Week 5 Tutorial

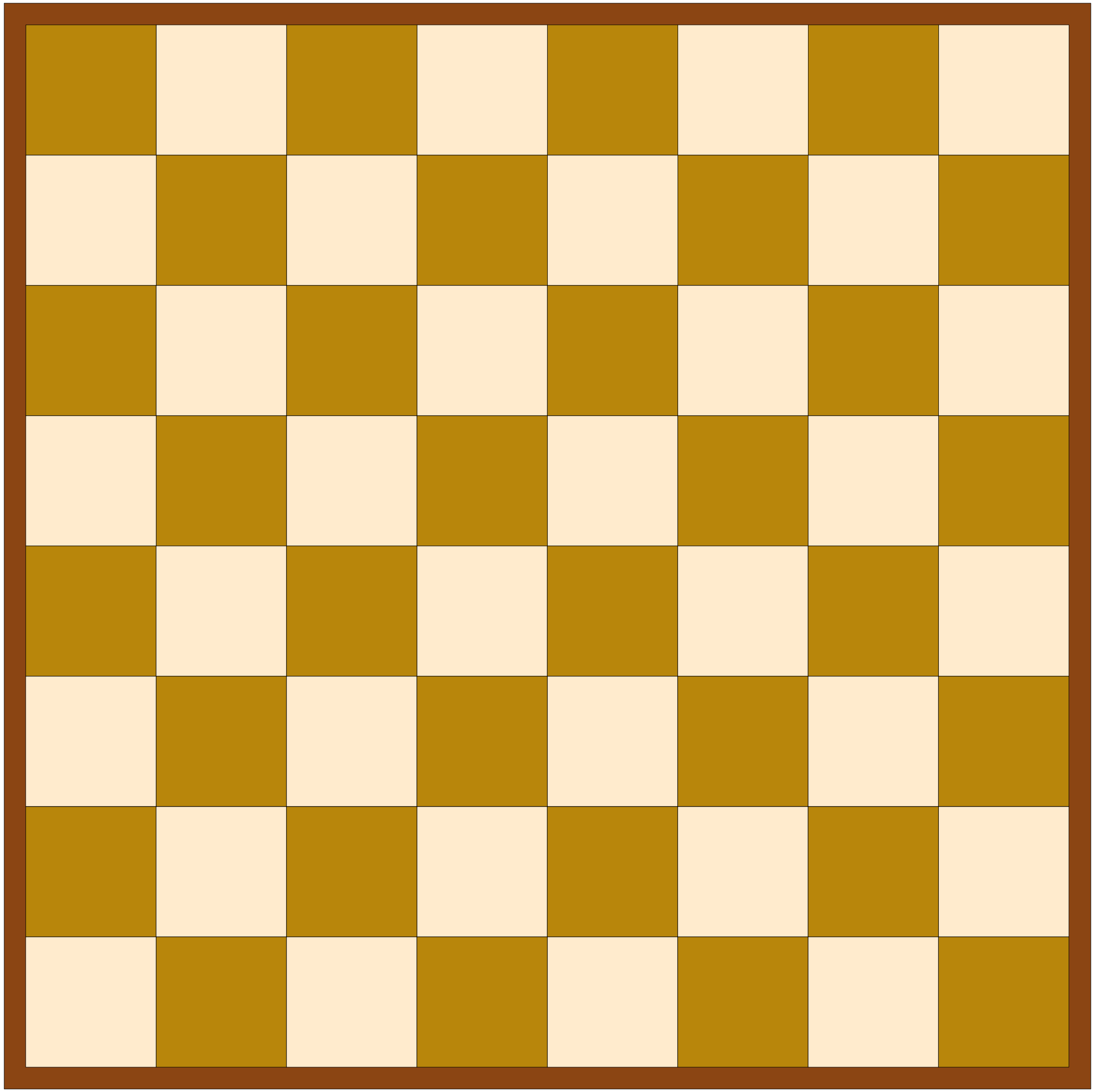
Graphs and the Pigeonhole Principle

Check-In Form:

https://forms.gle/PupBWQXJybWvR15s9

Part One: The Pigeonhole Principle

A Clash of Kings



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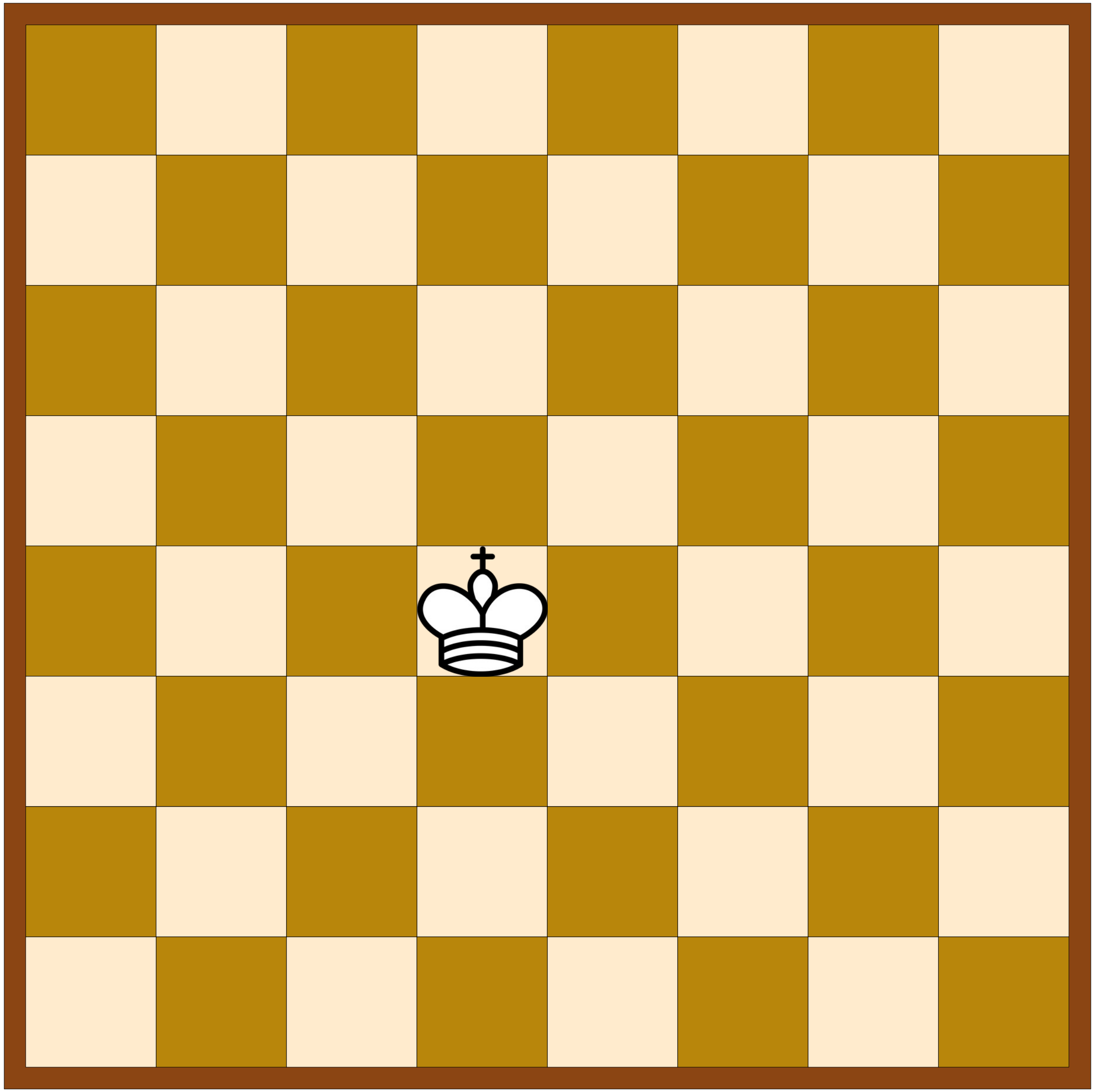

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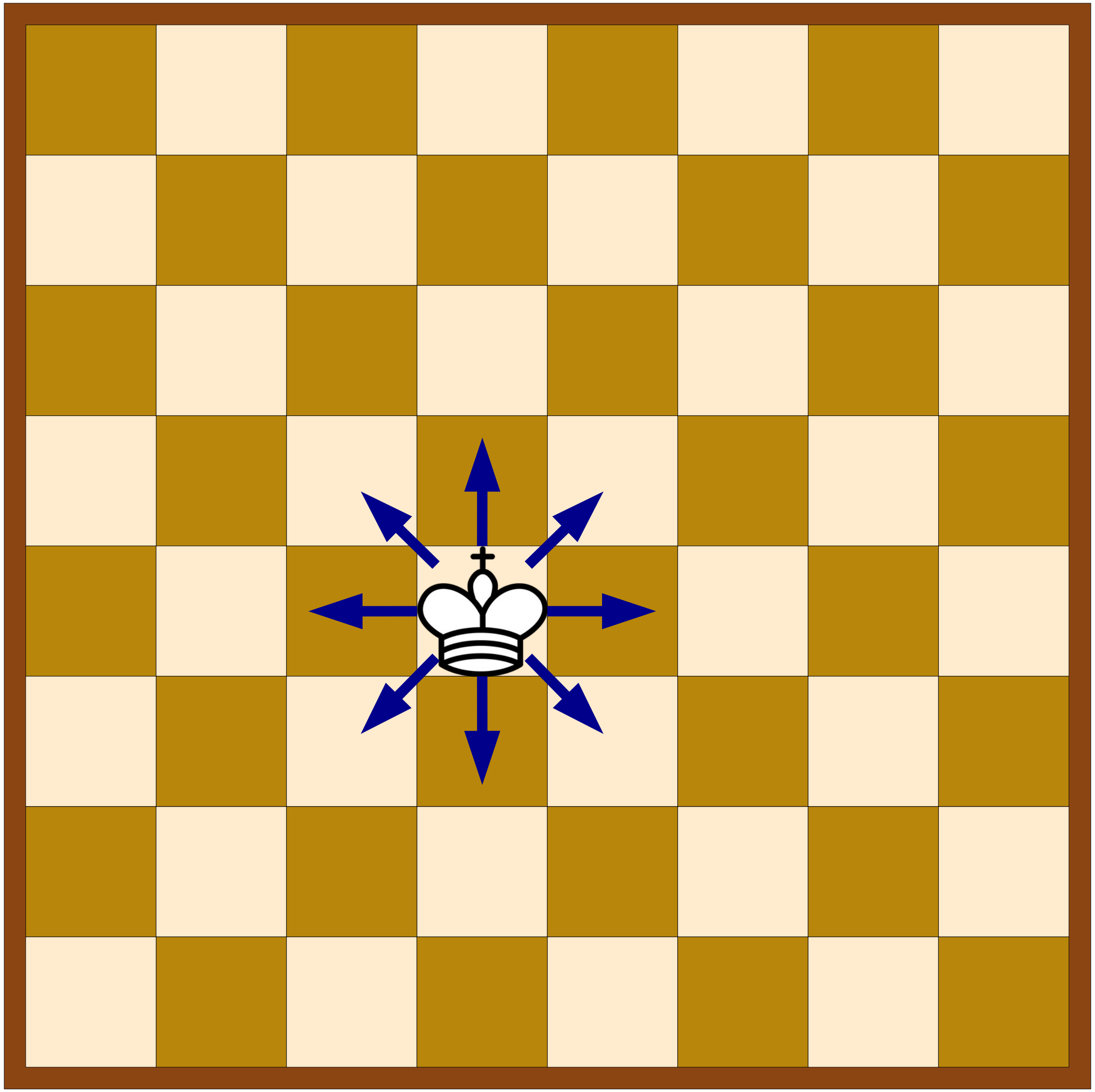

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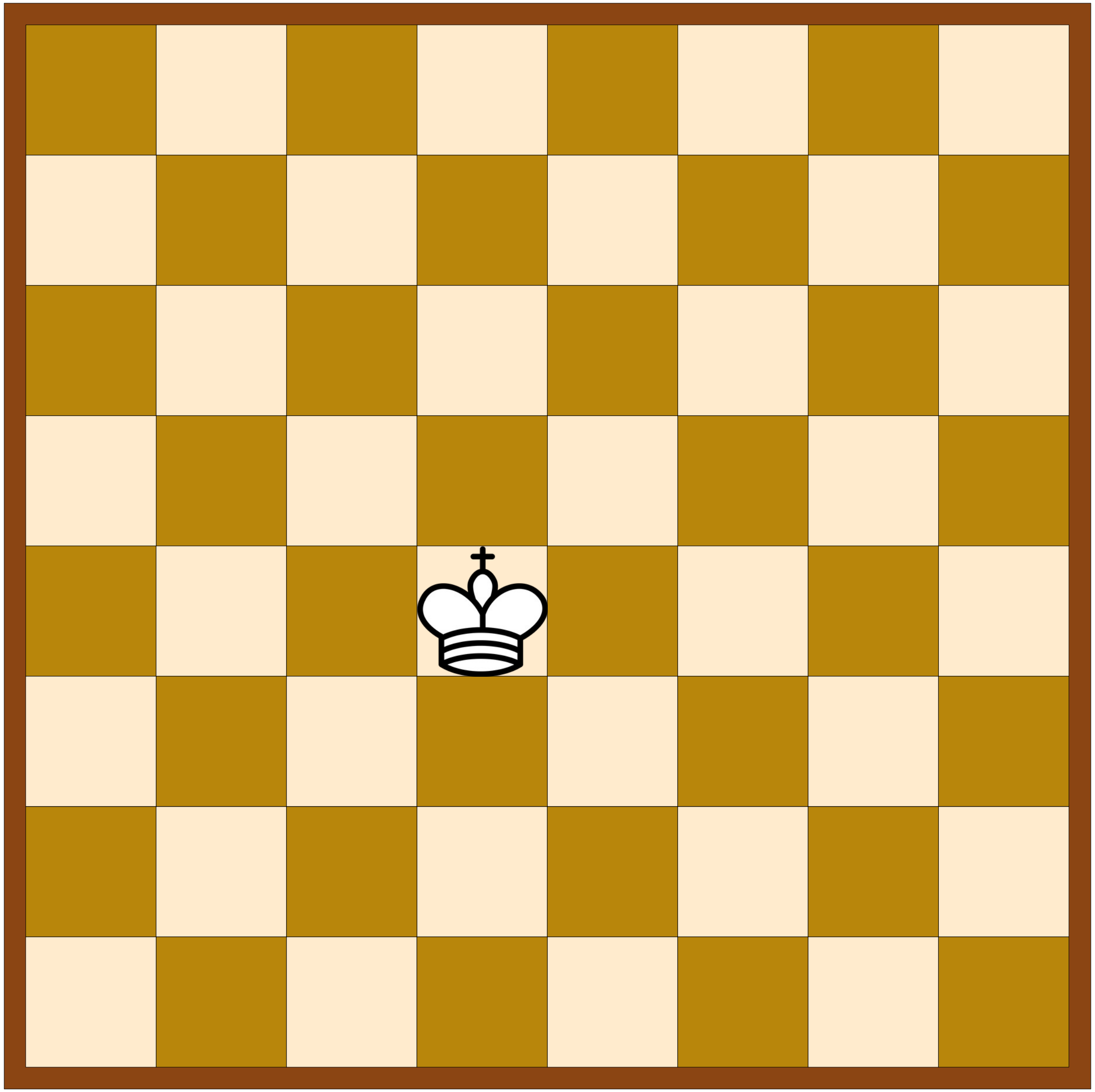

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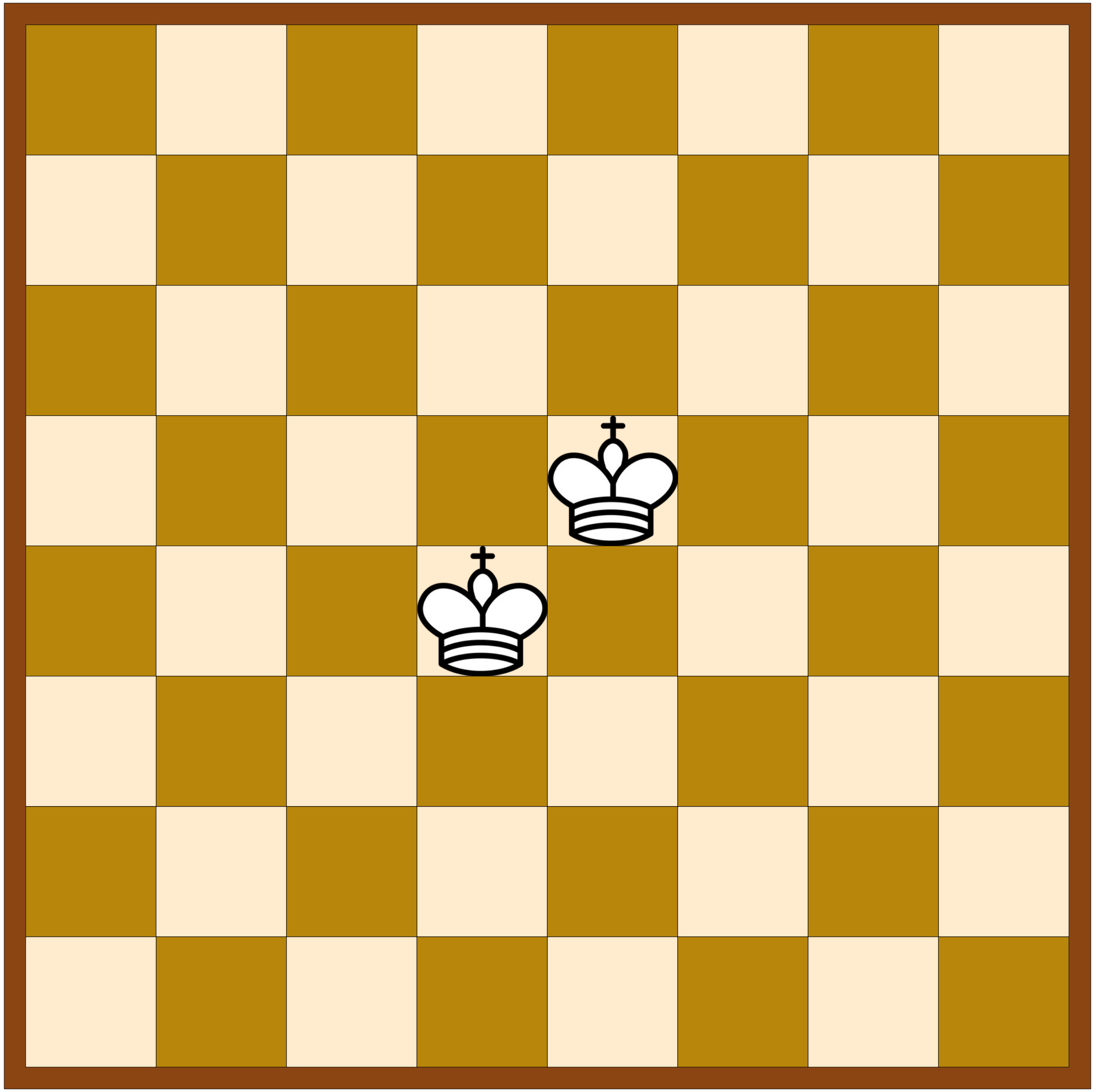

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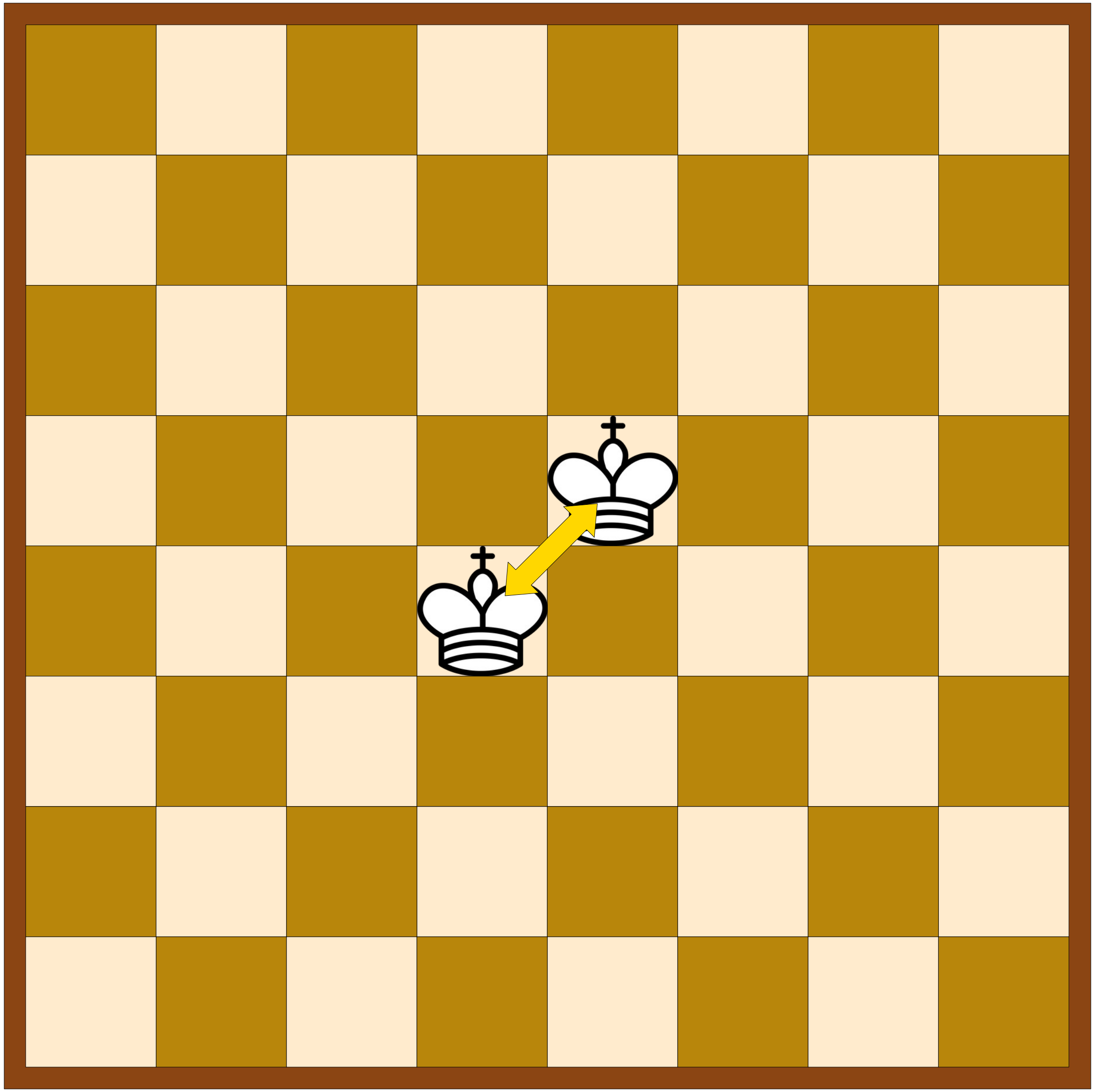

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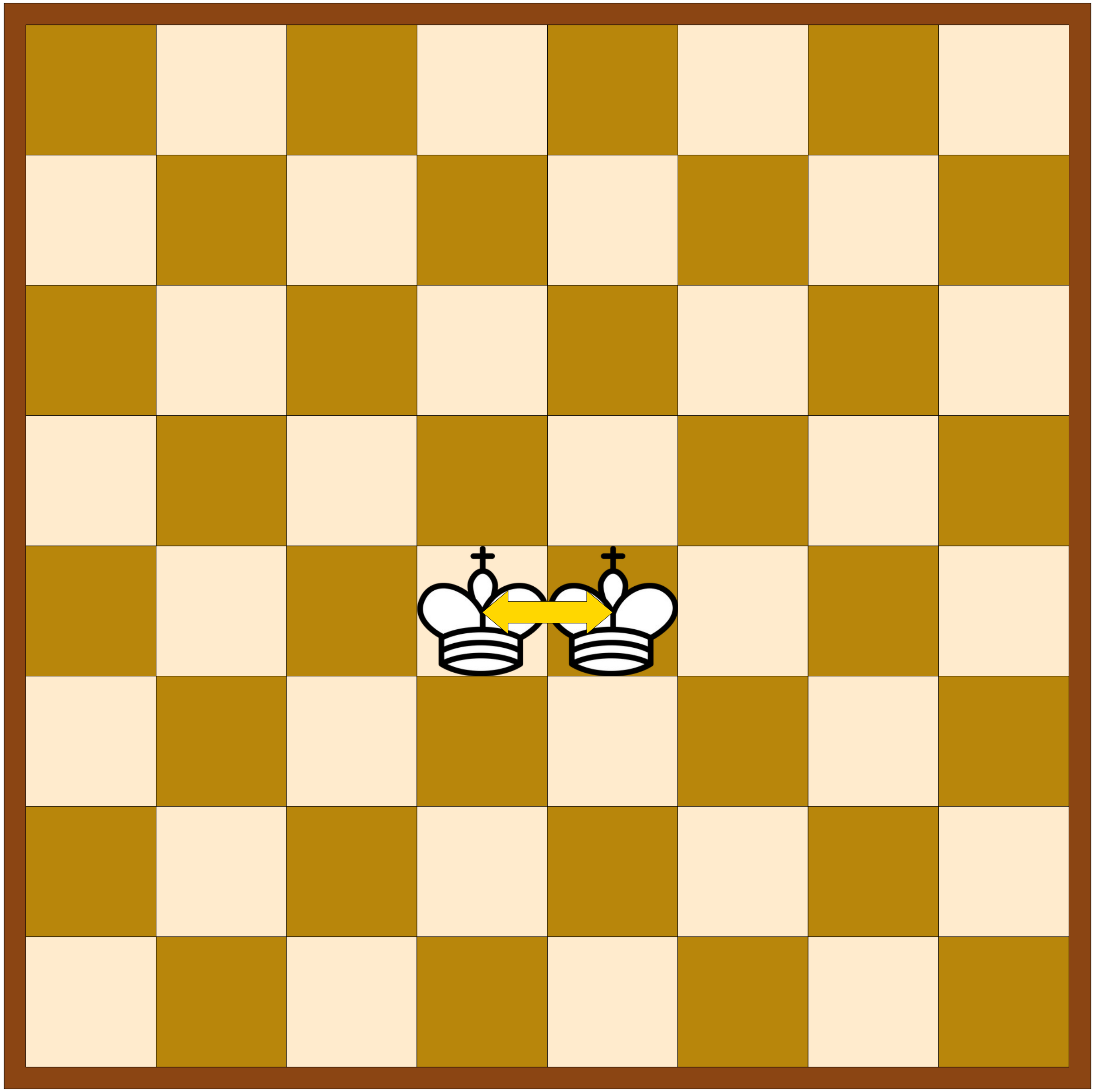

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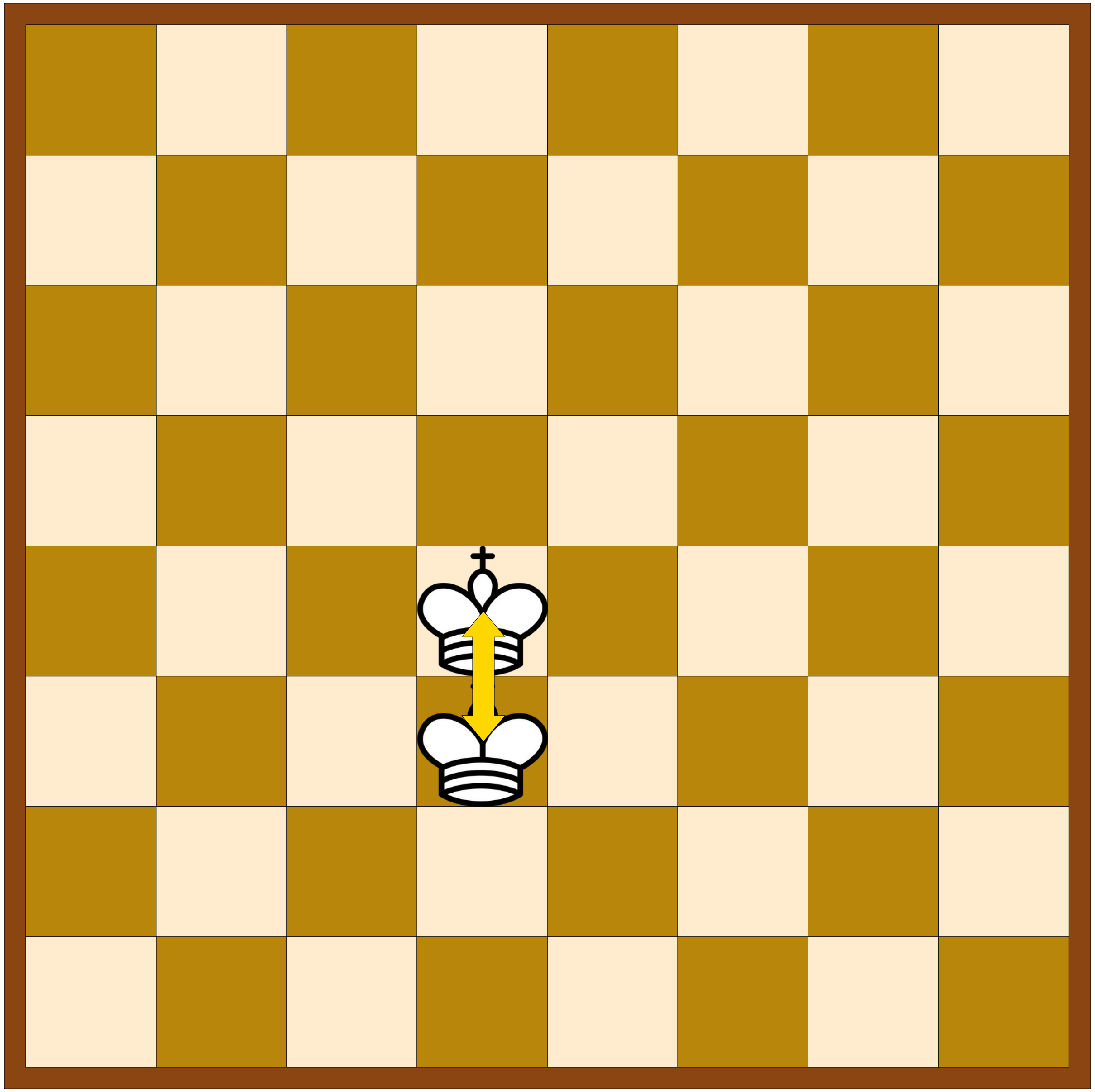

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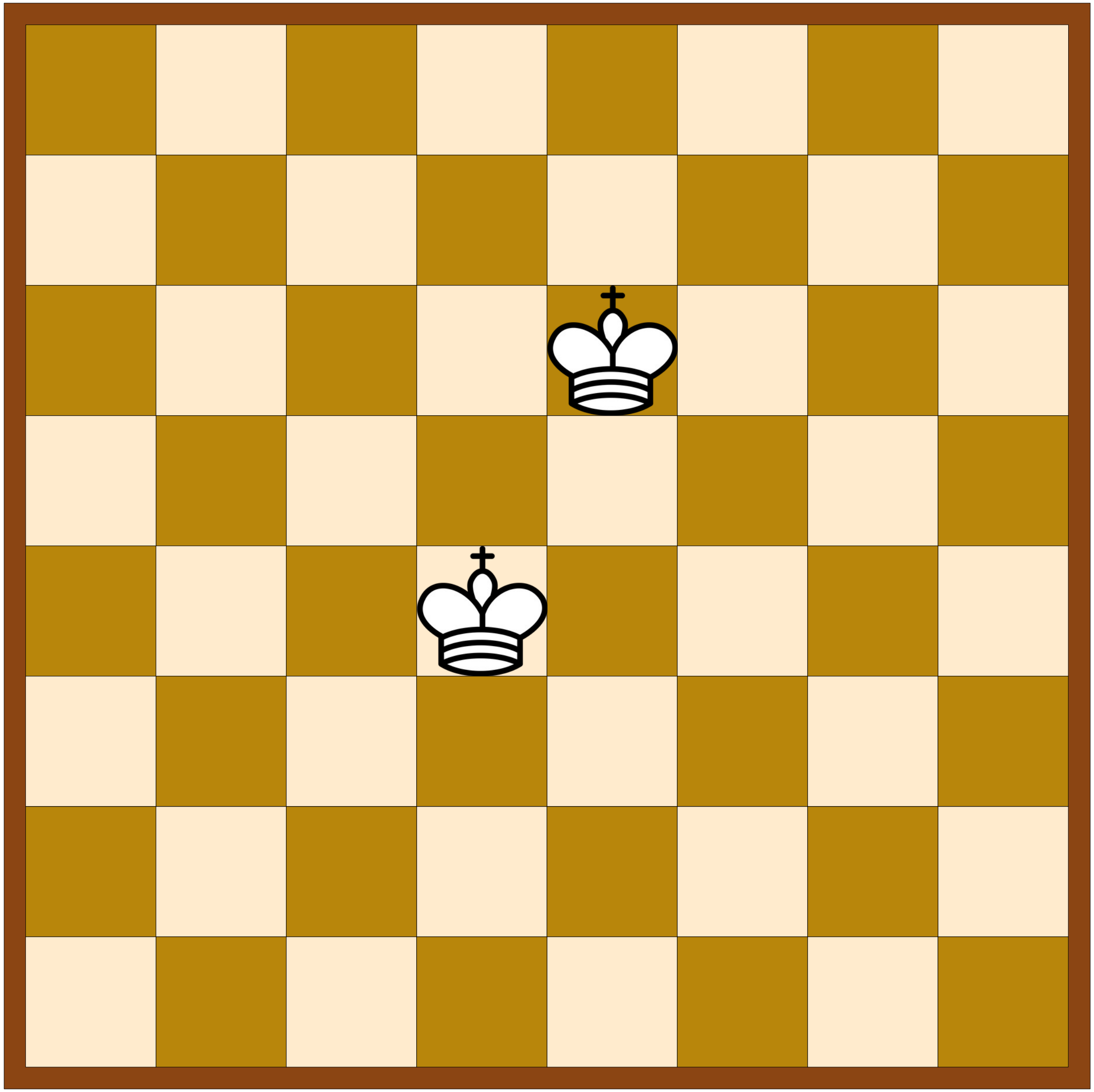

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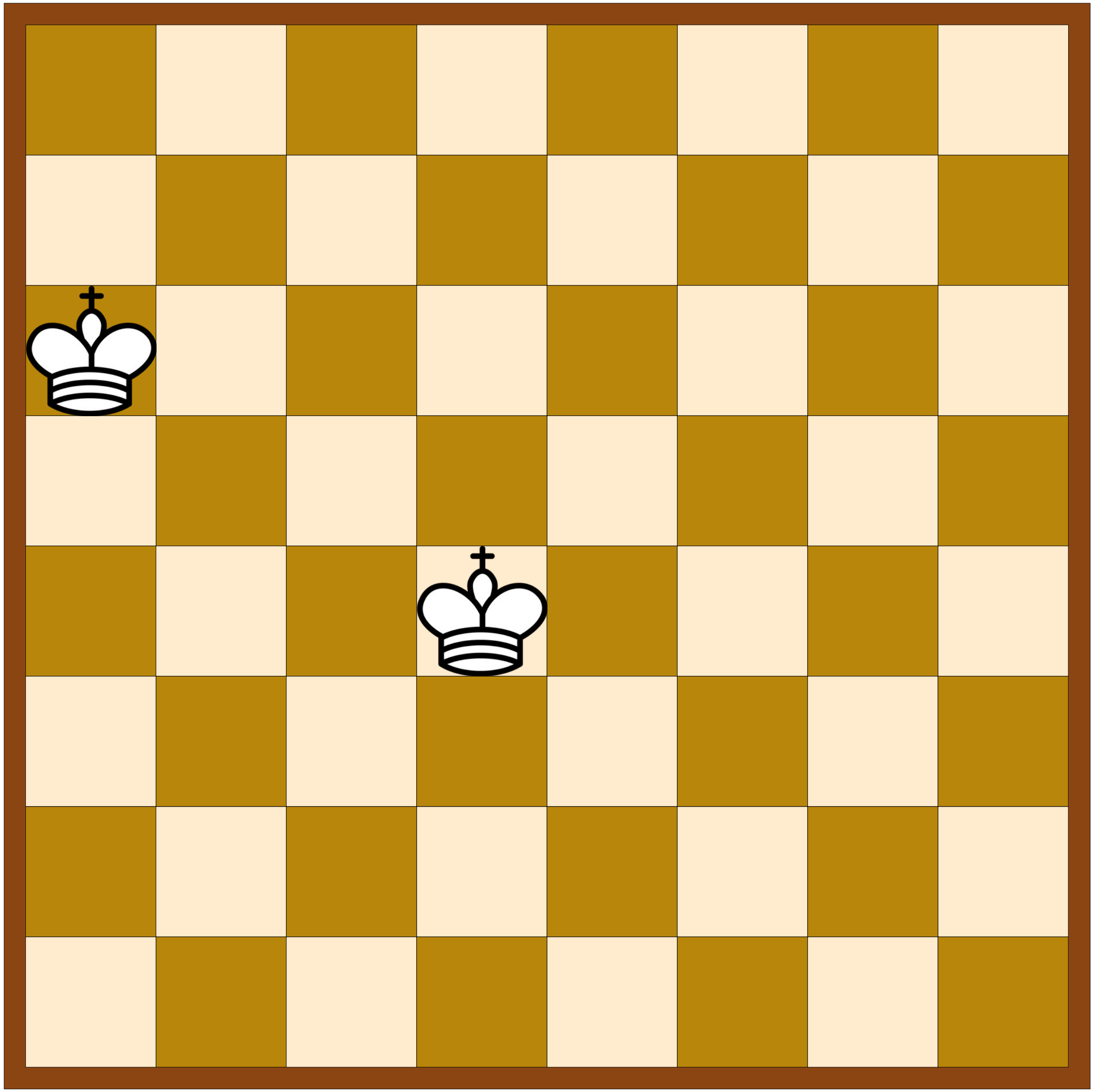

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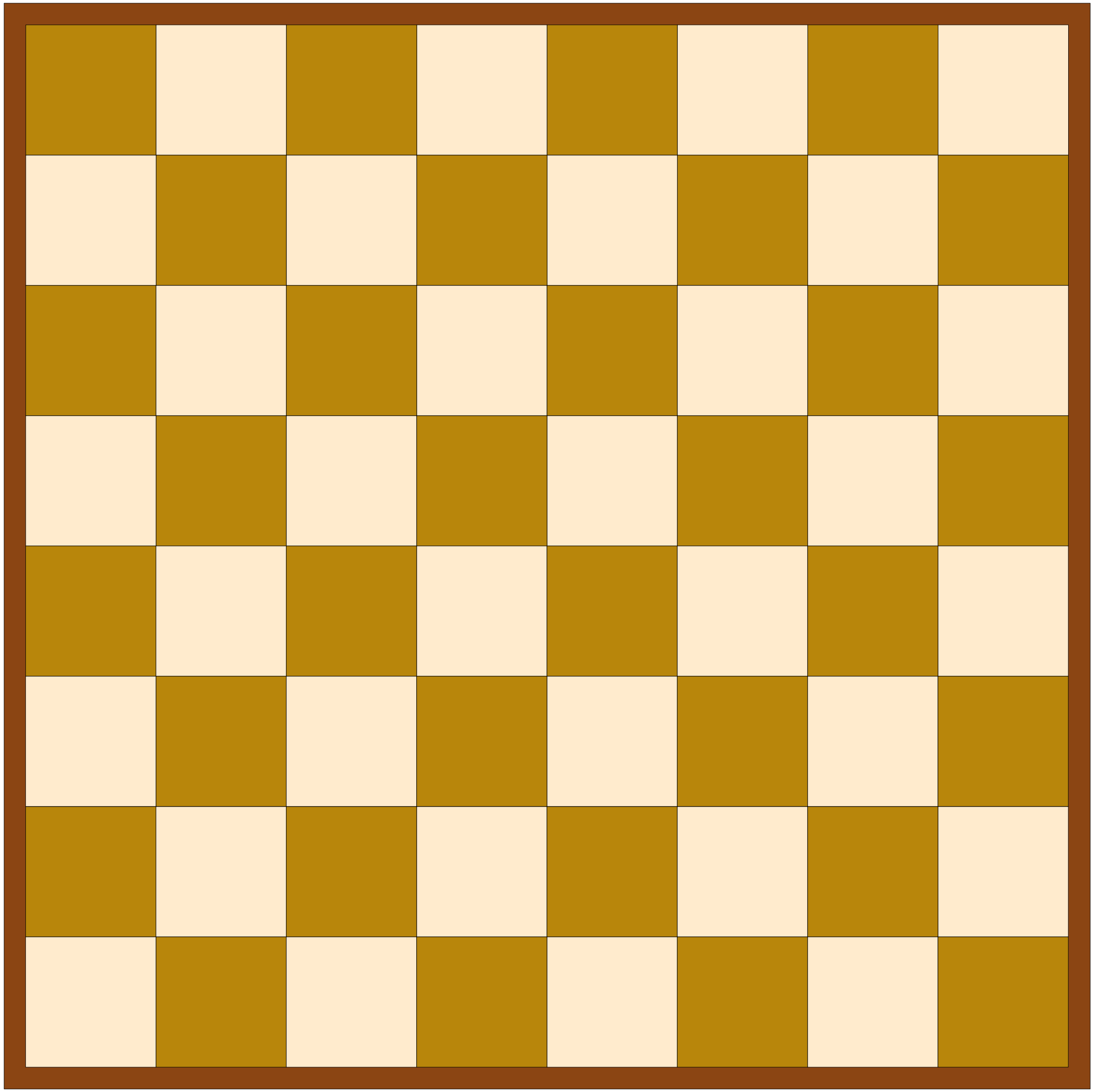
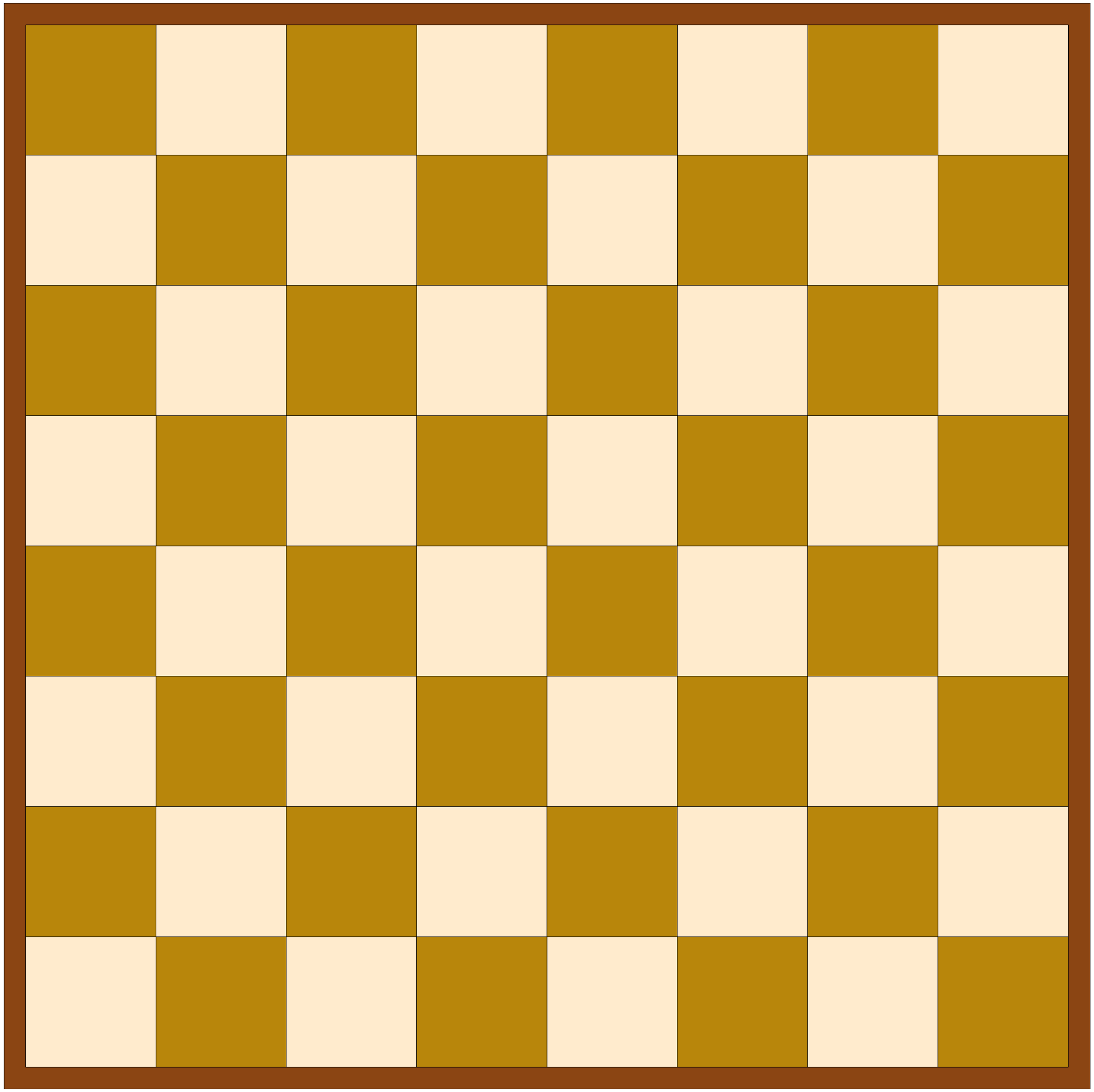
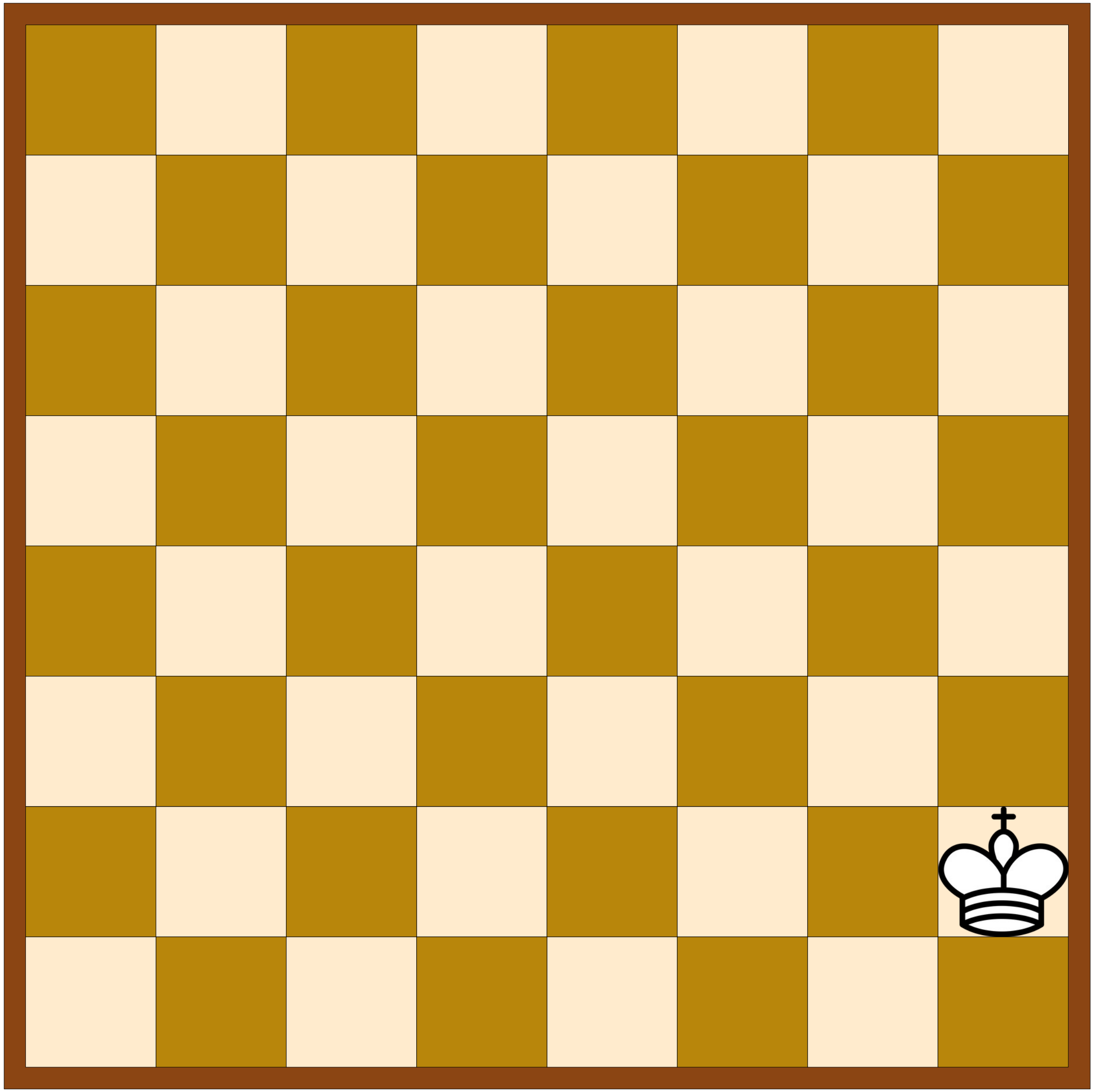


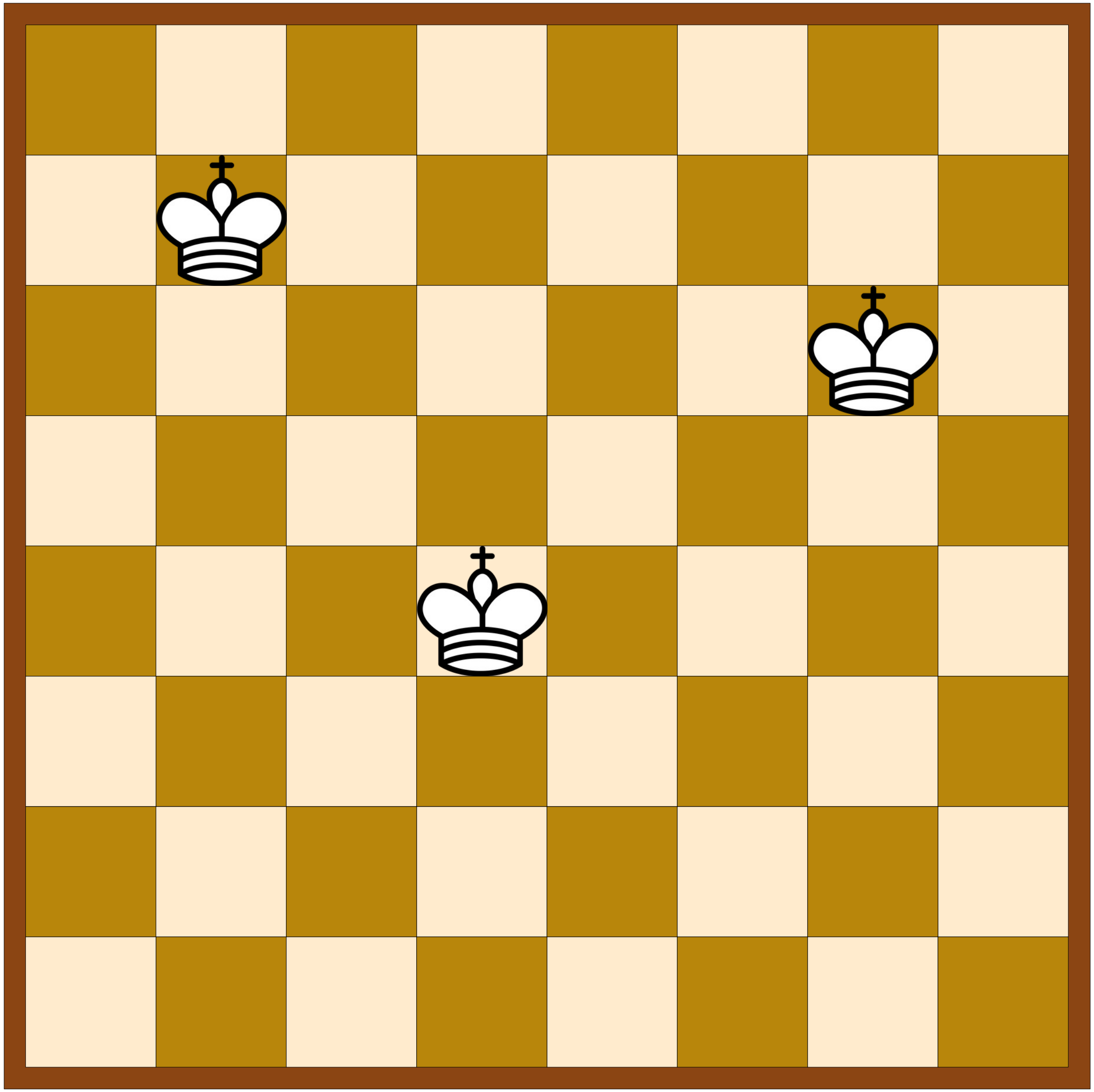
Question: How many non-attacking kings can you place on an 8×8 chessboard?



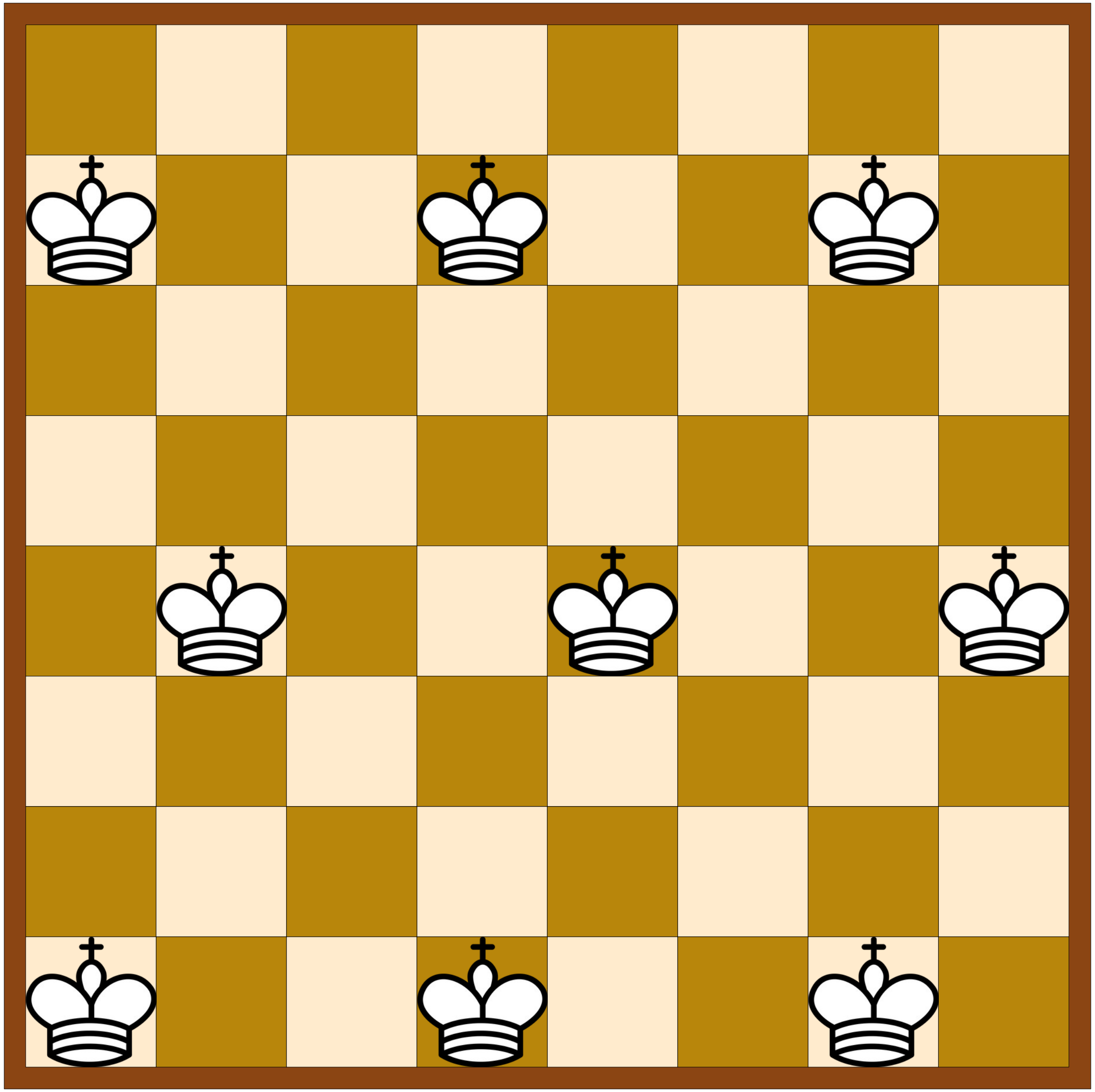
One Peaceful King



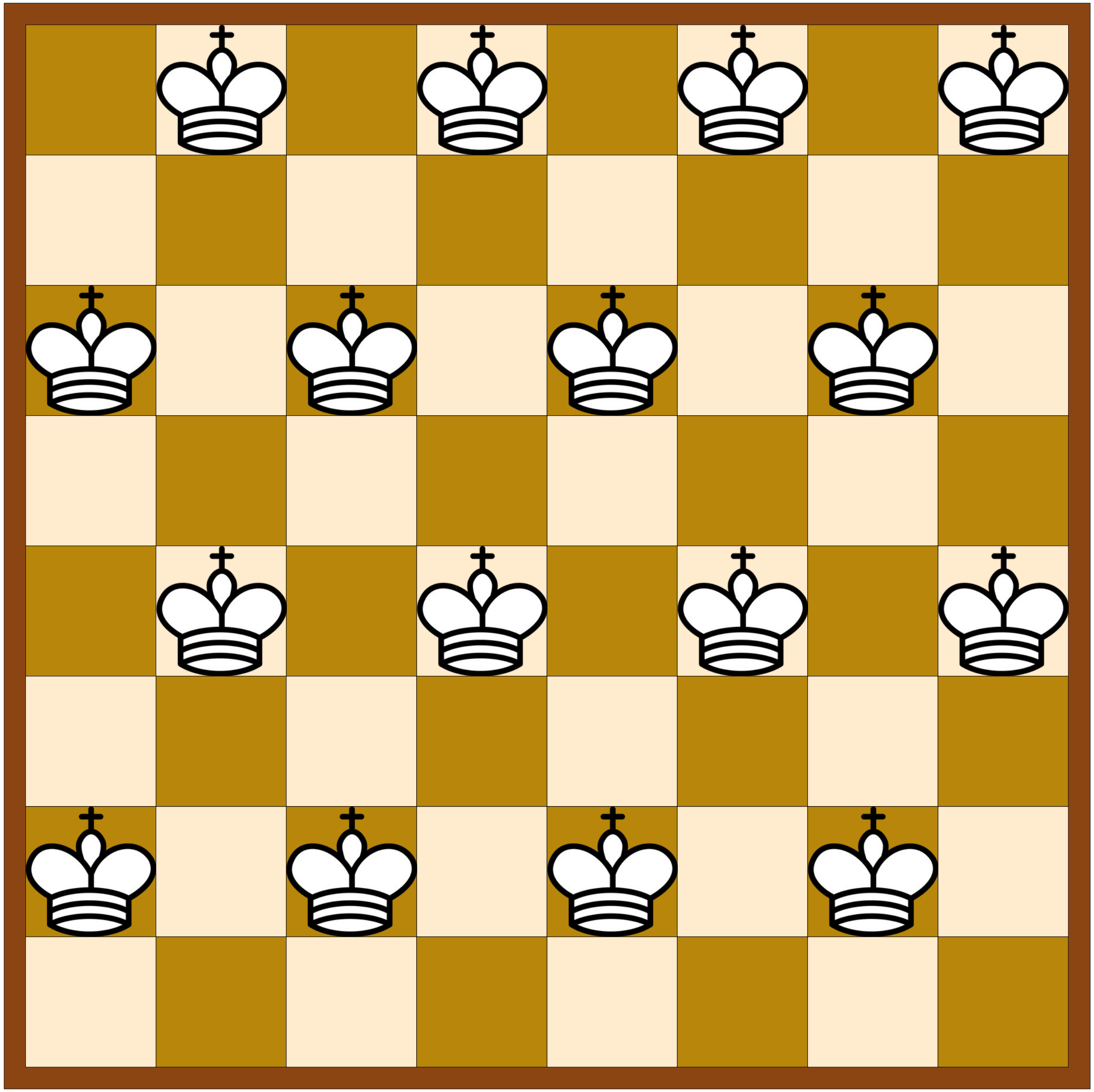
Three Peaceful Kings



Nine Peaceful Kings



Sixteen Peaceful Kings



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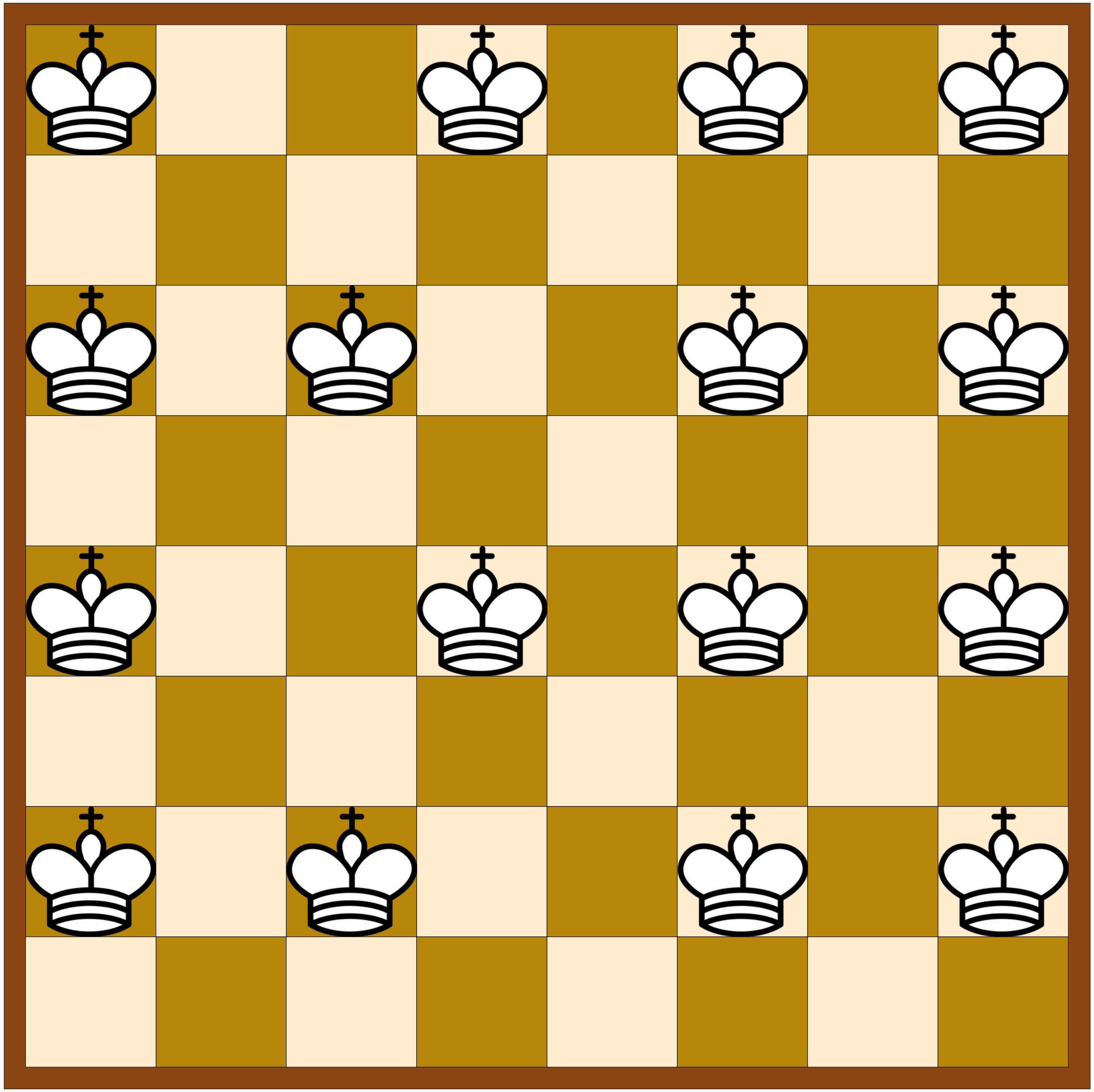

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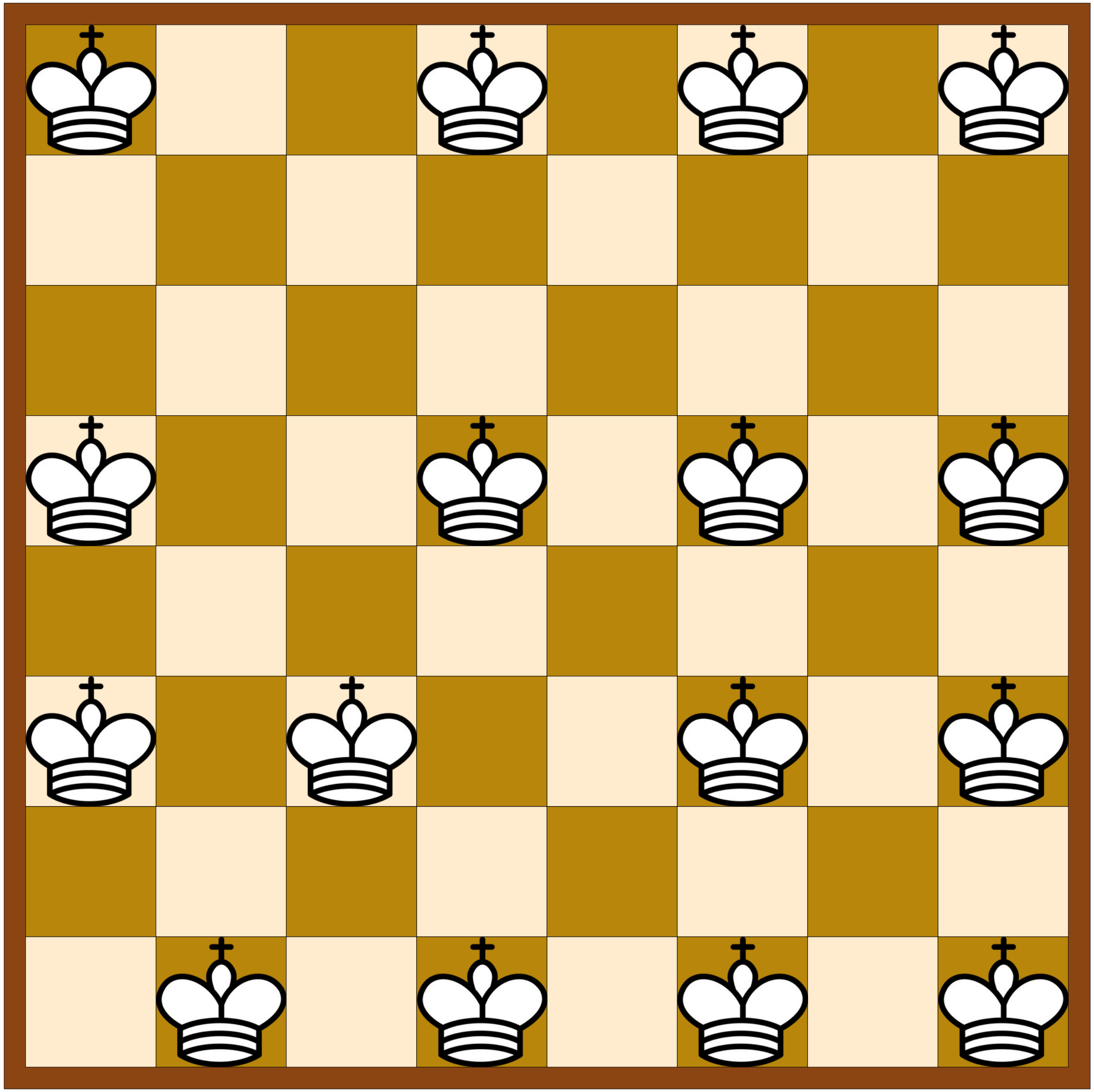
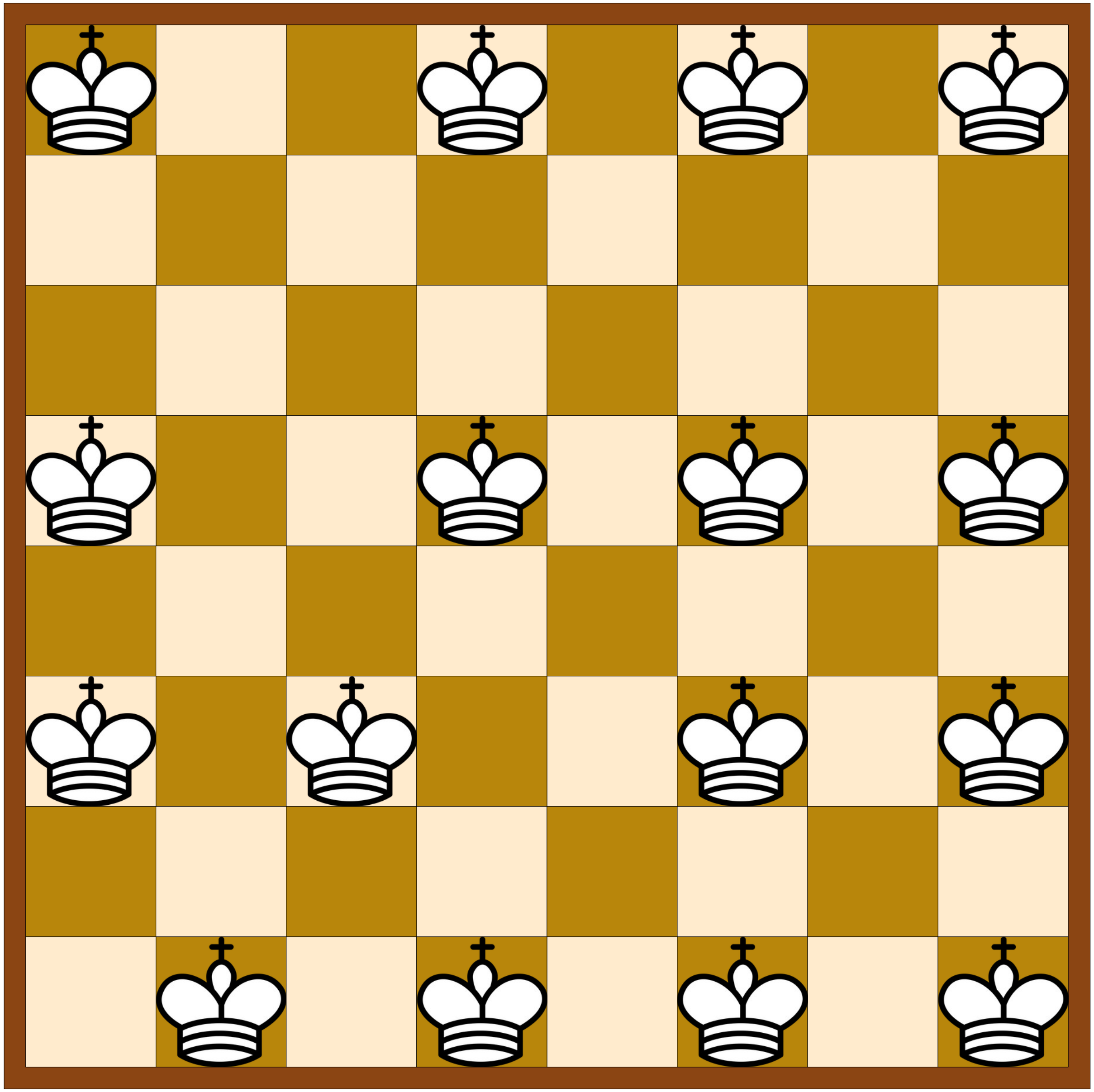


Theorem: In any arrangement of seventeen or more kings on an 8×8 chessboard, at least two of the kings attack one another.



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- ***Step Three:*** Get a collision.
  - Argue that $m > n$, then invoke the pigeonhole principle to get two items in the same box.
- ***Step Four:*** Explain why it matters.
  - This is specific to the problem at hand.

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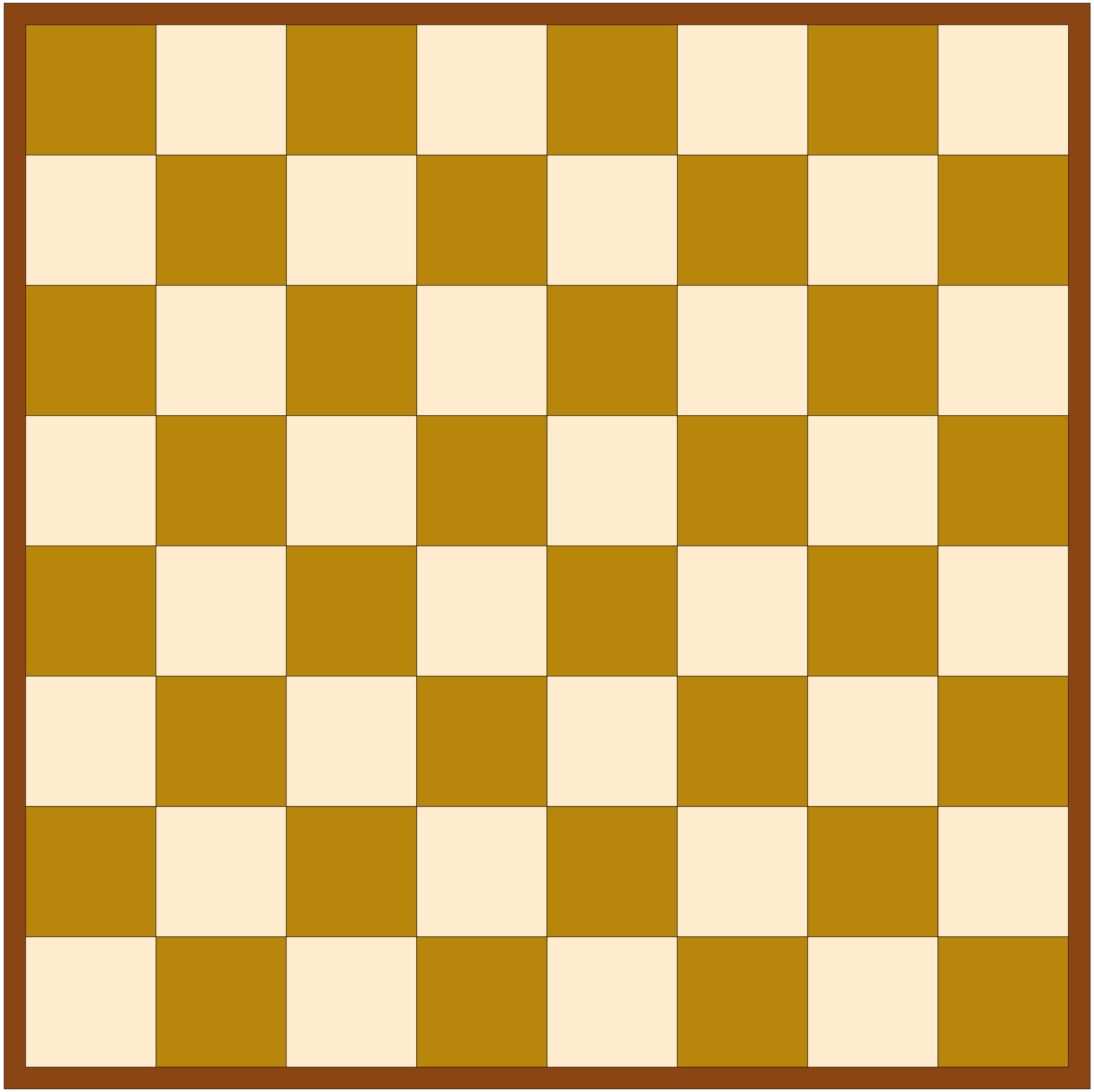

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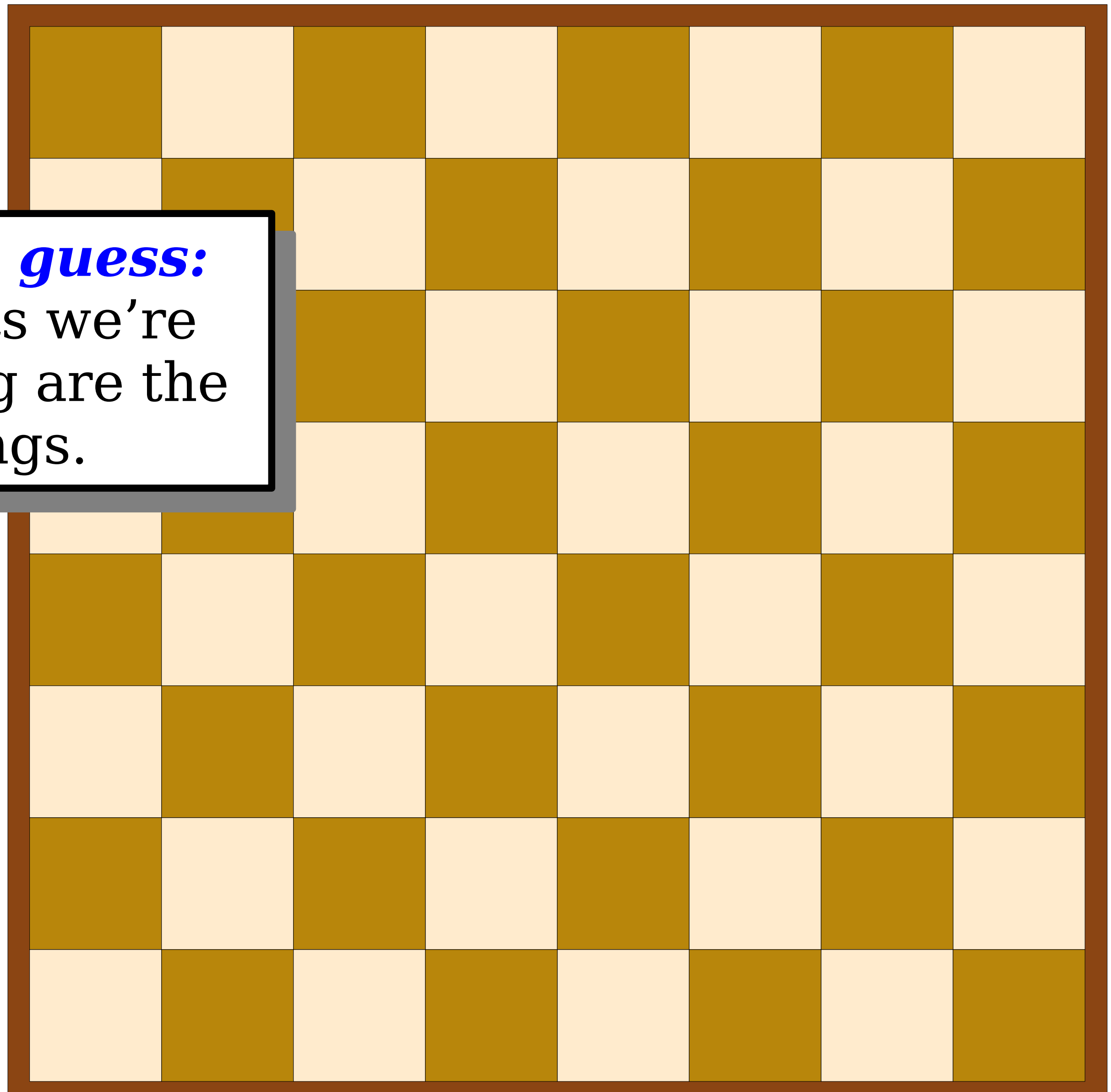

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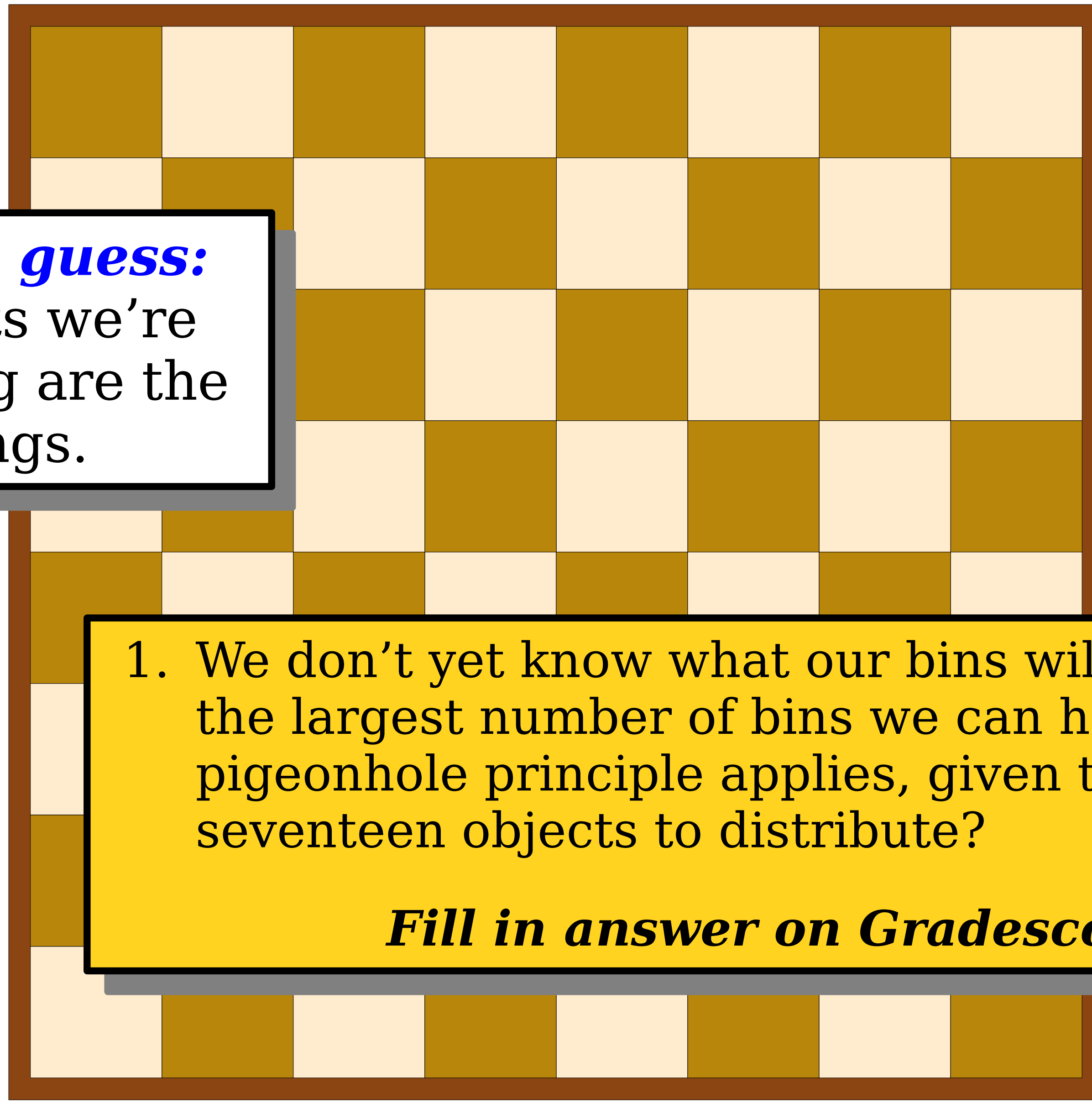

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1. We don't yet know what our bins will be. What is the largest number of bins we can have where the pigeonhole principle applies, given that we have seventeen objects to distribute?

Fill in answer on Gradescope!

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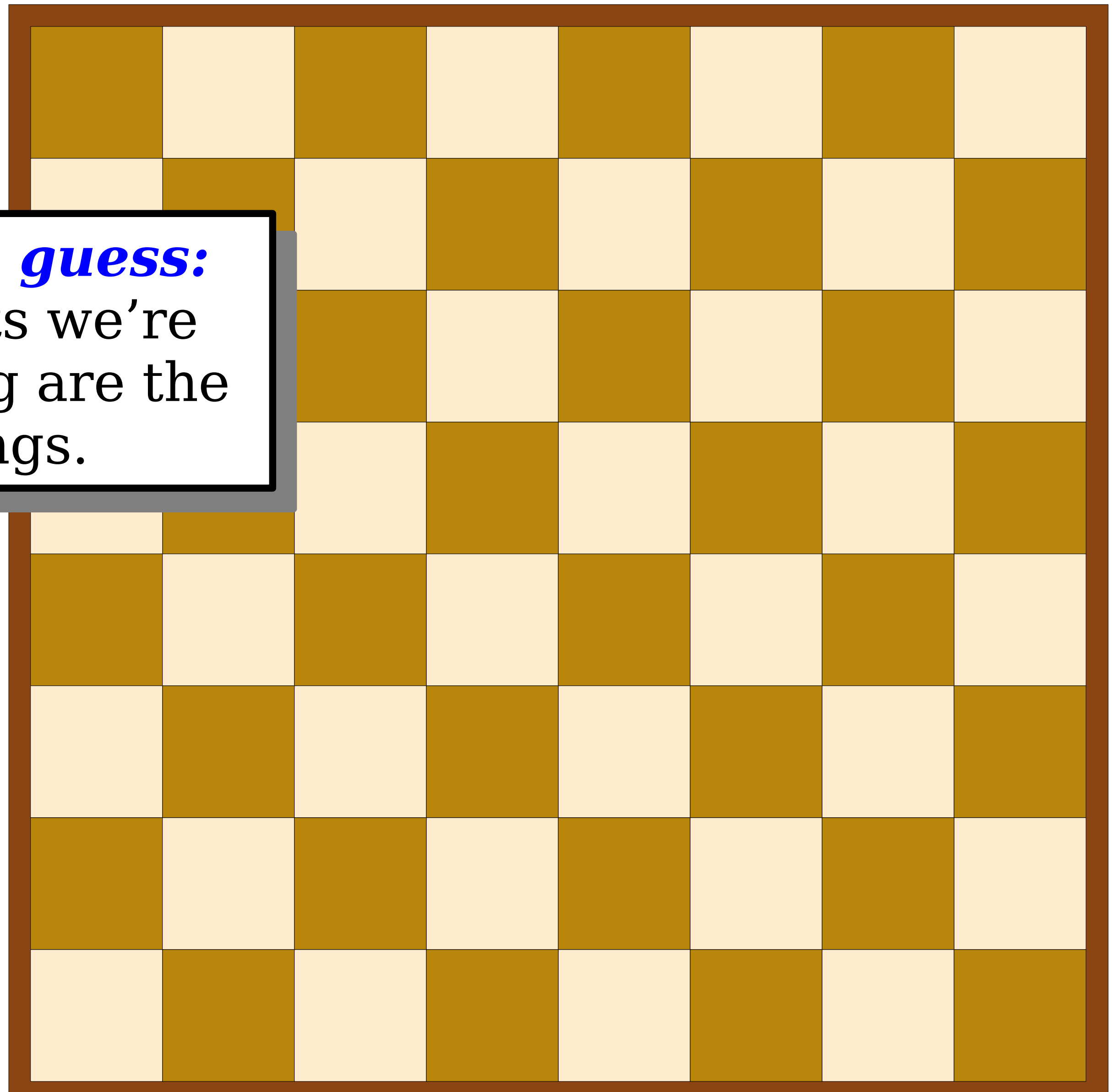

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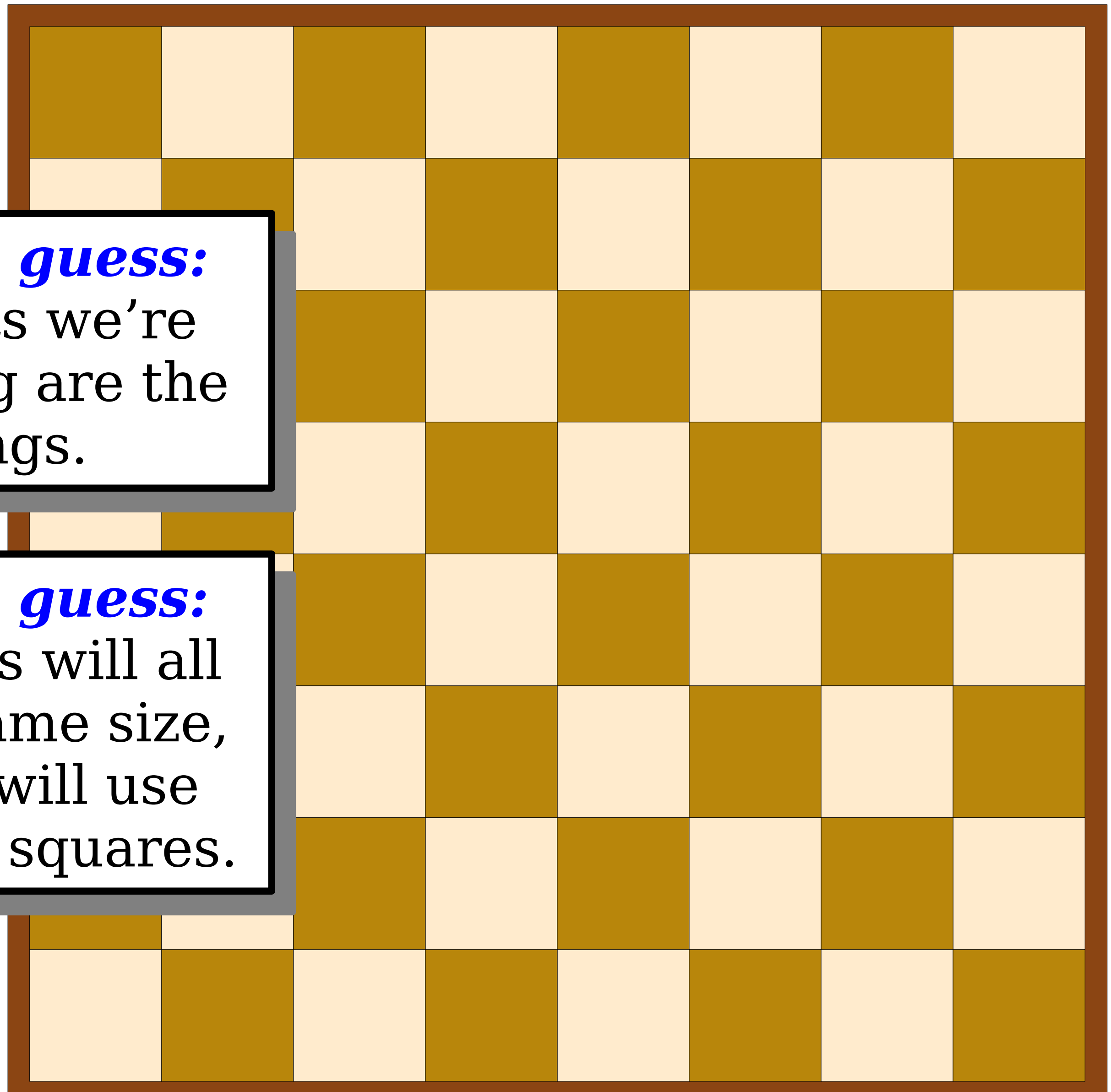

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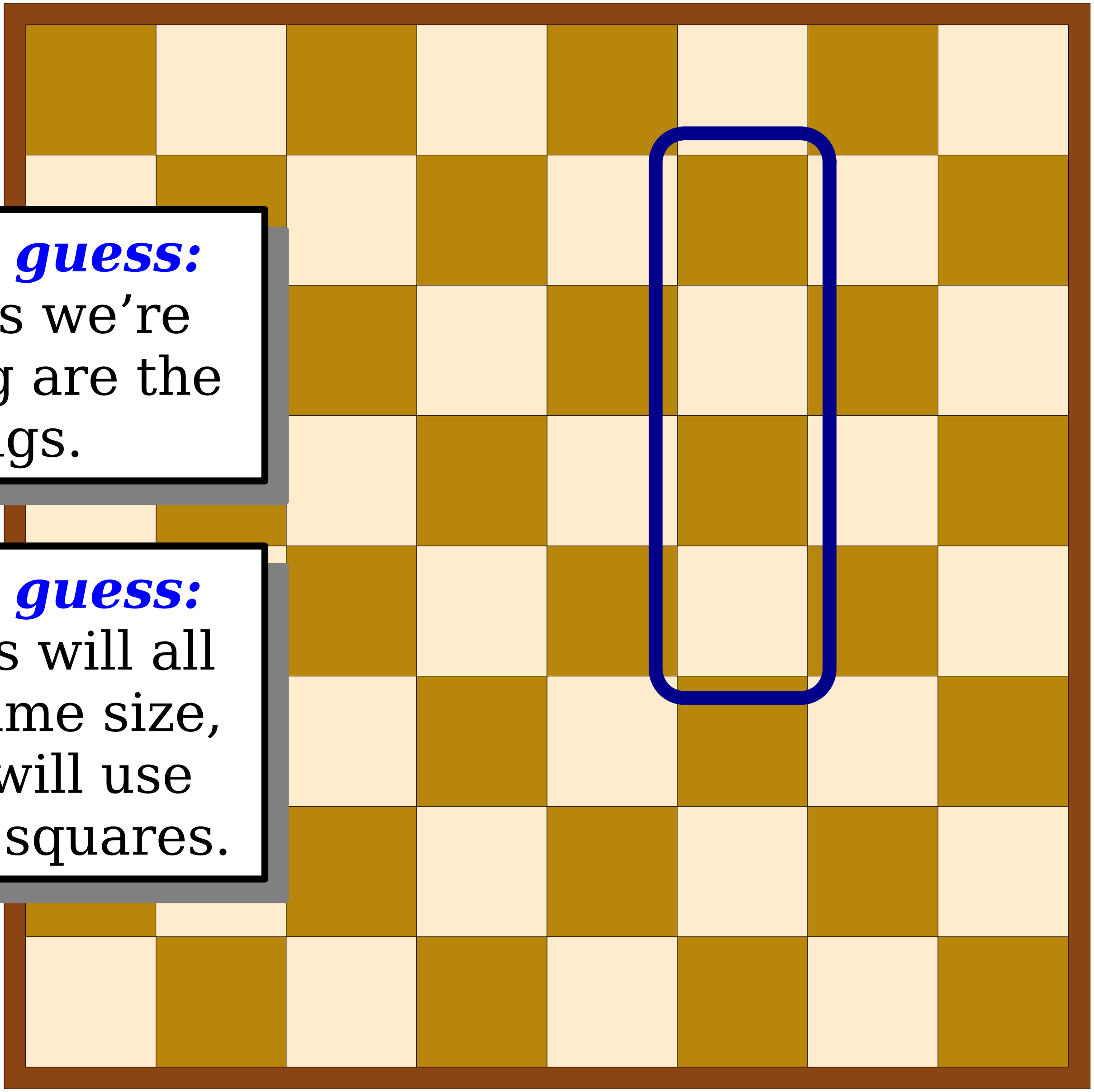

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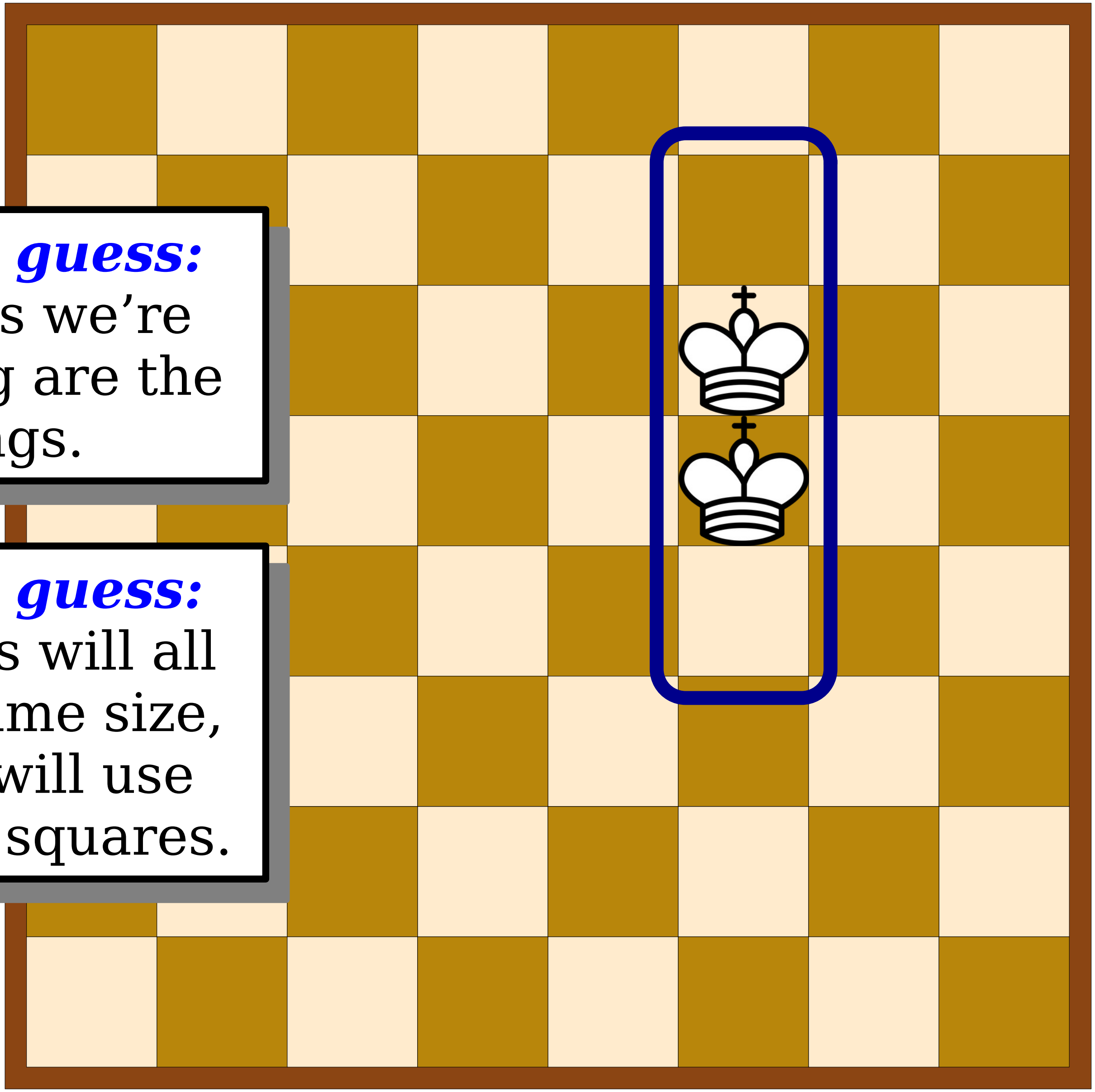

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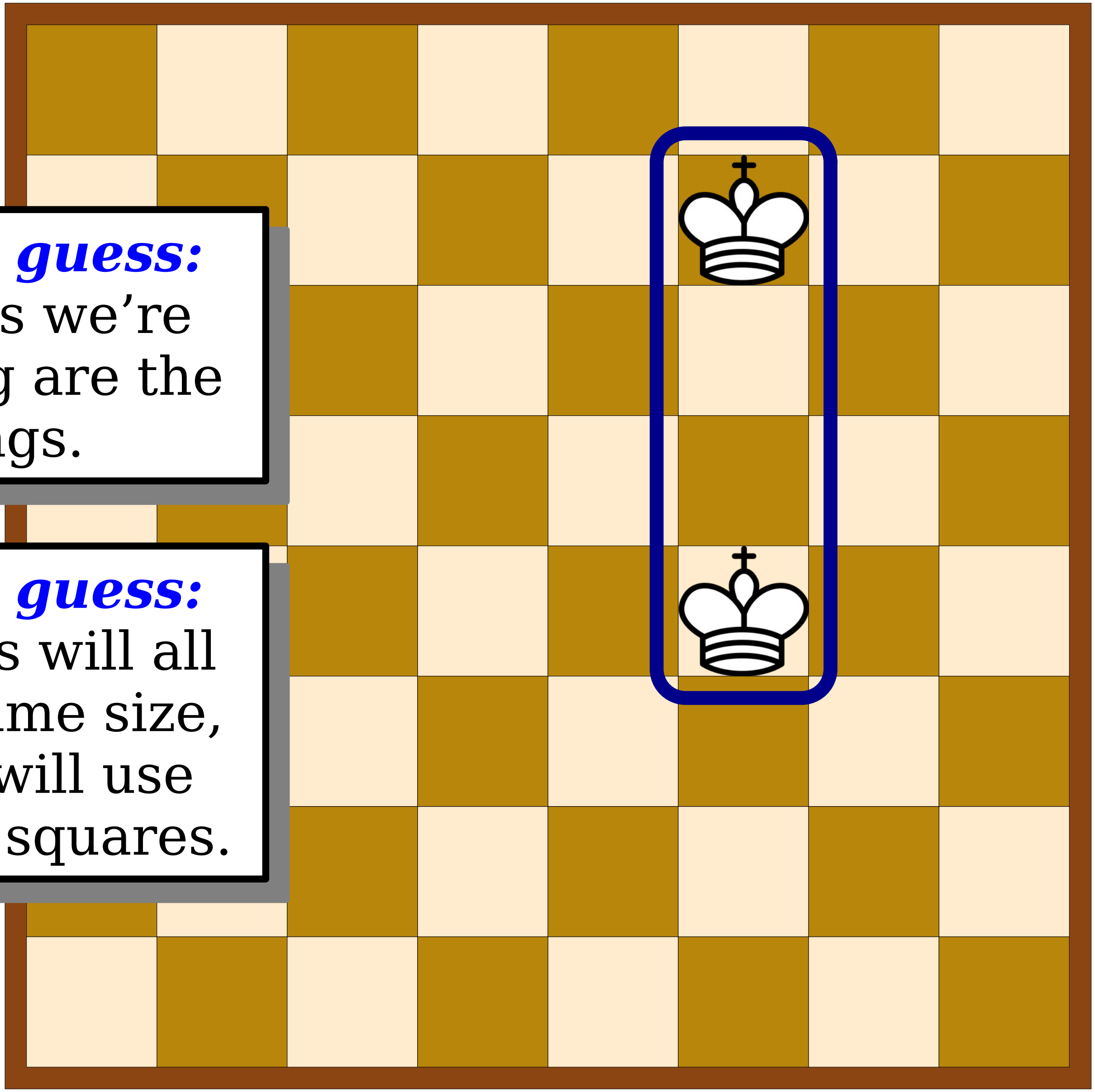

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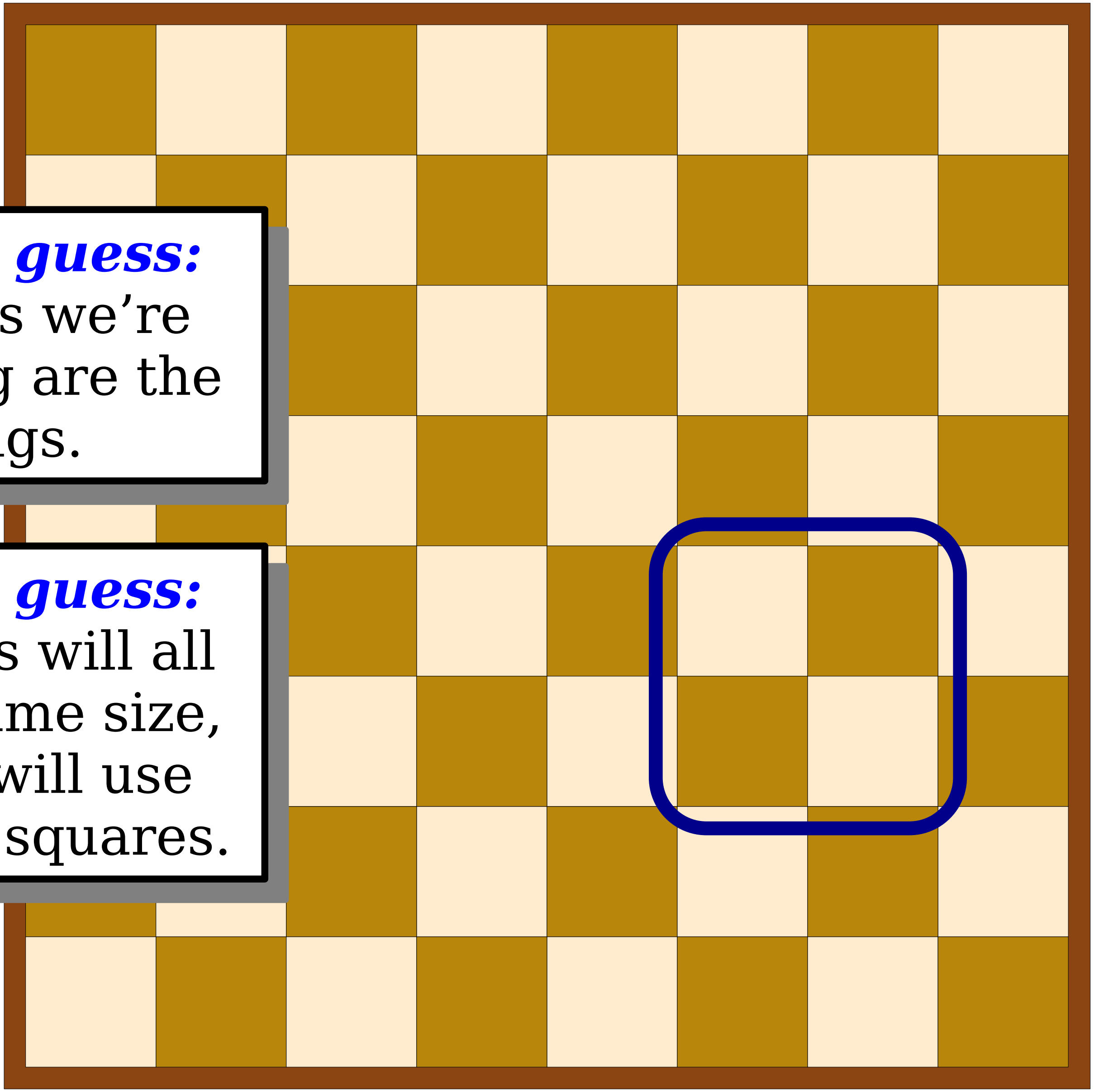

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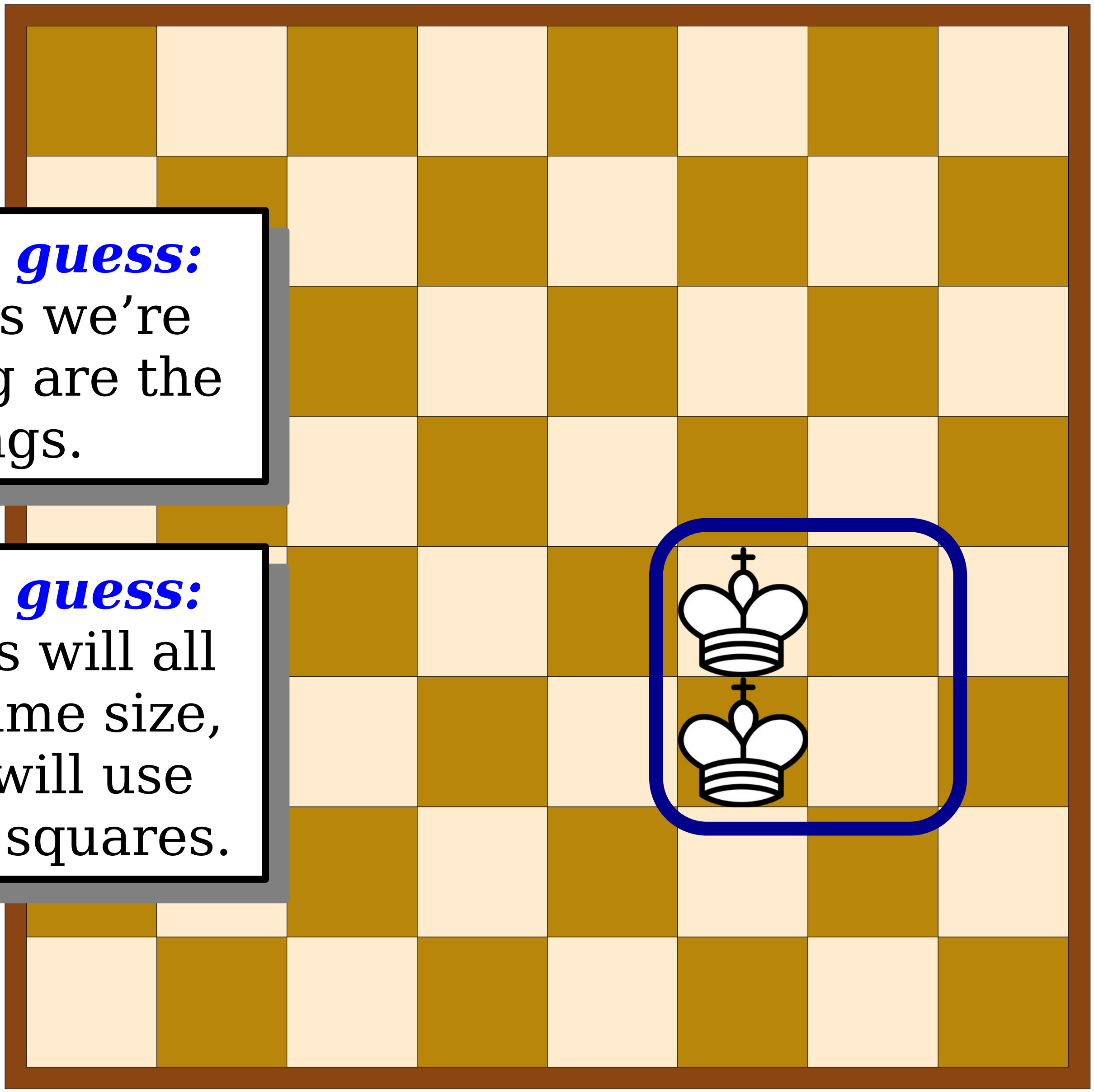

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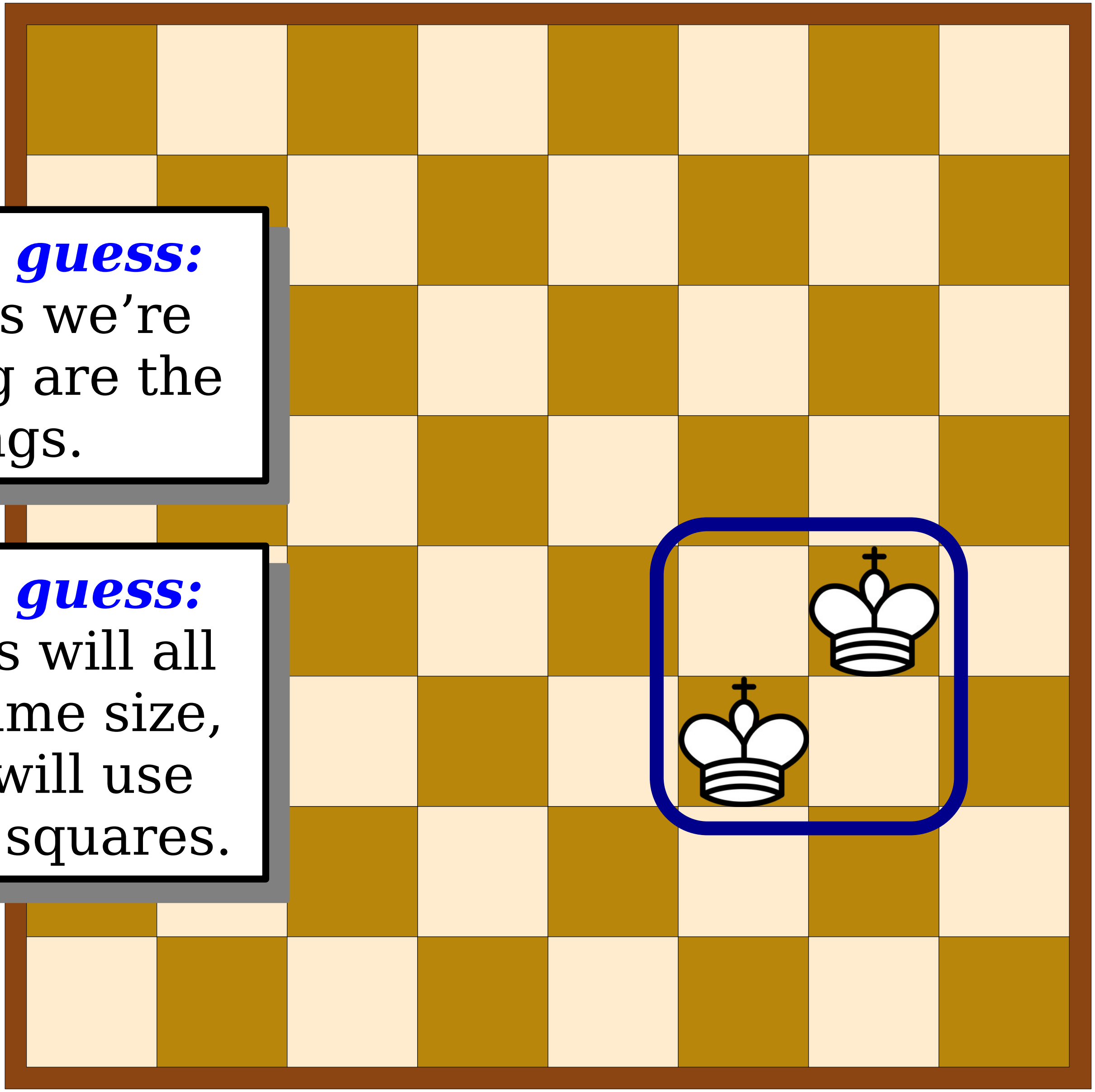

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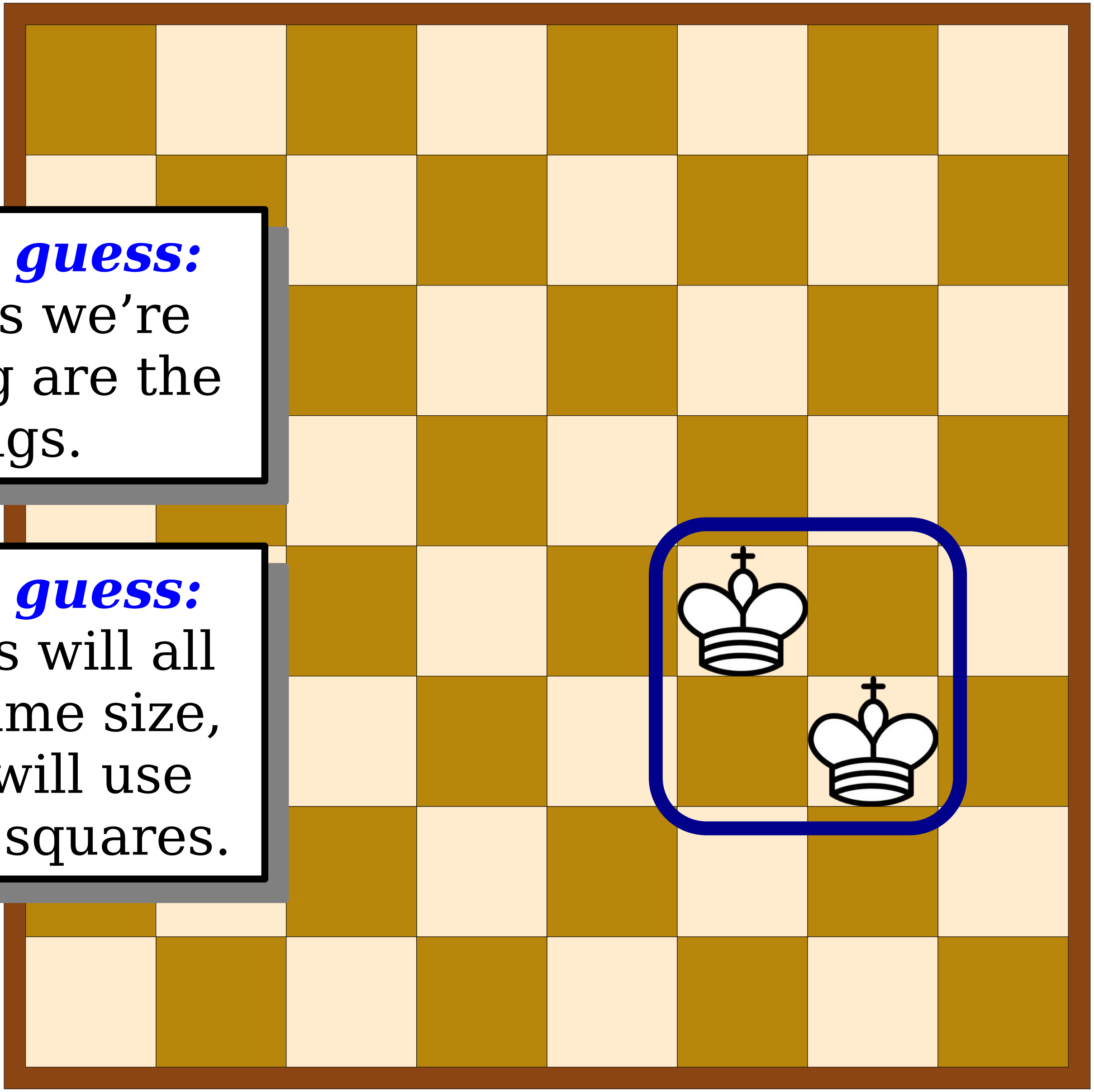

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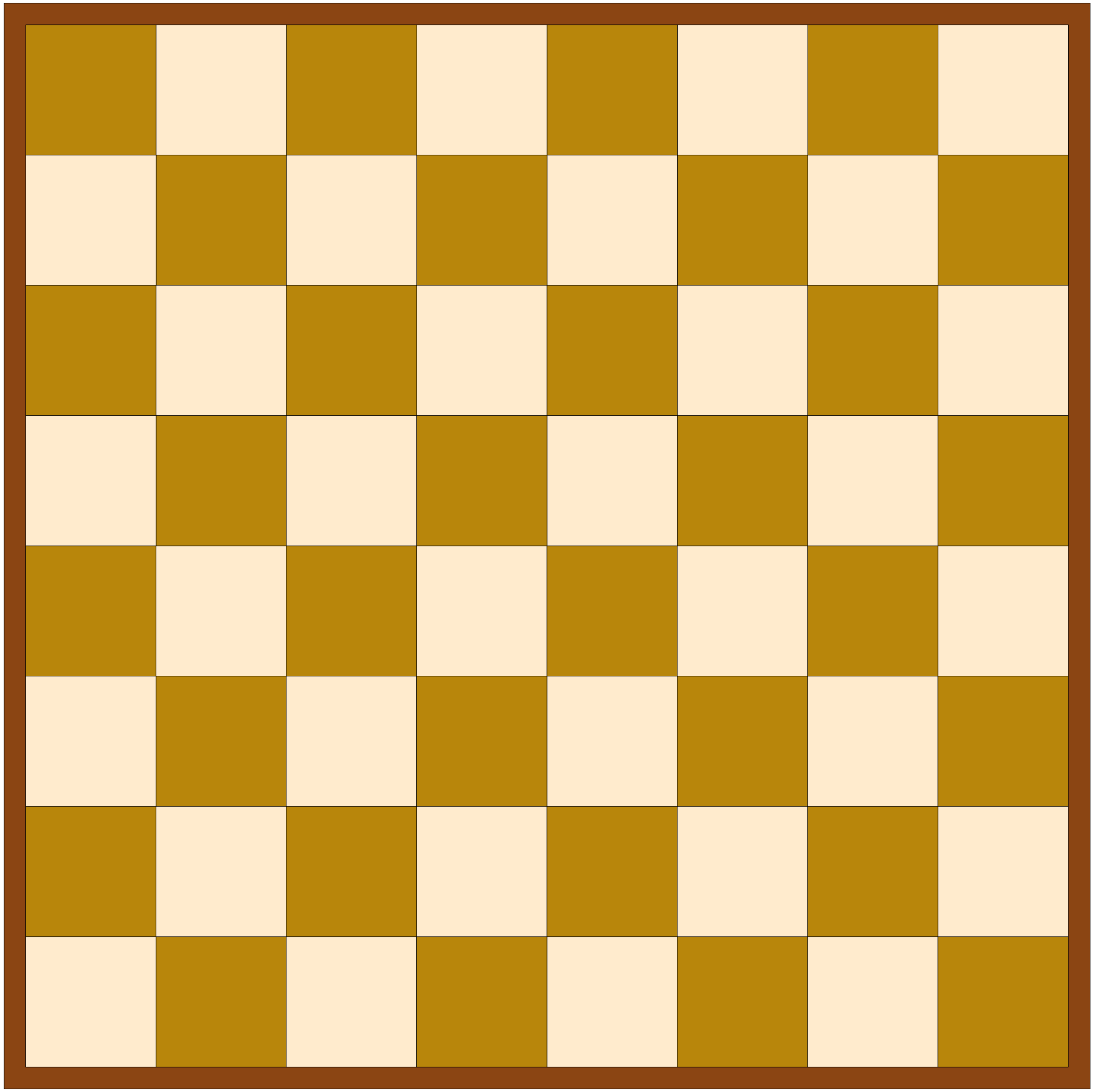

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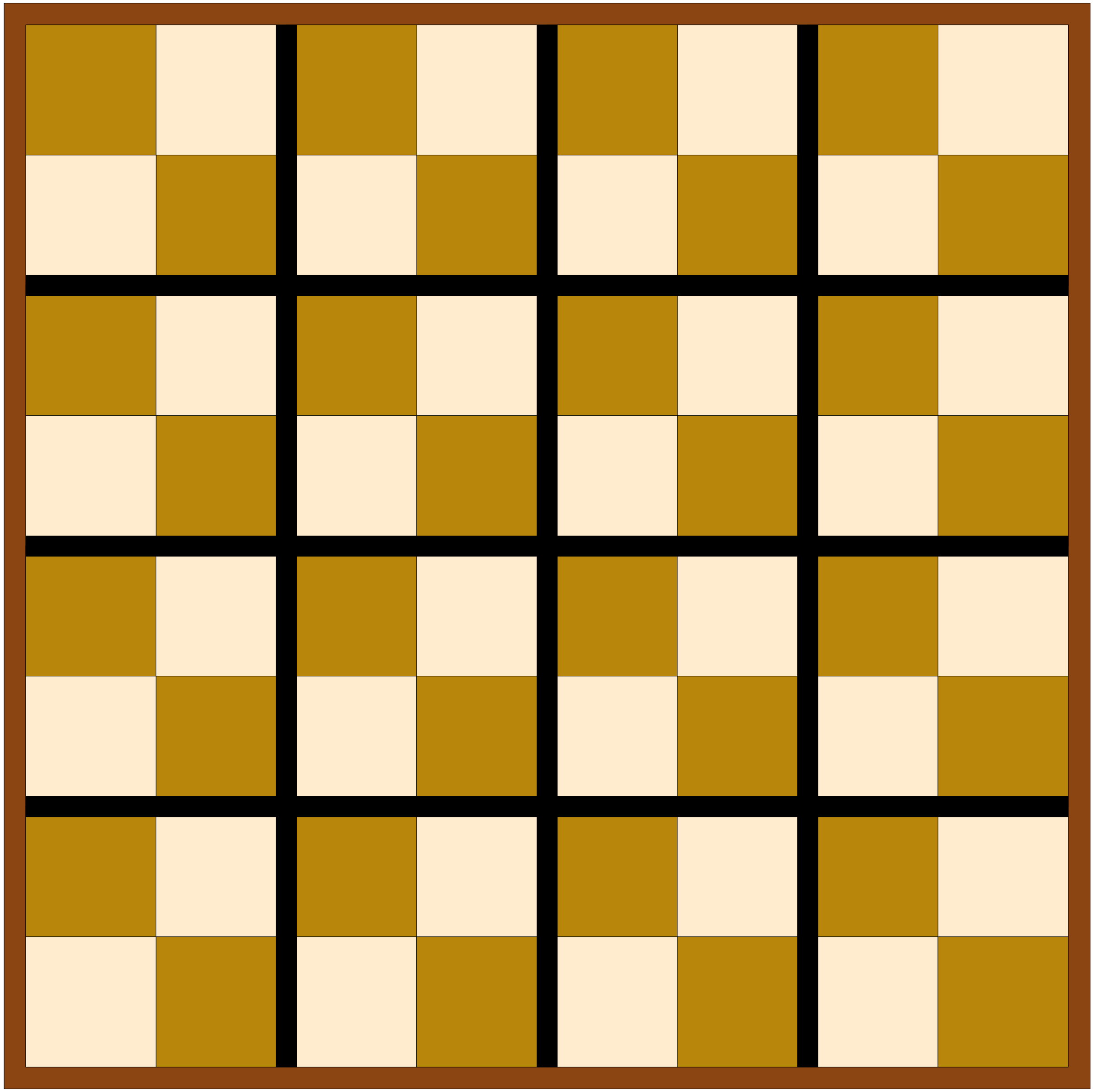

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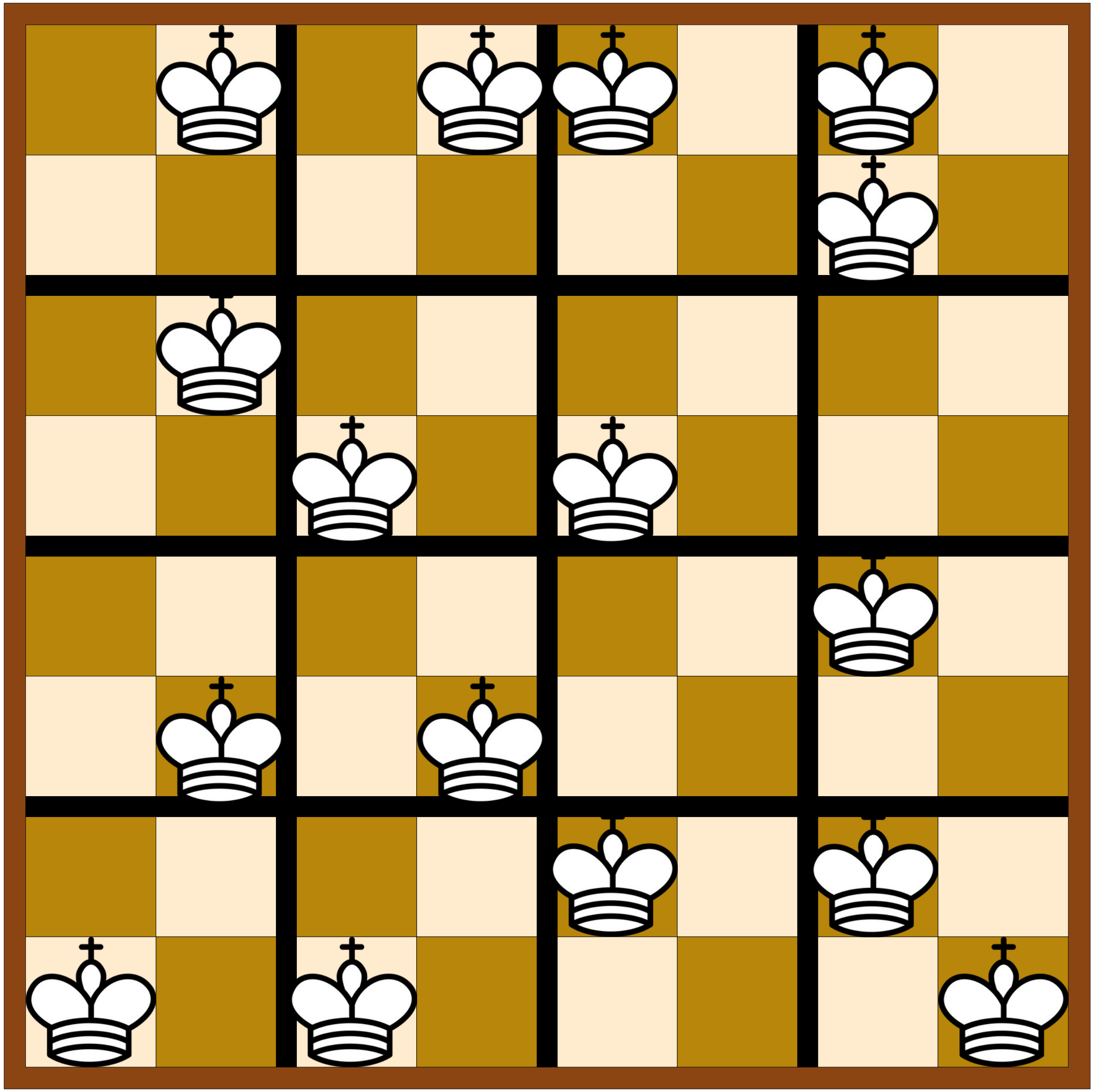

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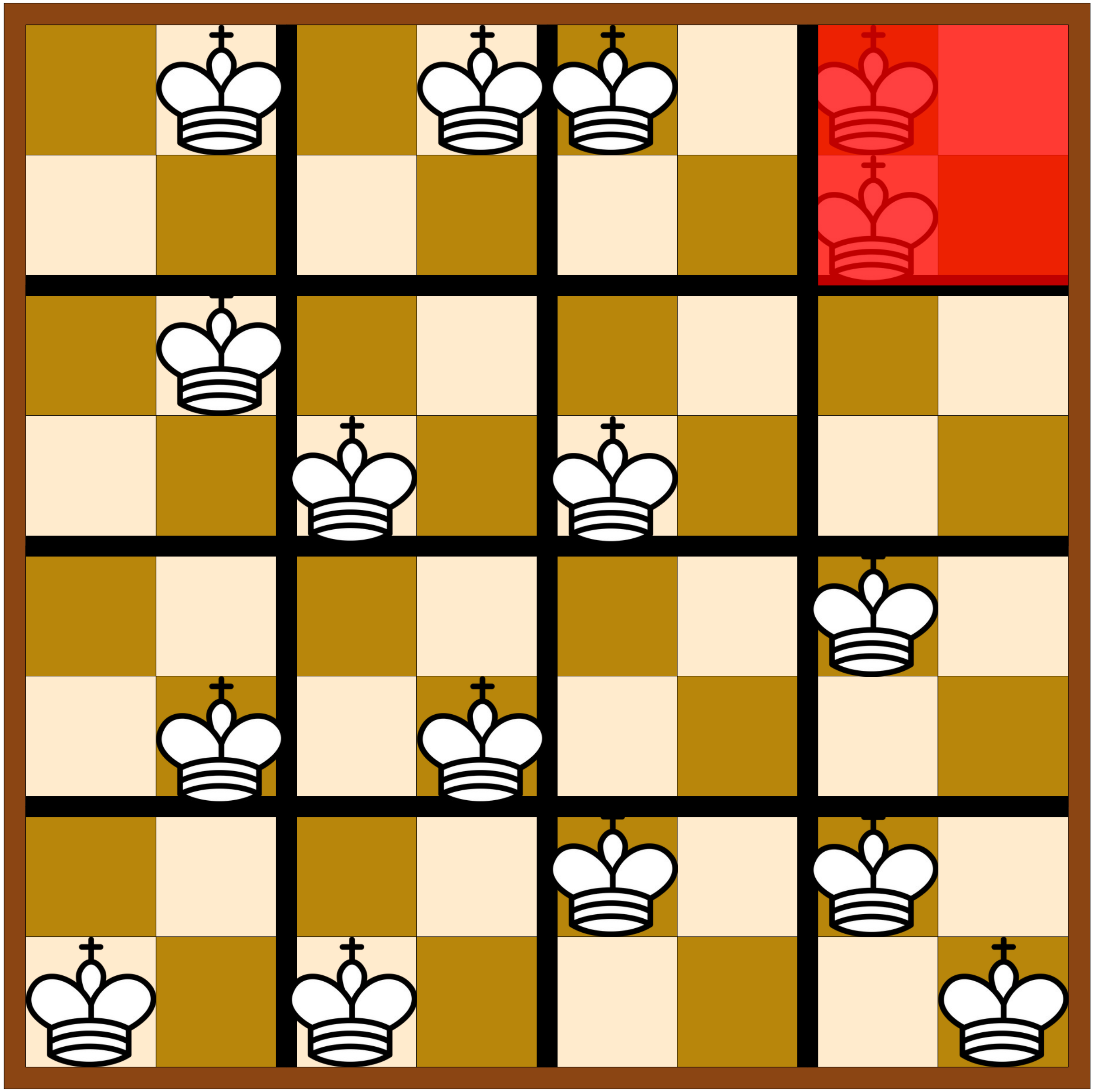
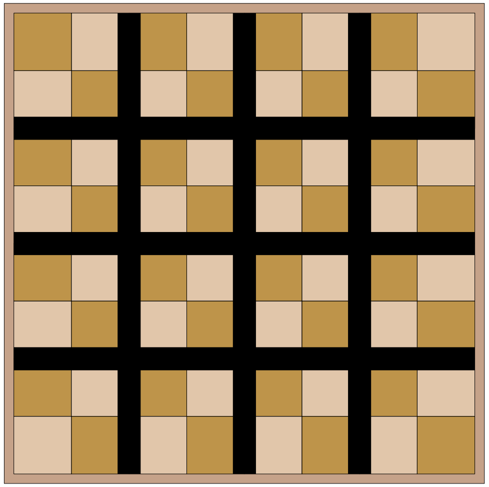


Theorem: In any arrangement of seventeen or more kings on an 8×8 chessboard, at least two of the kings attack one another.

Proof: Consider any such arrangement of kings. Subdivide the 8×8 chessboard into sixteen 2×2 square blocks, as shown here:



Since there are 17 kings and 16 blocks, by the pigeonhole principle there must be some block that contains at least two kings. These kings are then either adjacent horizontally, adjacent vertically, or adjacent diagonally. Therefore, they attack one another, as required. ■

Part Two: Graph Theory Warmup

Consider the following first-order logic statements about some graph $G = (V, E)$:

a) $\forall u \in V.\ \forall v \in V.\ \{u, v\} \notin E$

b) $\exists u \in V.\ \forall v \in V.\ (u \neq v \rightarrow \{u, v\} \in E)$

c) $\forall u \in V.\ \exists v \in V.\ (\{u, v\} \in E\ \land$
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2. Translate these FOL statements into English. Please use graph theory terms (adjacent, connected, degree, etc.) where appropriate.

Fill in answer on Gradescope!

$$\forall u \in V.\ \forall v \in V.\ \{u, v\} \notin E$$

(a)

$$\exists u \in V.\ \forall v \in V.\ (u \neq v \to \{u, v\} \in E)$$

(b)

$$
\begin{aligned}
&\forall u \in V.\ \exists v \in V.\ (\{u, v\} \in E\ \wedge \\
&\quad \forall w \in V.\ (w \neq v \to \{u, w\} \notin E) \\
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\end{aligned}
$$

(c)

Part Three: Graph Theory

Problem Set Four introduces independent sets in graphs. An ***independent set*** in a graph $G = (V, E)$ is a set $I \subseteq V$ where

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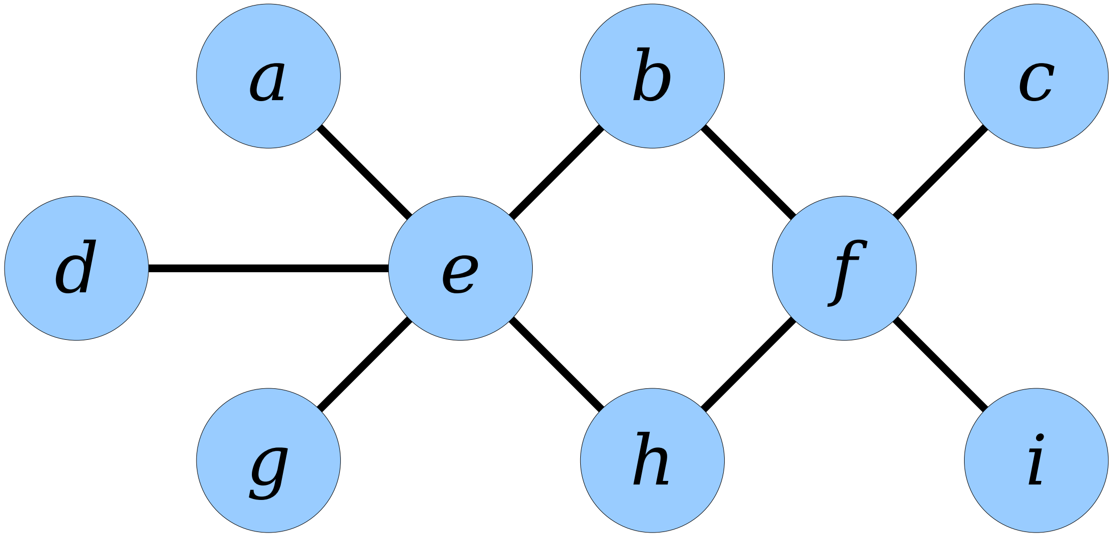

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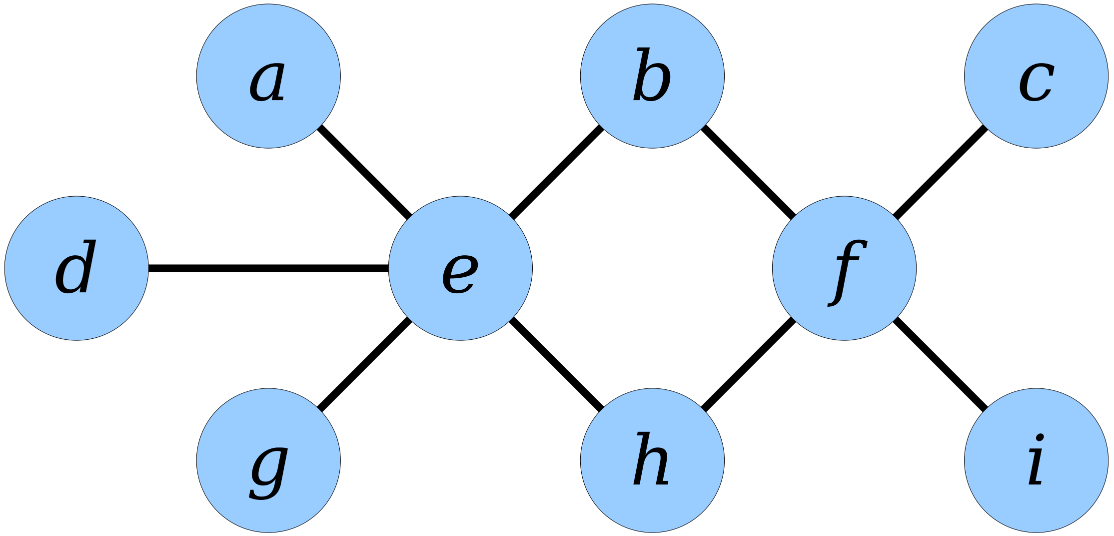
Now, a new definition: a ***vertex cover*** of a graph $G = (V, E)$ is a set $C \subseteq V$ where

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \quad \to \quad u \in C \lor v \in C).$$



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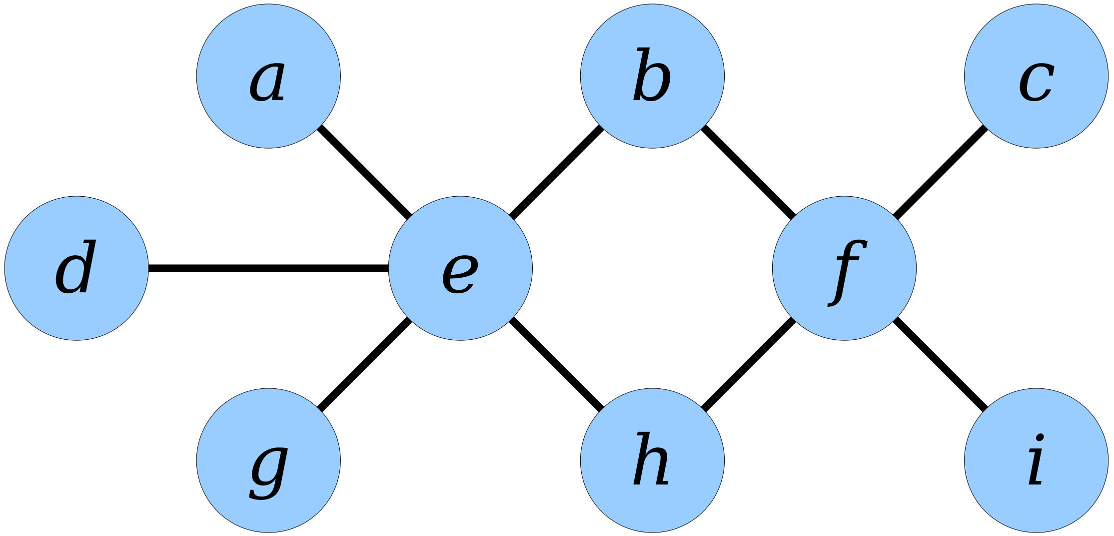

3a. What is the largest vertex cover of this graph?
3b. What is the smallest vertex cover of this graph?

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4. Fill in the details of the "Assume" and "Want to Show" sections. If you introduce new variables, indicate whether they're arbitrarily-chosen or whether they're unknowns to be solved for.

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Assume:	***Want to Show:***
Pick arbitrary graph $G = (V, E)$ Pick arbitrary independent set $I \subseteq V$. There are no edges between two nodes in I.	$V - I$ is a vertex cover of G.

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Now, a new definition: a ***vertex cover*** of a graph $G = (V, E)$ is a set $C \subseteq V$ where

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \quad \to \quad u \in C \lor v \in C).$$

Theorem: if I is an independent set of $G = (V, E)$, then $V - I$ is a vertex cover of G.

Assume:	***Want to Show:***
Pick arbitrary graph $G = (V, E)$	$V - I$ is a vertex cover of G.
Pick arbitrary independent set $I \subseteq V$. There are no edges between two nodes in I.	***Assume:***
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$V - I$ is a vertex cover of G.

Assume:

Pick arbitrary $u, v \in V$
Assume $\{u, v\} \in E$.

Want to Show:

$u \in V - I$ or $v \in V - I$.

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5. Explain why $u \notin I$ or $v \notin I$. (No formal proof is needed – just give the intuition.)

Fill in answer on Gradescope!

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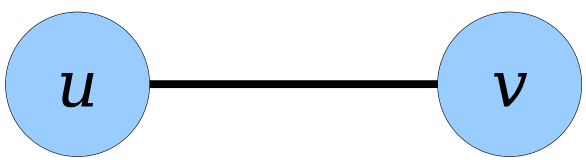

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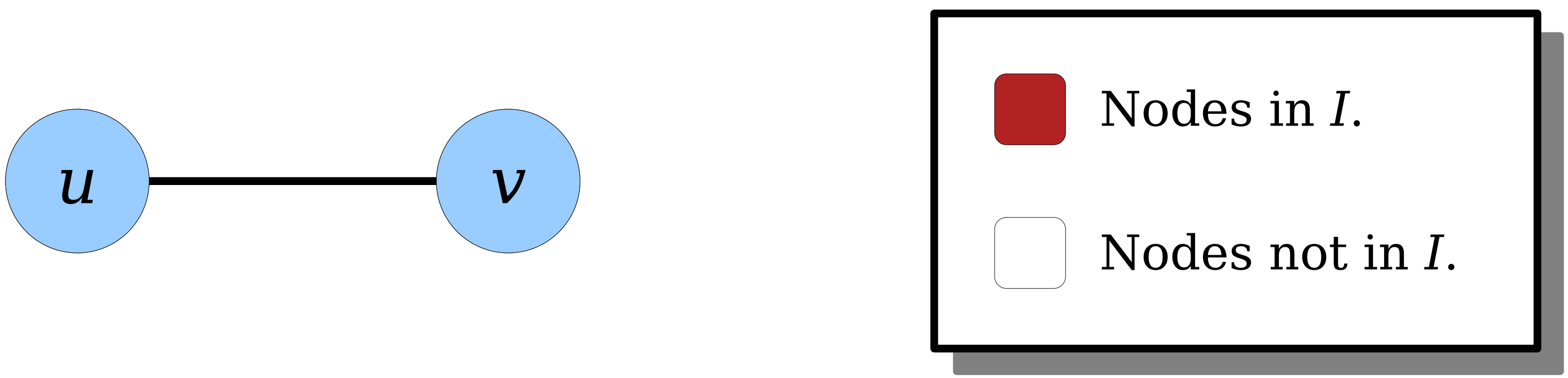

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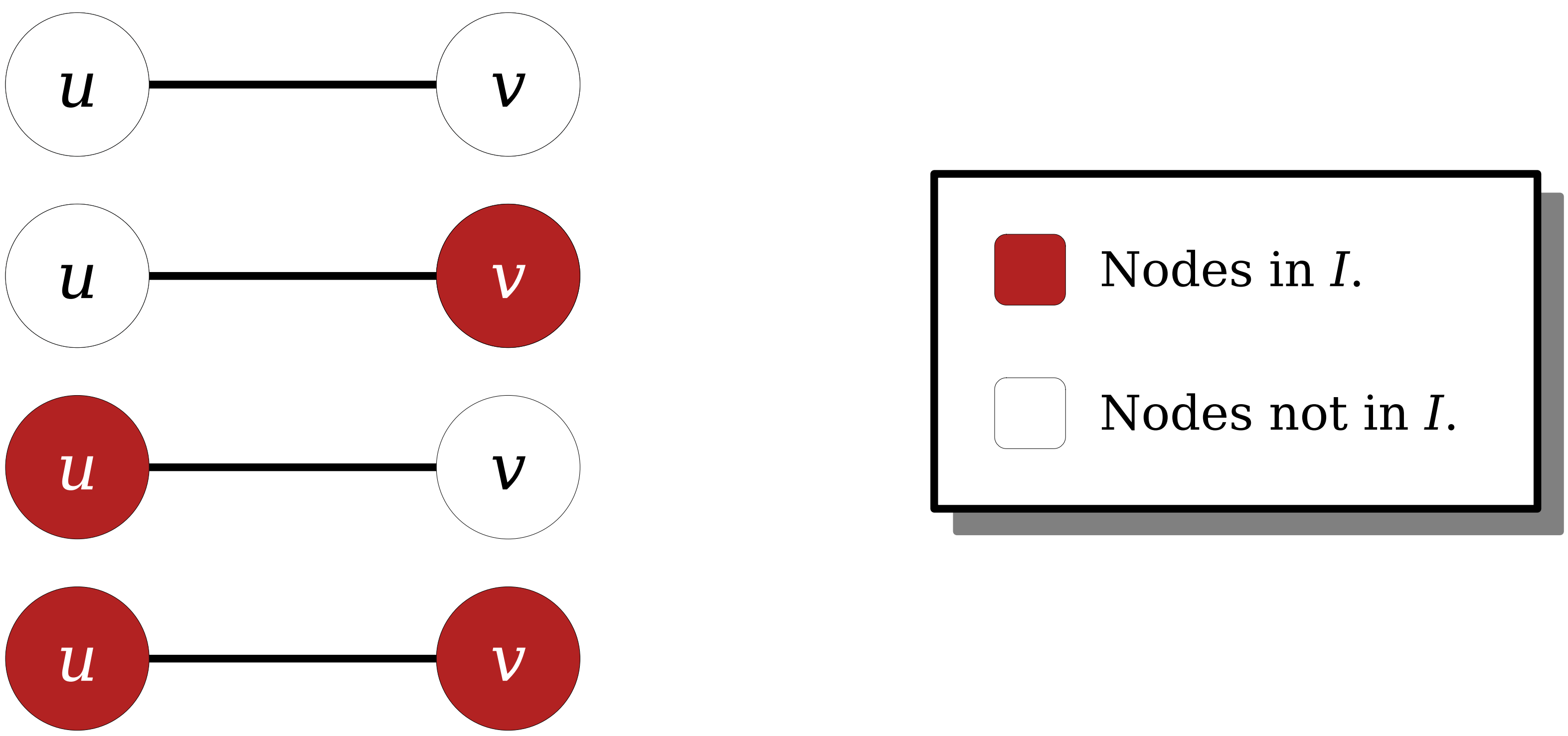

Theorem: If I is an independent set of a graph $G = (V, E)$, then $V - I$ is a vertex cover of G.

Proof: Let $G = (V, E)$ be a graph and $I \subseteq V$ be an independent set of G. We want to show that $V - I$ is a vertex cover of G. To do so, pick any $u, v \in V$ such that $\{u, v\} \in E$. We will show that $u \in V - I$ or that $v \in V - I$.

Suppose for the sake of contradiction that $u \notin V - I$ and that $v \notin V - I$. We already know that $u \in V$ and that $v \in V$, so from $u \notin V - I$ and $v \notin V - I$ we learn that $u \in I$ and $v \in I$. That in turn tells us that $\{u, v\} \notin E$, since by assumption I is an independent set. This contradicts the fact that $\{u, v\} \in E$.

We have reached a contradiction, so our assumption must have been wrong. Therefore, we see that $u \in V - I$ or that $v \in V - I$, which is what we needed to prove. ■

Thanks for Calling In!

Stay safe, stay healthy,
and have a good week!

See you next time.