
   (by assumption)
2. $y\boldsymbol{R}x$
   (definition of $\boldsymbol{R^{-1}}$)
3. $x\boldsymbol{R}y$
   ($\boldsymbol{R}$ is symmetric)
4. $y\boldsymbol{R^{-1}}x$
   (definition of $\boldsymbol{R^{-1}}$)

$\boldsymbol{R^{-1}}$ is transitive

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

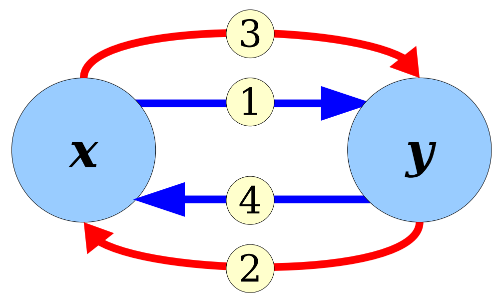
R^{-1} is reflexive



1 xRx
(R is reflexive)

2 $xR^{-1}x$
(definition of R^{-1})

R^{-1} is symmetric



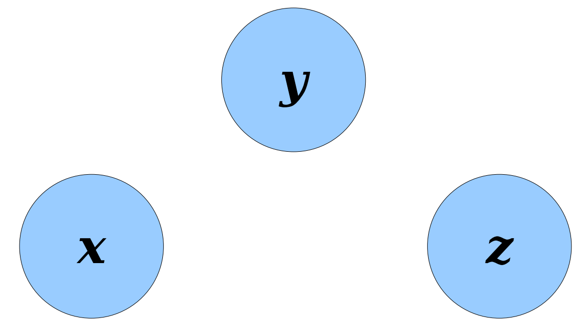
1 $xR^{-1}y$
(by assumption)

2 yRx
(definition of R^{-1})

3 xRy
(R is symmetric)

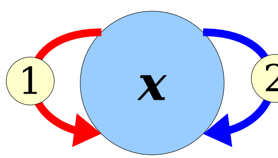
4 $yR^{-1}x$
(definition of R^{-1})

R^{-1} is transitive



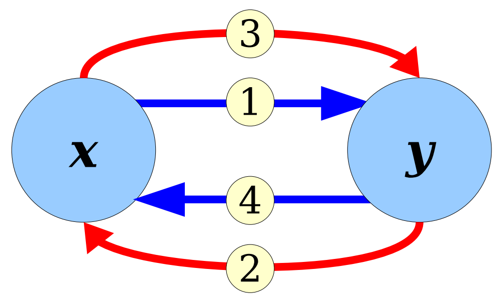
Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

$\boldsymbol{R^{-1}}$ is reflexive



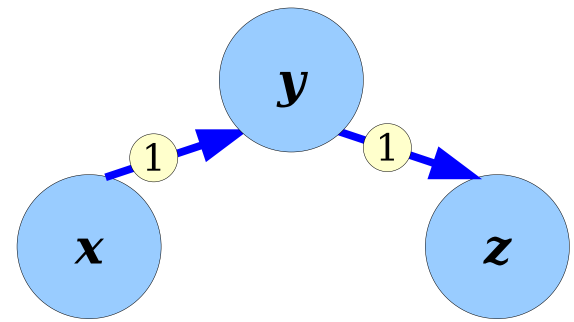
1. $x\boldsymbol{R}x$
   ($\boldsymbol{R}$ is reflexive)
2. $x\boldsymbol{R^{-1}}x$
   (definition of $\boldsymbol{R^{-1}}$)

$\boldsymbol{R^{-1}}$ is symmetric



1. $x\boldsymbol{R^{-1}}y$
   (by assumption)
2. $y\boldsymbol{R}x$
   (definition of $\boldsymbol{R^{-1}}$)
3. $x\boldsymbol{R}y$
   ($\boldsymbol{R}$ is symmetric)
4. $y\boldsymbol{R^{-1}}x$
   (definition of $\boldsymbol{R^{-1}}$)

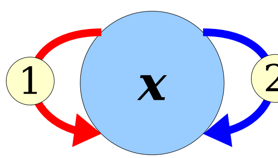
$\boldsymbol{R^{-1}}$ is transitive



1. $x\boldsymbol{R^{-1}}y$ and $y\boldsymbol{R^{-1}}z$
   (by assumption)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

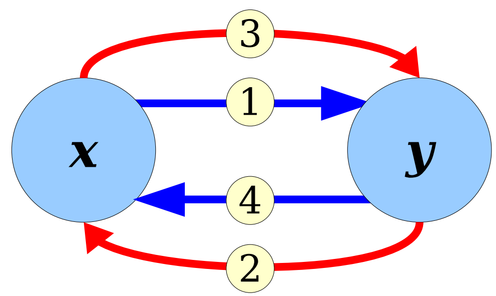
R^{-1} is reflexive



① xRx
(R is reflexive)

② $xR^{-1}x$
(definition of R^{-1})

R^{-1} is symmetric



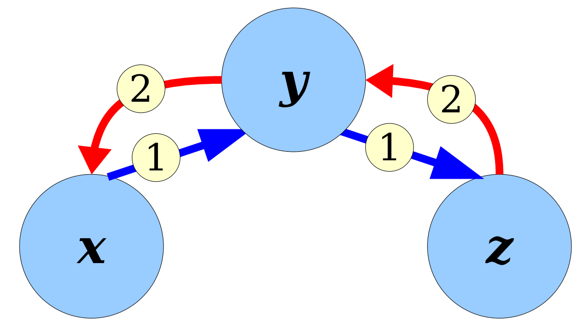
① $xR^{-1}y$
(by assumption)

② yRx
(definition of R^{-1})

③ xRy
(R is symmetric)

④ $yR^{-1}x$
(definition of R^{-1})

R^{-1} is transitive

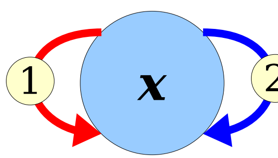


① $xR^{-1}y$ and $yR^{-1}z$
(by assumption)

② yRx and zRy
(definition of R^{-1})

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

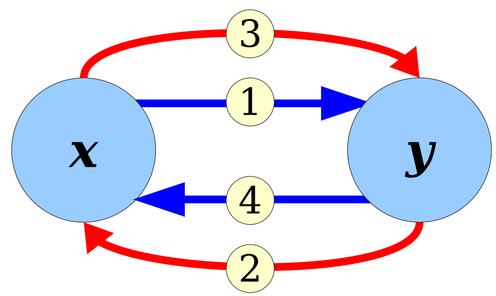
$\boldsymbol{R^{-1}}$ is reflexive



(1) $x\boldsymbol{R}x$
($\boldsymbol{R}$ is reflexive)

(2) $x\boldsymbol{R^{-1}}x$
(definition of $\boldsymbol{R^{-1}}$)

$\boldsymbol{R^{-1}}$ is symmetric



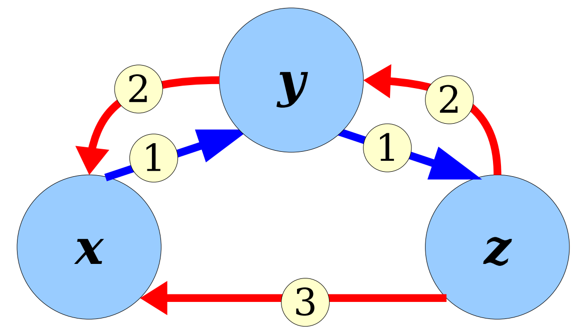
(1) $x\boldsymbol{R^{-1}}y$
(by assumption)

(2) $y\boldsymbol{R}x$
(definition of $\boldsymbol{R^{-1}}$)

(3) $x\boldsymbol{R}y$
($\boldsymbol{R}$ is symmetric)

(4) $y\boldsymbol{R^{-1}}x$
(definition of $\boldsymbol{R^{-1}}$)

$\boldsymbol{R^{-1}}$ is transitive



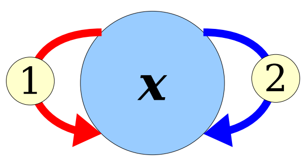
(1) $x\boldsymbol{R^{-1}}y$ and $y\boldsymbol{R^{-1}}z$
(by assumption)

(2) $y\boldsymbol{R}x$ and $z\boldsymbol{R}y$
(definition of $\boldsymbol{R^{-1}}$)

(3) $z\boldsymbol{R}x$
($\boldsymbol{R}$ is transitive)

Theorem: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

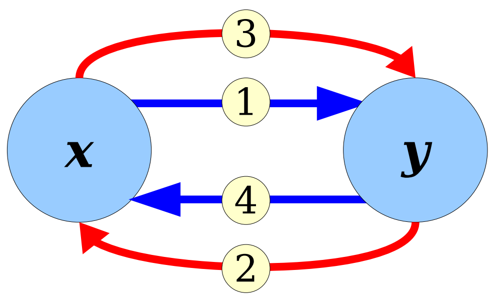
R^{-1} is reflexive



1 $x\mathbf{R}x$
($\mathbf{R}$ is reflexive)

2 $x\mathbf{R}^{-1}x$
(definition of $\mathbf{R}^{-1}$)

R^{-1} is symmetric



1 $x\mathbf{R}^{-1}y$
(by assumption)

2 $y\mathbf{R}x$
(definition of $\mathbf{R}^{-1}$)

3 $x\mathbf{R}y$
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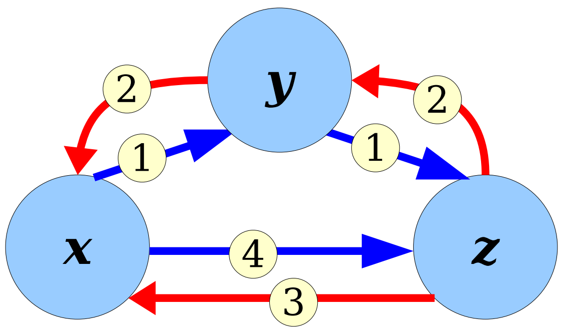
R^{-1} is transitive



1 $x\mathbf{R}^{-1}y$ and $y\mathbf{R}^{-1}z$
(by assumption)

2 $y\mathbf{R}x$ and $z\mathbf{R}y$
(definition of $\mathbf{R}^{-1}$)

3 $z\mathbf{R}x$
($\mathbf{R}$ is transitive)

4 $x\mathbf{R}^{-1}z$
(definition of $\mathbf{R}^{-1}$)

Theorem: If R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Proof: Let R be an equivalence relation over a set A. We want to show that R^{-1} is also an equivalence relation over A by proving that R^{-1} is reflexive, symmetric, and transitive.

To prove that R^{-1} is reflexive, consider any $x \in A$. We want to show that $xR^{-1}x$. By definition, this means that we want to show that xRx. And since R is reflexive, we know xRx holds.

To prove that R^{-1} is symmetric, consider any $x, y \in A$ where $xR^{-1}y$. We want to show that $yR^{-1}x$ holds. Since $xR^{-1}y$ holds, we know that yRx holds. Since R is symmetric and yRx is true, we know that xRy is true. Therefore by definition of R^{-1}, we know that $yR^{-1}x$ holds.

Finally, to prove that R^{-1} is transitive, consider any $x, y, z \in A$ where $xR^{-1}y$ and $yR^{-1}z$. We want to show that $xR^{-1}z$. Since $xR^{-1}y$ and $yR^{-1}z$, we know that yRx and that zRy. Since zRy and yRx, by transitivity of R we see that zRx. Thus by definition of R^{-1}, we know that $xR^{-1}z$ holds, as required. ■

Part 3: ***Functions***

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

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Let's ***draw some pictures*** to try and develop an intuitive feel for why this result is true.

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

This function is defined from some set A to itself, so we can draw that like this.

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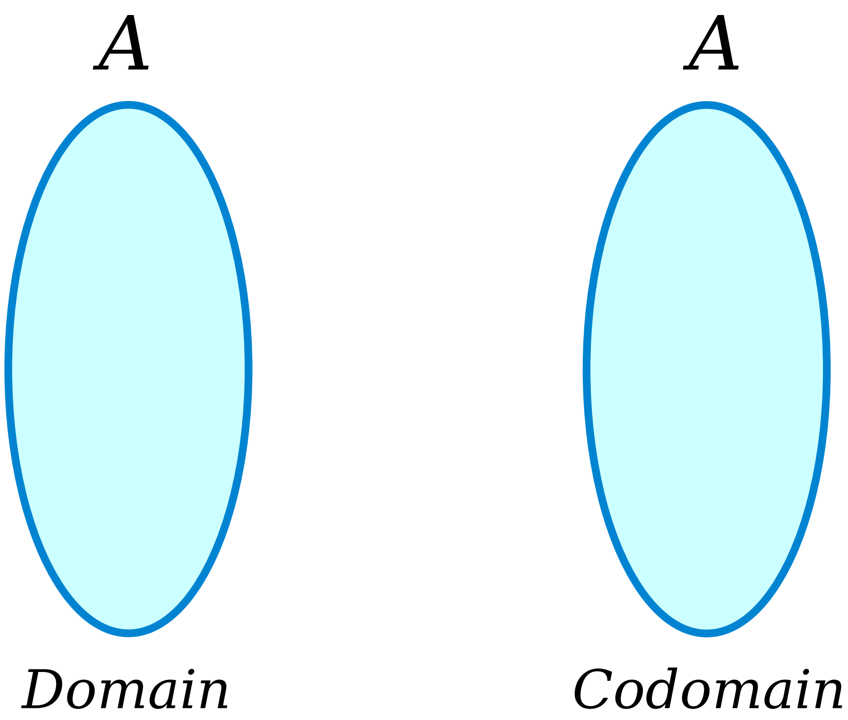

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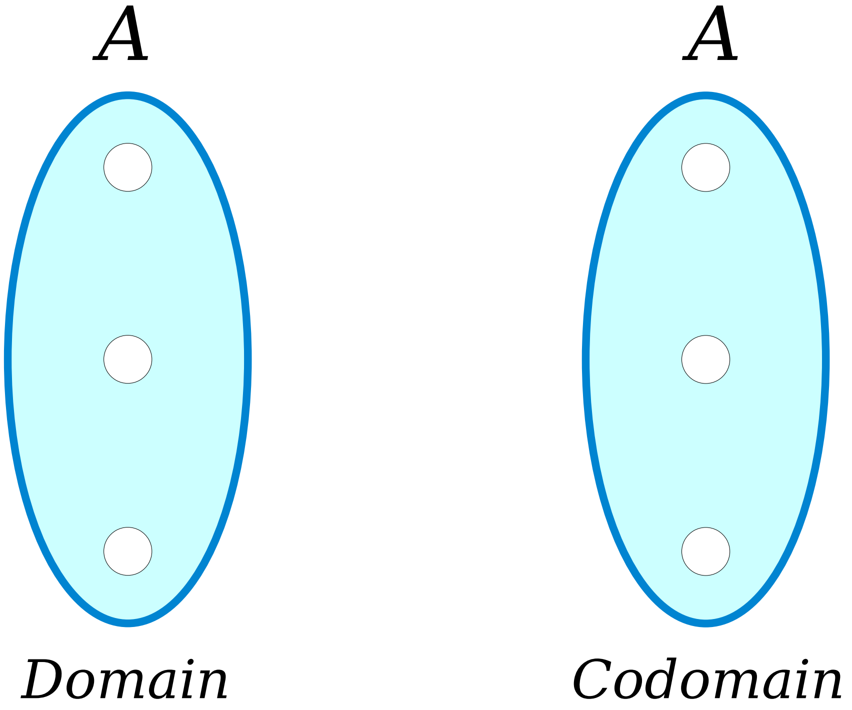
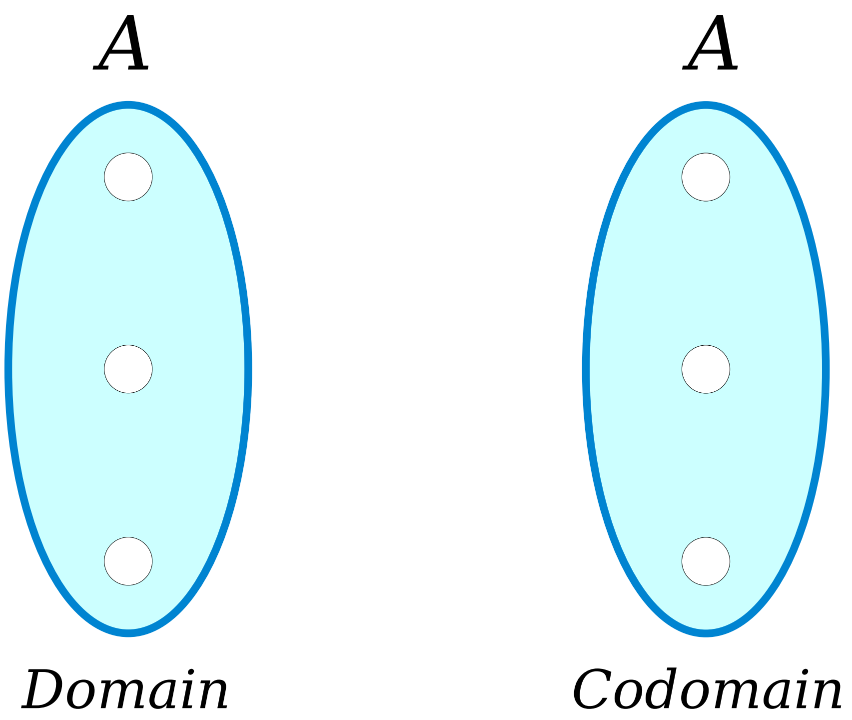


A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

Since this function goes from the set A back to itself, rather than drawing two copies of A, it might be easier to draw just one copy.



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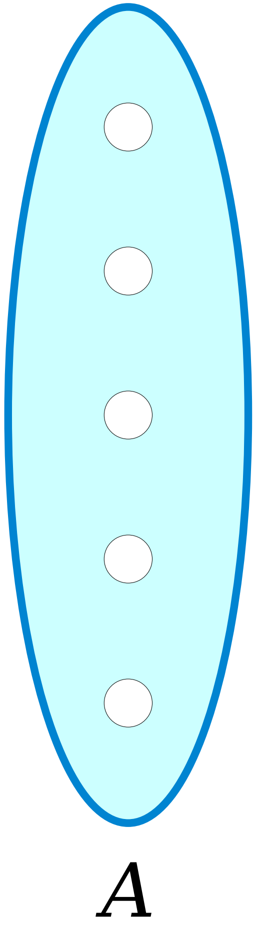

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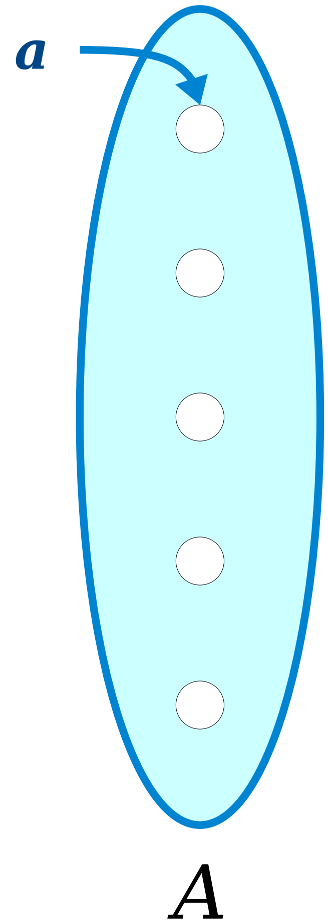

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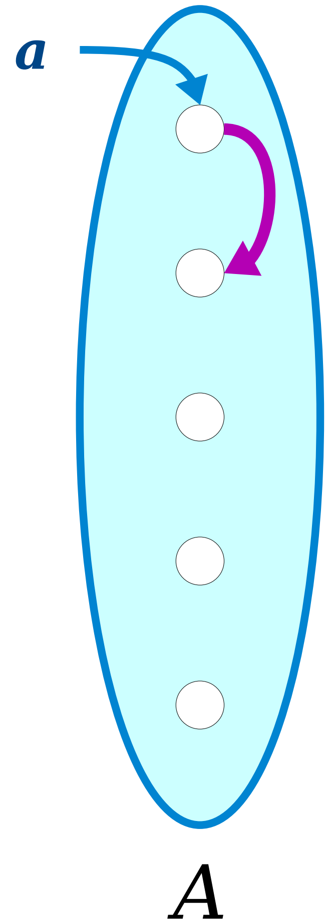

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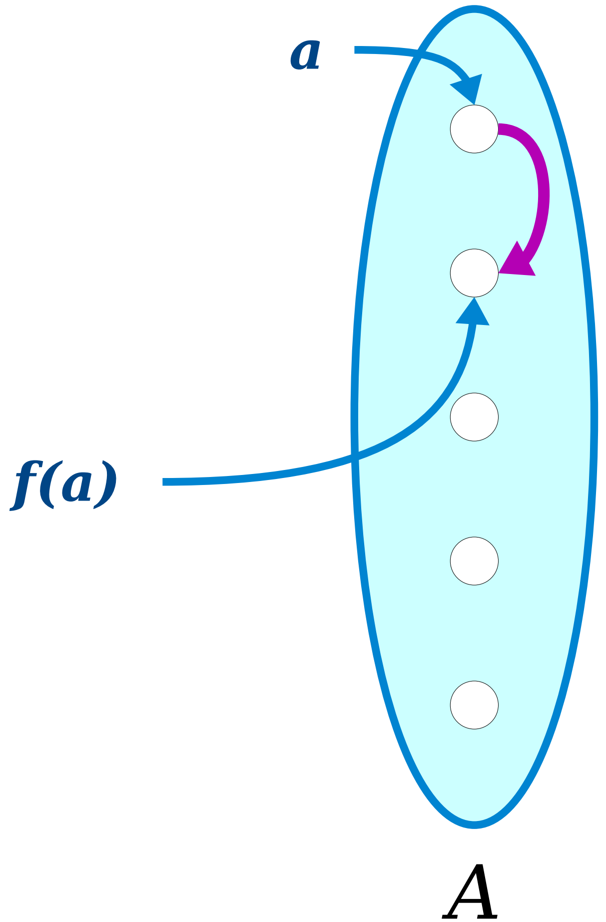

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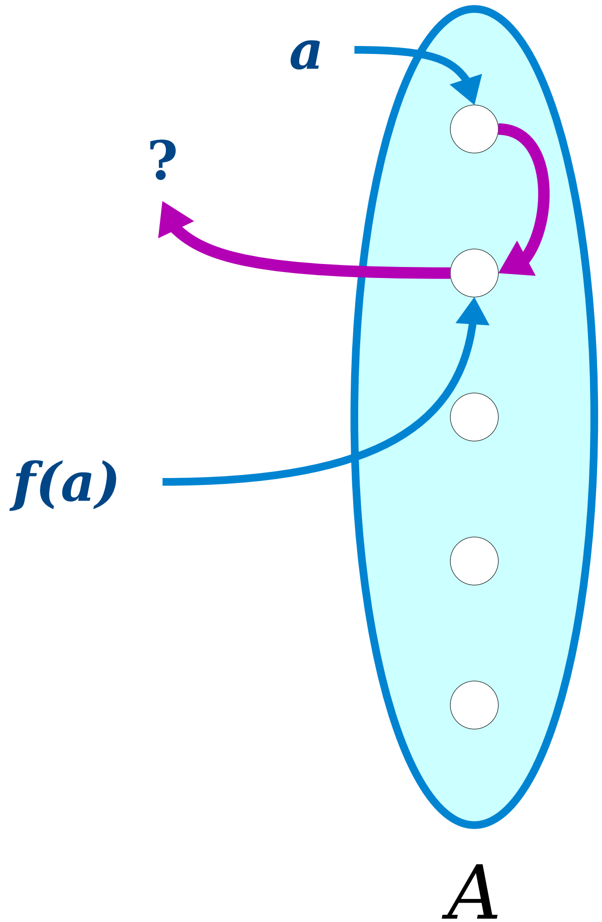
The function must be defined for each element of the set A. Can we say anything about where this arrow points?



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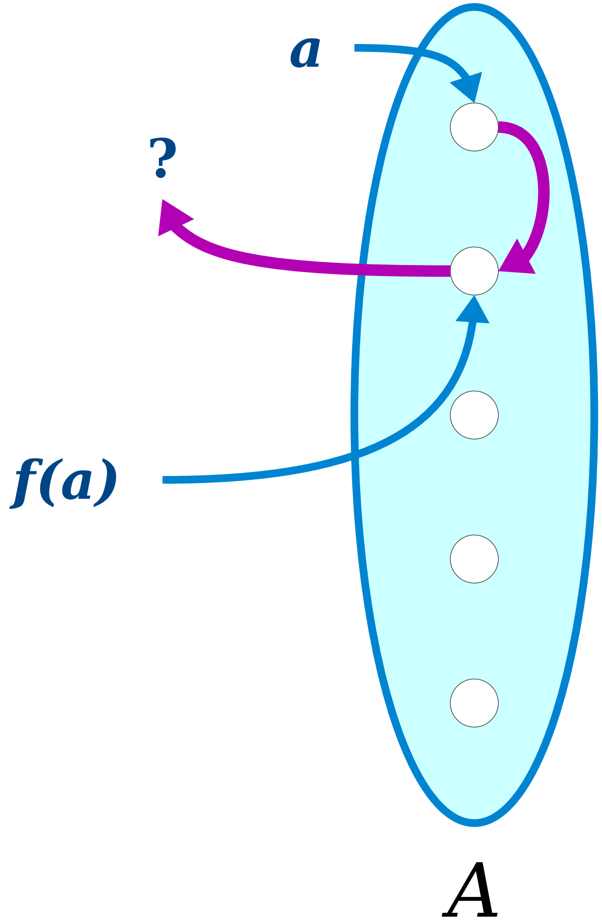
This part of the definition says that, in particular, we have to have f(f(a)) = a.



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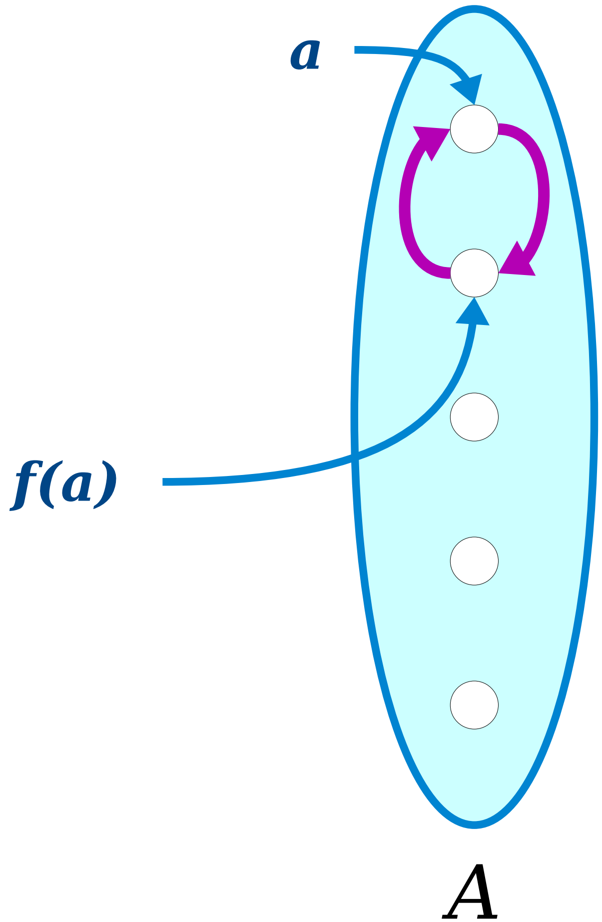

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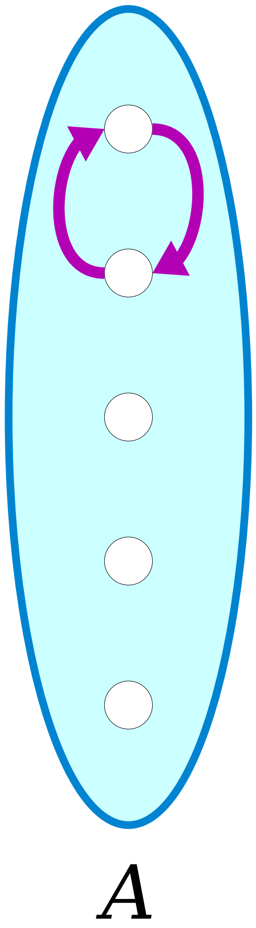

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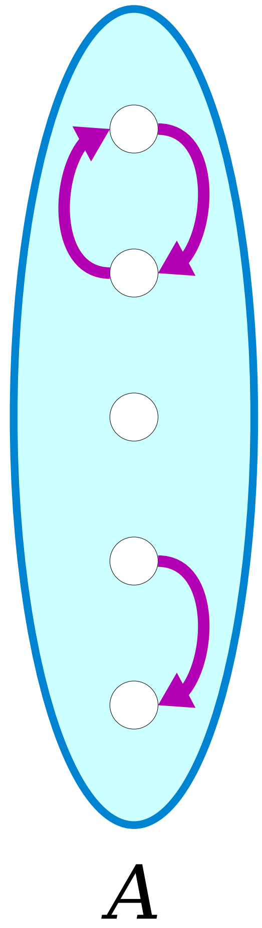

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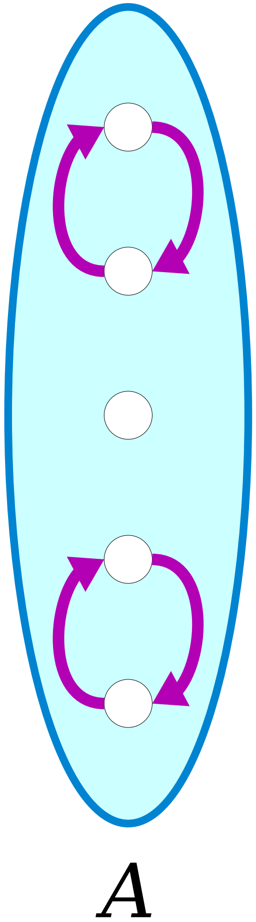

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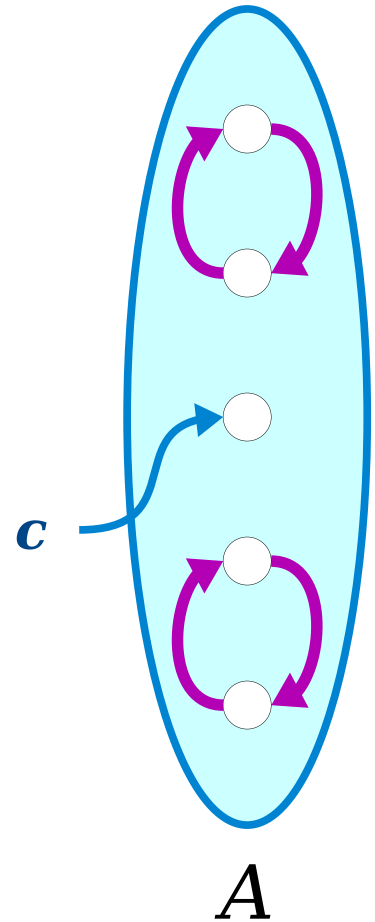
Where can the arrow from c go?



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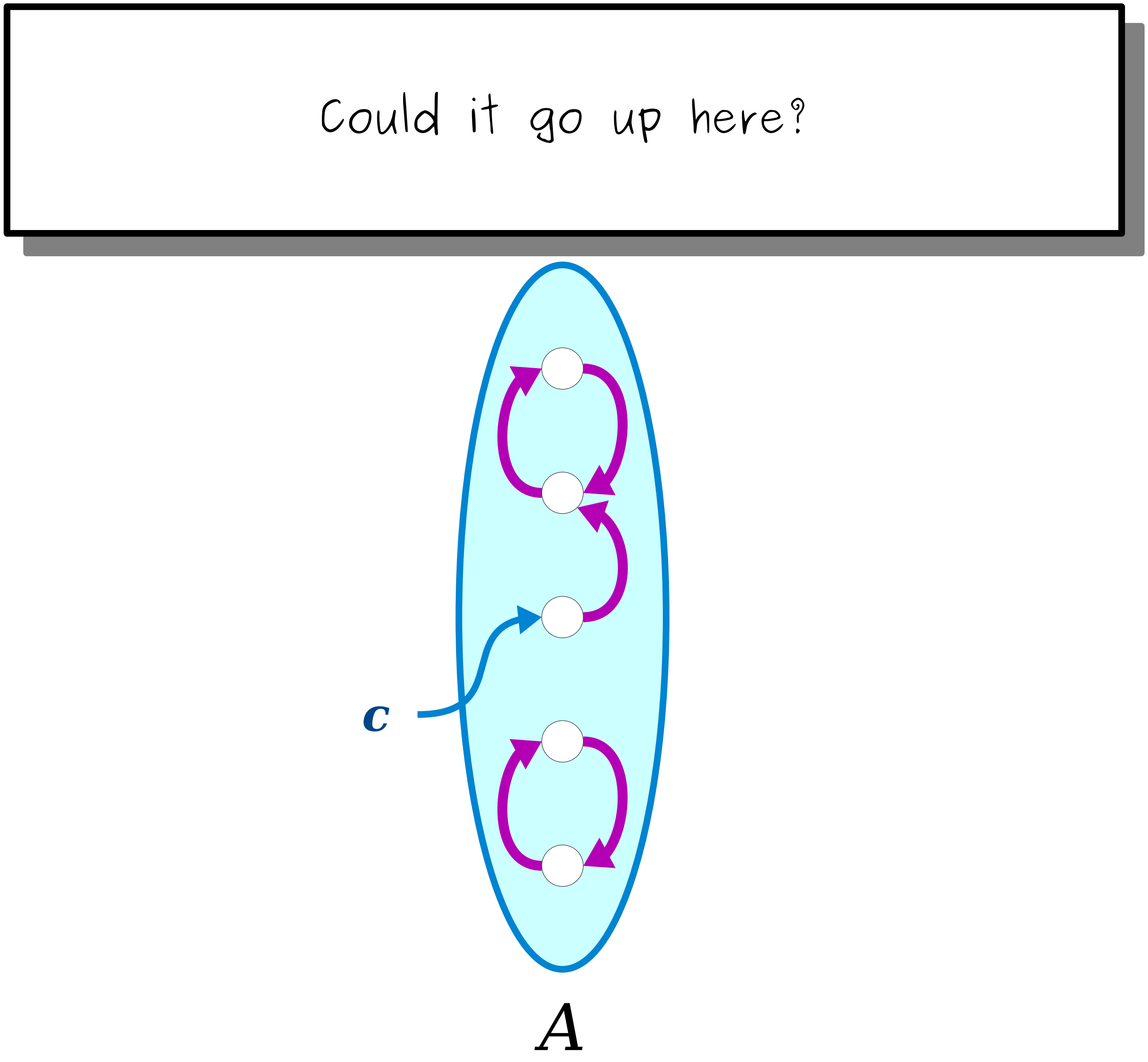
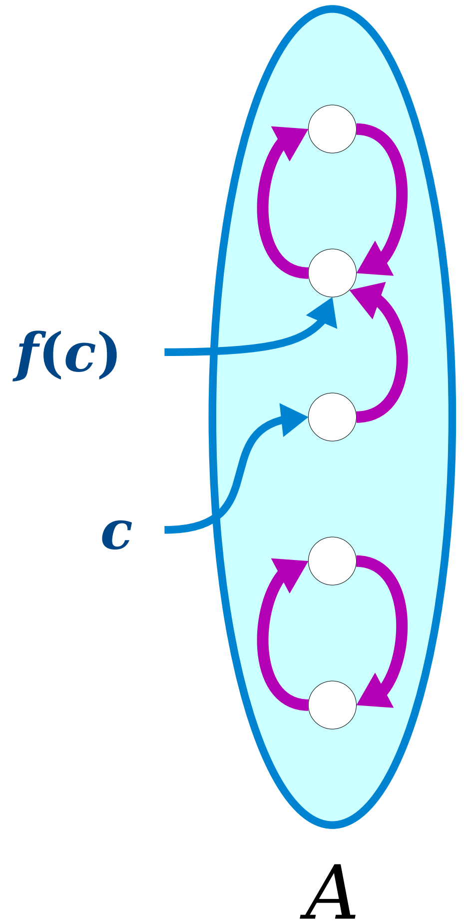


A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

If that's the case, then this element would be f(c).



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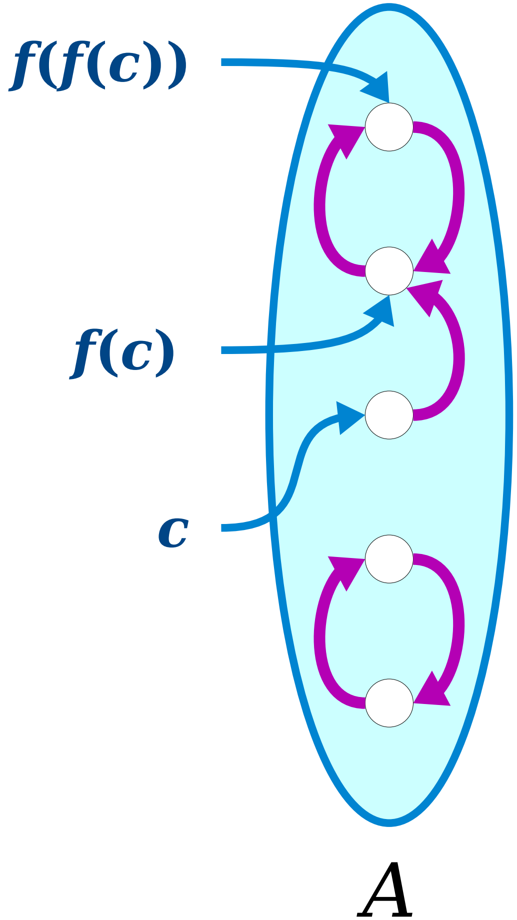
So this element would be $f(f(c))$.



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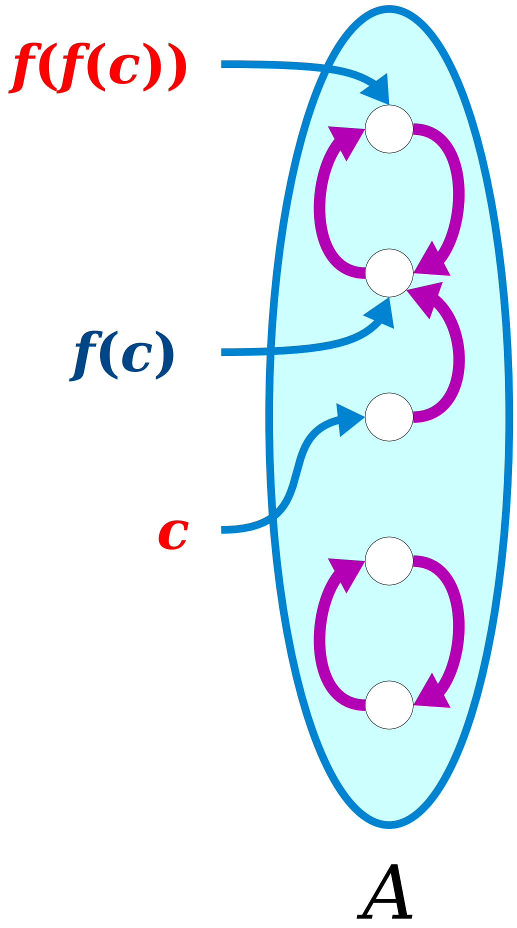
But that's not allowed, because we need to have f(f(c)) = c.



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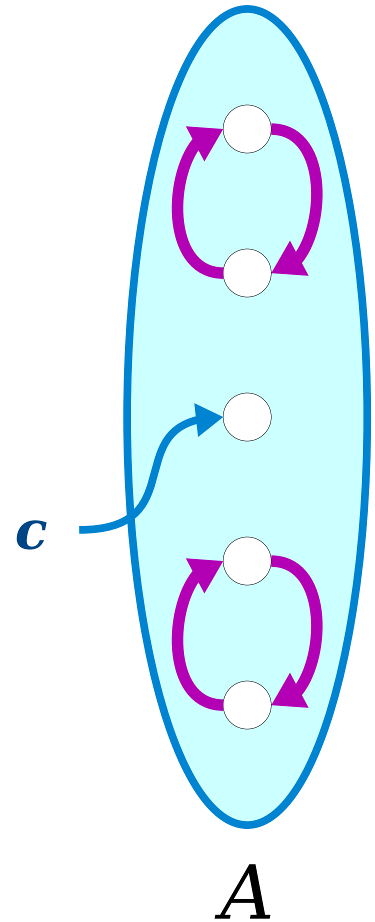
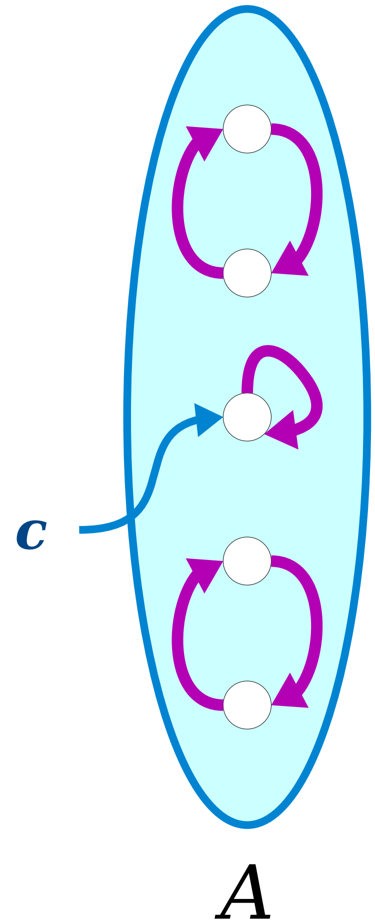
So we can't have the f(c) point to that element. And a similar argument rules out the other elements with arrows on them.



A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

But we could have the arrow point from c to itself, with c = f(c). Do you see why?



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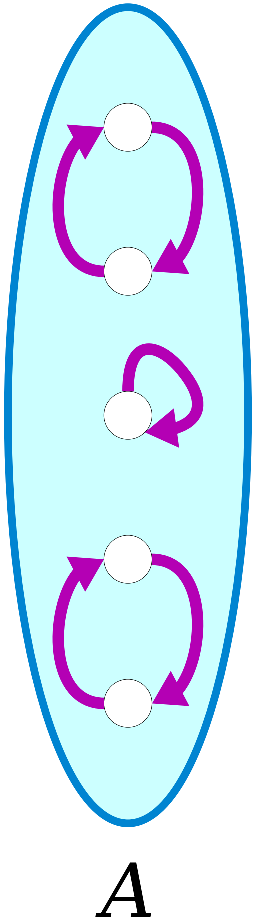

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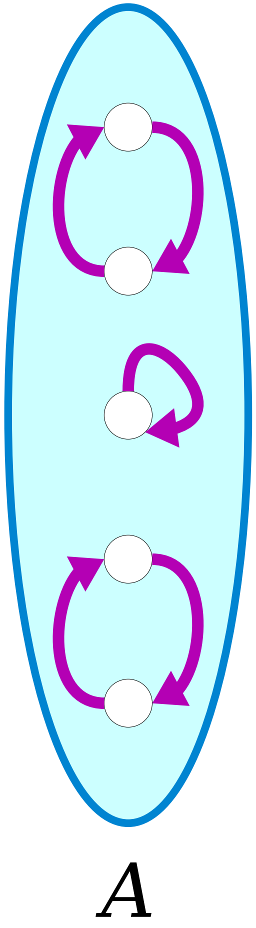


A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

3a) Complete the set-up for this proof by filling in the "assume" and "want to show" statements. Remember that the "assume" should include properly introducing any variables you need to state the assumption.

Assume: ______________________________

Want to show: __________________________

Fill in answer on Gradescope!

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

Assume:

Let $f : A \to A$ be an involution.

Want to Show:

We want to show that f is a bijection

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

Assume:

Let $f : A \to A$ be an involution.

Want to Show:

We want to show that f is a bijection by proving that

__________________________.

3b) Expand out the Want to Show one step further using the definition of a bijection. Then, write out the ***Assume*** and ***Want to Show*** for the two proofs we need to do in order to prove that f is a bijection.

Fill in answer on Gradescope!

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

Assume:

Let $f : A \to A$ be an involution.

Want to Show:

We want to show that f is a bijection by proving that it's both injective and surjective.

Injectivity:

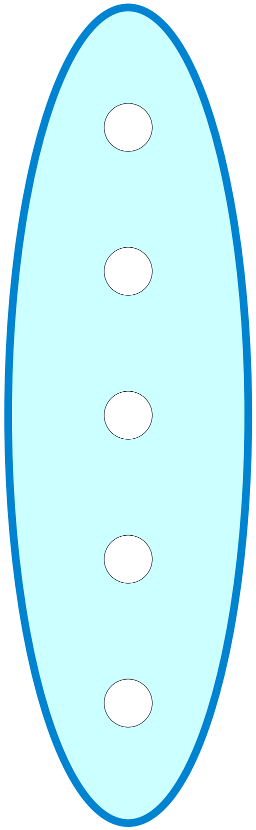
Assume: Pick $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$
Want to Show: $a_1 = a_2$

Surjectivity:

Assume: Pick an element $b \in A$.
Want to Show: There exists an $a \in A$ such that $f(a) = b$

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

Part 1: Injectivity



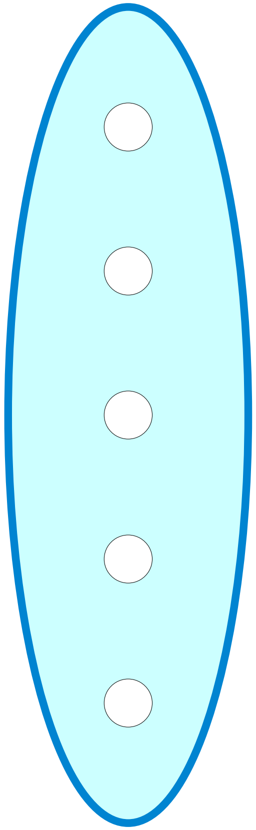
$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

("If the outputs are the same, the inputs are the same.")

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

Part 1: Injectivity

Pick arbitrary a_1 and a_2 from the domain



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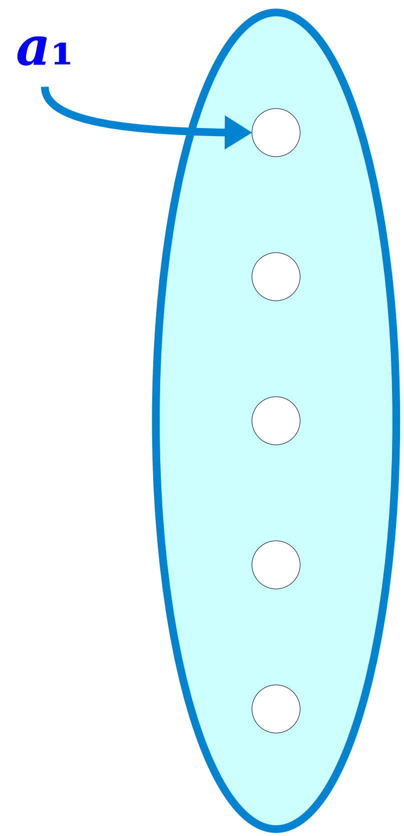

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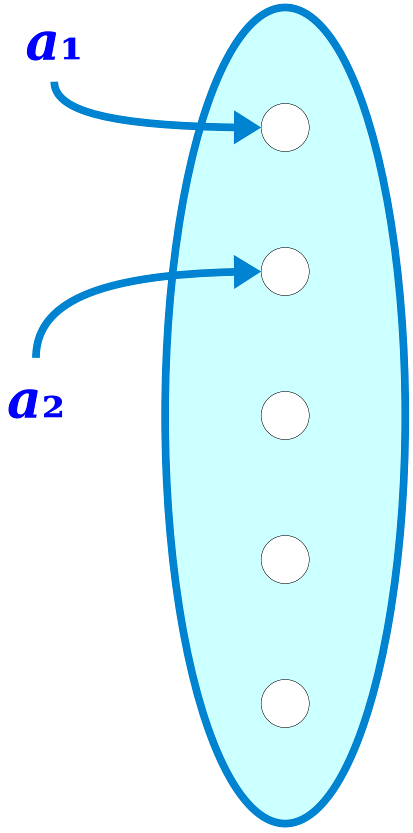

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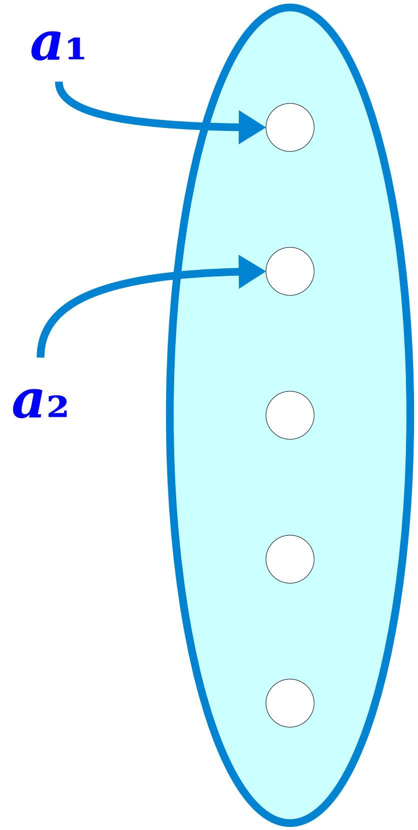

Assume that $f(a_1) = f(a_2)$

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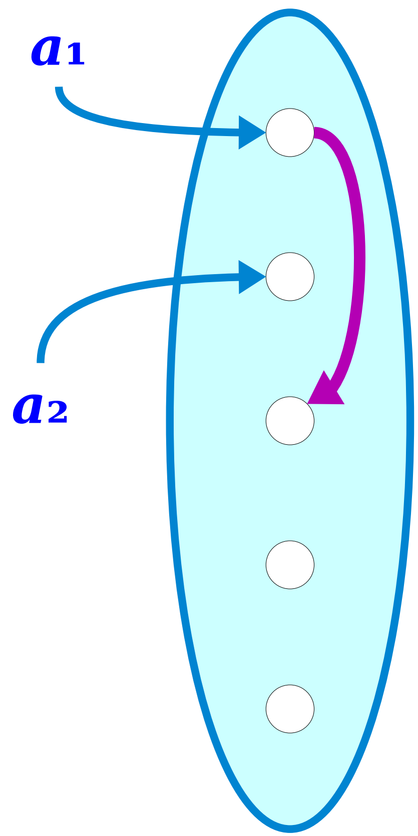

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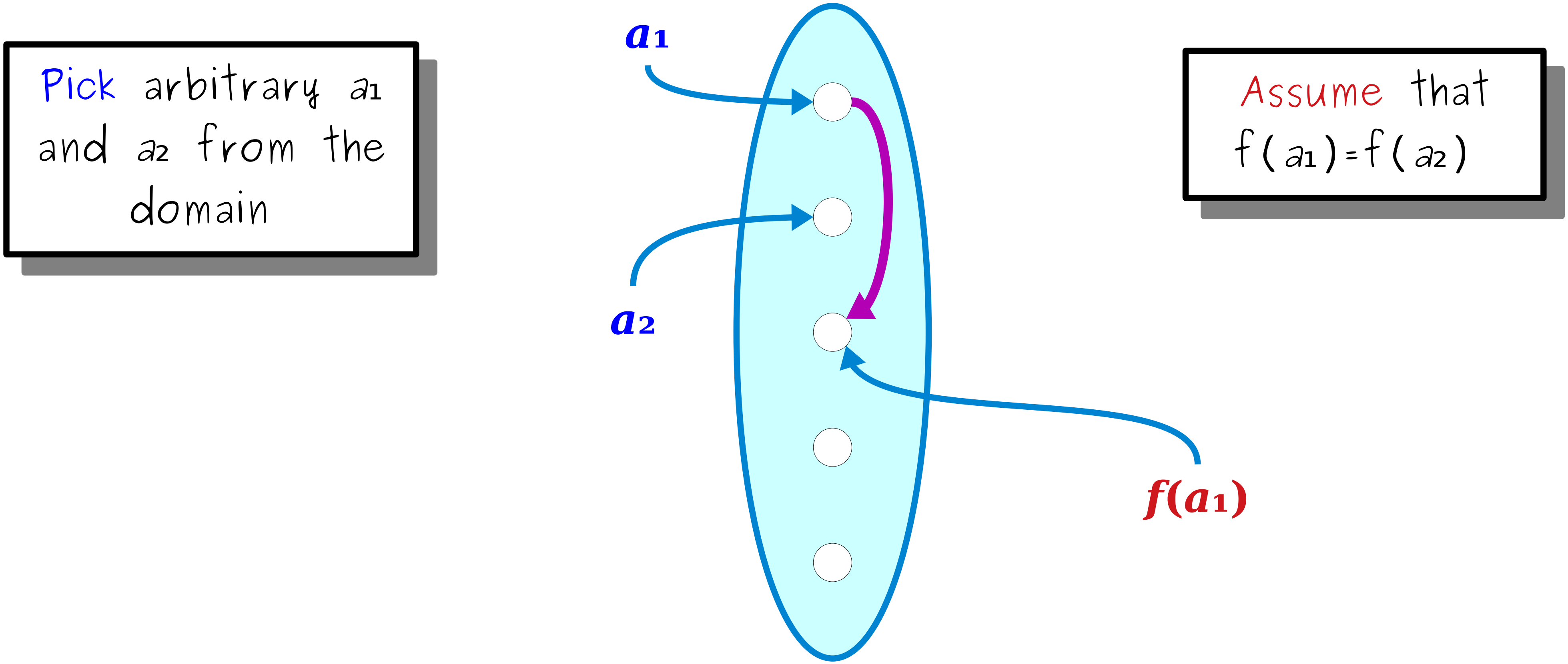

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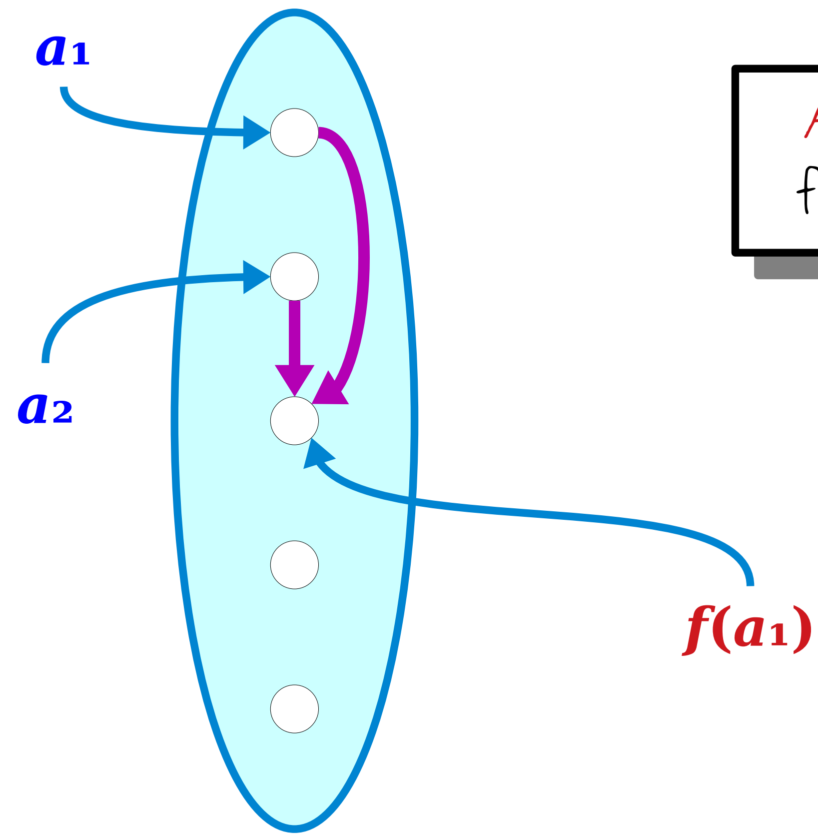

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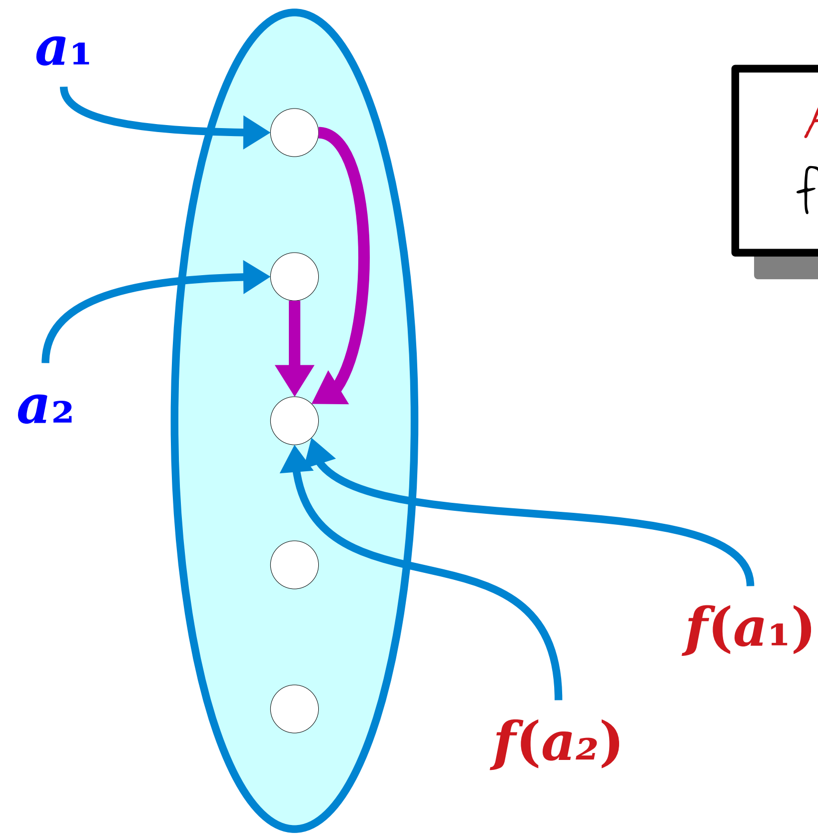

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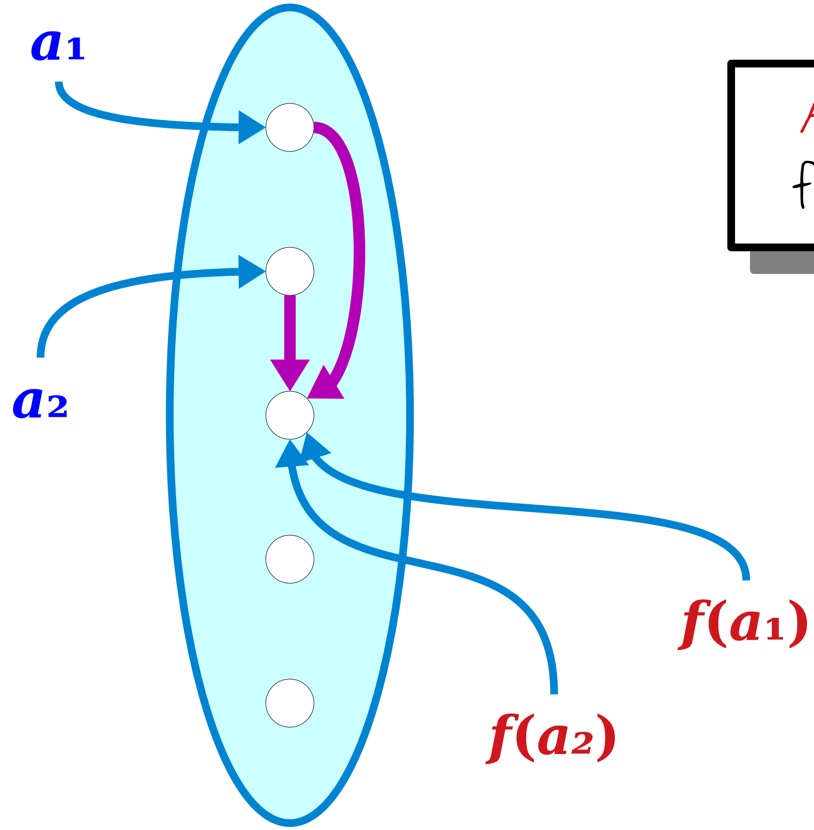
(*"If the outputs are the same, the inputs are the same."*)

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$. Prove that if f is an involution, then f is a bijection.

Part 1: Injectivity

Pick arbitrary a_1 and a_2 from the domain

Prove that $a_1 = a_2$



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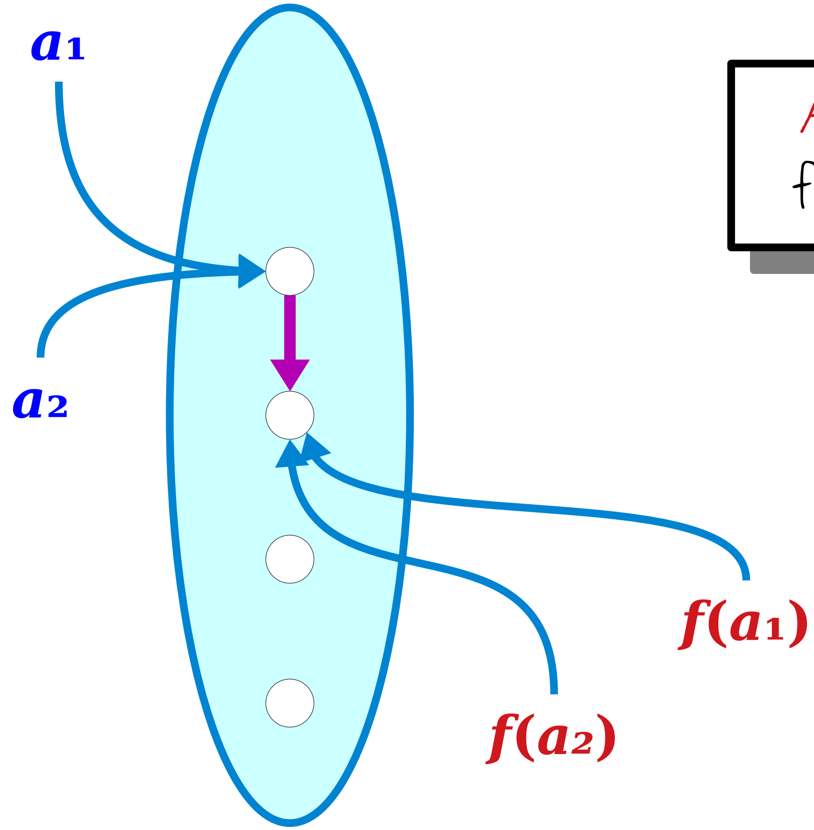

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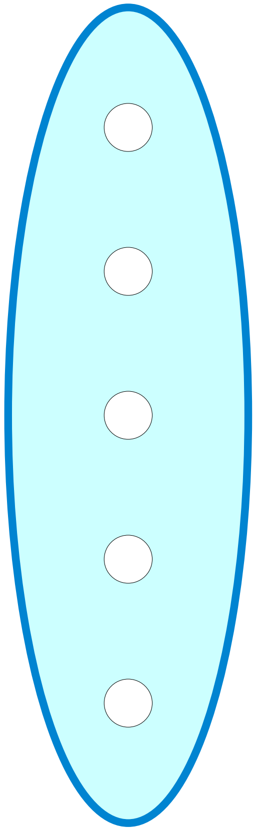

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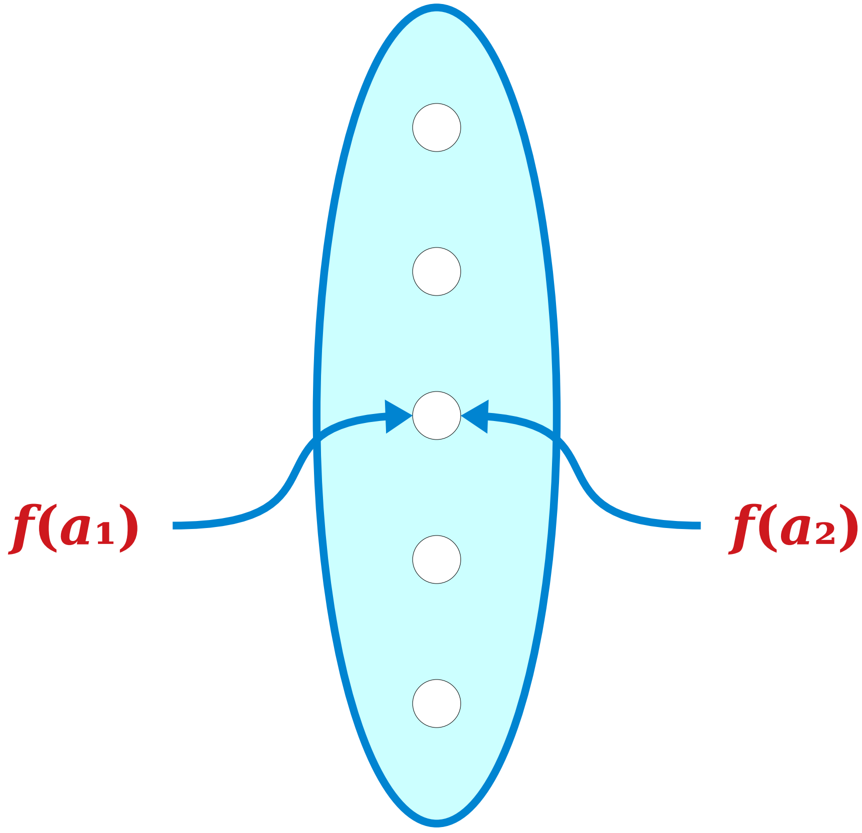

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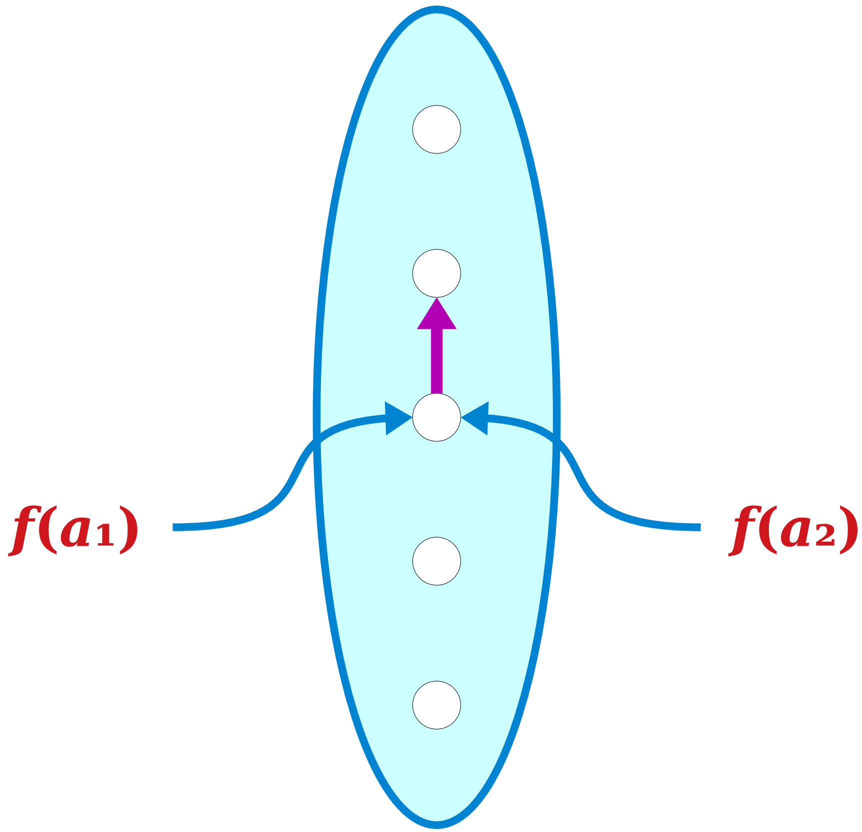

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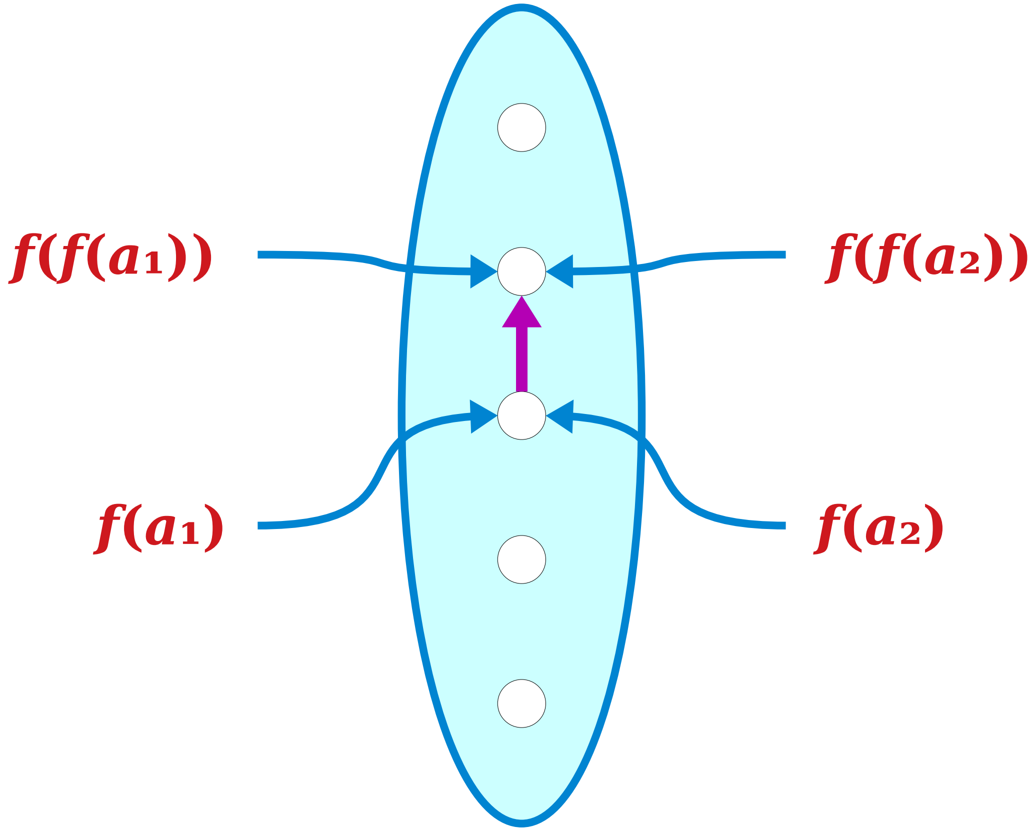

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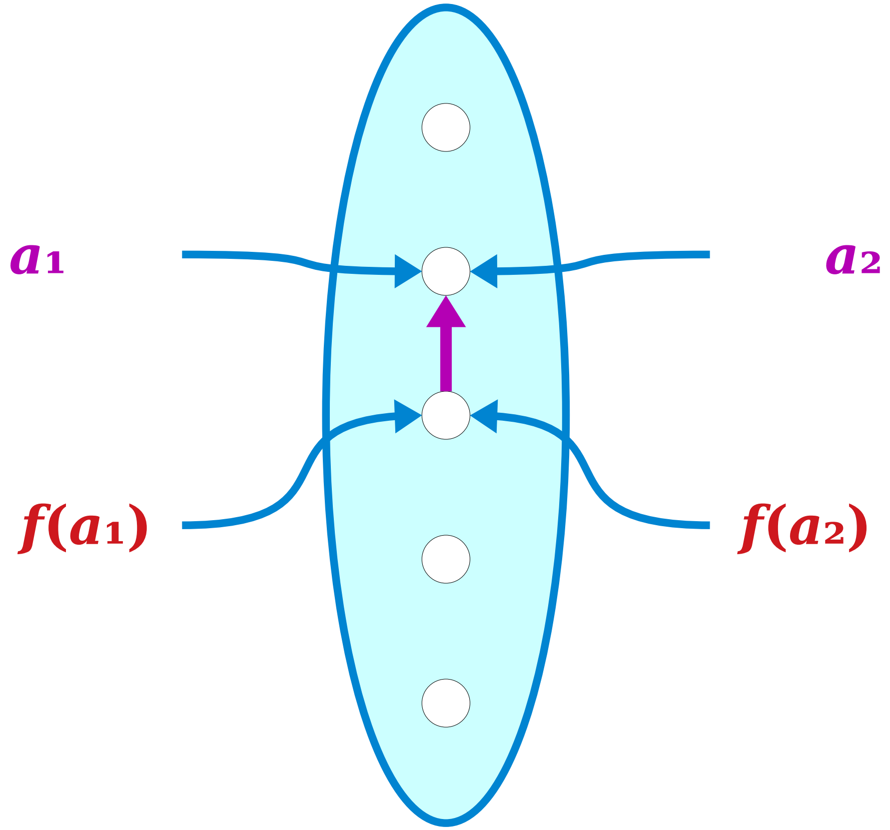

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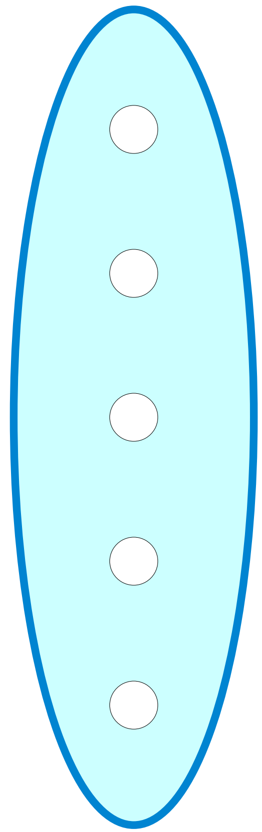

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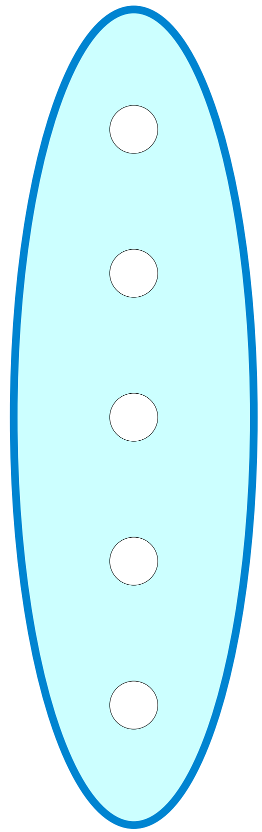

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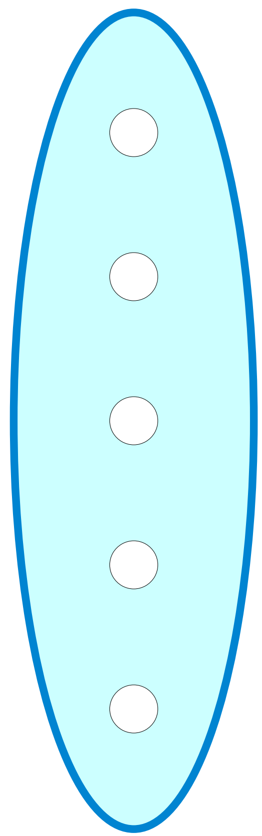
Part 2: Surjectivity

Prove that there exists an a in the domain (A) that maps to b



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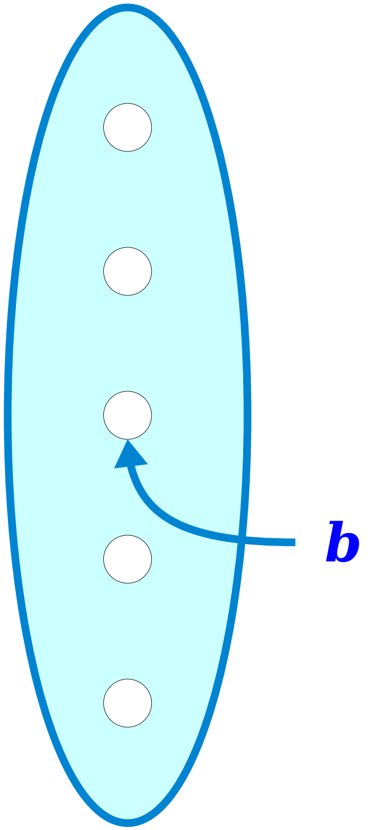

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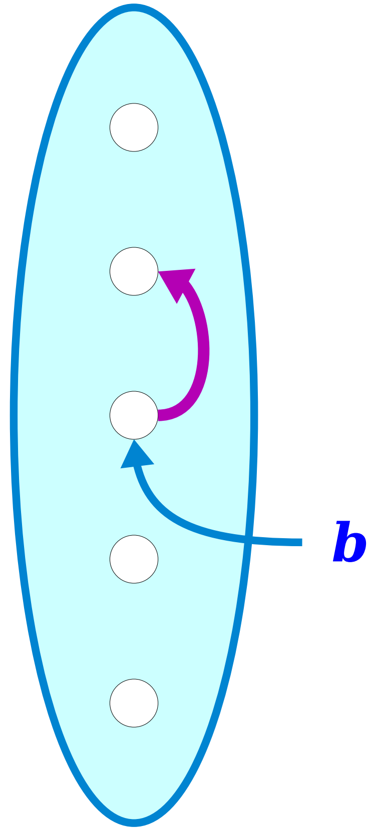

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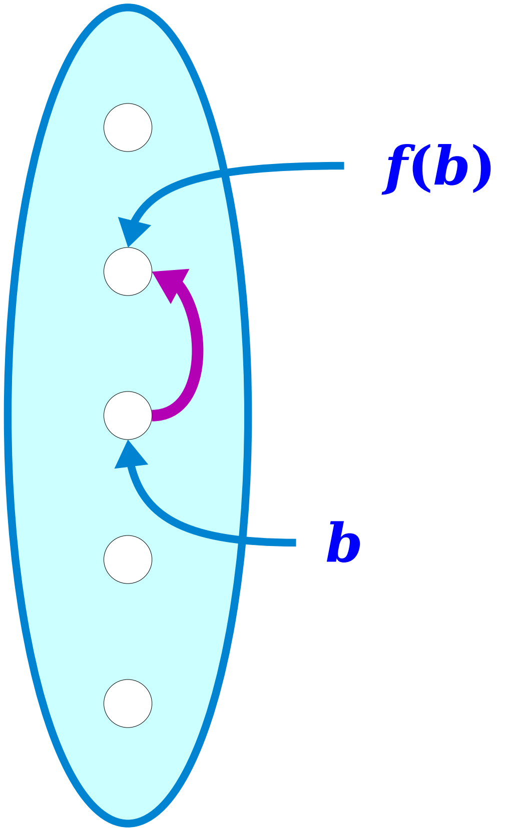

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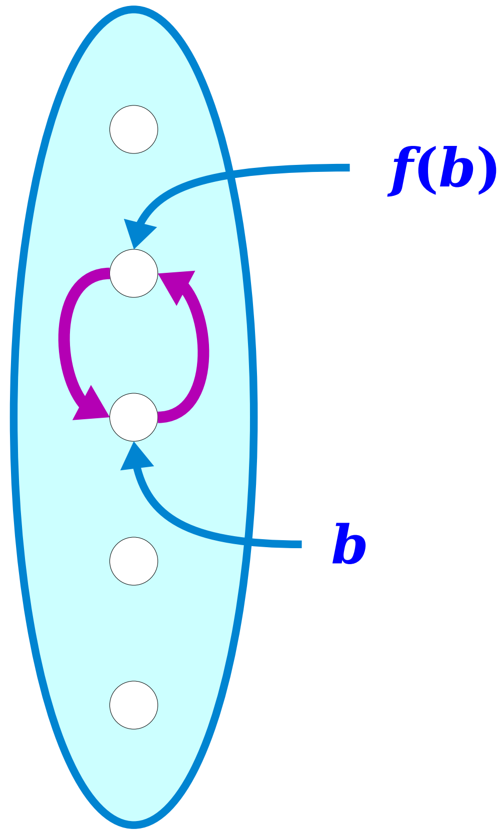

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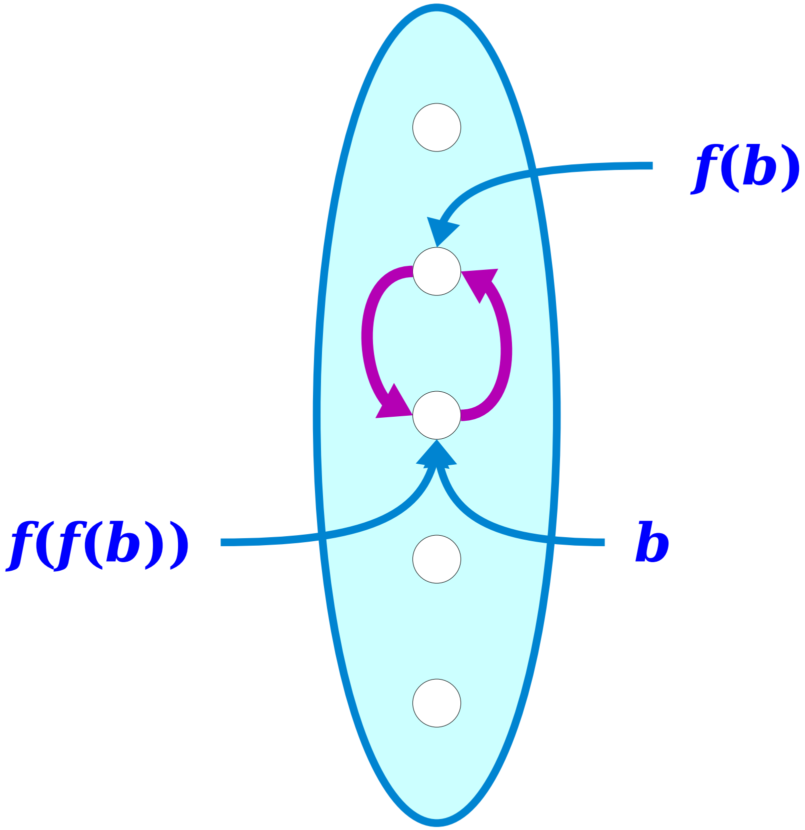

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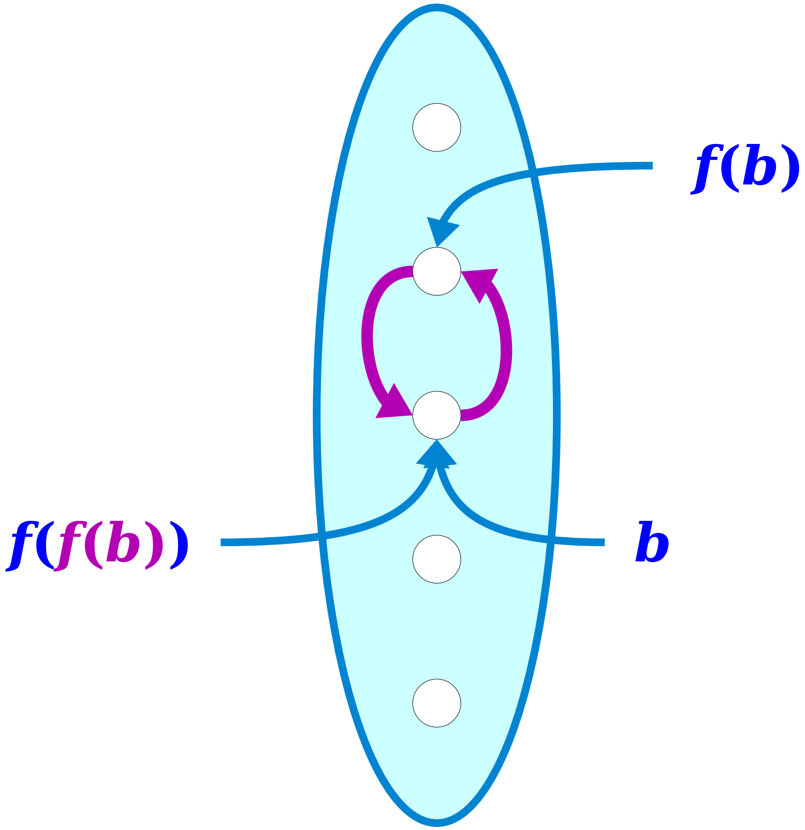

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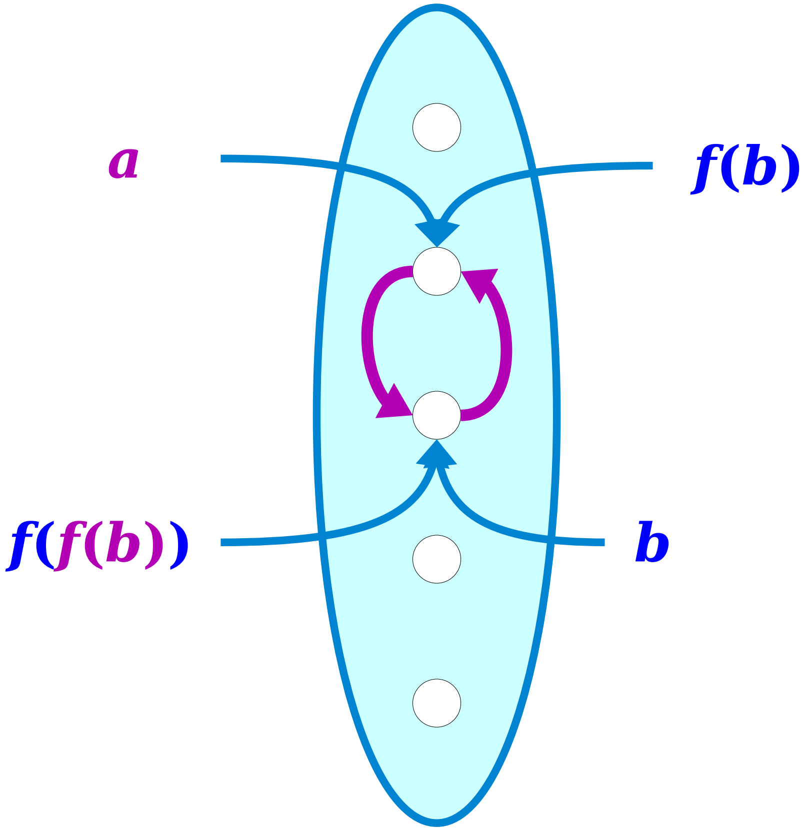

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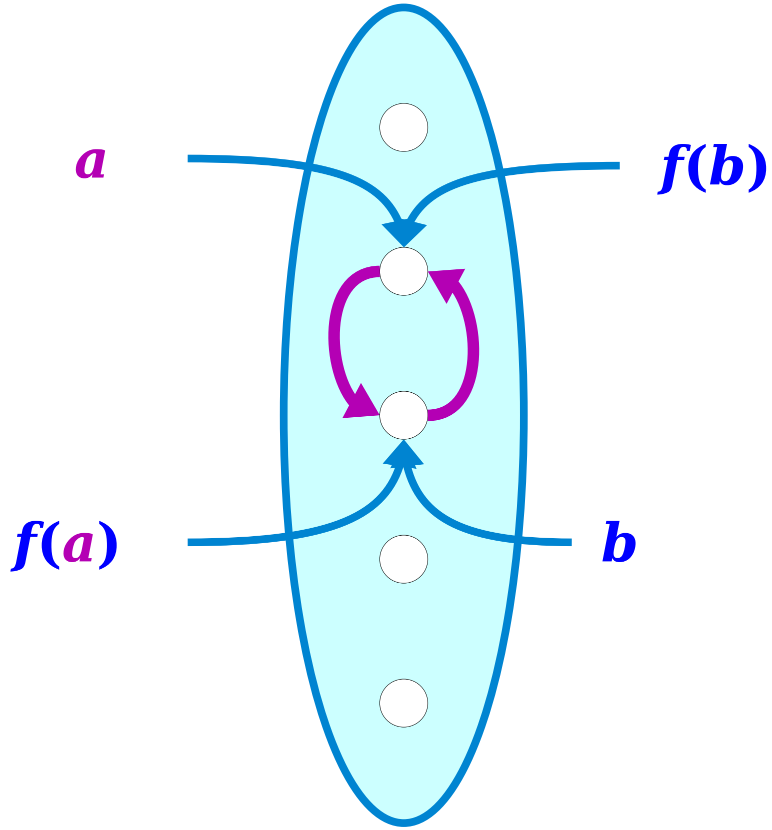

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Theorem: If $f : A \to A$ is an involution, then f is a bijection.

Proof: Let $f : A \to A$ be an involution. We want to show that f is a bijection by proving that it's both injective and surjective.

To prove that f is injective, consider any arbitrary $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We want to show that $a_1 = a_2$. To see this, start with $f(a_1) = f(a_2)$ and apply f to both sides of this equality. This tells us that $f(f(a_1)) = f(f(a_2))$. Since f is an involution, we know that $f(f(a_1)) = a_1$ and also that $f(f(a_2)) = a_2$, so we conclude that $a_1 = a_2$, as required.

To prove that f is surjective, consider any $b \in A$. We want to show that there is some $a \in A$ such that $f(a) = b$. To do so, let $a = f(b)$. Then, since f is an involution, we see that $f(a) = f(f(b)) = b$, as required. ■

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Which parts of this proof don't work if f is not an involution?

Thanks for Calling In!

Stay safe, stay healthy,
and have a good week!

See you next time.